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Piero D'Ancona, Roland Schnaubelt

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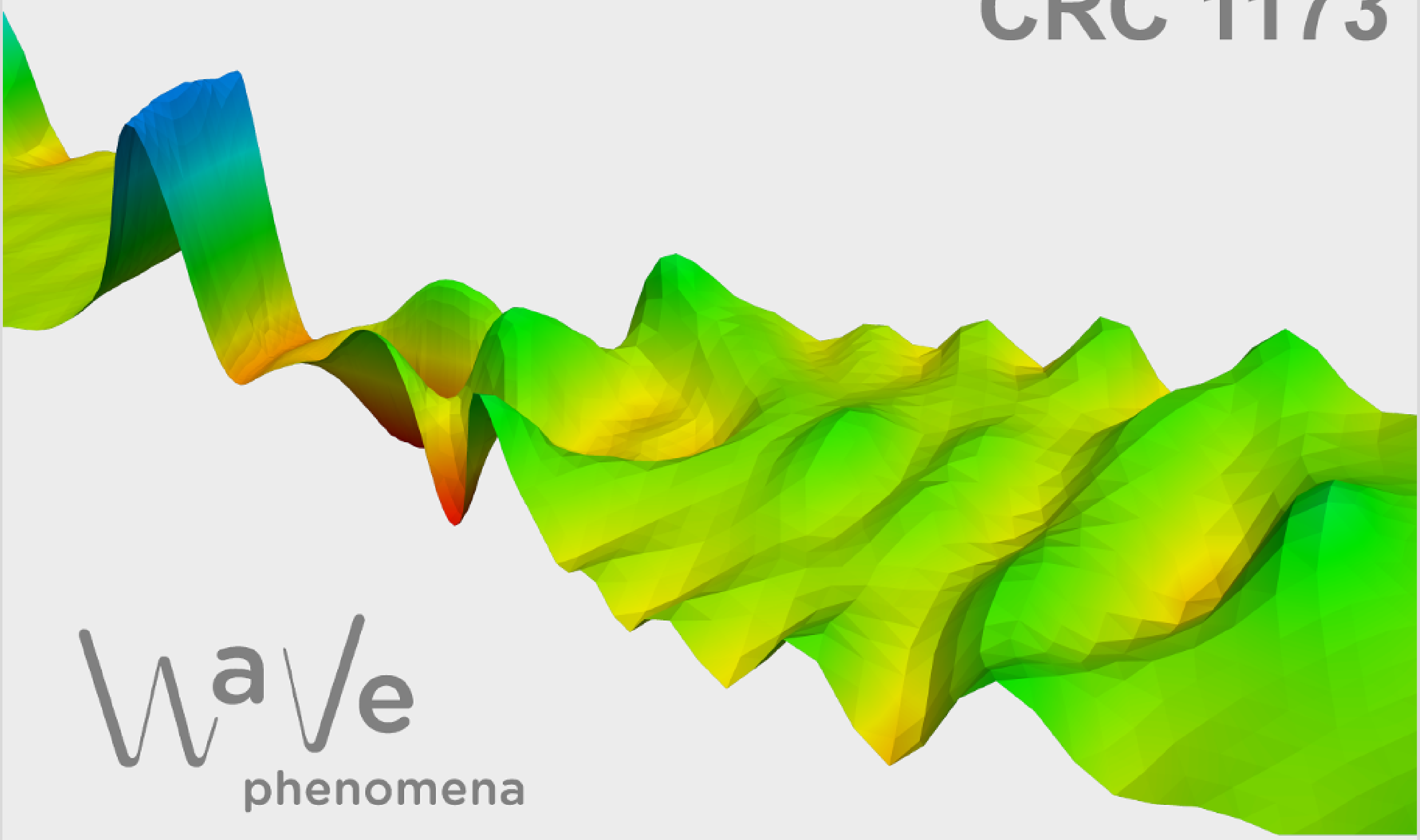

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GLOBAL STRICHARTZ ESTIMATES FOR AN INHOMOGENEOUS MAXWELL SYSTEM

PIERO D'ANCONA AND ROLAND SCHNAUBELT

Abstract. We show global-in-time Strichartz estimates for the isotropic Maxwell system with divergence free data. On the scalar permittivity and permeability we impose decay assumptions as $|x| \to \infty$ and a non-trapping condition. The proof is based on smoothing estimates in weighted L^2 spaces which follow from corresponding resolvent estimates for the underlying Helmholtz problem.

1. Introduction

This paper investigates a model for the propagation of electromagnetic waves in continuous media, the *Maxwell equations*

$$\mathbf{D}_t = \nabla \times \mathbf{H} - \mathbf{J}, \qquad \mathbf{B}_t = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = 0, \tag{1.1}$$

on $\mathbb{R}_t \times \mathbb{R}^3_x$ with *linear inhomogeneous* material laws

$$\mathbf{D} = \epsilon(x)\mathbf{E}, \qquad \mathbf{B} = \mu(x)\mathbf{H}, \tag{1.2}$$

and the (divergence free) current density $\mathbf{J} = \mathbf{J}(t, x)$. Here, $\mathbf{E}$ and $\mathbf{D}$ are the electric fields, $\mathbf{B}$ and $\mathbf{H}$ are the magnetic fields, and the permittivity ϵ and the permeability μ are positive scalar functions on $\mathbb{R}^3$. Hence the model is *isotropic*, i.e., the interaction of fields with matter depends on the location but not on the direction of the fields $\mathbf{D}, \mathbf{H}, \mathbf{E}, \mathbf{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$. We note that the divergence constraints follow from the evolution equations if the initial data $\mathbf{D}(0)$ and $\mathbf{B}(0)$ and the current $\mathbf{J}$ are divergence free.

The Maxwell system is the foundation of electromagnetic theory so that it is not necessary to recall the importance of model (1.1) and (1.2) in applications, including the classical case $\epsilon, \mu = \text{const}$. Despite the large literature devoted to the subject, see e.g. the monographs [8] and [17], many important questions are still unclear.

Global well posedness in Sobolev spaces H^s of the Cauchy problem for (1.1) follows from the general theory of hyperbolic systems, under rather weak conditions on the coefficients ϵ and μ. Here we are mainly interested in the asymptotic properties of solutions. Besides its inherent importance, information on the decay of the solutions is essential for the study of the corresponding nonlinear problems. In the constant coefficient case

$$\mathbf{E}_t = \nabla \times \mathbf{B} - \mathbf{J}, \qquad \mathbf{B}_t = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0,$$

with data

$$\mathbf{E}(0, x) = \mathbf{E}_0, \qquad \mathbf{B}(0, x) = \mathbf{B}_0,$$

solutions are easily seen to satisfy diagonal systems of wave equations

$$\Box \mathbf{E} = -\mathbf{J}_t, \qquad \Box \mathbf{B} = \nabla \times \mathbf{J}.$$

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Hence one can apply the well established theory on dispersive properties of wave equations. The strongest property is the pointwise decay

$$\|\mathbf{E}(t,\cdot)\|_{L^\infty} + \|\mathbf{B}(t,\cdot)\|_{L^\infty} \lesssim \left(\|\nabla \mathbf{E}_0\|_{L^1} + \|\nabla \mathbf{B}_0\|_{L^1}\right) \cdot |t|^{-1}, \tag{1.3}$$

where we set $\mathbf{J} = 0$. From (1.3) *Strichartz estimates* can be deduced. For all couples of *wave admissible* indices (p, q) and (r, s), that is to say

$$\tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{2}, \qquad p \in [2, \infty], \qquad q \in [2, \infty) \tag{1.4}$$

in dimension 3, we have

$$\||D|^{-\frac{2}{p}} D_{t,x}\mathbf{E}\|_{L^pL^q} + \||D|^{-\frac{2}{p}} D_{t,x}\mathbf{B}\|_{L^pL^q} \lesssim \|\nabla \mathbf{E}_0\|_{L^2} + \|\nabla \mathbf{B}_0\|_{L^2} + \|\mathbf{J}(0,\cdot)\|_{L^2} + \||D|^{\frac{2}{r}} D_{t,x}\mathbf{J}\|_{L^{r'}L^{s'}}$$

(see [19], [23]). Here we are using the notations $|D|^s u = \mathcal{F}^{-1}(|\xi|^s \widehat{u}(\xi))$, where $\mathcal{F}u = \widehat{u}$ is the Fourier transform, and $L^pL^q = L^p(\mathbb{R}; L^q(\mathbb{R}^3))$. An even weaker form of dispersion is expressed by the so called *smoothing estimates*

$$\|\langle x\rangle^{-1/2-}\mathbf{E}\|_{L^2L^2} + \|\langle x\rangle^{-1/2-}\mathbf{B}\|_{L^2L^2} \lesssim \|\mathbf{E}_0\|_{L^2} + \|\mathbf{B}_0\|_{L^2} \tag{1.5}$$

for $\mathbf{J} = 0$. (See e.g. [10] for a comprehensive framework for such estimates.)

Substantial work has been devoted in recent years to extend dispersive estimates to more general equations, including in particular equations with electromagnetic potentials or variable coefficients, and equations on manifolds (see among many others [20], [31], [33] for the Schrödinger equation; [9], [18], [14] for the wave equation; for wave equations with variable coefficients in highest order, [35], [34], [29]; concerning dispersive estimates, [36], [37], [38], [1], [21], [12]).

Astonishingly, only little is known about such estimates for the Maxwell system (1.1) and (1.2). In [15] *local-in-time* Strichartz estimates were shown for smooth scalar coefficients ϵ and μ being constant outside a compact set. For matrix valued coefficients the situation seems to be much more complicated, as already for constant matrices ϵ and μ the dispersive decay depends on the multiplicity of their eigenvalues, see [25], [26] and also [27]. Very recently, local-in-time Strichartz estimates with matrix valued (anisotropic) coefficients were shown in the two dimensional case, [32]. In the present work we are concerned with *global-in-time* Strichartz estimates for scalar ϵ and μ in C^2 under some decay assumptions as $|x| \to \infty$.

In our arguments we use a second-order formulation of (1.1) and (1.2). By a computation similar to the constant coefficient case, any solution $\mathbf{D}(t, x)$ to the problem (1.1) with (1.2) also solves the system

$$\mathbf{D}_{tt} + \nabla \times \tfrac{1}{\mu} \nabla \times \tfrac{1}{\epsilon}\mathbf{D} = -\mathbf{J}_t, \quad \nabla \cdot \mathbf{D} = 0, \qquad \mathbf{D}(0, x) = \mathbf{D}_0, \quad \mathbf{D}_t(0, x) = \nabla \times \tfrac{1}{\mu}\mathbf{B}_0 - \mathbf{J}(0). \tag{1.6}$$

The other fields satisfy similar equations, e.g., $\mathbf{B}$ satisfies an analogous system with ϵ and μ interchanged and modified data, namely

$$\mathbf{B}_{tt} + \nabla \times \tfrac{1}{\epsilon} \nabla \times \tfrac{1}{\mu}\mathbf{B} = \nabla \times \tfrac{1}{\epsilon}\mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0, \qquad \mathbf{B}(0, x) = \mathbf{B}_0, \quad \mathbf{B}_t(0, x) = -\nabla \times \tfrac{1}{\epsilon}\mathbf{D}_0. \tag{1.7}$$

The material laws (1.2) then imply

$$\mathbf{E}_{tt} + \tfrac{1}{\epsilon}\nabla \times \tfrac{1}{\mu} \nabla \times \mathbf{E} = -\tfrac{1}{\epsilon}\mathbf{J}_t, \quad \nabla \cdot (\epsilon\mathbf{E}) = 0, \quad \mathbf{E}(0) = \mathbf{E}_0, \quad \mathbf{E}_t(0) = \tfrac{1}{\epsilon}\nabla \times \mathbf{H}_0 - \tfrac{1}{\epsilon}\mathbf{J}(0), \tag{1.8}$$

$$\mathbf{H}_{tt} + \tfrac{1}{\mu}\nabla \times \tfrac{1}{\epsilon} \nabla \times \mathbf{H} = \tfrac{1}{\mu}\nabla \times \tfrac{1}{\epsilon}\mathbf{J}, \quad \nabla \cdot (\mu\mathbf{H}) = 0, \quad \mathbf{H}(0) = \mathbf{H}_0, \quad \mathbf{H}_t(0) = -\tfrac{1}{\mu}\nabla \times \mathbf{E}_0. \tag{1.9}$$

In this work we focus on (1.6). Equations (1.6) and (1.7) are essentially systems of wave equations with variable coefficients. Indeed, one can write

$$\epsilon\mu\nabla \times \tfrac{1}{\mu}\nabla \times \tfrac{1}{\epsilon}U = \nabla \times \nabla \times U - b(x, \partial)U$$

where $b(x, \partial)$ is the first-order matrix operator

$$b(x, \partial)U = (p + q) \times (\nabla \times U) + \nabla \times (p \times U) - (p + q) \times (p \times U) \tag{1.10}$$

with coefficients

$$p = \nabla \log \epsilon, \qquad q = \nabla \log \mu.$$

Here we heavily use that ϵ and μ are scalar. We also denote by $\widetilde{b}(x,\partial)$ the operator as in (1.10) with p and q interchanged:

$$\widetilde{b}(x,\partial)U = (p+q)\times(\nabla\times U) + \nabla\times(q\times U) - (p+q)\times(q\times U).$$

Since $\nabla\times\nabla\times\mathbf{D} = -\Delta\mathbf{D} + \nabla(\nabla\cdot\mathbf{D}) = -\Delta\mathbf{D}$, we see that (1.6) can be written as

$$\epsilon\mu\mathbf{D}_{tt} - \Delta\mathbf{D} - b(x,\partial)\mathbf{D} = -\epsilon\mu\mathbf{J}_t, \qquad \nabla\cdot\mathbf{D} = 0, \tag{1.11}$$

and similarly (1.7) is equivalent to

$$\epsilon\mu\mathbf{B}_{tt} - \Delta\mathbf{B} - \widetilde{b}(x,\partial)\mathbf{B} = \epsilon\mu\nabla\times\tfrac{1}{\epsilon}\mathbf{J}, \qquad \nabla\cdot\mathbf{B} = 0. \tag{1.12}$$

In other words, for scalar ϵ and μ the divergence constraint allows us to reduce (1.1) and (1.2) to a wave system with uncoupled principal part $(\epsilon\mu\partial_{tt} - \Delta)I_{3\times3}$.

