

On the Boussinesq equations with non-monotone temperature profiles

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ON THE BOUSSINESQ EQUATIONS WITH NON-MONOTONE TEMPERATURE PROFILES

CHRISTIAN ZILLINGER

ABSTRACT. In this article we consider the asymptotic stability of the two-dimensional Boussinesq equations with partial dissipation near a combination of Couette flow and temperature profiles $T(y)$. As a first main result we show that if T' is of size at most $\nu^{1/3}$ in a suitable norm, then the linearized Boussinesq equations with only vertical dissipation of the velocity but not of the temperature are stable. Thus, mixing enhanced dissipation can suppress Rayleigh-Bénard instability in this linearized case.

We further show that these results extend to the (forced) nonlinear equations with vertical dissipation in both temperature and velocity.

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1. INTRODUCTION

The Boussinesq equations are a standard approximate model of heat transfer in (viscous) fluids and are given by a coupled system of the Navier-Stokes equations and a dissipative transport equation for the temperature density:

$$(1) \quad \begin{aligned} \partial_t v + v \cdot \nabla v + \nabla p &= (\nu_x \partial_x^2 + \nu_y \partial_y^2)v + \theta e_2, \\ \partial_t \theta + v \cdot \nabla \theta &= (\mu_x \partial_x^2 + \mu_y \partial_y^2)\theta, \\ \nabla \cdot v &= 0. \end{aligned}$$

Here $v \in \mathbb{R}^2$ denotes the velocity, $p \in \mathbb{R}$ is the pressure, $\theta \in \mathbb{R}$ is the temperature and we consider the domain $\mathbb{T} \times \mathbb{R} \ni (x, y)$. The θe_2 term models buoyancy which causes hotter fluid to rise and colder fluid to sink.

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In Sections 2 and 3 of this article we consider the setting with only vertical dissipation of the velocity,

$$\nu_x = \mu_x = \mu_y = 0, \nu_y =: \nu > 0.$$

We refer to this setting as vertical dissipation. In Section 4 we assume vertical dissipation in both velocity and temperature, which we refer to as full vertical dissipation.

One readily observes that at least formally any pair of functions of the form

$$(2) \quad v = (\beta y, 0), \theta = T(y),$$

with $\beta \in \mathbb{R}$ and T smooth are automatically stationary solutions of the vertical dissipation problem (choosing $p = p(y)$ suitably). Here a particular focus in existing results has been on the case when T is affine and increasing, that is hotter fluid is on top of colder fluid, which is known as hydrostatic balance. A main aim of this article is to study more general profiles $T(y)$ and in particular answer how much T may oscillate if both shear and viscosity are available to counteract thermal instability.

More generally, the problem of partial dissipation has been an area of extensive research, where we in particular mention the recent works [EW15, Wid18, DWZZ18, YL18] and [WXZ19]. The question of global wellposedness has been addressed in series of works by Chae, Nam and Kim [CKN99, Cha06].

In this article, we will focus on questions of asymptotic stability close to specific families of solutions and how the interaction of mixing and temperature stratification may counteract instability.

In [YL18] Yang and Lin studied the stability of the linearized *inviscid* problem around the case where $T(y) = \alpha y$ is affine and showed that for some stability results it is necessary that $\alpha > 0$ and thus T is increasing. We recall these results in Section 2 and emphasize that the threshold with respect to α depends on whether one studies

- the vorticity ω , which is always unstable,
- the horizontal component of the velocity v_1 , which is stable if $\alpha > 0$ and unstable if $\alpha < 0$, or
- the vertical component of the velocity v_2 , which is stable if $\alpha > -2$ and unstable if $\alpha < -2$.

Thus, already in this case in a specific sense one may allow α to be *negative* if it is sufficiently small.

Recently, Masmoudi, Said-Houari and Zhao [MSHZ20] showed that the associated *nonlinear* problem near $T(y) = \alpha y$, $\alpha > 0$ without thermal diffusion but with viscous diffusion is asymptotically stable in Gevrey regularity. These results in particular show that this partial dissipation problem behaves similarly to the Euler equations [BM15] instead of the Navier-Stokes equations [BVW18]. If one instead considers full dissipation, in [Zil20b] we adapted the methods of [BVW18, Lis20] to establish nonlinear stability in Sobolev regularity.

This article extends the results of [Zil20b] to the case of *negative* α and partial dissipation. More precisely, we show for the linearized problem with vertical dissipation that the evolution is asymptotically stable provided

$$(3) \quad \alpha > -\frac{1}{100} \sqrt[3]{\nu}.$$

Similar results hold for $T(y)$ non-affine. Thus, mixing enhanced dissipation can suppress Rayleigh-Bénard instability with an enhanced dependence on ν .

For the nonlinear problem we further show that for affine T and full vertical dissipation the same stability results hold. As shown in [MSHZ20] in the case of vertical dissipation only in the vorticity a more careful analysis is required to control resonances, reminiscent of echoes in the Euler equations [DM18, DZ19a].

Our main results concerning the linearized problem are summarized in the following theorem.

Theorem 1.1. *Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a given temperature profile. Let $N \in \mathbb{N}$ and suppose that $T'(y) \in L^\infty$. We then consider linearized Boussinesq equations with vertical dissipation in the velocity only around $v = (y, 0)$, $\theta = T(y)$ in coordinates $(x + ty, y)$:*

$$\begin{aligned}\partial_t \omega &= \nu(\partial_y - t\partial_x)^2 \omega + \partial_x \theta, \\ \partial_t \partial_x \theta &= T'(y) \partial_x v_2, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}\end{aligned}$$

Then if the Fourier transform of T' satisfies the estimate

$$(4) \quad \sup_{\xi} \int |\mathcal{F}(T')(z - \xi)| \left(\frac{1 + |z|}{1 + |\xi|} + \frac{1 + |\xi|}{1 + |z|} \right)^N (1 + \min(\nu^{-\frac{2}{3}}, |z - \xi|^{\frac{2}{3}})) dz < \frac{1}{100} \nu^{1/3},$$

the initial value problem is stable in $H^N \times H^N$ in the sense that there exists a constant $C > 0$ such that for any initial data $(\omega_{in}, \partial_x \theta_{in}) \in H^N \times H^N$ the solution satisfies

$$\|\omega(t)\|_{H^N} + \nu \|\partial_x \theta\|_{H^N} \leq C \nu^{-\frac{2}{3}} (\|\omega_{in}\|_{H^N} + \nu \|\partial_x \theta_{in}\|_{H^N}).$$

In particular, if $T'(y) = \alpha y$, stability holds if $\alpha > -\frac{1}{100} \nu^{1/3}$.

The condition (4) is a sufficient condition to control commutators involving $T'(y)$ and is probably not optimal in its dependence on N . In the case where $T(y) = \alpha y$ is affine, it reduces to the condition $|\alpha| < C_N \nu^{1/3}$ and thus allows for α to be negative. See Theorem 3.2 for further discussion.

For the nonlinear problem with full vertical dissipation we obtain similar results.

Theorem 1.2. *Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a given temperature profile and consider the (forced) nonlinear problem around $v = (y, 0)$, $\theta = T(y)$ with vertical dissipation $\nu_y = \mu_y =: \nu > 0$ in coordinates $(x + ty, y)$:*

$$\begin{aligned}\partial_t \omega + v \cdot \nabla_t \omega &= \nu(\partial_y - t\partial_x)^2 \omega + \partial_x \theta, \\ \partial_t \theta + v \cdot \nabla_t \theta &= \nu(\partial_y - t\partial_x)^2 \theta + T'(y) v_2 \omega, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R},\end{aligned}$$

and suppose that T' satisfies the assumptions of Theorem 1.1. Then this problem is stable in Sobolev regularity. More precisely, for any $N \in \mathbb{N}$, $N \geq 5$ there exists $\epsilon_N = \epsilon_N(\nu)$ such that if initially

$$\|\omega\|_{H^N}^2 + \nu^{-1} \|\partial_x \theta\|_{H^N}^2 < \epsilon^2 < \epsilon_N^2,$$

then the solution remains bounded by $10\nu^{-2/3}\epsilon^2$ for all times.

We also obtain time integrability results for v , $(\partial_y - t\partial_x)\omega$ and $(\partial_y - t\partial_x)\theta$, which are stated in Sections 3 and 4 and omitted here for brevity.

- In the special case when $T(y) = \alpha y$ is affine the assumption reduces to $|\alpha| \leq \frac{1}{100}\nu^{1/3}$.
- We stress that α here is allowed to be *negative*. As a related result in Lemma 2.2 we remark that the *inviscid* results of [YL18] extend to $0 \geq \alpha > -2$ when considering the vertical component of the velocity v_2 .
- If there is no shear, then partial dissipation is not sufficient to restore stability of the vorticity for $\alpha < 0$ (see Lemma 2.1).
- A combination of shear and vertical dissipation suffices to restore stability of the vorticity. Moreover, in that case we obtain an enhanced threshold in terms of $-\nu^{1/3}$.
- These results further extend to the case of a non-affine, oscillating temperature profile $T(y)$. In particular, we do not rely on cancellations or conserved quantities available in the hydrostatic balance case.
- In addition to the linearized Boussinesq equations, we obtain results for the nonlinear small data problem, however, only with *full vertical* dissipation (considering $T(y)$ non-affine as a solution of the forced problem). As recently shown in [MSHZ20] this stronger assumption is probably necessary for stability in Sobolev regularity, since otherwise resonance chains may yield norm inflation.

The remainder of the article is structured as follows:

- In Section 2 we recall some results for the *inviscid* problem, first obtained in [YL18], to introduce instability mechanisms and to discuss in which sense (partial) dissipation is necessary for stability results. With these motivations we formulate four main questions Q1-Q4, which we address throughout the article.
- In Section 3.1 we begin by studying the special case when $T(y)$ is affine, where arguments are more transparent. In particular, we show that here the slope of the temperature profile can be allowed to be *negative* (colder fluid on top of hotter fluid) and that the size of the threshold depends on ν with an enhanced rate.
- In Section 3.2 we extend these linear results to the case of a general temperature profile $T(y)$ satisfying suitable smallness conditions. In particular, T is allowed to oscillate.
- Building on the linearized results, in Section 4 we study the nonlinear small data problem. Due to possible resonance chains, we here instead consider full vertical dissipation and consider $T(y)$ as a solution of the forced problem. This extends previous nonlinear results in [Zil20b] for the affine, increasing case to possibly oscillating profiles.

1.1. Notation. Throughout this article we consider solutions of the Boussinesq equations with vertical dissipation near the stationary solution

$$v = \begin{pmatrix} y \\ 0 \end{pmatrix}, \theta = T(y),$$

$$(t, x, y) \in (0, \infty) \times \mathbb{T} \times \mathbb{R}.$$

In this setting it is natural to work in coordinates moving with the flow

$$(x + ty, y)$$

and consider the equations satisfied by the perturbations in these coordinates.

If there is no possibility of confusion these perturbations are again denoted as ω and θ and the linearized problem studied in Section 3 is given by

$$\begin{aligned} \partial_t \omega &= \nu(\partial_y - t\partial_x)^2 \omega + \partial_x \theta, \\ \partial_t \theta &= T'(y)v_2, \\ v &= \nabla_t^\perp \Delta_t^{-1} \omega, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}. \end{aligned}$$

where

$$\nabla_t = \begin{pmatrix} \partial_x \\ \partial_y - t\partial_x \end{pmatrix}, \Delta_t = \partial_x^2 + (\partial_y - t\partial_x)^2$$

are the gradient and Laplacian in these coordinates.

In the nonlinear problem considered in Section 4 we additionally assume vertical dissipation also in the temperature and interpret $T(y)$ as a solution of the forced problem. The system satisfied by the perturbation in coordinates moving with the shear is then given by

$$\begin{aligned} \partial_t \omega &= \nu(\partial_y - t\partial_x)^2 \omega + \partial_x \theta - v \cdot \nabla_t \omega, \\ \partial_t \theta &= T'(y)v_2 - v \cdot \nabla_t \theta, \\ v &= \nabla_t^\perp \Delta_t^{-1} \omega, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}. \end{aligned}$$

We denote the Fourier transform of a function $u(x, y) \in L^2(\mathbb{T} \times \mathbb{R})$ by $\tilde{u}(k, \xi) \in L^2(\mathbb{Z} \times \mathbb{R})$ or $\mathcal{F}u$. Furthermore, we study several Fourier multipliers (see (33)), including

$$\begin{aligned} A(T, \xi, k) &= \exp\left(-2 \int_0^T \frac{1}{1 + (\frac{\xi}{k} - t)^2} dt\right), \\ B(T, k, \xi) &= \exp\left(-2 \int_0^T \frac{1}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} 1_I(t) dt\right), \end{aligned}$$

with I being a prescribed time interval/Fourier region (see (19)):

$$I = \{t \geq 0 : |\frac{\xi}{k} - t| \leq C\},$$

and C proportional to $\nu^{-1/3}$.