The main goal of the paper is to prove the following estimates, which apply in particular to the fields solving the Maxwell system (1.1) and (1.2).

Theorem 1.1. *Let $\epsilon(x), \mu(x) : \mathbb{R}^3 \to \mathbb{R}$ and assume for some $\delta \in (0, 1/2)$ that*

(1) $\inf \epsilon\mu > 0$ *and* $(\epsilon\mu)'_- \leq \frac{1}{4}(1-2^{-\delta})^{-1}\epsilon\mu\langle x\rangle^{-1-\delta}$,

(2) $|\epsilon - 1| + |\mu - 1| \lesssim \langle x\rangle^{-2-\delta}$, $|\nabla\epsilon| + |\nabla\mu| \lesssim \langle x\rangle^{-\frac{3}{2}-\delta}$, *and* $|D^2\epsilon| + |D^2\mu| \lesssim \langle x\rangle^{-\frac{5}{2}-\delta}$.

Let $\mathbf{D}_0 = \epsilon\mathbf{E}_0$, $\mathbf{B}_0 = \mu\mathbf{H}_0$, and $\mathbf{J}$ be divergence free. Then the solution $\mathbf{D}$ to (1.6) *satisfies the Strichartz estimate*

$$\||D|^{-\frac{2}{p}}D_{t,x}\mathbf{D}\|_{L^pL^q} \lesssim \|\nabla\mathbf{D}_0\|_{L^2} + \|\nabla\mathbf{B}_0\|_{L^2} + \|J(0)\|_{L^2} + \||D|^{\frac{2}{r}}\mathbf{J}_t\|_{L^{r'}L^{s'}}$$

for all wave admissible (p,q) and (r,s). The solution $\mathbf{B}$ to (1.7) *fulfill*

$$\||D|^{-\frac{2}{p}}D_{t,x}\mathbf{B}\|_{L^pL^q} \lesssim \|\nabla\mathbf{D}_0\|_{L^2} + \|\nabla\mathbf{B}_0\|_{L^2} + \||D|^{\frac{2}{r}}\nabla\mathbf{J}\|_{L^{r'}L^{s'}}.$$

Here we can replace $\mathbf{D}$ by $\mathbf{E}$ and $\mathbf{B}$ by $\mathbf{H}$, solving (1.8) *respectively* (1.9).

We briefly discuss the previous statements. In (1), the symbol $(a)'_- = \max\{-\partial_r a, 0\}$ denotes the negative part of the radial derivative, and $\langle x\rangle = (1+|x|^2)^{1/2}$. Wave admissible couples and the notations L^pL^q and $|D|^s$ have been defined above (see (1.4)).

The second assumption in (1) is our non-trapping condition. Note that this is a one–sided condition, affecting only the negative part of the radial derivative of $\epsilon\mu$; it is a kind of 'repulsivity' of the coefficients. It is well known that some hypothesis of this type is necessary to exclude trapped rays, which are an obstruction to global decay in time and even to the much weaker local energy decay. Many of our intermediate results are true under weaker decay assumptions than (2). For instance, our basic smoothing estimate (5.1) for the wave equation and the corresponding resolvent bound (4.6) are shown assuming condition (1), the decay

$$|\epsilon - 1| + |\mu - 1| + |D^2\epsilon| + |D^2\mu| \lesssim \langle x\rangle^{-2-\delta}, \qquad |\nabla\epsilon| + |\nabla\mu| \lesssim \langle x\rangle^{-1-\delta}, \tag{1.13}$$

and a non-resonance condition for the frequency $z = 0$ stated before Proposition 2.5. The extra decay in the above hypothesis (2) is needed to remove this non-resonance condition in Proposition 2.8, and also to establish certain Riesz-type bounds in Lemma 5.3 in (weighted) L^2 spaces which are crucial to derive the Strichartz estimates.

The proof of Theorem 1.1 is given at the end of the paper. It follows the general principle, pioneered in [31] and further developed in many works (e.g., [13], [16], [28], [34], [35]), that weak decay properties of solutions can be upgraded to much stronger decay, under suitable regularity and localization information on the coefficients. The main novelty of our paper is that we treat a system with variable coefficients in higher order terms. We explain our proofs in more detail.

For scalar wave equations, the paper [29] gives global Strichartz estimates if the coefficients are close to constants and decay as $|x| \to \infty$. (For derivatives the decay assumptions are similar to (1.13).) Moreover, local-in-time estimates are proven without the smallness condition. As we can put our problem in the form (1.11), we are able to apply these results after suitable localizations of our solution. Recall that the possibility to deduce global Strichartz estimates from local estimates combined with global local energy decay was discovered in [4]. The localization procedure introduces commutator terms which we must estimate in L^2L^2. These are controlled using the smoothing estimates in Propositions 5.4 and 5.5 which are based on (4.6). In this analysis, one must switch between homogeneous and inhomogeneous estimates; this requires TT^* arguments and suitable Riesz-type inequalities, see Lemma 5.3. To prove the latter, we use crucially the divergence conditions of the Maxwell system. On the other hand, we must avoid the usual TT^* argument since it would need Riesz' bounds in $\dot{H}_q^{-2/p}$ which are not available for our operator.

The necessary smoothing estimates are deduced directly from the resolvent bound (4.6) for the stationary problem, which also involves weighted L^2 norms, via Plancherel's Theorem. In principle, here we follow the general framework of Kato smoothing (see [10]). However we cannot apply the general theory since we have to work with the operator $L(z) = \epsilon\mu z^2+\Delta+b(x,\partial)$ without divergence constraint when showing the resolvent estimates. Since the operator $\Delta + b(x,\partial)$ is not self adjoint, the Kato theory can not be applied directly.

We prove the resolvent estimates by splitting into three different regimes: bounded frequencies, which we handle via compactness arguments, see Section 2; large frequences and large x, via Morawetz type estimates, see Section 3.1; and large frequences on a compact region of space via Carleman estimates, see Section 3.2. In the step for small frequencies one has to exclude eigenvectors and resonances of $L(z)$. Here it is crucial to show that such functions have to be divergence free, which is proved in the relevant Propositions 2.6, 2.7, and 2.8 using the structure of (1.11).

2. Low frequencies

We first prove a resolvent estimate which is valid for all values of the complex frequency, but with a constant $C(z)$ which may grow as $|z| \to \infty$. Hence, we will use this estimate only for z in a suitably chosen compact region. In the next section we shall prove a uniform estimate for large $|z|$. Except for the final result, in the present section the space dimension is $n \geq 3$, however in this paper we shall only need $n = 3$.

We shall apply a few variations of the following standard argument. Suppose a reference operator H_0 satisfies, for z in an open domain $\Omega \subseteq \mathbb{C}$, a resolvent estimate

$$\|R_0(z)v\|_{B_1} \leq C(z)\|v\|_{B_2}, \qquad R_0(z) = (H_0 + z)^{-1},$$

where B_1 and B_2 are some Banach spaces. Suppose also that

- H is a relatively compact perturbation of H_0, meaning that the operator $K(z) = (H - H_0)R_0(z)$ extends to a bounded and compact operator on B_2,
- $z \mapsto K(z)$ is continuous in the operator norm.

Then we can write

$$H + z = (H - H_0) + H_0 + z = (I + (H - H_0)R_0(z))(H_0 + z) = (I + K(z))(H_0 + z).$$

Let the operator $I + K : B_2 \to B_2$ be *injective*. Then it is also bijective since it is Fredholm. Moreover, the operator norm of $(I+K(z))^{-1}$ is locally bounded for $z \in \Omega$. This type of argument is classical on weighted L^2 spaces, see e.g. Theorem VI.14 in [30], and it holds more generally in Banach spaces (a fact likely rediscovered several times, see e.g Lemma 3.4 in [11]). As a consequence, we can invert $H + z$ for all values of $z \in \Omega$ and the resolvent estimate holds also for H, in the form

$$\|(H+z)^{-1}v\|_{B_1} \le C'(z)\|v\|_{B_2},$$

with a different $C'(z)$, which is locally bounded for $z \in \Omega$, but otherwise undetermined.

We first look at the operator without lower order terms $b(x,\partial)$ starting with a basic resolvent estimate outside the spectrum for

$$R(z) = (\Delta + az)^{-1}, \qquad z \in \mathbb{C} \setminus [0,+\infty).$$

Proposition 2.1. *Assume that $a \in L^\infty$, $a > 0$, $\lim_{|x|\to+\infty} a(x) = 1$ and $z \in \mathbb{D} = \mathbb{C} \setminus [0,+\infty)$. Then $\Delta + az : H^2 \to L^2$ is a bijection and $R(z) := (\Delta + az)^{-1}$ satisfies*

$$\|R(z)f\|_{H^2} \le C(z)\|f\|_{L^2}$$

for some continuous function $C : \mathbb{D} \to \mathbb{R}^+$.

Proof. Let $z \in \mathbb{D}$ and $R_0(z) = (z + \Delta)^{-1}$. We can write

$$\Delta + az = \Delta + z + (a-1)z = (I + (a-1)zR_0(z))(\Delta + z). \tag{2.1}$$

The operator $K(z) = (a-1)zR_0(z)$ is bounded and compact on L^2. We prove that $I + K(z)$ is injective for each $z \in \mathbb{D}$. Assume that $(I + K)u = 0$. Setting $v = R_0(z)u$, we have $v \in H^2$ and

$$(\Delta + za)v = 0 \quad \text{which implies} \quad \textstyle\int |\nabla v|^2 - z\int a|v|^2 = 0.$$

If $\Im z \neq 0$, taking the imaginary part we infer $v = 0$ and hence $u = (\Delta + z)v = 0$. If $\Im z = 0$ so that $z = -\lambda \in (-\infty, 0)$, we obtain

$$\textstyle\int |\nabla v|^2 + \lambda \int a|v|^2 = 0$$

and this implies again $v = 0$.