In Section 4 we study energy estimates on a given time interval $(0, T)$. Since T is fixed throughout this section, we omit it from our notation and for instance write

$$\|u\|_{L^p H^N} := \| \|u(t, \cdot)\|_{H^N} \|_{L^p((0, T))}.$$

We write $a \lesssim b$ if there exists a universal constant $C > 0$ such that $|a| \leq C|b|$.

2. MODEL CASES OF INSTABILITY

In order to introduce ideas and mechanisms, in this section we recall results available in the literature for

- The linearized viscous problem without shear around $\theta = \alpha y$, $v = 0$ and
- The linearized inviscid problem with shear around $\theta = \alpha y$, $v = (y, 0)$

Here, for simplicity we consider viscous dissipation in both horizontal and vertical direction, but no thermal dissipation. As a reference for the isolated mechanisms the interested reader is referred to the textbook by Frisch and Yaglom [Yag12, Section 2.8.3]. We emphasize that the results of this section are not new, but serve to motivate our questions Q1–Q4 stated at the end of this section, which we address in this article. Furthermore, they show that under weaker assumptions instabilities may form and that the conditions in Theorem 1.1 are in this sense optimal.

In the case without shear, explicit solutions are available and it is known that the slope of θ yields a sharp dichotomy between stability and exponential instability. The following basic lemma is reproduced from [Zil20b, Proposition 2.6].

Lemma 2.1. *Consider the Boussinesq equations in vorticity formulation linearized around*

$$v = (0, 0), \theta = \alpha y,$$

where $\alpha \in \mathbb{R}$:

$$(5) \quad \begin{aligned} \partial_t \omega &= \nu \Delta \omega + \partial_x \theta, \\ \partial_t \theta + \alpha v_2 &= \mu \Delta \theta. \end{aligned}$$

Here v_2 denotes the vertical component of the velocity field. Further suppose that at least one of ν or μ is zero. The evolution is stable if $\alpha > 0$ in the sense that for every $N \in \mathbb{N}$ the energy

$$(6) \quad \alpha \|\omega\|_{H^N}^2 + \|\nabla \theta\|_{H^N}^2$$

is decreasing. In contrast, if $\alpha < 0$, there exist solutions which grow exponentially in time.

As we show in Lemma 2.2 when adding shear the instability for $\alpha < 0$ is significantly reduced and the evolution of v_2 is even asymptotically stable if α is not too large.

Proof. In the interest of accessibility, we reproduce the main steps of the proof from [Zil20b].

We observe that equation (5) is a constant coefficient PDE and hence we obtain a decoupled system of ODEs for each Fourier mode with respect to x and y :

$$(7) \quad \partial_t \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} -\nu(k^2 + \xi^2) & ik \\ \frac{ik\alpha}{k^2 + \xi^2} & -\eta(k^2 + \xi^2) \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix},$$

where we use $\tilde{\omega}$ to denote the Fourier transform of the vorticity and $k \in \mathbb{Z}, \xi \in \mathbb{R}$ to denote the Fourier variables. In particular, we may study the problem at each frequency.

The case $\alpha > 0$: Let (k, ξ) and $\alpha > 0$ be given. Then we may reformulate the problem as

$$(8) \quad \partial_t \begin{pmatrix} \sqrt{\alpha\tilde{\omega}} \\ \sqrt{k^2 + \xi^2\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} -\nu(k^2 + \xi^2) & \frac{ik\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} \\ \frac{ik\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} & -\eta(k^2 + \xi^2) \end{pmatrix} \begin{pmatrix} \sqrt{\alpha\tilde{\omega}} \\ \sqrt{k^2 + \xi^2\tilde{\theta}} \end{pmatrix}.$$

Note that the off-diagonal entries are equal and purely imaginary. Therefore, if we denote the matrix by M it holds that $M + \overline{M}^T$ is a real-valued, negative semi-definite diagonal matrix. Hence it follows that

$$\frac{d}{dt} \left| \begin{pmatrix} \sqrt{\alpha\tilde{\omega}} \\ \sqrt{k^2 + \xi^2\tilde{\theta}} \end{pmatrix} \right|^2 = \overline{\begin{pmatrix} \sqrt{\alpha\tilde{\omega}} \\ \sqrt{k^2 + \xi^2\tilde{\theta}} \end{pmatrix}} \cdot (M + \overline{M}^T) \begin{pmatrix} \sqrt{\alpha\tilde{\omega}} \\ \sqrt{k^2 + \xi^2\tilde{\theta}} \end{pmatrix} \leq 0.$$

Integrating this estimate with respect to ξ and k (possibly with respect to a weight $\langle(k, \xi)\rangle^N$) it follows that

$$\alpha\|\tilde{\omega}\|_{L^2}^2 + \|\sqrt{k^2 + \xi^2}\tilde{\theta}\|_{L^2}^2$$

is non-increasing. The claimed result thus follows by Plancherel's theorem.

The case $\alpha < 0$: Let (k, ξ) with $k \neq 0$ and $\alpha < 0$ be given. Then the eigenvalues of the matrix

$$\begin{pmatrix} -\nu(k^2 + \xi^2) & ik \\ \frac{ik\alpha}{k^2 + \xi^2} & -\eta(k^2 + \xi^2) \end{pmatrix}$$

are given by

$$\begin{aligned} \lambda_{1,2} &= -\frac{\nu + \mu}{2}(k^2 + \xi^2) \pm \sqrt{\left(\frac{\nu + \mu}{2}(k^2 + \xi^2)\right)^2 - \nu\eta(k^2 + \xi^2)^2 - \alpha\frac{k^2}{k^2 + \xi^2}} \\ &= -\frac{\nu + \mu}{2}(k^2 + \xi^2) \pm \sqrt{\left(\frac{\nu - \mu}{2}(k^2 + \xi^2)\right)^2 - \alpha\frac{k^2}{k^2 + \xi^2}}, \end{aligned}$$

where we used the binomial formula $(a + b)^2 - 4ab = (a - b)^2$ in the last step.

We recall that by assumption (at least) one of ν, η vanishes. Therefore we define $C = \max(\nu, \eta)$ and observe that

$$(\eta + \nu)^2 = (\eta - \nu)^2 =: C^2$$

and that

$$\lambda_1 = -C(k^2 + \xi^2) + \sqrt{C^2(k^2 + \xi^2) + (-\alpha)\frac{k^2}{k^2 + \xi^2}},$$

is strictly positive, since $(-\alpha)\frac{k^2}{k^2 + \xi^2}$ is positive. This matrix thus has a positive eigenvalue and there exist solutions of (8) which grow exponentially in time. Given these exponentially growing solutions on single Fourier modes, we next construct exponentially growing solutions in H^N . We may pick a compact set in Fourier space, e.g. a ball, and construct initial data $(\omega_0, \theta_0) \in H^N \times H^{N+1}$ by prescribing the Fourier transform of the initial data to match these solutions (and vanish outside the ball). The corresponding solution then also exhibits exponential growth in time. \square

We remark that in the inviscid case, $\nu = \mu = 0$, these eigenvalues further simplify to

$$\pm \sqrt{-\alpha \frac{k^2}{k^2 + \xi^2}},$$

which are either purely imaginary if $\alpha > 0$ or positive and negative if $\alpha < 0$. Thus, if $\alpha > 0$ (hotter fluid is above) the evolution is not exponentially unstable. One speaks of *hydrostatic equilibrium*. The stability of this solution in the inviscid setting has recently been studied in [EW15, Wid18].

In contrast if $\alpha < 0$ (that is, the fluid is hotter below) then one eigenvalue is positive and the solution is exponentially unstable. This phenomenon is known as *Rayleigh-Bénard instability*. One main question in the following will then be whether a shear flow can suppress this instability.

Having discussed the effects of dissipation without shear. We next consider the effects of an affine shear in the inviscid problem, where again explicit solutions are available. The following results have been previously obtained in [YL18, Zil20b, MSHZ20] for $\alpha > 0$. By minor modifications of the proof the results further extend to negative α and higher Sobolev norms.

Lemma 2.2. *Consider the linearized inviscid Boussinesq equations in vorticity formulation around*

$$v = (y, 0), \theta = \alpha y$$

and coordinates $(x + ty, y)$ moving with the shear. Furthermore, define $c = \frac{1}{2}\Re(\sqrt{1-4\alpha}) \in [0, \infty)$. Then the velocity and temperature satisfy the following estimates

$$\begin{aligned} \|\theta\|_{H^N} &\lesssim t^{-1/2+c}(\|\omega_0\|_{H^{N+1}} + \|\theta_0\|_{H^{N+2}}), \\ \|v_1 - \langle v_1 \rangle\|_{H^N} &\lesssim t^{-1/2+c}(\|\omega_0\|_{H^{N+1}} + \|\theta_0\|_{H^{N+2}}), \\ \|v_2\|_{H^N} &\lesssim t^{-3/2+c}(\|\omega_0\|_{H^{N+2}} + \|\theta_0\|_{H^{N+3}}), \end{aligned}$$

and are thus stable if $c < \frac{1}{2}$ ($\alpha > 0$) and $c < \frac{3}{2}$ ($\alpha > -2$), respectively. They are unstable if $c > \frac{1}{2}$ ($\alpha < 0$) or $c > \frac{3}{2}$ ($\alpha < -2$).

The evolution of the vorticity in contrast is unstable for all α in the sense that there exists non-trivial initial data such that

$$\|\omega(t)\|_{H^N} \geq Ct^{1/2+c}(\|\omega_0\|_{H^N} + \|\partial_x \theta_0\|_{H^N})$$

as $t \rightarrow \infty$.

We emphasize that for v_2 we may allow $0 > \alpha > -2$ to be negative and that the evolution of the vorticity ω is unstable for any α .

We remark that this combination of stability and instability is consistent with the Orr mechanism. More precisely, by an integration by parts argument it holds that

$$\|v_1 - \int v_1 dx\|_{L^2} \leq Ct^{-1}\|\omega(t)\|_{H^1}.$$

Hence, if the velocity is asymptotically stable with a sharp decay rates of for instance $t^{-1/2}$, this implies that the vorticity is algebraically unstable in H^1 with a growth rate at least $t^{-1/2+1} = t^{1/2}$.

Proof. As in the proof of Lemma 2.1 we consider the Fourier formulation, now in coordinates $(k, \xi + kt)$ moving with the shear:

$$\partial_t \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} 0 & ik \\ \frac{ik\alpha}{k^2 + (\xi - kt)^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix}.$$

Due to the vanishing diagonal structure, we may decouple this problem as

$$\begin{aligned} \partial_t^2 \tilde{\omega} &= -\frac{\alpha k^2}{k^2 + (\xi - kt)^2} \tilde{\omega}, \\ \partial_t \tilde{\omega} &= ik \tilde{\theta}. \end{aligned}$$

After relabeling and shifting time by $\frac{\xi}{k}$, we observe that the first equation corresponds to a Schrödinger equation with potential:

$$(\partial_t^2 + \frac{\alpha}{1 + t^2})u = 0.$$

As observed in [YL18] this problem can be solved explicitly in terms of hypergeometric functions:

$$(9) \quad \begin{aligned} u(t) &= c_1 {}_2F^1\left(-\frac{1}{4} - \frac{1}{4}\sqrt{1-4\alpha}, -\frac{1}{4} + \frac{1}{4}\sqrt{1-4\alpha}, \frac{1}{2}, -t^2\right) \\ &+ c_2 t {}_2F^1\left(\frac{1}{4} - \frac{1}{4}\sqrt{1-4\alpha}, \frac{1}{4} + \frac{1}{4}\sqrt{1-4\alpha}, \frac{3}{2}, -t^2\right). \end{aligned}$$

As $t \rightarrow \infty$, it holds that ${}_2F^1(a, b, c, -t^2) \sim Ct^{-2a}$ (see [DLMF, 15.8(ii)]). The same asymptotic behavior is exhibited by the approximate problem

$$(\partial_t^2 + \frac{\alpha}{t^2})f = 0,$$

which we use to simplify discussion in the following. Making the ansatz $f = t^\beta$, we obtain that

$$(10) \quad \begin{aligned} f &= c_1 t^{\beta_1} + c_2 t^{\beta_2}, \\ \beta_{1,2} &= \frac{1}{2}(1 \pm \sqrt{1-4\alpha}), \end{aligned}$$

which matches the asymptotic behavior of the hypergeometric functions in (9). In particular, we observe that for any α , β_1 has positive real part which results in an algebraic instability of f and hence $\tilde{\omega}$. When considering the velocity and temperature, we recall that

$$ik\theta = \partial_t \omega \sim \partial_t f$$

and that the Biot-Savart law combined with the shear by $(y, 0)$ provides a gain of t^{-1} for $v_1 - \langle v_1 \rangle$ and by t^{-2} for v_2 by the Orr mechanism. Hence, we deduce that

$$\begin{aligned} \|v_1 - \langle v_1 \rangle\|_{H^N} + \|\theta\|_{H^N} &\sim t^{\beta_1-1}, \\ \|v_2\|_{H^N} &\sim t^{\beta_1-2} \end{aligned}$$

with β_1 as in (10). In particular, we observe that

$$\begin{aligned}\Re(\beta_1 - 1) &= c + \frac{1}{2} < 0 \text{ if } \alpha > 0, \\ \Re(\beta_1 - 1) &= c + \frac{1}{2} > 0 \text{ if } \alpha < 0, \\ \Re(\beta_1 - 2) &= c + \frac{3}{2} < 0 \text{ if } \alpha > -2, \\ \Re(\beta_1 - 2) &= c + \frac{3}{2} > 0 \text{ if } \alpha < -2,\end{aligned}$$

where we used that $\sqrt{1 - 4\alpha} = 1 \Leftrightarrow \alpha = 0$ and $\sqrt{1 - 4\alpha} = 3 \Leftrightarrow \alpha = -2$. \square

Given these (in)stability results our main questions in this article are the following:

- Q1 How much dissipation (and in which directions) needs to be added to restore linear stability?
- Q2 Can we allow α to be negative and how does the threshold depend on the dissipation?
- Q3 When considering the problem without thermal dissipation, it is natural to consider the more general problem around $v = (y, 0)$, $\theta = T(y)$. Under which conditions on T are such solutions linearly stable? For instance, can we allow T to oscillate?
- Q4 Do these results extend to the nonlinear small data regime and if so how do stability regions depend the dissipation coefficients (that is, what perturbations can be considered “small”)?

In this paper we focus on the case without thermal dissipation and $v = (y, 0)$, $\theta = T(y)$. The converse problem without viscous dissipation and $v = (U(y), 0)$, $\theta = \alpha y$ or time-dependent shear and temperature profile could be of future interest. We address questions Q1 and Q2 in Section 3.1 and Q3 in Section 3.2. The question Q4 of nonlinear stability is addressed in Section 4.

3. SHEAR CAN COUNTERACT HYDROSTATIC IMBALANCE

Building on the results of Lemma 2.2 for a combination of Couette flow and an unstable affine temperature profile, in this section we consider the problem with partial dissipation.

More precisely, we consider the nonlinear Boussinesq equations with vertical dissipation of the velocity and without thermal diffusion:

$$\begin{aligned}\partial_t v + v \cdot \nabla v + \nabla p &= \nu \partial_y^2 v + \begin{pmatrix} 0 \\ \theta \end{pmatrix}, \\ \partial_t \theta + v \cdot \nabla \theta &= 0.\end{aligned}$$

As remarked in the introduction, in Section 4 we additionally impose vertical thermal diffusion, but do not require it for the linear stability results of this section.

We observe that for any $\beta \in \mathbb{R}$ and any function $T(y)$, the collection

$$\begin{aligned} v &= \begin{pmatrix} \beta y \\ 0 \end{pmatrix}, \\ \theta &= T(y), \\ p &= \int^y T(s) ds, \end{aligned}$$

is a stationary solution of these equations. As remarked in Section 2 it is natural to ask about the stability of such solutions.

In Section 2 we studied some related special cases when $T(y)$ is affine:

- In Lemma 2.1 we studied the problem with trivial shear, that is $\beta = 0$. In this setting the flow turned out to be linearly stable if T is increasing and linearly exponentially unstable if T is decreasing, even if the slope is very small.
- In Lemma 2.2 we instead considered the case with shear but with trivial dissipation and saw that while the exponential instability is reduced to an algebraic one, the evolution of the vorticity is unstable.

The aim of this article is to understand how these results change when adding partial dissipation and whether they extend to more general profiles T . In this section we study the linearized problem first for the case of T affine (answering questions Q1, Q2) and then for general T in Section 3.2 (answering Q3). The nonlinear problem with full vertical dissipation is discussed in Section 4, which answers Q4. The author would like to thank Charlie Doering for raising the question of the stability of pairs $v = (U(y), 0), \theta = T(y)$ in a discussion.

3.1. Affine Temperature. In order to introduce ideas and mechanisms, we first study the case

$$T(y) = \alpha y,$$

where we allow $\alpha \in \mathbb{R}$ to be *negative* with a threshold depending on ν . More precisely, it turns out that for this special linearized problem we may allow α to be arbitrarily large, but for the nonlinear setting of Section 4 and the non-affine problem we require a bound by $\nu^{1/3}$. Shear enhanced dissipation suppresses Rayleigh-Bénard instability in this case, thus answering questions Q1 and Q2 of Section 2.

We remark that results for α positive have been previously established in [Zil20b]. As the main novelties of this article, we show that even if α is negative (but small) stability holds and that we may further allow T to be non-affine (see Section 3.2).

Theorem 3.1. *Consider the linearized Boussinesq equations around $v = (y, 0), \theta = \alpha y$ in coordinates*

$$(t, x - ty, y)$$

moving with the shear flow:

$$\begin{aligned} \partial_t \omega &= \nu(\partial_y - t\partial_x)\omega + \partial_x \theta, \\ \partial_t \theta &= \alpha v_2, \end{aligned}$$

on the domain $\mathbb{T} \times \mathbb{R}$. Then there exists $\alpha_ = -\frac{1}{100} \sqrt[3]{\nu} < 0$ such that the linearized evolution is stable at the level of the vorticity for any α with $|\alpha| < \alpha_*$. More*

precisely, for any $N \in \mathbb{N}$ there exists a constant $0 < C = C(\nu, \alpha)$ such that for all times $t > 0$ it holds that

$$\|\omega(t)\|_{H^N} + \|\partial_x \theta(t)\|_{H^N} \leq C\|\omega_0\|_{H^N} + C\|\partial_x \theta_0\|_{H^N},$$

where ω_0, θ_0 denote the initial data.

We stress that here we can allow α to be negative and that for $0 < \nu < 1$, the threshold $\nu^{1/3}$ is improved compared to the dissipative scale.

In particular, for this special setting we may even consider $\alpha \in \mathbb{R}$ arbitrary, but in view of later results focus on the case of small negative α .

Proof of Theorem 3.1. Similarly to the proof of Lemma 2.1 we may equivalently express the linearized Boussinesq equations around the affine temperature profile in Fourier variables as:

$$(11) \quad \partial_t \begin{pmatrix} \tilde{\omega} \\ k\tilde{\theta} \end{pmatrix} = \begin{pmatrix} -\nu(k^2 + (\xi - kt)^2) & i \\ \alpha \frac{i}{1 + (\frac{\xi}{k} - t)^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ k\tilde{\theta} \end{pmatrix},$$

where we consider coordinates $(k, \eta + kt)$ moving with Couette flow. Since the evolution of the x -averages of ω and θ decouples, in the following we without loss of generality only consider $k \neq 0$.

We stress that the coefficients here are time-dependent and hence this ODE system cannot anymore be explicitly solved in terms of a matrix exponential. However, a main advantage of the affine setting is that various estimates completely decouple, restrictions become trivial and operators commute, which makes this problem much simpler than the general profile case of Section 3.2 or the nonlinear problem of Section 4.

We note that the problem (11) decouples with respect to k and ξ , which we thus in the following treat as arbitrary but fixed. We then claim that for any $C > 1$, $\nu > 0$ and any $\alpha \in \mathbb{R}$ it holds that

$$(12) \quad |\tilde{\omega}(t)|^2 + k^2|\tilde{\theta}(t)|^2 \leq \left(1 + \frac{1}{|\alpha|}\right)(1 + C^2) \exp\left(\frac{|\alpha|}{\nu C^2}\right) (|\tilde{\omega}(0)|^2 + k^2|\tilde{\theta}(0)|^2)$$

and thus the solution at time t is controlled in terms of the initial data. We observe some special cases for the exponential:

- If we choose $C = 1$ we obtain a bound by

$$(13) \quad \exp\left(\frac{|\alpha|}{\nu}\right).$$

This bound holds for all α , but suggests a threshold $|\alpha| < \nu$.

- If we choose $C = \nu^{-1/4}$ we obtain a bound by

$$(14) \quad \exp(|\alpha|),$$

where only the algebraic prefactors depends on ν .

- If we choose $C = \nu^{-1/3}$ we obtain a bound by

$$(15) \quad \exp(|\alpha|\nu^{-1/3}),$$

where the exponent becomes uniformly bounded if we assume that $|\alpha| < \nu^{1/3}$.

The results of the theorem follow from the third case, where estimates in H^N are obtained by integrating the frequency-wise bound (12).

In order to introduce ideas and motivate the definition of C we first discuss the case $\alpha > 0$.

Step 1 (symmetrize): Let $\alpha > 0$ be given. That is, suppose we are in the setting of hydrostatic balance. Then one commonly exploited feature in the setting without shear is cancellation of the purely imaginary off-diagonal entries (compare [Zil20b, DWZZ18]).

Indeed, consider the rescaled problem

$$(16) \quad \partial_t \begin{pmatrix} \sqrt{\alpha\tilde{\omega}} \\ \sqrt{k^2 + (\xi - kt)^2} \tilde{\theta} \end{pmatrix} = \begin{pmatrix} -\nu(k^2 + (\xi - kt)^2) & \frac{i\sqrt{\alpha}}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} \\ \frac{i\sqrt{\alpha}}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} & \frac{t - \frac{\xi}{k}}{1 + (\frac{\xi}{k} - t)^2} \end{pmatrix} \begin{pmatrix} \sqrt{\alpha\tilde{\omega}} \\ \sqrt{k^2 + (\xi - kt)^2} \tilde{\theta} \end{pmatrix}$$

We observe that the off-diagonal entries are then exactly equal and imaginary and thus cancel under the matrix-valued map $M \mapsto M + \overline{M}^T$.

Therefore, if we denote the square of the Euclidean norm of the vector as

$$E(t) := \alpha|\tilde{\omega}|^2 + (k^2 + (\xi - kt)^2)|\tilde{\theta}|^2$$

it holds that

$$\begin{aligned} \partial_t E(t) &= -\nu(k^2 + (\xi - kt)^2)\alpha|\tilde{\omega}|^2 + \frac{t - \frac{\xi}{k}}{1 + (\frac{\xi}{k} - t)^2} (k^2 + (\xi - kt)^2)|\tilde{\theta}|^2 \\ &\leq \frac{\min(0, t - \frac{\xi}{k})}{1 + (\frac{\xi}{k} - t)^2} E(t) \end{aligned}$$

Integrating in time and using that

$$\int_0^t \frac{\min(0, t - \frac{\xi}{k})}{1 + (\frac{\xi}{k} - t)^2} \leq \ln(1 + t^2)$$

it follows that

$$(17) \quad E(t) \leq (1 + t^2)E(0).$$

Thus, irrespective of the size of $\alpha > 0$ and of $\nu \geq 0$ we have shown that the evolution in H^N is at most algebraically unstable.

Step 2(Using dissipation): Compared to our desired result, the estimate by (17) is not yet sufficient, since it is not uniform in time.