Thus by analytic Fredholm theory we can invert $I + K(z)$ on L^2 and the operator norm of $(I + K(z))^{-1}$ is locally bounded in $z \in \mathbb{D}$. The claim follows writing

$$(\Delta + az)^{-1} = R_0(z)(I + K(z))^{-1}$$

and using the elementary estimate

$$\|R_0(z)v\|_{H^2} \le C(z)\|v\|_{L^2}, \qquad C(z) = Cd(z,\mathbb{R}^+)^{-1},$$

and the bound on $(I + K(z))^{-1}$. Note that $C(z)$ blows up as $z \to \mathbb{R}^+$. □

The next step is a *limiting absorption principle* for $R(z)$, where the limits of $R(z)$ as $\pm\Im z \downarrow 0$ exist in a suitable topology. In the following, we commit a slight abuse of notation since for $\lambda \in \sigma(-\Delta) = [0,+\infty)$ there are *two* extensions $R_0(\lambda \pm i0)$ of the resolvent, and we shall denote both limits with the same notation $R_0(z)$ for the sake of terseness. The limiting absorption principle for the free Laplacian is expressed by the uniform estimate

$$\|R_0(z)f\|_X + |z|^{1/2}\|R_0(z)f\|_Y + \|\nabla R_0(z)f\|_{\dot Y} \le C\|f\|_{Y^*}$$

valid for all $z \in \mathbb{C}$, with a constant independent of z. Here the norms of X, Y and Y^* are defined as follows: Y^* is the predual of Y, while

$$\|v\|_X^2 := \sup_{R>0} \tfrac{1}{\langle R\rangle^2} \textstyle\int_{\{|x|=R\}} |v|^2 dS, \qquad \|v\|_Y^2 := \sup_{R>0} \tfrac{1}{\langle R\rangle} \int_{\{|x|\le R\}} |v|^2 dx.$$

We shall also need the (stronger) homogeneous norms

$$\|v\|_{\dot X}^2 = \sup_{R>0} \tfrac{1}{R^2} \textstyle\int_{\{|x|=R\}} |v|^2 dS, \qquad \|v\|_{\dot Y}^2 = \sup_{R>0} \tfrac{1}{R} \int_{\{|x|\le R\}} |v|^2 dx. \tag{2.2}$$

We note the equivalent expressions in terms of dyadic norms

$$\|v\|_Y \eqsim \|\langle x\rangle^{-\frac12} v\|_{\ell^\infty L^2}, \qquad \|v\|_{Y^*} \eqsim \|\langle x\rangle^{\frac12} v\|_{\ell^1 L^2}, \qquad \|v\|_X \eqsim \|\langle x\rangle^{-1} v\|_{\ell^\infty L^\infty L^2}, \tag{2.3}$$

writing (using polar coordinates in the last term)

$$\|v\|_{\ell^\infty L^2} = \sup_{j\geq 0} \|v\|_{L^2(A_j)}, \qquad A_0 = \{|x| \leq 1\}, \quad A_j = \{2^{j-1} \leq |x| \leq 2^j\},$$

$$\|v\|_{\ell^1 L^2} = \sum_{j\geq 0} \|v\|_{L^2(A_j)}, \qquad \|v\|_{\ell^\infty L^\infty L^2} = \sup_{j\geq 0} \|v\|_{L^\infty_{|x|} L^2_\omega(A_j)}.$$

These norms can be considered as sharp versions of weighted L^2 norms. Indeed, it is easy to check the inequalities

$$\begin{aligned} &\|\langle x\rangle^{-\frac{1}{2}-\delta} v\|_{L^2} \leq C(\delta)\|v\|_Y, \qquad \|v\|_{Y^*} \leq C(\delta)\|\langle x\rangle^{\frac{1}{2}+\delta} v\|_{L^2}, \\ &\|\langle x\rangle^{-\frac{3}{2}-\delta} v\|_{L^2} \leq C(\delta)\|v\|_X, \qquad \|\langle x\rangle^{-1} v\|_Y \leq \|v\|_X \end{aligned} \tag{2.4}$$

for all $\delta > 0$.

In the next lemma we collect the relevant estimates for the free Laplacian. We write them at the point z^2 with $\Im z \geq 0$, thus covering the entire complex plane for both sides of $[0, +\infty)$. (In later sections it will be convenient to use z^2.) We set $\widehat{x} = |x|^{-1}x$ for $x \in \mathbb{R}^n \setminus \{0\}$.

Lemma 2.2. *Let $z \in \mathbb{C}$ with $\Im z \geq 0$. Then we have, with constants independent of z,*

$$\|R_0(z^2)f\|_X + \|zR_0(z^2)f\|_Y + \|\nabla R_0(z^2)f\|_Y \leq C\|f\|_{Y^*}, \tag{2.5}$$

$$\|(\nabla - i\widehat{x}z)R_0(z^2)f\|_{L^2} \leq C\||x|f\|_{L^2}. \tag{2.6}$$

Moreover, for $s \in [\frac{1}{2}, 1]$ we have, with C independent of s and z,

$$\|\langle x\rangle^{s-1}(\nabla - i\widehat{x}z)R_0(z^2)f\|_{\ell^\infty L^2} \leq C\|\langle x\rangle^s f\|_{\ell^1 L^2}. \tag{2.7}$$

Proof. Estimate (2.5) is essentially the classical Agmon–Hörmander estimate, which is uniform in z in the special case of the operator Δ. See e.g. [7] for a complete proof.

Consider now (2.6). Take $f \in L^2$ with $|x|f \in L^2$. The restriction that $f \in L^2$ can be removed by approximation. Define $u = R_0(\lambda + i\eta)f$, so that $(\Delta + \lambda + i\eta)u = f$. We multiply this equation by $\overline{u}$, take the imaginary and the real part of the resulting identity, and integrate over $\mathbb{R}^n$. We then obtain (see (3.6) and (3.8) below for a similar computation)

$$\eta\|u\|_{L^2}^2 = \Im \textstyle\int f\overline{u}, \qquad \|\nabla u\|_{L^2}^2 = \lambda\|u\|_{L^2}^2 - \Re \textstyle\int f\overline{u}. \tag{2.8}$$

If $\lambda \leq 2|\eta|$, these equations imply

$$\|\nabla u\|_{L^2}^2 \leq 2|\eta|\|u\|_{L^2}^2 + \|f\overline{u}\|_{L^1} \leq 3\|f\overline{u}\|_{L^1},$$

and hence

$$(|\eta| + |\lambda|)\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \lesssim \|f\overline{u}\|_{L^1} \leq \||x|f\|_{L^2}\||x|^{-1}u\|_{L^2}.$$

Using Hardy's inequality $\||x|^{-1}u\|_{L^2} \lesssim \|\nabla u\|_{L^2}$, we conclude

$$(|\eta| + |\lambda|)\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \lesssim \||x|f\|_{L^2}^2.$$

We write this estimate in terms of $z^2 = \lambda + i\eta$. Note that if $\arg z \in [\frac{\pi}{8}, \pi - \frac{\pi}{8}]$, then $\arg z^2 \in [\frac{\pi}{4}, 2\pi - \frac{\pi}{4}]$, i.e., $\lambda \leq |\eta|$. We have thus proved

$$\|zR_0(z^2)f\|_{L^2} + \|\nabla R_0(z^2)f\|_{L^2} \lesssim \||x|f\|_{L^2} \quad \text{provided} \quad \arg z \in [\tfrac{\pi}{8}, \pi - \tfrac{\pi}{8}]. \tag{2.9}$$

This estimate obviously yields

$$\|(\nabla - i\widehat{x}z)R_0(z^2)f\|_{L^2} \lesssim \||x|f\|_{L^2} \tag{2.10}$$

for the same values of z. Next, we consider the region $\arg z \in [0, \frac{\pi}{8}] \cup [\pi - \frac{\pi}{8}, \pi]$, i.e., $\arg z^2 = \lambda + i\eta \in [0, \frac{\pi}{4}] \cup [2\pi - \frac{\pi}{4}, 2\pi]$ or equivalently $0 \leq |\eta| \leq \lambda$. Proposition 3.1 in [3] shows that

$$\|(\nabla - i\widehat{x}\sqrt{\lambda})R_0(\lambda + i\eta)f\|_{L^2} \lesssim \||x|f\|_{L^2} \tag{2.11}$$

with a constant independent of η and λ. Setting $u = R_0(\lambda + i\eta)f$ and $v = e^{-i|x|\sqrt{\lambda}}u$, we have $\nabla v = e^{-i|x|\sqrt{\lambda}}(\nabla - i\widehat{x}\sqrt{\lambda})u$. By Hardy's inequality, estimate (2.11) implies

$$\||x|^{-1}R_0(\lambda + i\eta)f\|_{L^2} = \||x|^{-1}u\|_{L^2} = \||x|^{-1}v\|_{L^2} \lesssim \|\nabla v\|_{L^2} \lesssim \||x|f\|_{L^2}.$$

From the first part of (2.8) we then deduce

$$|\eta|\,\|R_0(\lambda + i\eta)f\|_{L^2}^2 \leq \||x|f\|_{L^2}\||x|^{-1}u\|_{L^2} \lesssim \||x|f\|_{L^2}^2.$$

Observe that for $\lambda + i\eta = z^2$ and $0 \leq |\eta| \leq \lambda$ we have

$$|\sqrt{\lambda} - z| = |(\Re z^2)^{1/2} - z| \leq \sqrt{|\eta|}.$$

The previous estimates thus lead to

$$\|(\nabla - iz\widehat{x})R_0(z^2)f\|_{L^2} \leq \|(\nabla - i\sqrt{\lambda}\widehat{x})R_0(z^2)f\|_{L^2} + \sqrt{|\eta|}\|R_0(z^2)f\|_{L^2} \lesssim \||x|f\|_{L^2}.$$

Combined with (2.10), we see that (2.6) holds uniformly in z for all $\Im z \geq 0$.

For the last assertion, we note that (2.7) for $s = 1$ follows from (2.6). If $s = \frac{1}{2}$, inequalities (2.3) and (2.5) yield

$$\|\langle x\rangle^{-\frac{1}{2}}(\nabla - i\widehat{x}z)R_0(z^2)f\|_{\ell^\infty L^2} \leq C\|(\nabla - i\widehat{x}z)R_0(z^2)f\|_Y \leq C\|f\|_{Y^*} \leq C\|\langle x\rangle^{\frac{1}{2}}f\|_{\ell^1 L^2}.$$

Real interpolation between the cases $s = \frac{1}{2}$ and $s = 1$ then gives (2.7). □

We now prove the limiting absorption principle for $\Delta + az^2$. As for $R_0(z)$, the two extensions on the positive reals for $\Im z \downarrow 0$ and for $\Im z \uparrow 0$ are different, but for simplicity we will use the same notation $R(z)$ for both. The weighted L^2 space with norm $\|\langle x\rangle^s u\|_{L^2}$ is denoted by L^2_s.

Proposition 2.3. *Assume $\langle x\rangle^{2+\delta}(a-1) \in L^\infty$ for some $\delta > 0$. Then $R(z)$ satisfies the estimate*

$$\|R(z^2)f\|_X + \|zR(z^2)f\|_Y + \|\nabla R(z^2)f\|_Y \leq C(z)\|f\|_{Y^*} \tag{2.12}$$

for all $\Im z \geq 0$ and for some continuous $C(z)$. Let $s' < s$ in $(1/2, 1]$ and $\langle x\rangle^{s+\frac{3}{2}+\delta}(a-1) \in L^\infty$. We then have

$$\|\langle x\rangle^{s'-1}(\nabla - i\widehat{x}z)R(z^2)f\|_{L^2} \leq C(s', s, z)\|\langle x\rangle^s f\|_{L^2}. \tag{2.13}$$

Moreover, for $f \in L^2_s$ there exists $g \in L^2_s$ with $R(z^2)f = R_0(z^2)g$.

Proof. We shall use the inequalities

$$\|u\|_{Y^*} \lesssim \|\langle x\rangle^{1+\delta}u\|_Y, \qquad \|u\|_{Y^*} \lesssim \|\langle x\rangle^{2+\delta}u\|_X. \tag{2.14}$$

valid for any $\delta > 0$, see (2.4). Let $K(z) = (a-1)z^2R_0(z^2)$. The operator $\langle x\rangle^{-2-\delta}zR_0(z^2)$ is compact on Y^* and bounded uniformly in z, as it follows from estimates (2.5) and (2.14) (or as a special case of Lemma 3.1 in [11]). Writing $K(z) = \langle x\rangle^{2+\delta}(a-1)z \cdot \langle x\rangle^{-2-\delta}zR_0(z^2)$ we see that $K(z) : Y^* \to Y^*$ is also a compact operator for each $z \in \mathbb{C}$ whose operator norm is locally bounded in $z \in \mathbb{C}$.

We next prove that $I + K(z) : Y^* \to Y^*$ is injective. Thus assume $(I + K(z))v = 0$ for some $v \in Y^* \hookrightarrow L^2$. Let $u = R_0(z^2)v$ so that $u \in Y \cap H^2_{loc}$ if $z \neq 0$, $u \in X \cap H^2_{loc}$ if $z = 0$, and u satisfies $\Delta u + az^2u = 0$. If $z = 0$ this means that $u \in X$ is harmonic, hence $v = 0$. If $\Im z^2 \neq 0$ or $z^2 < 0$, we have $u = R_0(z^2)v \in H^2$. Proposition 2.1 now yields $u = 0 = v$. Finally, if $z^2 = \lambda > 0$ then u satisfies

$$(\Delta + \lambda)u + \lambda(a-1)u = 0.$$

Regarding $W(x) = \lambda(a-1)$ as a potential with $|x|^2\langle x\rangle^{\delta/2}W \in \ell^1 L^\infty$, Lemma 3.3 in [11] shows that $v = 0$. Then (2.12) follows from (2.5) as before by analytic Fredholm theory and the representation $R(z^2) = R_0(z^2)(I + K(z))^{-1}$.

Consider now the radiation estimate (2.13) assuming $\langle x\rangle^{s+\frac{3}{2}+\delta}(a-1) \in L^\infty$. We transfer estimate (2.7) for R_0 to the perturbed resolvent R, using the representation $R(z^2) = R_0(z^2)(I + K(z))^{-1}$. In view of (2.3) and (2.7), we only have to prove that $I + K(z)$ is an invertible operator on the weighted space L^2_s with norm $\|\langle x\rangle^s f\|_{L^2}$. Note that we have already shown that $I+K(z)$ is injective on the larger space Y^*. It thus it remains to check that $K(z)$ is compact on L^2_s. We can write

$$K(z) = \langle x\rangle^{\frac{3}{2}+\delta}(a-1)z\cdot\langle x\rangle^{\frac{1}{2}}\cdot\langle x\rangle^{-2-\delta}zR_0(z^2).$$

Observe that $\langle x\rangle^{\frac{3}{2}+\delta}(a-1)z$ is a bounded operator from L^2 to L^2_s since $\langle x\rangle^{s+\frac{3}{2}+\delta}(a-1)\in L^\infty$, $\langle x\rangle^{\frac{1}{2}}$ is bounded from Y^* to L^2 by (2.3), and $\langle x\rangle^{-2-\delta}R_0(z)$ is compact on Y^* because of (2.5) and (2.14). Summing up, $K(z): Y^*\to L^2_s$ is compact and due to the embedding $L^2_s\hookrightarrow Y^*$ it is also compact on L^2_s.

The final claim is a consequence of the representation $R(z^2)=R_0(z^2)(I+K(z))^{-1}$ and of the bijectivity of $I+K(z)$ on L^2_s for the above values of s. □

Note that writing $\Delta R(z)f = f - azR(z)f$, Proposition 2.3 also yields

$$\|\Delta R(z^2)f\|_Y \le \|f\|_Y + |z|C(z)\|f\|_{Y^*} \le C_1(z)\|f\|_{Y^*}$$

where we used the inequality $\|f\|_Y\lesssim\|f\|_{Y^*}$, cf. (2.3). This gives the complete estimate

$$\|R(z^2)f\|_X + \|zR(z^2)f\|_Y + \|\nabla R(z^2)f\|_Y + \|\Delta R(z^2)f\|_Y \le C(z)\|f\|_{Y^*}. \tag{2.15}$$

Finally we consider the case of the full operator

$$L(z) = \Delta + a(x)z^2 + b(x,\partial).$$

In the following, we actually treat a more general *matrix operator*

$$L(z) = I_3\Delta + I_3 a(x)z^2 + b(x,\partial).$$

Here I_3 is the 3×3 identity matrix so that the principal part is a diagonal Laplacian operator. Moreover, $b(x,\partial)$ is a 3×3 matrix first-order operator subject to conditions as in the scalar case. It will be clear from the proofs that in our setting no change is required in the matrix case.

In order to perform the usual injectivity step, we shall make the following spectral assumption saying that $L(z)$ has no resonances or eigenvalues. See Remark 2.4 and Propositions 2.6, 2.7, and 2.8 below for a closer examination of these conditions. There we show that these conditions only lead to mild extra conditions when establishing our main results on the Strichartz estimates for the Maxwell system. Actually, these extra conditions are only needed to exclude a resonance at $z=0$, see Proposition 2.8.

Spectral assumption (S). Let $\Im z\ge 0$. Then $L(z)u=0$ implies $u=0$, provided

(1) either $z\notin\mathbb{R}$ and $u\in H^2$ (no eigenvalues)
(2) or $z\in\mathbb{R}$ and $u=R_0(z^2)f$ for some $\langle x\rangle^{\frac{1}{2}+}f\in L^2$ (no embedded resonances).

Note that $u\in R_0(z^2)Y^*$ satisfies $\nabla u,\Delta u\in Y$, and $u\in X$ (and $u\in Y$ if $z\neq 0$) by Lemma 2.2.

We briefly discuss condition (2) for $z=0$ (no resonances at 0). It is necessary since the presence of resonances competes with dispersion, a well studied effect since [22]. If $\langle x\rangle^{\frac{1}{2}+\sigma}f\in L^2$ then $u=\Delta^{-1}f$ satisfies $\langle x\rangle^{-\frac{1}{2}-\sigma'}u\in L^2$ for all $\sigma'>0$, thus our non-resonance assumption is slightly weaker than the usual one.

Remark 2.4. Assumption (S) is satisfied for z sufficiently large with respect to the coefficients. This is a consequence of estimate (4.6) in the next section.

Moreover, the non-resonance assumption is *generic* in the following sense. We take a parameter $\omega\in\mathbb{R}\setminus 0$ and consider the modified operator $\Delta+\omega b$. Under the previous assumptions on ϵ and μ, then the set of values ω such that $\Delta+\omega b$ has a resonance at 0 is *discrete*. Indeed, one easily checks that 0 is a resonance for $\Delta+\omega b$ if and only if $-\omega^{-1}$ is an eigenvalue for the compact operator $b(x,\partial)\Delta^{-1}$ on the weighted L^2 space with weight $\langle x\rangle^{\frac{1}{2}+\sigma}$.

Proposition 2.5. *Let $L(z) = I_3\Delta + I_3 a(x)z^2 + b(x,\partial)$ with $|x|^2\langle x\rangle^\delta(a-1) \in L^\infty$ and $b(x,\partial)$ a first-order matrix differential operator satisfying*

$$|b(x,\partial)v| \leq C_b(\langle x\rangle^{-2-\delta}|v| + \langle x\rangle^{-1-\delta}|\nabla v|) \tag{2.16}$$

for some $C_b, \delta > 0$. Assume $L(z)$ satisfies the spectral assumption (S). Then for $\Im z \geq 0$ we have

$$\|u\|_X + \|zu\|_Y + \|\nabla u\|_Y + \|\Delta u\|_Y \leq C(z)\|\langle x\rangle^{\frac{1}{2}+}L(z)u\|_{L^2}. \tag{2.17}$$

Proof. As before we write

$$L(z) = (I + K(z))(\Delta + az^2), \qquad K(z) = b(x,\partial)R(z^2), \tag{2.18}$$

where $R(z) = (\Delta + az^2)^{-1}$ is the operator constructed in Proposition 2.3. Estimates (2.4) and (2.15) and the assumptions on the coefficients imply the compactness of $K(z)$ as an operator on $L^2_{1/2+}$ and the continuity of the map $z \mapsto K(z)$ in the operator norm.

To prove injectivity of $I+K(z)$, assume $f+K(z)f = 0$ for some $f \in L^2_{1/2+}$. Let $u = R(z^2)f$ so that u solves $L(z)u = 0$. Note that by the final claim of Proposition 2.3 we also have $u = R_0(z^2)g$ for some $g \in L^2_{1/2+}$. If $z \in \mathbb{R}$, assumption (S) yields $u = 0$ and hence $f = (\Delta + az^2)u = 0$. If $z \notin \mathbb{R}$, since $Y^* \subset L^2$ and $R(z^2) : L^2 \to H^2$, we see that u is actually an eigenfunction of $L(z)$, and again by (S) we deduce $u = 0$. The rest of the proof is similar to the previous ones. $\square$

The spectral assumption (S) holds if a and b have some additional structure that is present in our main goal, the Maxwell system in the second-order form (1.11). We first consider part (1) of (S) and exclude eigenvalues in the next result. Observe that the assumptions (2.19) and (2.25) imply condition (2.16) from Proposition 2.5, cf. (1.10). This fact is used below several times.

Proposition 2.6. *Assume that the coefficients in Proposition 2.5 have the form*

$$a(x) = \epsilon(x)\mu(x), \qquad b(x,\partial)u = \nabla\times\nabla\times u - \epsilon(x)\mu(x)\nabla\times(\tfrac{1}{\mu(x)}\nabla\times\tfrac{1}{\epsilon(x)}u), \tag{2.19}$$

where ϵ and μ are bounded and uniformly strictly positive. Then property (1) in the spectral assumption (S) is satisfied.

Proof. In the present case the equation $L(z)u = 0$ can be rewritten as

$$z^2\epsilon\mu u + \Delta u + \nabla\times\nabla\times u - \epsilon\mu\nabla\times(\tfrac{1}{\mu}\nabla\times\tfrac{1}{\epsilon}u) = 0$$

or equivalently

$$z^2 u + \tfrac{1}{\epsilon\mu}\nabla(\nabla\cdot u) - \nabla\times(\tfrac{1}{\mu}\nabla\times\tfrac{1}{\epsilon}u) = 0. \tag{2.20}$$

Assume that $z \notin \mathbb{R}$ and $u \in H^2$ is a solution of (2.20). By taking the divergence of the equation, we see that the function $\phi = \nabla\cdot u$ satisfies

$$z^2\phi + \nabla\cdot(\tfrac{1}{\epsilon\mu}\nabla\phi) = 0.$$

As $z \notin \mathbb{R}$, this equation implies $\phi = 0$ (i.e., u is divergence free) since the operator $\nabla\cdot(\frac{1}{\epsilon\mu}\nabla\phi)$ is selfadjoint and non negative as soon as the (real valued) coefficient $\epsilon\mu$ is bounded and strictly positive. Thus the equation $L(z)u = 0$ reduces to

$$z^2 u = \nabla\times(\tfrac{1}{\mu}\nabla\times\tfrac{1}{\epsilon}u), \qquad \nabla\cdot u = 0, \qquad u \in H^2. \tag{2.21}$$

It is now convenient to set

$$\mathbf{E} = u/\epsilon, \qquad \mathbf{H} = -(i\mu z)^{-1}\nabla\times\mathbf{E}, \tag{2.22}$$

so that $(\mathbf{E},\mathbf{H})$ are H^1 solutions of the stationary Maxwell system

$$i\epsilon z\mathbf{E} = \nabla\times\mathbf{H}, \quad i\mu z\mathbf{H} = -\nabla\times\mathbf{E}, \quad \nabla\cdot(\epsilon\mathbf{E}) = \nabla\cdot(\mu\mathbf{H}) = 0. \tag{2.23}$$

We integrate the identity

$$|\widehat{x}\times\mathbf{E}|^2+|\mathbf{H}|^2-|\widehat{x}\times\mathbf{E}+\mathbf{H}|^2=-2\Re(\widehat{x}\cdot\mathbf{E}\times\overline{\mathbf{H}})$$

over a sphere $|x|=R$. The divergence theorem then yields

$$\int_{|x|=R}[|\widehat{x}\times\mathbf{E}|^2+|\mathbf{H}|^2-|\widehat{x}\times\mathbf{E}+\mathbf{H}|^2]dS=-2\Re\int_{|x|\le R}\nabla\cdot(\mathbf{E}\times\overline{\mathbf{H}})dx.$$

Writing

$$\nabla\cdot(\mathbf{E}\times\overline{\mathbf{H}})=\overline{\mathbf{H}}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\overline{\mathbf{H}})=-iz\mu|\mathbf{H}|^2+i\overline{z}\epsilon|\mathbf{E}|^2,$$

we deduce

$$\int_{|x|=R}[|\widehat{x}\times\mathbf{E}|^2+|\mathbf{H}|^2]dS+2\Im z\int_{|x|\le R}[\epsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2]dx=\int_{|x|=R}|\widehat{x}\times\mathbf{E}+\mathbf{H}|^2dS. \tag{2.24}$$

If we integrate in R from 0 to $+\infty$, the RHS gives a finite contribution since $\mathbf{E},\mathbf{H}\in L^2$. As a consequence the second integral on the LHS must be 0 (recall that $\Im z>0$). We have proved that $\mathbf{E}=\mathbf{H}=0$ and in particular $u=0$. □

We next treat resonances at $z^2>0$ which requires more sophisticated tools.

Proposition 2.7. *Assume that the coefficients in Proposition 2.5 have the form* (2.19) *and satisfy* $\epsilon,\mu>0$ *as well as*

$$\langle x\rangle^{2+\delta}(|\epsilon-1|+|\mu-1|+|D^2\epsilon|+|D^2\mu|)+\langle x\rangle^{1+\delta}(|\nabla\epsilon|+|\nabla\mu|)\in L^\infty. \tag{2.25}$$

Then also property (2) in the spectral assumption (S) is satisfied if $z\in\mathbb{R}\setminus\{0\}$.

Proof. Let $z\in\mathbb{R}\setminus\{0\}$ so that $\lambda=z^2>0$. We take a solution u of $L(z)u=0$ of the form $u=R_0(\lambda)f$ for some $f\in L^2_{1/2+}\hookrightarrow Y^*$. In particular, from (2.15) with $a=1$ we know that $u,\nabla u,\Delta u\in Y$.

Proceeding as in the previous proposition, we see that $\phi=\nabla\cdot u\in Y$ satisfies

$$\lambda\phi+\nabla\cdot(\tfrac{1}{\epsilon\mu}\nabla\phi)=0$$

which can be written as

$$(\Delta+\lambda)\phi-\nabla\beta\cdot\nabla\phi+\lambda(\epsilon\mu-1)\phi=0,\qquad \beta=\ln(\epsilon\mu).$$

Setting $\phi=\sqrt{\epsilon\mu}\psi$, this equation is transformed into

$$(\Delta+\lambda)\psi+c(x)\psi=0,\qquad c(x)=\tfrac12\Delta\beta-\tfrac14|\nabla\beta|^2+\lambda(\epsilon\mu-1). \tag{2.26}$$

Condition (2.25) for some $\delta'>\delta$ implies that $|x|^2\langle x\rangle^\delta c(x)\in\ell^1L^\infty$ and $c\psi\in Y^*$. Lemma 3.3 in [11] thus yields $\psi=0$ and hence $\phi=0$.