In the following we hence modify the definition of E to also make use of the dissipation. More precisely, we introduce a cut-off

$$(18) \quad C > 1$$

to be specified later and define the resonant time interval

$$(19) \quad I := \{t \geq 0 : |\frac{\xi}{k} - t| \leq C\}.$$

Then it holds that

$$(20) \quad \begin{aligned} & \partial_t \left(\frac{\sqrt{\alpha\tilde{\omega}}}{\sqrt{k^2 + \min((\xi - kt)^2, C^2)}\tilde{\theta}} \right) \\ &= \left(\begin{array}{cc} -\nu(k^2 + (\xi - kt)^2) & \frac{ik\sqrt{\alpha}}{\sqrt{k^2 + \min((\xi - kt)^2, C^2)}} \\ \frac{ik\sqrt{\alpha}}{\sqrt{k^2 + \min((\xi - kt)^2, C^2)}} & \frac{t - \frac{\xi}{k}}{1 + (\frac{\xi}{k} - t)^2} 1_I(t) \end{array} \right) \left(\frac{\sqrt{\alpha\tilde{\omega}}}{\sqrt{k^2 + \min((\xi - kt)^2, C^2)}\tilde{\theta}} \right), \end{aligned}$$

where $1_I(t) \in \{0, 1\}$ denotes the indicator function of I . We then define the modified energy as

$$(21) \quad E(t) := \alpha \|\omega\|_{H^N}^2 + \|\partial_x \theta\|^2 + \|\min(\xi - kt, C)\tilde{\theta}\|_{L^2_N}^2$$

Step 2a (resonant region): If $t \in I$ and $\alpha > 0$ the problem and the definition of $E(t)$ are identical to the one considered in Step 1 and it follows that

$$(22) \quad \partial_t E(t) \leq \frac{\min(0, t - \frac{\xi}{k})}{1 + (\frac{\xi}{k} - t)^2} E(t).$$

However, by definition of the interval I it holds that

$$\int_I \frac{\min(0, t - \frac{\xi}{k})}{1 + (\frac{\xi}{k} - t)^2} dt \leq \ln(1 + C^2)$$

and thus the growth of E during the resonant time is bounded by $(1 + C^2)$.

Step 2b (non-resonant region): Next suppose that $t \notin I$ and thus $|\frac{\xi}{k} - t|$ is large. In particular, $(\xi - kt)^2 \leq k^2 + (\xi - kt)^2 \leq 2(\xi - kt)^2$ and thus vertical dissipation is comparable to full dissipation.

Then the off-diagonal entries in (20) can be estimated as

$$\begin{aligned} & \left| \frac{i\sqrt{\alpha}}{\sqrt{1 + \min((\frac{\xi}{k} - t)^2, C^2)}} \right| = \frac{\sqrt{\alpha}}{C^2}, \\ & \left| \frac{i\sqrt{\alpha}}{\sqrt{1 + \min((\frac{\xi}{k} - t)^2, C^2)}} \frac{\sqrt{k^2 + \min((\xi - kt)^2, C^2)}}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} \right| \leq \frac{\sqrt{\alpha}}{C^2}. \end{aligned}$$

Thus, using Young's inequality with

$$\frac{\sqrt{\nu}\sqrt{k^2 + (\xi - kt)^2}}{\sqrt{\nu}\sqrt{k^2 + (\xi - kt)^2}},$$

we deduce that

$$(23) \quad \begin{aligned} \partial_t E(t) &\leq -\frac{\nu}{2}(k^2 + (\xi - kt)^2)\alpha|\tilde{\omega}|^2 + \frac{\alpha}{\nu C^4} \frac{1}{k^2 + (\xi - kt)^2} |\sqrt{k^2 + \min((\xi - kt)^2, C^2)}\tilde{\theta}|^2 \\ &\leq \frac{\alpha}{\nu C^4} \frac{1}{k^2 + (\xi - kt)^2} E(t). \end{aligned}$$

We note that the factor on the right-hand-side is integrable in time.

Step 2c (Conclusion for $\alpha > 0$) Combining the resonant estimate (22) and the non-resonant estimate (23), we deduce that

$$(24) \quad \begin{aligned} \partial_t E(t) &\leq (1_I(t) \frac{\min(0, t - \frac{\xi}{k})}{1 + (\frac{\xi}{k} - t)^2} + (1 - 1_I(t)) \frac{\alpha}{\nu C^4} \frac{1}{k^2 + (\xi - kt)^2}) E(t) \\ &\Rightarrow E(t) \leq (1 + C^2) \exp(\frac{\alpha}{\nu C^4}) E(0), \end{aligned}$$

with $E(t)$ defined in (21). The claimed estimate (12) for $\alpha > 0$ then follows by comparing $E(t)$ with the squares of the H^N norms. It remains to discuss the case of negative α .

Step 3 (negative α) Let now $\alpha < 0$ be given and consider the problem rescaled by $\sqrt{|\alpha|}$ instead. Then the evolution equation (20) reads

(25)

$$(26) \quad \begin{aligned} &\partial_t \left(\frac{\sqrt{|\alpha|} \tilde{\omega}}{\sqrt{k^2 + \min((\xi - kt)^2, C^2)} \tilde{\theta}} \right) \\ &= \left(\begin{array}{cc} -\nu(k^2 + (\xi - kt)^2) & \frac{i\sqrt{|\alpha|}}{\sqrt{1 + \min((\frac{\xi}{k} - t)^2, C^2)}} \\ -\frac{i\sqrt{|\alpha|}}{\sqrt{1 + \min((\frac{\xi}{k} - t)^2, C^2)}} \frac{\sqrt{1 + \min((\frac{\xi}{k} - t)^2, C^2)}}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} & \frac{t - \frac{\xi}{k}}{1 + (\frac{\xi}{k} - t)^2} 1_I(t) \end{array} \right) \left(\frac{\sqrt{|\alpha|} \tilde{\omega}}{\sqrt{k^2 + \min((\xi - kt)^2, C^2)} \tilde{\theta}} \right). \end{aligned}$$

We thus define the energy as

$$E(t) = |\alpha| |\tilde{\omega}|^2 + (k^2 + \min((\xi - kt)^2, C^2)) |\tilde{\theta}|^2,$$

which agrees with the previous definition if $\alpha > 0$.

Step 3a (non-resonant region): Suppose that $t \notin I$. We observe that in Step 2b we did not make use of the sign of α but only used Young's inequality. Furthermore, in that region $|\xi - kt| \geq |k|$ and thus in this region vertical dissipation dominates full dissipation.

Hence, by the same argument we may deduce that also for our extended definition of $E(t)$ it holds that

$$(27) \quad \partial_t E(t) \leq \frac{\alpha}{\nu C^4} \frac{1}{k^2 + (\xi - kt)^2} E(t).$$

Step 3b (resonant region): Suppose that $t \in I$. Then we observe that off-diagonal terms in (25) are of the same size but have the opposite sign and hence do not cancel anymore. However, we may use Young's inequality to still bound

$$\begin{aligned} \partial_t E(t) &\leq -\nu(k^2 + (\xi - kt)^2) |\alpha| |\tilde{\omega}|^2 + \frac{|\alpha|}{\sqrt{1^2 + (\frac{\xi}{k} - t)^2}} E(t) + \frac{t - \frac{\xi}{k}}{1 + (\frac{\xi}{k} - t)^2} E(t) \\ &\leq \frac{1 + |\alpha|}{\sqrt{1^2 + (\frac{\xi}{k} - t)^2}} E(t), \end{aligned}$$

which yields a bound on the total growth by

$$(1 + C^2)^{1 + |\alpha|}.$$

Combining the estimates in the resonant and non-resonant region, we deduce that

$$(28) \quad E(t) \leq (1 + C^2)^{1+|\alpha|} \exp(\pi \frac{|\alpha|}{\nu C^4}) E(0).$$

In particular, choosing $C = \nu^{-1/3}$ and supposing that $|\alpha| < \min(\nu^{1/3}, 1)$, this estimate reduces to

$$E(t) \leq (1 + \nu^{-2/3})^2 e^\pi E(0),$$

which implies the result.

We remark that in the proof for negative α we have not relied on cancellation but only on smallness of $\sqrt{|\alpha|}$ in combination with Young's inequality. Hence, we may consider a modification of the energy $E(t)$ as

$$|\hat{\alpha}|\tilde{\omega}|^2 + (k^2 + \min((\xi - kt)^2, C^2))|\tilde{\theta}|^2$$

with $\hat{\alpha} = \max(|\alpha|, \nu^{1/3})$ and repeat the same proof, since $\frac{|\alpha|}{\sqrt{\hat{\alpha}}} < \nu^{1/6}$ and $\sqrt{\hat{\alpha}} < \nu^{1/6}$ satisfy the desired inequalities. \square

We remark that in the proof of this affine case we can allow α to be arbitrarily large and are also free to choose C arbitrarily. As we discuss in the following, if T' is non-constant or if we study the nonlinear problem, smallness of α is required in the proof. In view of resonances in the related linear inviscid damping problem [DZ19b] some form of smallness condition is probably necessary.

3.2. Non-affine Temperature. Having discussed the setting of affine hydrostatic (im)balance, we next consider $T(y)$ non-affine and address the question Q3 of Section 2 under which conditions on T in terms of ν such solutions are stable. Here the problem does not decouple in frequency anymore and we thus employ a by now classical Cauchy-Kowalewskaya or ghost energy approach (compare [MV11, BM15, Zil16]).

The linearized system around $\theta = T(y)$ in Lagrangian coordinates is given by:

$$(29) \quad \begin{aligned} \partial_t \omega &= \nu(\partial_y - t\partial_x)^2 \omega + \partial_x \theta, \\ \partial_t \partial_x \theta &= -T'(y) \partial_x^2 \Delta_t^{-1} \omega, \end{aligned}$$

where we applied a derivative in x to the second equation. Since the evolution of the x -averages decouples, we assume without loss of generality that

$$(30) \quad \int \omega dx = 0 = \int \theta dx$$

throughout this section.

Our main results are summarized in the following theorem.

Theorem 3.2. *Let $T(y)$ be a given temperature profile, $N \in \mathbb{N}$ and consider the linearized Boussinesq equations (29) with vertical dissipation $\nu > 0$. Further suppose that the Fourier transform of T' satisfies*

$$(31) \quad \int (1 + |\xi|)^{N+5} |\mathcal{F}(T')(\xi)| \leq 4^{-N} \nu^{1/3},$$

then for any initial data $\omega_0, \theta_0 \in H^N \times H^{N+1}$ it holds that

$$\begin{aligned} & \|\omega(t)\|_{H^N}^2 + \|\partial_x \theta(t)\|_{H^N}^2 \\ & \leq C(1 + \nu^{-2/3})^2 (\|\omega_0\|_{H^N}^2 + \|\theta_0\|_{H^{N+1}}^2). \end{aligned}$$

We remark that (31) here is a sufficient condition to control several commutators. We expect that in particular for large N it is far from sufficient and that it for instance would suffice to assume smallness for small N and only a finite norm for large N (compare [Zil19]). Furthermore, if T happens to be strictly increasing, stability is expected also for large norms of T' . The main focus of this theorem thus lies on cases where T may be oscillating. In the case $T(y) = \alpha y$, the (tempered) Fourier transform is given by a Dirac measure and the condition (31) reduces to $|\alpha| < \nu^{1/3}$, as in Section 3.1.

Proof. In Section 3.1 we had seen that in the special case when $T'(y) = \alpha y$ is affine, the functions θ, ω satisfy the frequency-wise bound

$$(32) \quad \begin{aligned} & \partial_t (|\alpha| |\tilde{\omega}|^2 + (k^2 + \min((\xi - kt)^2, \nu^{-2/3})) |\tilde{\theta}|^2) \\ & \leq \left(\frac{1}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} 1_I + \frac{1}{1 + (\frac{\xi}{k} - t)^2} \right) (|\alpha| |\tilde{\omega}|^2 + (k^2 + \min((\xi - kt)^2, \nu^{-2/3})) |\tilde{\theta}|^2). \end{aligned}$$

Similarly to the (linear) inviscid damping problem in the Euler equations, while this frequency-wise bounds fail in the general setting, an integrated version can be shown to hold more generally. More precisely, we define two Fourier weights

$$(33) \quad A(T, \xi, k) = \exp\left(-2 \int_0^T \frac{1}{1 + (\frac{\xi}{k} - t)^2} dt\right),$$

$$(34) \quad B(T, k, \xi) = \exp\left(-2 \int_0^T \frac{1}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} 1_I dt\right),$$

where we included a factor 2 to have additional flexibility to absorb errors.

Then in this affine case the estimate (32) implies that if we define the energy

$$(35) \quad E(t) = \alpha \|AB\omega\|_{H^N}^2 + \|AB\mathcal{F}^{-1}(k^2 + \min((\xi - kt)^2, \nu^{-2/3}))\mathcal{F}\theta\|_{H^N}^2,$$

where ω, θ is a solution for $T(y) = \alpha y$, then $E(t)$ is non-increasing and moreover satisfies the decay estimate

$$\partial_t E(t) \leq - \iint \left(\frac{1}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} 1_I + \frac{1}{1 + (\frac{\xi}{k} - t)^2} \right) (|\alpha| |AB\tilde{\omega}|^2 + (k^2 + \min((\xi - kt)^2, \nu^{-2/3})) |AB\tilde{\theta}|^2).$$

In particular, $E(t)$ is non-increasing and the inequality $E(t) \leq E(0)$ implies the result of the theorem for the special case when T is affine.

Let now $T(y)$ be given and for simplicity of notation define

$$(36) \quad \sigma = \mathcal{F}^{-1} \sqrt{k^2 + \min((\xi - kt)^2, \nu^{-2/3})} \mathcal{F}.$$

and introduce the constant α in terms of operator norms:

$$(37) \quad \alpha := \|\|\nabla_t\|^{-1} B T' B^{-1} \|\nabla_t\|_{H^N \mapsto H^N} + \|(\partial_x^2 \Delta_t^{-1})^{-3/2} B T'(y) B^{-1} B(\partial_x^2 \Delta_t^{-1})^{3/2}\|_{H^N \mapsto H^N}.$$

As the last step of this proof we will show that by (31) it follows that $\alpha \leq \nu^{1/3}$. Similarly as in Theorem 3.1, we remark that in all the following estimates we may replace α by $\hat{\alpha} = \max(\alpha, \nu^{1/3})$ if $\alpha < \nu^{1/3}$.

We now claim that if $E(t)$ is defined by the same formula as in (35) but with ω, θ being solutions of the linearized problem with temperature profile T , then $E(t)$ is non-increasing. This then implies the desired estimate by controlling B and α (or $\hat{\alpha}$) in terms of ν .

We hence have to estimate

$$\begin{aligned} \partial_t E(t)/2 &= -\nu\alpha\|\nabla_t AB\omega\|_{H^N}^2 + \alpha\langle AB\omega, AB\partial_x\theta\rangle \\ &\quad + \langle AB\sigma\theta, ABT'(y)B^{-1}B\partial_x^2\Delta_t^{-1}\omega\rangle \\ &\quad + \alpha\langle (\dot{A}B + A\dot{B})\omega, AB\omega\rangle \\ &\quad + \langle (\dot{A}B + A\dot{B})\sigma\theta, AB\sigma\theta\rangle \\ &\quad + \langle AB\dot{\sigma}\theta, AB\sigma\theta\rangle. \end{aligned}$$

Here the dissipation terms and derivatives of AB yield non-negative contributions and are thus beneficial and \dot{B} was defined in such a way to control

$$\langle AB\dot{\sigma}\theta, AB\sigma\theta\rangle.$$

More precisely, we note that inside the resonant interval I ,

$$\partial_t\sigma^2 = -2k(\xi - kt) = \frac{-2(\frac{\xi}{k} - t)}{1 + (\frac{\xi}{k-t})^2}\sigma^2$$

can be controlled by $\dot{B}B$.

It thus remains to estimate

$$(38) \quad E_\omega := \alpha\langle AB\omega, AB\partial_x\theta\rangle$$

and

$$(39) \quad E_\theta := \langle AB\sigma\theta, ABT'(y)B^{-1}B\partial_x^2\Delta_t^{-1}\omega\rangle$$

Estimating E_ω : Since the evolution equation for ω does not involve $T'(y)$ we may argue as in the affine case and control E_θ frequency-wise. More precisely, for any given frequency (k, ξ) we need to control

$$|\alpha(AB)^2(t, k, \xi)\bar{\omega}(t, k, \xi)ik\tilde{\theta}(t, k, \xi)|$$

Resonant region: If t, k, ξ are such that $|\frac{\xi}{k} - t| \leq \nu^{-1/3}$ we may bound by this by

$$\frac{\sqrt{\alpha}}{\sqrt{1 + (\frac{\xi}{k} - t)}}(AB)^2(t, k, \xi) \left[\alpha|\tilde{\omega}|^2 + \left| \sqrt{1 + (\frac{\xi}{k} - t)^2}\tilde{\theta} \right|^2 \right],$$

which can be absorbed into

$$\langle A\dot{B}\omega, AB\omega\rangle + \langle A\dot{B}\sigma\theta, AB\sigma\theta\rangle$$

by construction of B .

Non-resonant region: If instead t, k, ξ are such that $|\frac{\xi}{k} - t| \geq \nu^{-1/3}$, then \dot{B} vanishes and we instead make use of the vertical dissipation. That is, we estimate

$$\begin{aligned} &|\alpha(AB)^2(t, k, \xi)\bar{\omega}(t, k, \xi)ik\tilde{\theta}(t, k, \xi)| \\ &= \alpha(AB)^2|\tilde{\omega}| \frac{\sqrt{\nu}\sqrt{k^2 + (\xi - kt)^2}}{\sqrt{\nu}\sqrt{k^2 + (\xi - kt)^2}} \frac{1}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} \sqrt{k^2 + \nu^{-2/3}}|\tilde{\theta}| \end{aligned}$$

by the dissipation term

$$-\alpha(AB)^2(\sqrt{\nu}\sqrt{(\xi - kt)^2}|\tilde{\omega}|)^2$$

and

$$(AB)^2 \frac{1}{1 + (\frac{\xi}{k} - t)^2} |\sqrt{k^2 + \nu^{-2/3}}|\tilde{\theta}|^2.$$

Here we used that in the non-resonant region $(\xi - kt)^2$ controls the full dissipation.

The latter term can then be absorbed into

$$\langle \dot{A}B\sigma\theta, AB\sigma\theta \rangle$$

provided

$$(40) \quad \frac{\sqrt{\alpha}}{\sqrt{\nu}\sqrt{k^2 + (\xi - kt)^2}}$$

is less than 1. Since we are in the non-resonant region (40) can be bounded from above by

$$\sqrt{\alpha}\nu^{-1/2+1/3} = (\alpha\nu^{-1/3})^{1/2},$$

which is small since $\alpha < \nu^{1/3}$ by assumption.

Estimating E_θ : In order to estimate the contribution (39)

$$E_\theta = \langle AB\sigma\theta, ABT'(y)B^{-1}B\partial_x^2\Delta_t^{-1}\omega \rangle$$

we follow a similar argument as in the affine case. However, as T' is non-constant we further have to control an interaction term between the resonant and non-resonant regions.

More precisely, for any given time t we define the Fourier set

$$\Omega(t) = \{(k, \xi) : k \neq 0, |\frac{\xi}{k} - t| \leq \nu^{-1/3}\}.$$

That is, instead of time interval I associated to given frequencies, we consider frequencies for a given time t . We then split

$$\begin{aligned} \omega &= 1_\Omega\omega + (1 - 1_\Omega)\omega =: \omega_{\text{in}} + \omega_{\text{out}}, \\ \theta &= 1_\Omega\theta + (1 - 1_\Omega)\theta =: \theta_{\text{in}} + \theta_{\text{out}}. \end{aligned}$$

We then split the contributions as

$$\begin{aligned} &\langle AB\sigma\theta, ABT'(y)B^{-1}B\partial_x^2\Delta_t^{-1}\omega_{\text{out}} \rangle \\ &\langle AB\sigma\theta_{\text{in}}, ABT'(y)B^{-1}B\partial_x^2\Delta_t^{-1}\omega_{\text{in}} \rangle \\ &\langle AB\sigma\theta_{\text{out}}, ABT'(y)B^{-1}B\partial_x^2\Delta_t^{-1}\omega_{\text{in}} \rangle. \end{aligned}$$

We remark that in the affine case the third term identically vanished due to the disjoint Fourier support of ω_{in} and θ_{out} , but that this orthogonality is lost in the general case.

Step 2a (ω_{out}): We argue as in the affine case. Since ω_{out} is supported in Ω it holds that

$$\|\nabla_t B\partial_x^2\Delta_t^{-1}\omega_{\text{out}}\|_{H^N} \leq \nu^{2/3}\|(\partial_y - t\partial_x)B\omega\|_{H^N}.$$

For θ we do not need a further control of the support and may bound

$$\| |\nabla_t|^{-1}AB\sigma\theta \|_{H^N}$$

by the time decay of A , provided

$$\| |\nabla_t|^{-1} B T'(y) B^{-1} |\nabla_t| \|_{H^N \rightarrow H^N} \leq \sqrt{\alpha} \nu^{1/6}.$$

By our choice of α the left-hand-side is bounded by α and this estimate is therefore satisfied provided $\alpha < \nu^{1/3}$, as assumed.

Step 2b ($\theta_{\text{in}}, \omega_{\text{in}}$): Similarly as in the proof of Theorem 3.1 we use the time decay of \overline{B} to control this contribution. More precisely, we may bound this contribution in terms of

$$\begin{aligned} & \alpha \langle AB\omega_{\text{in}}, \sqrt{\partial_x^2 \Delta_t^{-1}} AB\omega_{\text{in}} \rangle \\ & + \langle AB\sigma\theta_{\text{in}}, \sqrt{\partial_x^2 \Delta_t^{-1}} AB\sigma\theta_{\text{in}} \rangle, \end{aligned}$$

provided

$$\| \sqrt{\partial_x^2 \Delta_t^{-1}} B T'(y) B^{-1} \sqrt{\partial_x^2 \Delta_t^{-1}}^{-1} \|_{H^N \rightarrow H^N} \leq \sqrt{\alpha}.$$

We remark that $\sqrt{\partial_x^2 \Delta_t^{-1}} = |\partial_x| |\nabla_t|^{-1}$ and that $B T'(y) B^{-1}$ does not depend on x . Hence this estimate is equivalent to the one of step 2a.

Step 2c ($\theta_{\text{out}}, \omega_{\text{in}}$): As T' is non-constant the contribution

$$\langle AB\sigma\theta_{\text{out}}, AB T'(y) B^{-1} B \partial_x^2 \Delta_t^{-1} \omega_{\text{in}} \rangle$$

generally does not vanish. However, since θ_{out} is supported away from the resonant region, we may insert an identity operator $(\partial_x^2 \Delta_t^{-1})^{3/2-3/2}$ and bound

$$\| AB (\partial_x^2 \Delta_t^{-1})^{3/2} \sigma\theta_{\text{out}} \|_{H^N} \leq \nu^{2/3} \sqrt{-\langle \dot{A} B \sigma\theta_{\text{out}}, AB \sigma\theta_{\text{out}} \rangle}$$

and estimate

$$\| (\partial_x^2 \Delta_t^{-1})^{-3/2} AB T'(y) B^{-1} B \partial_x^2 \Delta_t^{-1} \omega_{\text{in}} \|_{H^N}$$

by

$$\| (\partial_y - t\partial_x) AB \omega \|_{H^N}.$$

This contribution can thus be absorbed by the same argument as in Step 2a, provided

$$\| (\partial_x^2 \Delta_t^{-1})^{-3/2} B T'(y) B^{-1} B (\partial_x^2 \Delta_t^{-1})^{3/2} \|_{H^N \rightarrow H^N} \leq \nu^{1/6} \sqrt{\alpha},$$

which by our definition of α reduces to $\alpha < \nu^{1/3}$.

Step 4 (controlling α): It remains to be shown that the estimate (31) controls α . Here we make use of Schur's test, which controls the L^2 operator norm of a map

$$u(x) \mapsto \int K(x, y) u(y) dy$$

by the square root of

$$\sup_x \int |K(x, y)| dy \sup_y \int |K(x, y)| dy.$$

More precisely, we may express the map $u \mapsto |\nabla_t| B T' B^{-1} |\nabla_t|^{-1} u$ as integration against a kernel on the Fourier side:

$$\tilde{u}(k, \xi) \mapsto \int \sqrt{k^2 + (\xi - kt)^2} B(t, k, \xi) \tilde{T}'(\xi - \zeta) B^{-1}(t, k, \zeta) \sqrt{k^2 + (\zeta - kt)^2} \tilde{u}(k, \zeta) d\zeta.$$

Since we are further interested in a map on H^N we add an additional weight

$$\frac{1 + |\xi|^N}{1 + |\zeta|^N}.$$

Then Schur's test asks us to control

$$\sup_{\xi} \int \sqrt{k^2 + (\xi - kt)^2} B(t, k, \xi) \tilde{T}'(\xi - \zeta) B^{-1}(t, k, \zeta) \sqrt{k^2 + (\zeta - kt)^2} \frac{1 + |\xi|^N}{1 + |\zeta|^N} d\zeta \leq C_1$$

and

$$\sup_{\zeta} \int \sqrt{k^2 + (\xi - kt)^2} B(t, k, \xi) |\tilde{T}'(\xi - \zeta)| B^{-1}(t, k, \zeta) \sqrt{k^2 + (\zeta - kt)^2} \frac{1 + |\xi|^N}{1 + |\zeta|^N} d\xi \leq C_2,$$

which then bounds the L^2 operator norm by $\sqrt{C_1 C_2}$.