We next show some decay of u. Since u is divergence free, as in Proposition 2.6 the equation $L(z)u=0$ is reduced to (2.21) with $z^2=\lambda$. Defining $(\mathbf{E},\mathbf{H})$ as in (2.22), with $\sqrt{\lambda}$ in place of z, we see that $(\mathbf{E},\mathbf{H})$ satisfy the Maxwell system (2.23) with $z=\sqrt{\lambda}>0$. Since $\Im z=0$, equations (2.24) and (2.22) imply

$$\begin{aligned}\int_{|x|=R}[|\widehat{x}\times\mathbf{E}|^2+|\mathbf{H}|^2]dS&=\int_{|x|=R}|\widehat{x}\times\mathbf{E}+\mathbf{H}|^2dS\\&=\int_{|x|=R}|\mu\sqrt{\lambda}|^{-2}|\nabla\times\mathbf{E}-i\mu\sqrt{\lambda}\widehat{x}\times\mathbf{E}|^2dS.\end{aligned}$$

Multiplying both sides by the (radial) function $\langle x\rangle^{s-1}$ and integrating in the radial variable, we arrive at

$$\|\langle x\rangle^{s-1}\widehat{x}\times\mathbf{E}\|_{L^2}+\|\langle x\rangle^{s-1}\mathbf{H}\|_{L^2}\le C(\mu)\lambda^{-1/2}\|\langle x\rangle^{s-1}(\nabla\times\mathbf{E}-i\mu\sqrt{\lambda}\widehat{x}\times\mathbf{E})\|_{L^2}.$$

Now the radiation estimate (2.7) with $s=\frac12+$ yields

$$\|\langle x\rangle^{-\frac12+}(\nabla-i\widehat{x}\sqrt{\lambda})R_0(\lambda)f\|_{L^2}\le C\|\langle x\rangle^{\frac12+}f\|_{L^2}. \tag{2.27}$$

By means of $\mathbf{E} = u/\epsilon$, we write

$$\nabla \times \mathbf{E} - i\mu\sqrt{\lambda}\widehat{x} \times \mathbf{E} = (\nabla \tfrac{1}{\epsilon}) \times u + \tfrac{i\sqrt{\lambda}}{\epsilon}(1-\mu)\widehat{x} \times u + \tfrac{1}{\epsilon}(\nabla \times u - i\sqrt{\lambda}\widehat{x} \times u). \tag{2.28}$$

We know that $u = R_0(\lambda)f$ for some $f \in Y^*$, so that $u \in X$ and $\sqrt{\lambda}u \in Y$ by (2.5). Condition (2.25) and (2.4) then imply that the first two terms on the RHS of $\langle x\rangle^{-\frac{1}{2}+}$ times (2.28) are bounded by $\|\langle x\rangle^{\frac{1}{2}+} f\|_{L^2}$. Using also (2.27), we derive

$$\|\langle x\rangle^{-\frac{1}{2}+}\widehat{x} \times \mathbf{E}\|_{L^2} + \|\langle x\rangle^{-\frac{1}{2}+}\mathbf{H}\|_{L^2} \le C(\epsilon,\mu,\lambda)\|\langle x\rangle^{\frac{1}{2}+} f\|_{L^2} < \infty.$$

This proves that $\langle x\rangle^{-\frac{1}{2}+}\mathbf{H}$ and hence $\langle x\rangle^{-\frac{1}{2}+}\nabla \times \mathbf{E}$ are contained in L^2. On the other hand, $\mathbf{E} = \epsilon^{-1}R_0(\lambda)f$ satisfies $\langle x\rangle^{-1/2-}\mathbf{E} \in L^2$ by (2.4). The condition $\nabla \cdot (\epsilon\mathbf{E}) = 0$ and the decay of $\nabla\epsilon$ thus give $\langle x\rangle^{-\frac{1}{2}+}\nabla \cdot \mathbf{E} \in L^2$. It follows that $\langle x\rangle^{-\frac{1}{2}+}\nabla\mathbf{E}$ is an element of L^2, which leads to $\langle x\rangle^{-\frac{1}{2}+}\nabla u \in L^2$ and the estimate

$$\|\langle x\rangle^{-\frac{1}{2}+}\nabla u\|_{L^2} \le C(\epsilon,\mu,\lambda)\|\langle x\rangle^{\frac{1}{2}+} f\|_{L^2} < \infty.$$

Recalling the original equation satisfied by u, we have

$$(\Delta + \lambda)u = -g$$

with $g = \lambda(a-1)u + b(x,\partial)u$ and $a = \epsilon\mu$. Since $u, \nabla u \in Y$, the decay assumption (2.25) and (2.4) yield $\langle x\rangle^{\frac{1}{2}+} g \in L^2$. By the radiation estimate (2.7) for $R_0(\lambda)$, we obtain that $\langle x\rangle^{-\frac{1}{2}+}(\nabla u - i\sqrt{\lambda}\widehat{x}u) \in L^2$ and in conclusion $\langle x\rangle^{-\frac{1}{2}+}u \in L^2$. Note that also $|x|^{-\frac{1}{2}+}u$ belongs to L^2.

To prove that $u = 0$, we use a Carleman estimate from Proposition 5 of [24] for the special case of the operator $\Delta + \lambda$ and of a function with $|x|^{-1/2+}u \in L^2$. There it is shown that

$$\|w\rho u\|_{L^2} + \|\tfrac{|x|w\rho}{h'(\ln|x|)+|x|}\nabla u\|_{L^2} \lesssim \|w(x)\rho^{-1}(\Delta + \lambda)u\|_{L^2}$$

where $w(x) = e^{h(\ln|x|)}$, $\varepsilon, \tau_1 > 0$ are small but fixed, and

$$h'(t) = \tau_1 + (\tau e^{t/2} - \tau_1)\tfrac{\tau^2}{\tau^2 + \varepsilon e^t}, \qquad \rho(|x|) = \Big(\tfrac{h'(\ln|x|)}{|x|^2}\Big(1 + \tfrac{h'(\ln|x|)^2}{|x|^2}\Big)\Big)^{1/4}.$$

The estimate is uniform in $\tau \ge \hat{\tau}$ for some $\hat{\tau} \ge 1$. We further set $\varphi(r) = h(\ln r)$ and note

$$\rho(r) = r^{-\frac{1}{4}}(\varphi'(r) + \varphi'(r)^3)^{\frac{1}{4}}.$$

We can write

$$\|w\rho^{-1}(\Delta + \lambda)u\|_{L^2} \lesssim \|w\rho^{-1}(\tfrac{1}{\epsilon\mu}\Delta + \lambda)u\|_{L^2} + \lambda\|w\rho^{-1}(\epsilon\mu - 1)u\|_{L^2}$$

and also

$$-\tfrac{1}{\epsilon\mu}\Delta u = \nabla \times \tfrac{1}{\mu}\nabla \times \tfrac{1}{\epsilon}u + L.O.T. = \lambda u + L.O.T.$$

Here the lower order terms are bounded by $\langle x\rangle^{-2-\delta}|u| + \langle x\rangle^{-1-\delta}|\nabla u|$ due to (1.10) and (2.25). We obtain

$$\|w\rho u\|_{L^2} + \|\tfrac{|x|w\rho}{h'(\ln|x|)+|x|}\nabla u\|_{L^2} \lesssim \|w\rho^{-1}(L.O.T.)\|_{L^2} + \lambda\|w\rho^{-1}(\epsilon\mu - 1)u\|_{L^2}.$$

To absorb the RHS by the left, we have to prove that the functions $m_1 = \rho^{-2}\langle x\rangle^{-2-\delta}$ and $m_2 = \rho^{-2}(1+\varphi')\langle x\rangle^{-1-\delta}$ are smaller than a certain constant $\eta > 0$ uniformly in x for a fixed large τ. This will yield $u = 0$ and thus the result. Let $r = |x|$. We first observe that

$$\begin{aligned}
&\varphi'(r) = \frac{h'(\ln r)}{r} = \frac{\tau^3 + \tau_1\varepsilon r^{\frac{1}{2}}}{\tau^2 r^{\frac{1}{2}} + \varepsilon r^{\frac{3}{2}}},\\
&m_1(x) \le \langle x\rangle^{-\frac{3}{2}-\delta}(\varphi'(r) + \varphi'(r)^3)^{-\frac{1}{2}} \le \langle x\rangle^{-\frac{3}{2}-\delta}\varphi'(r)^{-\frac{1}{2}} \le \langle x\rangle^{-\frac{1}{2}-\delta}\varphi'(r)^{-\frac{1}{2}},\\
&m_2(x) \le \langle x\rangle^{-\frac{1}{2}-\delta}\tfrac{1+\varphi'(r)}{(\varphi'(r)+\varphi'(r)^3)^{\frac{1}{2}}} = \langle x\rangle^{-\frac{1}{2}-\delta}\varphi'(r)^{-\frac{1}{2}} \le C\langle x\rangle^{-\frac{1}{2}-\delta}\tfrac{\tau r^{\frac{1}{4}} + \varepsilon^{\frac{1}{2}} r^{\frac{3}{4}}}{\tau^{\frac{3}{2}} + (\tau_1\varepsilon)^{\frac{1}{2}} r^{\frac{1}{4}}} =: m(x).
\end{aligned}$$

Let $r \geq r_0$ for some $r_0 \geq 1$ to be fixed below. We compute

$$m(x) \leq \frac{r^{-\frac{1}{4}-\delta}}{\tau^{\frac{1}{2}}} + \frac{\varepsilon^{\frac{1}{2}} r^{\frac{1}{4}-\delta}}{(\tau_1\varepsilon)^{\frac{1}{2}} r^{\frac{1}{4}}} \lesssim \tau^{-\frac{1}{2}} + r_0^{-\delta}.$$

uniformly for $\tau \geq 1$ and $r \geq r_0$. We can fix $r_0 \geq 1$ and $\tau_0 \geq \hat{\tau}$ such that $m(x) \leq \eta$ for all $\tau \geq \tau_0$ and $|x| \geq r_0$. Let now $|x| = r \leq r_0$. In similar way we estimate

$$m(x) \leq \frac{\tau r^{\frac{1}{4}} + \varepsilon^{\frac{1}{2}} r^{\frac{3}{4}}}{\tau^{\frac{3}{2}} + (\tau_1\varepsilon)^{\frac{1}{2}} r^{\frac{1}{4}}} \leq \tau^{-\frac{1}{2}} r_0^{\frac{1}{4}} + \tau^{-\frac{3}{2}} \varepsilon^{\frac{1}{2}} r_0^{\frac{3}{4}}.$$

Fixing a large $\tau \geq \tau_0$, we conclude that $m(x) \leq \eta$ and hence $m_1(x), m_2(x) \leq \eta$ for all x. □

It is possible to exclude also a resonance at $z^2 = 0$, provided the first derivatives of the coefficients decay a bit faster. We now use that the space dimension is $n = 3$ which did not play a role so far.

Proposition 2.8. *Assume the real-valued coefficients $\epsilon, \mu > 0$ satisfy* (2.19) *and*

$$|\epsilon - 1| + |\mu - 1| + |D^2\mu| + |D^2\epsilon| \lesssim \langle x\rangle^{-2-\delta}, \qquad |\nabla\epsilon| + |\nabla\mu| \lesssim \langle x\rangle^{-\frac{3}{2}-\delta} \tag{2.29}$$

for some $\delta \in (0, \frac{1}{2})$. Let $L(0)u = 0$ for some $u = \Delta^{-1}f$ and $f \in L^2_{1/2+}$. Then $u = 0$, so that spectral assumption (S) is true in view of Propositions 2.6 and 2.7.