We claim that this kernel can be bounded by $|\tilde{T}'(\xi - \zeta)|(1 + |\xi - \zeta|)^{N+5}$, at which point (31) implies that $C_1 = C_2 = \nu^{1/3}$, which concludes the proof.

Indeed, by construction of B , we can control

$$B(t, k, \xi) B^{-1}(t, k, \zeta) \leq \sqrt{1 + |\xi - \zeta|^2}.$$

Similarly, if $|\xi| \leq 3|\zeta|$, we may simply control

$$\frac{1 + |\xi|^N}{1 + |\zeta|^N} \leq 1 + 3^N.$$

If instead $|\xi| \geq 3|\zeta|$, then

$$|\xi| \leq |\xi - \zeta| + \frac{1}{3}|\xi| \Leftrightarrow |\xi| \leq \frac{3}{2}|\xi - \zeta|.$$

and thus

$$\frac{1 + |\xi|^N}{1 + |\zeta|^N} \leq \left(\frac{3}{2}\right)^N (1 + |\xi - \zeta|^N).$$

Finally, we need to control

$$\frac{k^2 + (\xi - kt)^2}{k^2 + (\zeta - kt)^2} = \frac{1 + \left(\frac{\xi}{k} - t\right)^2}{1 + \left(\frac{\zeta}{k} - t\right)^2}.$$

Here we may simply estimate

$$\left(\frac{\xi}{k} - t\right)^2 \leq 2\left(\frac{\zeta}{k} - t\right)^2 + 2(\xi - \zeta)^2.$$

The first term cancels with the numerator, while for the second we simply bound by $|\xi - \zeta|$.

Thus, in total it suffices to bound

$$\sup_{\zeta} \int |\tilde{T}'(\xi - \zeta)|(1 + |\xi - \zeta|)^{N+5} d\xi < C_1 = C_2,$$

which is the assumption of our theorem. \square

We remark that in the case when T' is increasing stronger results are possible, for instance allowing α to be much larger, by using additional cancellations as in Section 3.1. The main advantage of this theorem hence lies in the fact that we can allow T' to be decreasing or oscillating.

4. THE NONLINEAR EQUATIONS WITH VERTICAL DISSIPATION

Given the results for the linearized problem, it is natural to ask whether they extend to the nonlinear perturbed problem:

$$\begin{aligned} \partial_t \omega + y \partial_x \omega + v \cdot \nabla \omega &= \nu \partial_y^2 \omega + \partial_x \theta, \\ \partial_t \theta + y \partial_x \theta + T'(y) v_2 + v \cdot \nabla \theta &= 0, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}, \end{aligned}$$

and, if so, how this depends on ν . As shown recently by Masmoudi, Said-Houari and Zhao [MSHZ20], this problem may exhibit an instability reminiscent of echo chains in the Vlasov-Poisson equations [Bed20, Zil20a] and Euler equations [DM18, DZ19a]. For this reason, we do not expect results in Sobolev regularity to extend (without strong modification). Therefore, in this section we instead consider the more viscous problem

$$(41) \quad \begin{aligned} \partial_t \omega + y \partial_x \omega + v \cdot \nabla \omega &= \nu \partial_y^2 \omega + \partial_x \theta, \\ \partial_t \theta + y \partial_x \theta + T'(y) v_2 + v \cdot \nabla \theta &= \nu \partial_y^2 \theta, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}, \end{aligned}$$

where we impose *full vertical dissipation* and view $T(y)$ as a solution of the forced problem. Similarly to results for the case of hydrostatic balance with shear studied in [Zil20b] our aim here is to extend the linear (asymptotic) stability results to the nonlinear equations with small data and thus answer question Q4 of Section 2.

Theorem 4.1. *Let $N \geq 5$ and suppose that the temperature profile $T(y)$ satisfies the linear stability assumptions of Theorem 3.2. Let further $0 < \epsilon < \nu^2$ and suppose that the initial data satisfies*

$$\|\omega_0\|_{H^N} + \nu^{-1/2} \|\partial_x \theta_0\|_{H^N} \leq \epsilon.$$

The the unique global solution with this initial data satisfies

$$\begin{aligned} \|\omega\|_{L^\infty H^N} + \nu \|(\partial_y - t \partial_x) \omega\|_{L^2 H^N} + \|v_\neq\|_{L^2 H^N} &\leq 10 \nu^{-1/3} \epsilon, \\ \|\partial_x \theta\|_{L^\infty H^N} &\leq 10 \epsilon, \end{aligned}$$

where $L^p H^N := L^p((0, \infty); H^N)$ and $v_\neq = v - \int v dx$ denotes the non-shear component of the velocity.

Remark 1. • *The nonlinear problem with vertical dissipation but without shear has been previously studied in [ACW10] and [LT16].*

- *The threshold $\epsilon < \nu^2$ here is imposed to control losses of powers $\nu^{1/3}$ in enhanced dissipation estimates encoded in our Fourier multiplier B .*
- *The nonlinear problem without thermal dissipation has been recently studied in [MSHZ20]. In particular, they require Gevrey regularity to control resonances, which suggests that stability in Sobolev regularity may either fail or require non-trivial modification [DZ19a, DM18].*
- *In a previous work [Zil20b] we studied the special case where $T(y)$ is affine (with positive slope) and with full dissipation. The present result allows for possibly oscillating profiles and only requires vertical dissipation.*
- *We remark that we here estimate $\sigma \theta$ instead of $\nabla_t \theta$ or $\partial_x \theta$. This is in view to the results of Section 3.1, for which we do not expect control of $\nabla_t \theta$.*

- *In view of the partial dissipation results of Section 3 we here omit questions of enhanced dissipation.*
- *There has been extensive work on various partial dissipation regimes as well as on the inviscid problem. We discuss some of this literature in the introductory Section 1.*

Proof. We follow a classical bootstrap argument approach [MV11, BVW18, Lis20] in the spirit of Cauchy-Kowalewskaya. As in [Zil20b] we here make use of multipliers constructed in [BVW18] and [Lis20] for the Navier-Stokes and MHD problems, respectively, and adapt them to the problem at hand. In contrast to these works we do not aim to derive (enhanced) dissipation estimates. However, we show that vertical dissipation is sufficient to employ these bootstrap methods (see also the discussion of echo chains [MSHZ20] in Section 1). We remark that in Section 3.2 we have derived estimates for the associated linearized problem, which we use as a basis for our estimates in the following. A main challenge in the control of various contributions here will be that we can only control vertical dissipation and hence will have to separately consider regimes where horizontal dissipation would be large.

In our bootstrap construction we consider $L^p H^N$ norms on a time interval $(0, T)$, $T > 0$, which incorporate a time-dependent Fourier multiplier M with

$$\nu^{1/3} \leq M \leq 1,$$

to be specified later (see equation (44)). We then consider the maximal time $T > 0$ such that the following *bootstrap estimates* are satisfied:

$$(42) \quad \begin{aligned} & \|M\omega_{\neq}\|_{L_t^\infty H^N}^2 + \nu \|(\partial_y - t\partial_x)M\omega_{\neq}\|_{L^2 H^N}^2 + \nu \|1_{|\xi-kt|\leq|k|}M\omega_{\neq}\|_{L^2 H^N}^2 \\ & \quad + \|\nabla_t \Delta_t^{-1}M\omega_{\neq}\|_{L^2 H^N}^2 \leq 16\epsilon^2, \\ & \|\sigma M\theta_{\neq}\|_{L^\infty H^N}^2 + \nu \|(\partial_y - t\partial_x)\sigma M\theta_{\neq}\|_{L^2 H^N}^2 + \nu \|1_{|\xi-kt|\leq|k|}\sigma M\theta_{\neq}\|_{L^2 H^N}^2 \\ & \quad + \|\nabla_t \Delta_t^{-1}\sigma M\theta_{\neq}\|_{L^2 H^N}^2 \leq 16\nu\epsilon^2, \\ & \|\omega_{=}\|_{L_t^\infty H^N}^2 + \nu \|\partial_y \omega_{=}\|_{L^2 H^N}^2 \leq 16\epsilon^2, \\ & \|\sigma \theta_{=}\|_{L^\infty H^N}^2 + \nu \|\sigma \partial_y \omega_{=}\|_{L^2 H^N}^2 \leq 16\nu\epsilon^2, \end{aligned}$$

where $\omega_{=}, \theta_{=}$ denote the x -averages and $\omega_{\neq}, \theta_{\neq}$ their orthogonal complement and σ is the Fourier multiplier

$$\sigma = \mathcal{F}^{-1}(k^2 + (\min(\xi - kt)^2, \nu^{-2/3}))\mathcal{F}.$$

By local well-posedness and the assumed existence of a solution, there exists some positive time $T > 0$ such that (42) holds with $L^2(\mathbb{R}_+; \cdot)$ and $L^\infty(\mathbb{R}_+; \cdot)$ replaced by $L^2((0, T); \cdot)$ and $L^\infty((0, T); \cdot)$. If the maximal time T with this property is infinity, this yields the results of the theorem in view of the bounds on M .

In the following we thus assume for the sake of contradiction that $T < \infty$ is maximal. We will then show that at the time T none of the estimates in (42) attain equality. Therefore, by continuity the estimates are still satisfied for a slightly larger time, which contradicts the maximality and thus implies the result.

In order to introduce ideas let us first consider the x -averages. We remark that in the linearized results of Section 3 their evolution decoupled and reduced to heat evolution. Thus, in the following we have to control the effects of the nonlinearity, where the lack of full dissipation requires us to introduce some additional splittings.

Estimating $\omega_=_$: We observe that $\partial_x \theta$, $v_{\neq} \cdot \nabla_t \omega_=_$ and $v_=_ \cdot \nabla_t \omega_{\neq}$ all possess a vanishing x -average and thus obtain the following evolution equation for $\omega_=_$:

$$\partial_t \omega_=_ + 0 + (v_{\neq} \cdot \nabla_t \omega_{\neq})_=_ = \nu \partial_y^2 \omega_=_ + 0.$$

Testing this equation with $\omega_=_$ and integrating in time, we deduce that

$$(43) \quad \|\omega_=(T)\|_{H^N}^2 + 2\nu \int_0^T \|\partial_y \omega_=\|_{H^N}^2 = \|\omega_=(0)\|_{H^N}^2 + \int_0^T \langle \omega_=_ , v_{\neq} \cdot \nabla_t \omega_{\neq} \rangle.$$

We recall that by assumption the initial data is of size much smaller than $\sqrt{8}\epsilon$. Thus, if we can show that the integral on the right-hand-side is bounded by ϵ , this implies that equality in (42) is indeed not attained here.

As we assume only vertical dissipation, we first discuss the part involving y derivatives of ω_{\neq} :

$$\begin{aligned} & \int_0^T \langle \omega_=_ , v_{\neq}^2 (\partial_y - t\partial_x) \omega_{\neq} \rangle \\ & \leq \|\omega_=\|_{L^\infty H^N} \|v_{\neq}^2\|_{L^2 H^N} \|(\partial_y - t\partial_x) \omega_{\neq}\|_{L^2 H^N} \\ & \leq 4\epsilon 4\nu^{-1/3} \epsilon \nu^{-1/2} 4\nu^{-1/3} \epsilon = 4\nu^{-7/6} \epsilon 16\epsilon^2, \end{aligned}$$

where v_{\neq}^2 denotes the vertical component of the velocity field, and the loss of factors $\nu^{-1/3}$ is due to the multiplier M . Since by assumption $4\nu^{-7/6}\epsilon$ is much smaller than 1, this term is too small to help achieve equality.