Proof. 1) We have $\Delta u = f \in L^2_{1/2+} \hookrightarrow Y^*$ and hence $D^2 u \in L^2$. Moreover, Lemma 2.2 yields $\nabla u \in Y$ and $u \in X$. As before, we first consider the function $\phi = \nabla \cdot u$ which now fulfills the equation

$$\nabla \cdot (\tfrac{1}{\epsilon\mu}\nabla\phi) = 0, \qquad \text{i.e.,} \qquad \Delta\phi = \nabla\beta \cdot \nabla\phi, \qquad \beta = \ln(\epsilon\mu).$$

Starting from $\nabla\phi \in L^2$, we get $\Delta\phi \in L^2$ and then $\nabla\phi \in H^2_{loc}$, so that $\phi \in C^1$. By (2.29),

$$g = \nabla\beta \cdot \nabla\phi \quad \text{satisfies} \quad g \in L^2_{\frac{3}{2}-\frac{\delta}{2}}, \quad \nabla g \in L^2_{\frac{3}{2}+\delta}.$$

Note that this implies $\langle x\rangle^{\frac{3}{2}+\delta} g \in L^6$ because of

$$\|\langle x\rangle^{\frac{3}{2}+\delta} g\|_{L^6} \lesssim \|\nabla(\langle x\rangle^{\frac{3}{2}+\delta} g)\|_{L^2} \lesssim \|\langle x\rangle^{\frac{1}{2}+\delta} g\|_{L^2} + \|\langle x\rangle^{\frac{3}{2}+\delta}\nabla g\|_{L^2} < \infty.$$

Since $\phi = \Delta^{-1} g$, we can estimate

$$|\phi(x)| \lesssim \int \tfrac{|g(y)|}{|x-y|} dy \lesssim \|\langle x\rangle^{\frac{3}{2}-\delta/2} g\|_{L^2} (\int \tfrac{dy}{\langle y\rangle^{3-\delta}|x-y|^2})^{1/2} \lesssim |x|^{-1+\delta/2}$$

using the standard inequality

$$\int_{\mathbb{R}^n} \tfrac{dy}{\langle y\rangle^a |x-y|^b} \lesssim |x|^{n-a-b}$$

for $a, b \in (0, n)$ with $a + b > n$. In a similar way we obtain

$$|\nabla\phi(x)| \lesssim \int \tfrac{|g(y)|}{|x-y|^2} dy \lesssim \|\langle x\rangle^{\frac{3}{2}+\delta} g\|_{L^6} (\int \tfrac{dy}{\langle y\rangle^{(3+2\delta)3/5}|x-y|^{12/5}})^{5/6} \lesssim |x|^{-1-\delta}.$$

Together we have proved the decay

$$|\phi(x)| \lesssim \langle x\rangle^{-1+\delta/2}, \qquad |\nabla\phi(x)| \lesssim \langle x\rangle^{-1-\delta}. \tag{2.30}$$

Let χ be a radial cut-off function equal to 1 on $B(0,1)$ and with support in $B(0,2)$. Set $\chi_R(x) = \chi(R^{-1}x)$ for $R \geq 1$ and $\phi_R = \chi_R\phi$. We compute

$$\nabla(\alpha\nabla\phi_R) = 2\alpha\nabla\chi_R \cdot \nabla\phi + \nabla\alpha \cdot \nabla\chi_R\phi + \alpha\phi\Delta\chi_R, \qquad \alpha = (\epsilon\mu)^{-1}.$$

Multiply by ϕ_R and integrate by parts. The above estimates then imply

$$\int_{|x|\leq R} \alpha|\nabla\phi_R|^2 \lesssim \int_{R\leq|x|\leq 2R} (R^{-1}\langle x\rangle^{-2-\delta/2} + \langle x\rangle^{-1-\delta} R^{-1}\langle x\rangle^{-2+\delta} + \langle x\rangle^{-2+\delta} R^{-2})$$

(we used again (2.29)) and we deduce that for $R\to\infty$

$$\int_{|x|\le R}\alpha|\nabla\phi_R|^2\lesssim R^{-\delta/2}\to 0. \tag{2.31}$$

We conclude that $\nabla\phi=0$, and by the decay of ϕ we have $\nabla\cdot u=\phi=0$.

2) Using $\nabla\cdot u=0$, as in Proposition 2.6 the equation $L(0)u=0$ is reduced to

$$\nabla\times(\tfrac{1}{\mu}\nabla\times\tfrac{1}{\epsilon}u)=0\qquad\text{or equivalently}\qquad \Delta u=-b(x,\partial)u=:F. \tag{2.32}$$

We can write

$$|F|=|bu|\lesssim(|\nabla\mu|^2+|\nabla\epsilon|^2+|D^2\epsilon|)|u|+(|\nabla\epsilon|+|\nabla\mu|)|\nabla u|. \tag{2.33}$$

We have $\nabla u\in Y\subset L^2_{-1/2-\sigma}$, $u\in X\subset L^2_{-3/2-\sigma}$ (see (2.4)), and by assumption $\Delta u\in L^2_{1/2+\sigma}$, for some $\sigma>0$. Hypothesis (2.29) then yields that

$$\langle x\rangle^{\frac12+\delta-\sigma}|\Delta u|\lesssim\langle x\rangle^{-\frac32-\sigma}|u|+\langle x\rangle^{-\frac12-\sigma}|\nabla u|\ \in L^2.$$

(Actually, we only use condition (1.13) here.) We fix numbers $\frac{\delta}{2}>\gamma'>\gamma>\sigma$. By Hölder's inequality, $\Delta u=F$ belongs L^p with

$$\frac1p=\frac12+\frac{\frac12+\gamma'-\sigma}{3}<1.$$

Sobolev's embedding thus implies that

$$\nabla u\in L^q\quad\text{with}\quad\frac1q=\frac13+\frac{\gamma'-\sigma}{3}<\frac12,\qquad u\in L^r\quad\text{with}\quad\frac1r=\frac{\gamma'-\sigma}{3}.$$

We infer $\langle x\rangle^{\gamma-\sigma-\frac12}\nabla u\in L^2$ and $\langle x\rangle^{\gamma-\sigma-\frac32}u\in L^2$, so that $\langle x\rangle^{\frac12+\delta+\gamma-\sigma}\Delta u\in L^2$ by (1.13).

We can repeat the argument replacing $\frac12-\sigma+\delta$ by $\frac12-\sigma+\gamma+\delta$, and derive that $\langle x\rangle^{2\gamma-\sigma-\frac12}\nabla u$ and $\langle x\rangle^{2\gamma-\sigma-\frac32}u$ belong to L^2. This procedure can be started as long as $\frac12-\sigma+k\gamma+\gamma'<\frac32$. If $\frac12-\sigma+k\gamma+\gamma'\ge1$ we obtain $\nabla u\in L^2$ where the improvement stops for ∇u. For u we can achieve $\langle x\rangle^{-\frac12-\frac{\delta}{2}}u\in L^2$.

Assumption (2.29) now gives $F\in L^2_{(3+\delta)/2}$ and

$$\langle x\rangle^{\frac32+\delta}|F|\lesssim\langle x\rangle^{-\frac12}|u|+|\nabla u|.$$

The second term at the right belongs to L^6 since $\|\nabla u\|_{L^6}\lesssim\|\Delta u\|_{L^2}$ and $\Delta u\in L^2_{1/2+}$. For the first term we have

$$\|\langle x\rangle^{-\frac12}u\|_{L^6}\lesssim\|\nabla(\langle x\rangle^{-\frac12}u)\|_{L^2}\lesssim\|\langle x\rangle^{-\frac32}u\|_{L^2}+\|\langle x\rangle^{-\frac12}\nabla u\|_{L^2}<\infty$$

by the above decay properties. We infer that $\langle x\rangle^{\frac32+\delta}F\in L^6$. Thus we can repeat the argument in Step 1) and we obtain

$$|u(x)|\lesssim\langle x\rangle^{-1+\delta/2},\qquad|\nabla u(x)|\lesssim\langle x\rangle^{-1-\delta}. \tag{2.34}$$

For χ_R as above, the map $u_R=\chi_R u$ satisfies

$$\nabla\times(\tfrac1\mu\nabla\times\tfrac1\epsilon u_R)=\nabla\times(\tfrac1a\nabla\chi_R\times u)+\tfrac1\mu\nabla\chi_R\times(\nabla\times\tfrac1\epsilon u)$$

because of (2.32). Similar to (2.31), we conclude that

$$\int_{|x|\le R}|\nabla\times\tfrac1\epsilon u_R|^2\tfrac1\mu dx\lesssim R^{-\delta/2}$$

and hence $\nabla\times\frac1\epsilon u=0$. The Helmholtz decomposition thus yields $\frac1\epsilon u=\nabla\varphi$ with the potential

$$\varphi=\Delta^{-1}\nabla\cdot\tfrac1\epsilon u=\Delta^{-1}(\nabla\tfrac1\epsilon\cdot u),$$

where we employed again $\nabla \cdot u = 0$. Estimates (2.29) and (2.34) imply

$$|\varphi(x)| \lesssim \int \frac{dy}{\langle y\rangle^{\frac{5}{2}+\frac{\delta}{2}}|x-y|} \lesssim |x|^{-\frac{1}{2}-\frac{\delta}{2}}, \qquad |\nabla\varphi(x)| \lesssim \int \frac{dy}{\langle y\rangle^{\frac{5}{2}+\frac{\delta}{2}}|x-y|^2} \lesssim |x|^{-\frac{3}{2}-\frac{\delta}{2}}.$$

On the other hand, we have $0 = \nabla\cdot u = \nabla\cdot(\epsilon\nabla\varphi)$ which leads to

$$\int_{|x|\le R} \epsilon|\nabla\varphi|^2 dx = \Big|\int_{|x|=R} \epsilon\overline{\varphi}\widehat{x}\cdot\nabla\varphi dS\Big| \lesssim R^2 R^{-\frac{1}{2}-\frac{\delta}{2}} R^{-\frac{3}{2}-\frac{\delta}{2}} \lesssim R^{-\delta}.$$

As $R\to\infty$, we infer that $u = \epsilon\nabla\varphi = 0$. □

3. High frequencies

In the high frequency regime $|z| \gg 1$ we can prove more precise estimates, with the correct dependence on z of the constants. This will require a splitting of space variables in two domains: for large x we can use a Morawetz type estimate since lower order terms are small there, while for bounded x a modified Carleman estimate is sufficient. This kind of splitting has been used by several authors (see e.g. [6]).

3.1. **Morawetz estimate.** Assume $a(x) > 0$ and let

$$f = \Delta v + z^2 a(x) v, \qquad z^2 = \lambda + i\eta. \tag{3.1}$$

Here we may assume $\eta \ge 0$ since the case $\eta < 0$ is reduced to the first one by conjugating the equation. Then for all real valued ϕ and ψ we have the well known identities

$$\begin{aligned}\Re\nabla\cdot\{Q+P\} = &-\tfrac{1}{2}\Delta(\Delta\psi+\phi)|v|^2 + 2\partial_j v\partial_j\partial_k\psi\partial_k\overline{v} - \lambda a(x)\phi|v|^2 + \lambda\nabla\psi\cdot\nabla a|v|^2 \\ &+ \phi|\nabla v|^2 + 2\eta a(x)\Im[v\nabla\psi\cdot\nabla\overline{v}] + \Re([\Delta,\psi]+\phi)\overline{v}f],\end{aligned} \tag{3.2}$$

$$\nabla\cdot P = \phi|\nabla v|^2 - z^2 a(x)|v|^2\phi + f\overline{v}\phi - \tfrac{1}{2}\Delta\phi|v|^2 + i\Im(\overline{v}\nabla v\cdot\nabla\phi) \tag{3.3}$$

for the functions

$$\begin{aligned}Q &= \nabla v\ [\Delta,\psi]\overline{v} - \tfrac{1}{2}\nabla\Delta\psi|v|^2 - \nabla\psi|\nabla v|^2 + \nabla\psi a(x)\lambda|v|^2,\\ P &= \nabla v\phi\overline{v} - \frac{1}{2}\nabla\phi|v|^2.\end{aligned}$$

The quick way to check these identities is by expanding the derivatives of P and Q at the left hand side. In these computations we assume that the functions are sufficiently regular, and below we also need some integrability; these technical assumptions can be removed by approximation arguments. We rewrite (3.2) in the form

$$\Re\nabla\{Q+P\} + I_\eta + I_f = I_{\nabla v} + I_v \tag{3.4}$$

where

$$\begin{aligned}I_{\nabla v} &= 2\partial_j v\,(\partial_j\partial_k\psi)\,\overline{\partial_k v} + \phi|\nabla v|^2, \qquad I_v = -\tfrac{1}{2}\Delta(\Delta\psi+\phi)|v|^2 - \lambda a(x)\phi|v|^2 + \lambda\nabla\psi\cdot\nabla a|v|^2,\\ I_f &= -\Re(f\,[\Delta,\psi]\overline{v} + f\,\overline{v}\,\phi), \qquad I_\eta = -2\eta a(x)\Im(\overline{v}\,\nabla\psi\cdot\nabla v).\end{aligned}$$

1) We first deduce from (3.3) some easy estimates, where we now work in three space dimensions for simplicity. We take the imaginary part in (3.3) and integrate on $\mathbb{R}^3$. It follows

$$\eta\textstyle\int a(x)|v|^2\phi = \Im\int f\overline{v}\phi + \Im\int \overline{v}\nabla v\cdot\nabla\phi. \tag{3.5}$$

Choosing $\phi = 1$, we infer

$$\eta\|a^{1/2}v\|^2 = \Im\textstyle\int f\overline{v}. \tag{3.6}$$

Similarly, the real part of (3.3) yields

$$\textstyle\int\phi|\nabla v|^2 = \lambda\int a|v|^2\phi - \Re\int f\overline{v}\phi + \frac{1}{2}\int\Delta\phi|v|^2 \tag{3.7}$$

and with $\phi = 1$

$$\|\nabla v\|^2 = \lambda \|a^{1/2}v\|^2 - \Re \int f\overline{v}. \tag{3.8}$$

In order to estimate the term I_η in (3.4), we use (3.6) and (3.8) to deduce

$$\int I_\eta \leq 2|\eta| \|a^{1/2}\nabla\psi\|_{L^\infty} \|a^{1/2}v\|_{L^2} \|\nabla v\|_{L^2} \leq C|\eta|^{1/2} (\int |f\overline{v}|)^{1/2} (|\lambda| \|a^{1/2}v\|_{L^2}^2 + \int |f\overline{v}|)^{1/2}$$

with $C = 2\|a^{1/2}\nabla\psi\|_{L^\infty}$. Equation (3.6) then leads to

$$\int I_\eta \leq C(\int |f\overline{v}|)^{1/2} (|\lambda| \int |f\overline{v}| + |\eta| \int |f\overline{v}|)^{1/2},$$

and we arrive at the estimate

$$\int I_\eta \leq 2\|a^{1/2}\nabla\psi\|_{L^\infty} (|\lambda| + |\eta|)^{1/2} \|f\overline{v}\|_{L^1}. \tag{3.9}$$