For the term involving x -derivatives, we introduce a Fourier multiplier χ which corresponds to the projection onto the set

$$\{(k, \xi) : |\xi - kt| \geq |k|\}.$$

Then by construction it holds that

$$\|\chi \partial_x \omega_{\neq}\|_{L^2 H^N} \leq \|\chi (\partial_y - t\partial_x) \omega_{\neq}\|_{L^2 H^N} \leq \|(\partial_y - t\partial_x) \omega_{\neq}\|_{L^2 H^N},$$

which thus allows for an estimate of the same form as for the part involving y derivatives.

Finally, we estimate

$$\begin{aligned} & \int_0^T \langle \omega_=_ , v_{\neq}^1 \partial_x (1 - \chi) \omega_{\neq} \rangle \\ & = - \int_0^T \langle \omega_=_ , (\partial_x v_{\neq}^1) (1 - \chi) \omega_{\neq} \rangle \\ & \leq \|\omega_=\|_{L^\infty H^N} \|v_{\neq}^1\|_{L^2 H^N} \|(1 - \chi) \omega_{\neq}\|_{L^2 H^N} \\ & \leq 4\epsilon \nu^{-7/6} 4\epsilon 4\epsilon, \end{aligned}$$

where we lose several powers of ν due to the enhanced multiplier B discussed in Section 3.2. As this contribution is also much smaller than $16\epsilon^2$, we conclude that

$$\|\omega_=(T)\|_{H^N}^2 + \nu \int_0^T \|\partial_y \omega_=\|_{H^N}^2 < 8\epsilon^2$$

and thus equality in (42) is not attained.

Estimating $\theta_=_$: Before discussing $\sigma\theta_=_$, we consider $\theta_=_$, where we can argue analogously to the case of $\omega_=_$. We may test the equation

$$\partial_t \theta_=_ + (v_{\neq} \cdot \nabla_t \theta_{\neq})_=_ = \nu \partial_y^2 \theta_=_$$

with $\theta_=_$ and integrate in time to again derive an integral estimate. We then estimate the contribution

$$\int_0^T \langle \theta_=_ , v_{\neq} \cdot \nabla_t \theta_{\neq} \rangle$$

by

$$\begin{aligned} & \|\theta_=_\|_{L^\infty H^N} \|v_{\neq}^2\|_{L^2 H^N} \|(\partial_y - t\partial_x)\theta_{\neq}\|_{L^2 H^N} \\ & + \|\theta_=_\|_{L^\infty H^N} \|v_{\neq}^1\|_{L^2 H^N} \|(\partial_y - t\partial_x)\chi\theta_{\neq}\|_{L^2 H^N} \\ & + \|\theta_=_\|_{L^\infty H^N} \|\partial_x v_{\neq}^1\|_{L^2 H^N} \|(1-\chi)\theta_{\neq}\|_{L^2 H^N}. \end{aligned}$$

By the bootstrap assumptions this sum can be controlled in terms of $\nu^{-7/6}\epsilon^3$, which is much smaller than ϵ^2 .

Estimating $\sigma\theta_=_$: We may extend the definition of σ to purely y -dependent functions as the Fourier multiplier $\sigma = \mathcal{F}^{-1} \min(|\xi|, \nu^{-1/3}) \mathcal{F}$. We note that the operator norm of σ is bounded by $\nu^{-1/3}$ and thus $\sigma\theta_=_$ could be controlled in terms of $\theta_=_$. However, in this way we would pass from a bound by ϵ^2 to one by $\nu^{-2/3}\epsilon^2$, which is insufficient for our bootstrap approach. Instead we aim to show that by a similar argument as above $\|\sigma\theta_=_\|_{L^\infty H^N}$ can be controlled by ϵ^2 , where the loss of powers of ν only factors into the smallness conditions on ϵ used to control nonlinear interaction terms.

We may control

$$\begin{aligned} & \|\sigma\theta_=(T)\|_{H^N}^2 + \nu \|\partial_y \sigma\theta_=\|_{L^2 H^N}^2 \\ & = \|\sigma\theta_=(0)\|_{H^N}^2 + \int_0^T \langle \sigma\theta_=_ , \sigma v_{\neq} \cdot \nabla_t \theta_{\neq} \rangle \\ & \leq \|\sigma\theta_=(0)\|_{H^N}^2 + \|\sigma\theta_=_\|_{L^\infty H^N} \nu^{-1/3} (\|v_{\neq}^2\|_{L^2 H^N} \|(\partial_y - t\partial_x)\theta_{\neq}\|_{L^2 H^N} \\ & \quad + \|v_{\neq}^1\|_{L^2 H^N} \|(\partial_y - t\partial_x)\chi\theta_{\neq}\|_{L^2 H^N} + \|\partial_x v_{\neq}^1\|_{L^2 H^N} \|(1-\chi)\theta_{\neq}\|_{L^2 H^N}). \end{aligned}$$

Thus, by assumption on ϵ and the initial data, equality in (42) is also not achieved for $\sigma\theta_=_$.

Estimating ω_{\neq} and θ_{\neq} : Having discussed the control of the x -averages, we now turn to control $\omega_{\neq}, \sigma\theta_{\neq}$. Here we will first focus on contributions due to $T(y)$ and the x -averages and finally discuss the control of the nonlinearity involving v_{\neq} .

We recall that ω_{\neq} and θ_{\neq} satisfy the system

$$\begin{aligned} \partial_t \omega_{\neq} &= \nu(\partial_y - t\partial_x)^2 \omega_{\neq} + \partial_x \theta_{\neq} - v_{\neq}^1 \partial_x \omega_{\neq} - v_{\neq}^2 \partial_y \omega_{\neq} - (v_{\neq} \cdot \nabla_t \omega_{\neq})_{\neq}, \\ \partial_t \theta_{\neq} &= \nu(\partial_y - t\partial_x)^2 \omega_{\neq} + T'(y)v_{\neq}^2 - v_{\neq}^1 \partial_x \theta_{\neq} - v_{\neq}^2 \partial_y \theta_{\neq} - (v_{\neq} \cdot \nabla_t \theta_{\neq})_{\neq}, \end{aligned}$$

where we consider $\omega_=_$ and $\theta_=_$ as given functions.

In the linearized problem of Section 3.2 we could without loss of generality assume that $\omega_=_ = \theta_=_ = 0$ and constructed a non-increasing energy functional. In the following we build on these estimates and integrate them in time to show that the control (42) is stable under small nonlinear perturbations.

We recall the multipliers A, B defined in (33) in Section 3.2:

$$A(t, k, \xi) = \exp(c \arctan(\frac{\xi}{k} - t)),$$

$$B(t, k, \xi) = \exp\left(-\int_0^t \frac{1}{\sqrt{1 + (\frac{\xi}{k} - \tau)^2} 1_{|\frac{\xi}{k} - \tau| \leq \nu^{-1/3}}} d\tau\right),$$

and for simplicity of notation write

$$(44) \quad M := AB.$$

Let us first study the time-derivative of

$$\|M\omega_{\neq}\|_{H^N}^2.$$

Then it hold that

$$(45) \quad \begin{aligned} & \|M\omega_{\neq}(T)\|_{H^N}^2 - \int_0^T \langle M\omega_{\neq}, \dot{M}\omega_{\neq} \rangle + \nu \int_0^T \|M(\partial_y - t\partial_x)\omega_{\neq}\|_{H^N}^2 \\ &= \|M\omega_{\neq}(0)\|_{H^N}^2 + \int_0^T \langle M\omega_{\neq}, M(v_{\neq}^1 \partial_x \omega_{\neq}) \rangle \\ & \quad + \int_0^T \langle M\omega_{\neq}, M(v_{\neq}^2 \partial_y \omega_{\neq}) \rangle \\ & \quad + \int_0^T \langle M\omega_{\neq}, M\partial_x \theta_{\neq} \rangle \\ & \quad + \int_0^T \langle M\omega_{\neq}, M(v_{\neq} \cdot \nabla_t \omega_{\neq}) \rangle \\ &=: \|M\omega_{\neq}(0)\|_{H^N}^2 + \mathcal{T}_{v_{\neq}^1} + \mathcal{T}_{\omega_{\neq}} + \mathcal{T}_{\omega_{\neq}, \theta_{\neq}} + \mathcal{T}_{v_{\neq}}. \end{aligned}$$

By assumption $\|M\omega_{\neq}(0)\|_{H^N}^2$ is much smaller than ϵ^2 , so if we can show that the various terms \mathcal{T} on the right-hand-side can be controlled by the left-hand-side and higher powers of ϵ , we can show that the left-hand-side remains smaller than 4ϵ for all times.

Estimating $\mathcal{T}_{v_{\neq}^1}$: In order to estimate $\mathcal{T}_{v_{\neq}^1}$ we make use of cancellation in an integration parts, following a similar argument as in [Zil20b] with additional adjustments to account for partial dissipation. More precisely, given the multiplier M , we note that by Parseval's identity

$$\begin{aligned} & \langle M\omega_{\neq}, M(v_{\neq}^1 \partial_x \omega_{\neq}) \rangle \\ &= \langle M\omega_{\neq}, M(v_{\neq}^1 \partial_x \omega_{\neq}) - v_{\neq}^1 \partial_x M\omega_{\neq} \rangle \\ &= \sum \int \int M(t, k, \xi) \omega_{\neq}(k, \xi) \omega_{\neq}(k, \xi + \zeta) (M(t, k, \xi) - M(t, k, \xi + \zeta)) v_{\neq}(\zeta). \end{aligned}$$

This cancellation is required to control $v_{\neq}(\zeta) = \frac{1}{i\zeta} \omega_{\neq}(\zeta)$ in terms of ω_{\neq} . In particular, if $|\zeta| \geq 1$ this control is trivial, while for $|\zeta| \leq 1$ we observe that $M(t, k, z)$ is Lipschitz with respect to z uniformly in t and $k \in \mathbb{Z} \setminus \{0\}$:

$$|M(t, k, \xi) - M(t, k, \xi + \zeta)| \leq C|\zeta|.$$

Hence, we can control $\mathcal{T}_{v_{\neq}^1}$ by

$$\|M\omega_{\neq}\|_{L^2 H^N} \|\omega_{\neq}\|_{L^2 H^N} \|\omega_{\neq}\|_{L^\infty H^N}.$$

The last factor is controlled by the preceding argument. For the first two factors, we make the observation that

$$\nu^{1/3} \leq \nu(k^2 + (\xi - kt)^2) + \frac{1}{\sqrt{1 + (\frac{\xi}{k} - t)^2}} 1_{|\frac{\xi}{k} - t| \leq \nu^{-1/3}}$$

and hence $\|M\omega_{\neq}\|_{H^N}^2$ (and $\nu^{2/3}\|\omega_{\neq}\|_{H^N}^2$) can be estimated in terms of the dissipation and the decay due to \dot{M} , at a loss of a factor $\nu^{1/3}$.

Estimating $\mathcal{T}_{\omega_{=}}$: We next discuss

$$\mathcal{T}_{\omega_{=}} = \langle M\omega_{\neq}, M(v_{\neq}^2 \partial_y \omega_{=}) \rangle.$$

Here we may easily estimate by

$$\nu^{-2/3} \|M\omega_{\neq}\|_{L^\infty H^N} \|Mv_{\neq}^2\|_{L^2 H^N} \|\partial_y \omega_{=}\|_{L^2 H^N},$$

where the factor of $\nu^{-2/3}$ corresponds to a rough bound of the operator norm of M . All factors are controlled in terms of the bootstrap assumption and thus $\mathcal{T}_{\omega_{=}}$ is much smaller than ϵ^2 provided ϵ^3 is much smaller than ϵ^2 in terms of powers of ν .

Estimating $\mathcal{T}_{\omega_{\neq}, \theta_{\neq}}$: As one of the main results of Section 3.2 we have shown that $M = AB$ is constructed in just such a way that

$$|\langle AB\omega_{\neq}, AB\partial_x \theta_{\neq} \rangle| \leq -\langle M\omega_{\neq}, \dot{M}\omega_{\neq} \rangle - \alpha^{-1} \langle M\sigma\theta_{\neq}, \dot{M}\sigma\theta_{\neq} \rangle$$

with $\alpha = \max(\|T'\|, \nu^{1/3})$ (see Theorem 3.2 for the precise definition). Hence, we can absorb this contribution into the left-hand-side of (45), provided we can control $\dot{M}\sigma\theta_{\neq}$, which will be the left-hand-side of a later equation (47).

Estimating $\mathcal{T}_{v_{\neq}, \theta_{\neq}}$: It remains to discuss the main nonlinearity, where a key challenge is given by the lack of horizontal dissipation.