2) In (3.5) we choose ϕ as

$$\phi(x) = 1 \quad \text{if } |x| \leq R, \quad \phi(x) = 2 - \tfrac{|x|}{R} \quad \text{if } R \leq |x| \leq 2R, \quad \phi(x) = 0 \quad \text{if } |x| \geq 2R. \tag{3.10}$$

We compute

$$\begin{aligned} |\eta| \int_{|x|\leq R} a|v|^2 &\leq \int_{|x|\leq 2R} |f\overline{v}| + \tfrac{1}{R} \int_{R\leq |x| \leq 2R} |v||\nabla v| \\ &\lesssim \int_{|x|\leq 2R} |f\overline{v}| + R\|v\|_{\dot{X}} \|\nabla v\|_{\dot{Y}}. \end{aligned} \tag{3.11}$$

Observe that we have used the homogeneous norms (2.2). Dividing by R and taking the supremum over $R > 0$, we obtain the estimate

$$|\eta| \|a^{1/2}v\|_{\dot{Y}}^2 \lesssim \||x|^{-1}f\overline{v}\|_{L^1}^2 + \|v\|_{\dot{X}} + \|\nabla v\|_{\dot{Y}}^2. \tag{3.12}$$

Next, take $\phi = \frac{1}{|x|\vee R}$ and note that

$$\Delta\phi = -\tfrac{1}{R^2}\delta_{|x|=R}.$$

For this ϕ, equation (3.7) implies

$$\int \tfrac{|\nabla v|^2 - \lambda a|v|^2}{|x|\vee R} dx + \tfrac{1}{2R^2} \int_{|x|=R} |v|^2 dS \leq \int \tfrac{|f\overline{v}|}{|x|\vee R} \leq \||x|^{-1}f\overline{v}\|_{L^1}. \tag{3.13}$$

To proceed, we have to distinguish three cases for λ. First, let $\lambda \leq 0$. We deduce

$$\tfrac{1}{R} \int_{|x|\leq R} (|\nabla v|^2 + a|\lambda||v|^2) dx + \tfrac{1}{2R^2} \int_{|x|=R} |v|^2 dS \leq \||x|^{-1}f\overline{v}\|_{L^1},$$

and thus, taking the supremum over $R > 0$,

$$\|\nabla v\|_{\dot{Y}}^2 + |\lambda| \|a^{1/2}v\|_{\dot{Y}}^2 + \|v\|_{\dot{X}}^2 \lesssim \||x|^{-1}f\overline{v}\|_{L^1}.$$

Combined with (3.12), this relation shows

$$\|\nabla v\|_{\dot{Y}}^2 + \|za^{1/2}v\|_{\dot{Y}}^2 + \|v\|_{\dot{X}}^2 \lesssim \||x|^{-1}f\overline{v}\|_{L^1} \qquad \text{for} \quad \lambda \leq 0. \tag{3.14}$$

If $\lambda \geq 0$, with a similar computation, from (3.13) we infer the inequality

$$\|\nabla v\|_{\dot{Y}}^2 + \|v\|_{\dot{X}}^2 \leq C_0(\||x|^{-1}f\overline{v}\|_{L^1} + |\lambda| \|a^{1/2}v\|_{\dot{Y}}^2) \qquad \text{for} \quad \lambda \geq 0$$

for a suitable constant $C_0 > 0$. Let now $\lambda \leq (2C_0)^{-1}|\eta|$. As $\|\cdot\|_{\dot{Y}} \geq \|\cdot\|_Y$, the above estimate, (3.12) and (3.14) imply

$$\|\nabla v\|_Y^2 + \|za^{1/2}v\|_Y^2 + \|v\|_{\dot{X}}^2 \lesssim \||x|^{-1}f\overline{v}\|_{L^1} \qquad \text{for} \quad \lambda \leq C_1|\eta| \tag{3.15}$$

where $C_1 = (2C_0)^{-1}$.

Recall now that $f = (\Delta + z^2 a)v$. In the desired result we also have a first-order operator $b = b(x, \partial)$ satisfying (3.27) below, with a sufficiently small constant σ. To include this term, we write $f = (\Delta + z^2 a + b)v - bv$. We can control the new term with bv via

$$\||x|^{-1}\overline{v}b(x,\partial)v\|_{L^1} \lesssim \sigma \||x|^{-1/2}\langle x\rangle^{-1-\delta/2}v\|_{L^2}^2 + \sigma\|\langle x\rangle^{-(1+\delta)/2}\nabla v\|_{L^2} \|\langle x\rangle^{-(1+\delta)/2}|x|^{-1}v\|_{L^2}$$

so that (a variant of) (2.4) shows

$$\||x|^{-1}\overline{v}b(x,\partial)v\|_{L^1} \lesssim \sigma\|v\|_{\dot X}^2 + \sigma\|\nabla v\|_Y^2.$$

These terms can be absorbed at the left if $\sigma > 0$ is small enough. Inserting $f = (\Delta + z^2a + b)v - bv$ in (3.15), we conclude

$$\|\nabla v\|_Y^2 + \|za^{1/2}v\|_Y^2 + \|v\|_{\dot X}^2 \lesssim \||x|^{-1}\overline{v}(\Delta + z^2a + b)v\|_{L^1} \quad \text{for} \quad \lambda \le C_1|\eta|. \tag{3.16}$$

Observe that

$$\||x|^{-1}g\overline{v}\|_{L^1} \le \|g\|_{Y^*}\||x|^{-1}v\|_{\dot Y} \lesssim \|g\|_{Y^*}\|v\|_{\dot X}.$$

again by a variant of (2.4). Absorbing a $\|v\|_{\dot X}^2$ term, we conclude

$$\|\nabla v\|_Y + \|za^{1/2}v\|_Y + \|v\|_{\dot X} \lesssim \|(\Delta + z^2a + b)v\|_{Y^*} \qquad \text{for} \quad \lambda \le C_1|\eta|. \tag{3.17}$$

3) It remains to consider the case $0 \le C_1|\eta| \le \lambda$, for which we need (3.4). For arbitrary $R > 0$, we now employ the functions

$$\psi = \frac{R^2 + |x|^2}{2R}\mathbf{1}_{|x|\le R} + |x|\mathbf{1}_{|x|>R}, \qquad \phi = -\frac{1}{R}\mathbf{1}_{|x|\le R}. \tag{3.18}$$

One calculates

$$\begin{aligned} \psi' &= \frac{|x|}{|x| \vee R}, &\qquad \psi'' &= \frac{1}{R}\mathbf{1}_{|x|\le R}, \\ \Delta\psi + \phi &= \frac{2}{|x| \vee R}, &\qquad \Delta(\Delta\psi + \phi) &= -\tfrac{2}{R^2}\delta_{|x|=R}. \end{aligned} \tag{3.19}$$

We assume

$$0 < \alpha \le a(x) \le M, \qquad \|\langle x\rangle a'_- a^{-1}\|_{\ell^1 L^\infty} \le \tfrac{1}{4}.$$

Using these relations and the inequality

$$\textstyle\int \nabla\psi \cdot \nabla a|v|^2 \ge -\|a'_-|v|^2\|_{L^1} \ge -2\|a^{1/2}v\|_Y^2\|\langle x\rangle a'_- a^{-1}\|_{\ell^1 L^\infty},$$

cf. (2.3), we derive

$$\textstyle\sup_{R>0}\int I_v \ge \|v\|_{\dot X}^2 + \frac{\lambda}{2}\|a^{1/2}v\|_Y^2. \tag{3.20}$$

(Recall (2.2).) Since ψ is radial, we can write

$$2\partial_j v\,(\partial_j\partial_k\psi)\,\overline{\partial_k v} = 2\psi''\,|\widehat{x}\cdot\nabla v|^2 + 2\tfrac{\psi'}{|x|}\left[|\nabla v|^2 - |\widehat{x}\cdot\nabla v|^2\right] \ge \tfrac{2}{R}\mathbf{1}_{|x|\le R}|\nabla v|^2,$$

so that

$$\textstyle\sup_{R>0}\int I_{\nabla v} \ge \|\nabla v\|_Y^2. \tag{3.21}$$

Integrating (3.4), the lower bounds (3.20) and (3.21) show

$$\textstyle\|v\|_{\dot X}^2 + \lambda\|a^{1/2}v\|_Y^2 + \|\nabla v\|_Y^2 \lesssim \sup_{R>0}\int I_f + \sup_{R>0}\int I_\eta. \tag{3.22}$$

In view of $|\Delta\psi + \phi| \le 2/|x|$ and $|\nabla\psi| \le 1$, we have

$$\textstyle\int I_f \le 2\||x|^{-1}f\overline{v}\|_{L^1} + 2\|f\nabla\overline{v}\|_{L^1}.$$

Because of $0 \le C_1|\eta| \le \lambda$, estimate (3.9) for the above ψ yields

$$\textstyle\int I_\eta \lesssim (M\lambda)^{1/2}\|f\overline{v}\|_{L^1}. \tag{3.23}$$

for every $R > 0$. We thus arrive at

$$\|v\|_{\dot X}^2 + \lambda\|a^{1/2}v\|_Y^2 + \|\nabla v\|_Y^2 \lesssim \||x|^{-1}f\overline{v}\|_{L^1} + \|f\nabla\overline{v}\|_{L^1} + (M\lambda)^{1/2}\|f\overline{v}\|_{L^1} \qquad \text{for} \quad \lambda \ge C_1|\eta|.$$

We now use the inequalities

$$\||x|^{-1}f\overline{v}\|_{L^1} \le \|f\|_{Y^*}\|v\|_{\dot X}, \qquad \|f\overline{v}\|_{L^1} \le \|f\|_{Y^*}\|v\|_Y$$

as well as $|\eta| \le \frac{1}{C_1}\lambda$ and $a \ge \alpha$, to obtain

$$\|v\|_{\dot X}^2 + \|zv\|_Y^2 + \|\nabla v\|_Y^2 \le C(M,\alpha)\|f\|_{Y^*}^2. \tag{3.24}$$

Recall that $f = (\Delta + z^2 a(x))v$. As in (3.16), in (3.24) one can now add and subtract the term bv on the right hand side and absorb error terms for a small $\sigma > 0$ (w.r.t. α and M). We conclude that

$$\|v\|_{\dot X}^2 + \|zv\|_Y^2 + \|\nabla v\|_Y^2 \le c(\alpha,M)\|(\Delta + z^2 a + b)v\|_{Y^*}^2. \tag{3.25}$$

Putting the pieces together, (3.17) and (3.25) we have proved the following uniform resolvent estimate under a smallness condition on the coefficients of $b(x,\partial)$.

Proposition 3.1. *Let $z \in \mathbb{C}$ with $\Im z \ge 0$. Assume that for some $M, \alpha > 0$*

$$\alpha \le a(x) \le M, \qquad \|\langle x\rangle a^{-1} a'_-\|_{\ell^1 L^\infty} \le \tfrac{1}{4}, \tag{3.26}$$

while the first-order operator $b(x,\partial)$ satisfies for some $\sigma, \delta > 0$

$$|b(x,\partial)v| \le \sigma(\langle x\rangle^{-2-\delta}|v| + \langle x\rangle^{-1-\delta}|\nabla v|). \tag{3.27}$$

Let σ be sufficiently small with respect to α and M. We then have

$$\|v\|_{\dot X} + \|zv\|_Y + \|\nabla v\|_Y \le c(\alpha, M, \sigma, \delta)\|(\Delta + z^2 a + b)v\|_{Y^*}. \tag{3.28}$$

We now localize estimate (3.28) to a region $|x| \ge S$, where $S > 1$ is fixed but arbitrary. We shall assume that condition (3.27) is satisfied only in this region:

$$|b(x,\partial)v| \le \sigma(\langle x\rangle^{-2-\delta}|v| + \langle x\rangle^{-1-\delta}|\nabla v|) \quad \text{for} \quad |x| > S. \tag{3.29}$$

Let χ_0 be a real valued radial cutoff equal to 0 for $|x| \le 1$ and equal to 1 for $|x| \ge 2$, with a non negative radial derivative $\chi_0' \ge 0$. Set $\chi(x) = \chi_0(x/S)$ with the parameter $S > 1$. Note that

$$|\nabla\chi| \lesssim S^{-1}\mathbf{1}_{|x|\sim S}, \qquad |\Delta\chi| \lesssim S^{-2}\mathbf{1}_{|x|\sim S}$$

where $|x| \sim S$ is a shortcut for $S \le |x| \le 2S$. We consider $w = \chi v$, $L = \Delta + az^2 + b$, $z^2 = \lambda + i\eta$, and

$$f = Lv, \qquad g = Lw = \chi f + [L,\chi]v, \qquad [L,\chi]v = 2\nabla\chi\cdot\nabla v + \Delta\chi v + [b,\chi]v.$$

Assumption (3.29) yields

$$|[b(x,\partial),\chi]v| \le |v|\sigma\langle x\rangle^{-1-\delta}|\nabla\chi| \lesssim \sigma S^{-2-\delta}|v|\mathbf{1}_{|x|\sim S},$$

where we can assume w.l.o.g. $\sigma \le 1$. We thus obtain

$$|[L,\chi]v| \le cS^{-1}(|v| + |\nabla v|)\mathbf{1}_{|x|\sim S} \tag{3.30}$$

for some constant $c = c(\sigma, M)$. We now prove a version of (3.28) for $v_S = \chi v$.