If we had full dissipation at our disposal, this estimate would reduce to controlling by

$$\|\omega_{\neq}\|_{L^\infty H^N} \|v_{\neq}\|_{L^2 H^N} \|\nabla_t \omega_{\neq}\|_{L^2 H^N}.$$

However, as we only require vertical dissipation the last factor is not easily controlled anymore. We thus have to invest additional effort to control this contribution.

As v_{\neq} is divergence-free, we observe that

$$\begin{aligned} \langle M\omega_{\neq}, M(v_{\neq} \cdot \nabla_t \omega_{\neq}) \rangle &= \langle M\omega_{\neq}, M(v_{\neq} \cdot \nabla_t \omega_{\neq}) - v_{\neq} \cdot \nabla_t M\omega_{\neq} \rangle \\ &= \sum \iiint M(k, \xi) \tilde{\omega}_{\neq}(k, \xi) (M(k, \xi) - M(k-l, \xi-\zeta)) \tilde{v}_{\neq}(l, \zeta) \\ &\quad \cdot \left(\frac{k-l}{\xi+\zeta-(k-l)t} \right) \tilde{\omega}_{\neq}(k-l, \xi-\zeta). \end{aligned}$$

We observe that if

$$|k-l| \leq \nu^{-1} |\xi+\zeta-(k-l)t|$$

the last gradient can simply be controlled by the vertical dissipation, which yields an estimate in terms of

$$\|\omega_{\neq}\|_{L^\infty H^N} \|v_{\neq}\|_{L^2 H^N} \|(\partial_y - t\partial_x)\omega_{\neq}\|_{L^2 H^N}$$

and can hence be controlled. Similarly, if

$$|k - l| \leq \nu^{-1}|l|$$

we can control in terms of

$$\|\omega_{\neq}\|_{L^\infty H^N} \|\partial_x v_{\neq}\|_{L^2 H^N} \|\omega_{\neq}\|_{L^2 H^N}.$$

It thus only remains to discuss the region where

$$(46) \quad \left|t - \frac{\xi + \zeta}{k - l}\right| \leq \nu, \\ |l| \leq \nu|k|.$$

Here, we make use of cancellations in M . More precisely, we note that $M(k, \xi)$ does not depend on k and ξ individually, but only on $\frac{\xi}{k}$ and that uniformly in time

$$\begin{aligned} |M(k, \xi) - M(k - l, \xi - \zeta)| &\leq C \left| \frac{\xi}{k} - \frac{\xi - \zeta}{k - l} \right| \\ &= C \left| \frac{\xi - kt}{k} - \frac{\xi - \zeta - (k - l)t}{k - l} \right| \\ &\leq C \frac{1}{1 + \nu} \frac{1}{|k - l|} (|\xi - kt| + |\xi - \zeta - (k - l)t|), \end{aligned}$$

where we used (46). We thus can control $\mathcal{T}_{v_{\neq}, \theta_{\neq}}$ in that region by

$$\|\omega_{\neq}\|_{L^\infty H^N} \|v_{\neq}\|_{L^2 H^N} \|(\partial_y - t\partial_x)\omega_{\neq}\|_{L^2 H^N},$$

which concludes the argument.

Controlling $\sigma\theta_{\neq}$ We next turn to controlling $\sigma\theta_{\neq}$, where we study the time derivative of

$$\|M\sigma\theta_{\neq}\|_{H^N}^2.$$

Integrating in time, we have to control

$$(47) \quad \begin{aligned} &\|M\sigma\theta_{\neq}(T)\|_{H^N}^2 - \int_0^T \langle M\sigma\theta_{\neq}, \dot{M}\sigma\theta_{\neq} \rangle + \nu \int_0^T \|M(\partial_y - t\partial_x)\sigma\theta_{\neq}\|_{H^N}^2 \\ &= \|M\sigma\theta_{\neq}(0)\|_{H^N}^2 + \int_0^T \langle M\sigma\theta_{\neq}, M\sigma T^l(y)v_{\neq}^2 \rangle + \int_0^T \langle M\sigma\theta_{\neq}, M\sigma v_{\neq}^1 \partial_x \theta_{\neq} \rangle \\ &\quad + \int_0^T \langle M\sigma\theta_{\neq}, M\sigma v_{\neq}^2 \partial_y \theta_{\neq} \rangle \\ &\quad + \int_0^T \langle M\sigma\theta_{\neq}, M\sigma v_{\neq} \cdot \nabla_t \theta_{\neq} \rangle \\ &=: \|M\sigma\theta_{\neq}(0)\|_{H^N}^2 + \mathcal{T}_T + \mathcal{T}_{v_{\neq}^1} + \mathcal{T}_{\theta_{\neq}} + \mathcal{T}_{v_{\neq}, \sigma\theta_{\neq}}. \end{aligned}$$

Here the aim again is to show that that all \mathcal{T} contributions add up to something smaller than ϵ^2 and hence equality is not attained.

Estimating \mathcal{T}_T As one of the main results of Section 3.2 we have shown that \mathcal{T}_T can be controlled in terms of the decay of the multipliers M and the vertical dissipation of ω only. Thus this contribution can be estimated in terms of the left-hand-side of (47) and (45).

Estimating $\mathcal{T}_{v_{\neq}^1}$: Here we may argue analogously as for ω_{\neq} , expect that M has been replaced by σM . We thus obtain an estimate by

$$\|M\sigma\theta_{\neq}\|_{L^2 H^N} \|\theta_{\neq}\|_{L^2 H^N} \|\omega_{\neq}\|_{L^\infty H^N}.$$

Estimating $\mathcal{T}_{\theta=}$: Here we may argue again analogously as for ω_{\neq} and control by

$$\|M\sigma\theta_{\neq}\|_{L^\infty H^N} \|v_{\neq}^2\|_{L^2 H^N} \|\partial_y \theta_{=}\|_{L^2 H^N}.$$

Estimating $\mathcal{T}_{v_{\neq}, \sigma\theta_{\neq}}$: We recall that in this theorem we assume vertical dissipation also for the temperature (in contrast to Section 3.2 and the problem considered in [MSHZ20]). Therefore, in this estimate we argue largely analogously to the estimate of $\mathcal{T}_{v_{\neq}, \omega_{\neq}}$. However, since σM also depends on k , we need some additional control in the region where the horizontal dissipation is not easily controlled.

More precisely, by the preceding arguments for $\mathcal{T}_{v_{\neq}, \omega_{\neq}}$ it suffices to consider

$$\begin{aligned} & \sum \iint (\sigma\theta_{\neq})(k, \xi) \frac{1}{\sigma(k-l, \xi-\zeta)} (\sigma M(k, \xi) - \sigma M(k-l, \xi-\zeta)) \tilde{v}_{\neq}(l, \zeta) \\ & \cdot \left(\frac{k-l}{\xi-\zeta + (k-l)t} \right) (\sigma\theta_{\neq})(k-l, \xi-\zeta) \end{aligned}$$

in the regions where $\xi - \zeta$ is very close to resonant and l is much smaller than k .

However, in that case we may split into differences in M and in σ and observe that

$$\begin{aligned} & \frac{\sqrt{k^2 + (\xi - kt)^2} - \sqrt{(k-l)^2 + (\xi + \zeta - (k-l)t)^2}}{\sqrt{(k-l)^2 + (\xi + \zeta - (k-l)t)^2}} \\ & \approx \frac{\sqrt{k^2} - \sqrt{(k-l)^2}}{\sqrt{(k-l)^2}} \approx \frac{l}{|k-l|}, \end{aligned}$$

where we could neglect $\xi - kt$ and $\xi + \zeta - (k-l)t$ since these terms could otherwise be controlled in terms of the vertical dissipation. Hence, over all we can control by

$$\|\sigma\theta_{\neq}\|_{L^\infty H^N} \|\partial_x v_{\neq}\|_{L^2 H^N} \|\sigma\theta_{\neq}\|_{L^2 H^N},$$

which concludes the proof. \square

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REFERENCES

- [ACW10] Dhanapati Adhikari, Chongsheng Cao, and Jiahong Wu. The 2d Boussinesq equations with vertical viscosity and vertical diffusivity. *Journal of Differential Equations*, 249(5):1078–1088, 2010.
- [Bed20] Jacob Bedrossian. Nonlinear echoes and Landau damping with insufficient regularity. *Tunisian Journal of Mathematics*, 3:121–205, 2020.
- [BM15] Jacob Bedrossian and Nader Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publ. Math. Inst. Hautes Études Sci.*, 122:195–300, 2015.
- [BVW18] Jacob Bedrossian, Vlad Vicol, and Fei Wang. The Sobolev stability threshold for 2D shear flows near Couette. *Journal of Nonlinear Science*, 28(6):2051–2075, 2018.
- [Cha06] Dongho Chae. Global regularity for the 2d Boussinesq equations with partial viscosity terms. *Advances in Mathematics*, 203(2):497–513, 2006.

- [CKN99] Dongho Chae, Sung-Ki Kim, and Hee-Seok Nam. Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. *Nagoya Mathematical Journal*, 155:55–80, 1999.
- [DLMF] *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.0.24 of 2019-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [DM18] Yu Deng and Nader Masmoudi. Long time instability of the Couette flow in low Gevrey spaces. *arXiv preprint arXiv:1803.01246*, 2018.
- [DWZZ18] Charles R Doering, Jiahong Wu, Kun Zhao, and Xiaoming Zheng. Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion. *Physica D: Nonlinear Phenomena*, 376:144–159, 2018.
- [DZ19a] Yu Deng and Christian Zillinger. Echo chains as a linear mechanism: Norm inflation, modified exponents and asymptotics. *arXiv preprint arXiv:1910.12914*, 2019.
- [DZ19b] Yu Deng and Christian Zillinger. On the smallness condition in linear inviscid damping: Monotonicity and resonance chains. *arXiv preprint arXiv:1911.02066*, 2019.
- [EW15] Tarek M Elgindi and Klaus Widmayer. Sharp decay estimates for an anisotropic linear semigroup and applications to the surface quasi-geostrophic and inviscid Boussinesq systems. *SIAM Journal on Mathematical Analysis*, 47(6):4672–4684, 2015.
- [Lis20] Kyle Liss. On the Sobolev stability threshold of 3d Couette flow in a uniform magnetic field. *Communications in Mathematical Physics*, pages 1–50, 2020.
- [LT16] Jinkai Li and Edriss S Titi. Global well-posedness of the 2d Boussinesq equations with vertical dissipation. *Archive for Rational Mechanics and Analysis*, 220(3):983–1001, 2016.
- [MSHZ20] Nader Masmoudi, Belkacem Said-Houari, and Weiren Zhao. Stability of Couette flow for 2d Boussinesq system without thermal diffusivity. *arXiv preprint arXiv:2010.01612*, 2020.
- [MV11] Clément Mouhot and Cédric Villani. On Landau damping. *Acta mathematica*, 207(1):29–201, 2011.
- [Wid18] Klaus Widmayer. Convergence to stratified flow for an inviscid 3d Boussinesq system. *Communications in Mathematical Sciences*, 16(6):1713–1728, 2018.
- [WXZ19] Jiahong Wu, Xiaojing Xu, and Ning Zhu. Stability and decay rates for a variant of the 2d Boussinesq–Bénard system. *Communications in Mathematical Sciences*, 17(8):2325–2352, 2019.
- [Yag12] Akiva M Yaglom. *Hydrodynamic instability and transition to turbulence*, volume 100. Springer Science & Business Media, 2012.
- [YL18] Jincheng Yang and Zhiwu Lin. Linear inviscid damping for Couette flow in stratified fluid. *Journal of Mathematical Fluid Mechanics*, 20(2):445–472, 2018.
- [Zil16] Christian Zillinger. Linear inviscid damping for monotone shear flows in a finite periodic channel, boundary effects, blow-up and critical Sobolev regularity. *Arch. Ration. Mech. Anal.*, 221(3):1449–1509, 2016.
- [Zil19] Christian Zillinger. Linear inviscid damping in Sobolev and Gevrey spaces. *arXiv preprint arXiv:1911.00880*, 2019.
- [Zil20a] Christian Zillinger. On echo chains in Landau damping: Self-similar solutions and Gevrey 3 as a linear stability threshold. *arXiv preprint arXiv:2001.00513*, 2020.
- [Zil20b] Christian Zillinger. On enhanced dissipation for the Boussinesq equations. *arXiv preprint arXiv:2004.08125*, 2020.

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