1) It is sufficient to consider $\eta \ge 0$ as the case $\eta < 0$ follows by conjugation. First, let $-\infty < \lambda \le C_1\eta < +\infty$. We can here apply estimate (3.16) with w in place of v, i.e.,

$$\|\nabla w\|_Y^2 + \|zw\|_Y^2 + \|w\|_{\dot X}^2 \le c\||x|^{-1}\overline{w}Lw\|_{L^1}.$$

(Since $w = 0$ for $|x| \le S$, it is sufficient to assume the localized condition (3.29) on the lower order terms.) Writing $Lw = \chi Lv + [L,\chi]v$ and using the estimate (3.30), we compute

$$\||x|^{-1}\overline{w}\chi Lw\|_{L^1} \le \||x|^{-1}\overline{w}\chi Lv\|_{L^1} + cS^{-1}\|(|v| + |\nabla v|)v\|_{L^1(|x|\sim S)}$$

for some $c = c(\sigma, M)$. The space $\ell^\infty L^\infty L^2$ was introduced after (2.3). Analogously, we define $\ell^1 L^1 L^2$ and control its norm by

$$\|u\|_{\ell^1 L^1 L^2} := \sum_{j\ge 0}\int_{2^j}^{2^{j+1}} \Big(\int_{|x|=r}|u|^2 dS\Big)^{\frac12} dr \le \sum_{j\ge 0} 2^{\frac{j}{2}}\Big(\int_{A_j}|u|^2 dx\Big)^{\frac12} \lesssim \|u\|_{Y^*},$$

employing (2.3) in the last step. By means of a variant of (2.3), we thus obtain

$$\||x|^{-1}\overline{w}\chi Lv\|_{L^1} \le \||x|^{-1}w\|_{\ell^\infty L^\infty L^2}\|\chi Lv\|_{\ell^1 L^1 L^2} \lesssim \|w\|_{\dot X}\|\chi Lv\|_{Y^*},$$

We conclude

$$\|\nabla w\|_Y^2 + \|zw\|_Y^2 + \|w\|_{\dot X}^2 \le c\|\chi Lv\|_{Y^*}^2 + cS^{-1}\|(|v| + |\nabla v|)v\|_{L^1(|x|\sim S)} \tag{3.31}$$

for $-\infty < \lambda \le C_1\eta < +\infty$.

2) Let now $\lambda \ge C_1\eta \ge 0$. For this case we resort to (3.22) with $w = \chi v$ in place of v and $h = (\Delta + z^2 a(x))w$ in place of f which gives

$$\|w\|_{\dot X}^2 + \lambda\|w\|_Y^2 + \|\nabla w\|_Y^2 \lesssim \sup_{R>0}\textstyle\int \widetilde{I}_h + \sup_{R>0}\int \widetilde{I}_\eta \tag{3.32}$$

where ψ and ϕ are given by (3.18) as well as

$$\widetilde{I}_h = -\Re\big((2\nabla\psi\cdot\nabla\overline{w} + \Delta\psi\overline{w} + \phi\overline{w})h\big), \qquad \widetilde{I}_\eta = -2\eta a(x)\Im(\overline{w}\,\nabla\psi\cdot\nabla w).$$

By (3.19) we have $|\nabla\psi| \le 1$ and hence

$$\textstyle\int \widetilde{I}_\eta \le 2M\eta\|\overline{w}\nabla w\|_{L^1} \lesssim 2M\eta\|\overline{w}\nabla v\|_{L^1} + 2M\eta S^{-1}\||v|^2\|_{L^1(|x|\sim S)} \tag{3.33}$$

for all $R > 0$. Next, identities (3.6) and (3.8) imply the estimates

$$\eta\|v\|^2 \le \alpha^{-1}\|f\overline{v}\|_{L^1}, \qquad \|\nabla v\|^2 \le M\lambda\|v\|^2 + \|f\overline{v}\|_{L^1}$$

where $\|\cdot\| = \|\cdot\|_{L^2}$. Taking into account $S > 1$ and $\lambda \ge C_1\eta$, we infer

$$\begin{aligned}
&\eta S^{-1}\|\mathbf{1}_{|x|\sim S}v\|^2 \le \alpha^{-1}\|f\overline{v}\|_{L^1},\\
&\quad\eta\|\overline{v}\nabla v\|_{L^1} \le \eta\lambda^{1/2}\|v\|^2 + \eta\lambda^{-1/2}\|\nabla v\|^2 \le \alpha^{-1}\lambda^{1/2}\|f\overline{v}\|_{L^1} + M\eta\lambda^{1/2}\|v\|^2 + \eta\lambda^{-1/2}\|f\overline{v}\|_{L^1}\\
&\qquad\qquad\qquad \le (\alpha^{-1} + M\alpha^{-1} + C_1^{-1})\lambda^{1/2}\|f\overline{v}\|_{L^1}.
\end{aligned}$$

So (3.33) leads to

$$\textstyle\int \widetilde{I}_\eta \le C(\alpha, M)(1+\lambda)^{1/2}\|f\overline{v}\|_{L^1} \le C(\alpha, M, \rho)\|Lv\|_{Y^*}^2 + \rho^2(1+\lambda)\|v\|_Y^2 \tag{3.34}$$

for all $\rho > 0$. On the other hand, $\widetilde{I}_h$ can be written as

$$\begin{aligned}
\widetilde{I}_h &= -\Re\big((2\chi\psi'\overline{v}_r + 2\chi'\psi'\overline{v} + (\Delta\psi + \phi)\overline{w})\cdot(2\chi' v_r + \Delta\chi v + \chi Lv - \chi b(x,\partial)v)\big)\\
&= N + I + II + III + IV
\end{aligned}$$

for the summands

$$\begin{aligned}
N &= -4\chi\chi'\psi'|v_r|^2,\\
I &= -\Re(2\chi'\psi'\overline{v} + (\Delta\psi + \phi)\overline{w})\cdot 2\chi' v_r,\\
II &= -\Re\big((2\nabla\psi\cdot\nabla\overline{w} + (\Delta\psi + \phi)\overline{w})\Delta\chi v\big),\\
III &= -\Re\big((2\nabla\psi\cdot\nabla\overline{w} + (\Delta\psi + \phi)\overline{w})\chi Lv\big),\\
IV &= \Re\big((2\nabla\psi\cdot\nabla\overline{w} + (\Delta\psi + \phi)\overline{w})(b(x,\partial)w - [b,\chi]v)\big).
\end{aligned}$$

The term N is negative and can be dropped. For the remaining terms, we recall from (3.19) that $|\nabla\psi| \le 1$ and $|\Delta\psi + \phi| \le 2/\langle x\rangle$ on the support of χ, independently of $R > 0$. Moreover, the definition of χ yields

$$\chi' \le cS^{-1}\mathbf{1}_{|x|\sim S} \le c\langle x\rangle^{-1} \qquad \text{and} \qquad |\Delta\chi| \le cS^{-2}\mathbf{1}_{|x|\sim S} \le c\langle x\rangle^{-2}$$

for $S > 1$. We thus obtain

$$\begin{aligned}
I + II &\lesssim S^{-2}|v|(|\nabla v| + |v|)\mathbf{1}_{|x|\sim S},\\
III &\lesssim |\chi Lv|(|\nabla w| + \langle x\rangle^{-1}|w|),
\end{aligned}$$

$$\|III\|_{L^1} \le c'\|\chi Lv\|_{Y^*}^2 + \frac{1}{10}\|\langle x\rangle^{-1}w\|_Y^2 + \frac{1}{10}\|\nabla w\|_Y^2.$$

Note that $\|\langle x\rangle^{-1}w\|_Y \le \|w\|_{\dot X}$, cf. (2.4). For IV we use (3.29) and get

$$\begin{aligned} IV &\le c(|\nabla w| + \langle x\rangle^{-1}|w|)\sigma(\langle x\rangle^{-1-\delta}|\nabla w| + \langle x\rangle^{-2-\delta}|w| + S^{-2-\delta}|v|\mathbf{1}_{|x|\sim S}) \\ &\le c\sigma\langle x\rangle^{-1-\delta}(|\nabla w| + \langle x\rangle^{-1}|w|)^2 + c\sigma S^{-2}|v|^2\mathbf{1}_{|x|\sim S}, \end{aligned}$$

estimating $[b,\chi]$ as in (3.30). Invoking (2.4), it follows

$$\|IV\|_{L^1} \lesssim \sigma\|w\|_{\dot X}^2 + \sigma\|\nabla w\|_Y^2 + \sigma S^{-2}\|v^2\|_{L^1(|x|\sim S)}.$$

Thus if σ is small enough we derive

$$\textstyle\int \widetilde{I}_h \le c\|\chi Lv\|_{Y^*}^2 + cS^{-2}\|(|\nabla v| + |v|)v\|_{L^1(|x|\sim S)} + \frac{1}{5}\|w\|_{\dot X}^2 + \frac{1}{5}\|\nabla w\|_Y^2.$$

Plugging this estimate and (3.34) in (3.32) and absorbing some terms at the LHS, we arrive at

$$\|w\|_{\dot X}^2 + \lambda\|w\|_Y^2 + \|\nabla w\|_Y^2 \le C\|Lv\|_{Y^*}^2 + CS^{-1}\|(|v| + |\nabla v|)v\|_{L^1(|x|\sim S)} + \rho^2(1+\lambda)\|v\|_Y^2. \quad (3.35)$$

By the condition $0 \le \eta \le C_1\lambda$ we can replace λ by $|\lambda + i\eta| = |z|^2$ on the LHS of the inequality. Combining (3.31) and (3.35), we have proved the following uniform resolvent estimate for functions localized outside a ball, provided that the lower order coefficients are small in that region.

Proposition 3.2. *Let $M, \alpha, \sigma, \delta > 0$ and $S > 1$. Assume that $a(x)$ satisfies* (3.26)*, while the first-order operator $b(x,\partial)$ satisfies*

$$|b(x,\partial)v| \le \sigma(\langle x\rangle^{-2-\delta}|v| + \langle x\rangle^{-1-\delta}|\nabla v|) \quad \text{for all } |x| \ge S. \quad (3.36)$$

Let $\sigma > 0$ be sufficiently small with respect to α and M. Then for all $z \in \mathbb{C}$ the function $v_S = v\mathbf{1}_{|x|\ge 2S}$ satisfies

$$\|v_S\|_X + \|zv_S\|_Y + \|\nabla v_S\|_Y \le C\|Lv\|_{Y^*} + \tfrac{C}{S}\|(|v| + |\nabla v|)v\|_{L^1(|x|\sim S)}^{1/2} + \rho(1+\lambda)^{1/2}\|v\|_Y \quad (3.37)$$

for all $\rho > 0$, where $\lambda = \Re z^2$, $L(z) = \Delta + z^2a(x) + b(x,\partial)$ and $C = C(\alpha, M, \sigma, \delta, \rho)$.

3.2. **Carleman estimate.** We shall combine estimate (3.37) with a Carleman estimate in a compact subset of $\mathbb{R}^3$, in order to handle coefficients which may be large on a bounded subset of $\mathbb{R}^3$. Our goal is an estimate for (large) frequencies $z^2 = \lambda + i\eta$ belonging to a suitable parabolic region, which is needed for our later investigations. In the following computations we consider functions $u \in H^2$ which decay fast enough, actually the result will be applied to functions with compact support.

First, let $\Re z^2 = \lambda < 0$. Integration by parts yields

$$\textstyle\int |\nabla v|^2 - \Re z^2 \int a(x)|v|^2 = -\Re\int((\Delta v + z^2 av)\overline{v}), \qquad \Im z^2 \int a(x)|v|^2 = \Im\int((\Delta v + z^2av)\overline{v}).$$

These identities lead to

$$\begin{aligned} \|\nabla v\|_{L^2}^2 + |\lambda|\|a^{1/2}v\|_{L^2}^2 &\le \|\Delta v + z^2av\|_{L^2}\|v\|_{L^2} \le \tfrac{1}{|\lambda|}\|\Delta v + z^2av\|_{L^2}^2 + \tfrac{|\lambda|}{4}\|a^{1/2}v\|_{L^2}^2 \\ |\eta|\|a^{1/2}v\|_{L^2}^2 &\le \tfrac{1}{|\lambda|}\|\Delta v + z^2av\|_{L^2}^2 + \tfrac{|\lambda|}{4}\|a^{1/2}v\|_{L^2}^2. \end{aligned}$$

Using $\alpha \le a(x)$, we obtain the elliptic estimate

$$\|\nabla v\|_{L^2}^2 + \|zv\|_{L^2}^2 \le \|\nabla v\|_{L^2}^2 + \tfrac{1}{\alpha}\|za^{1/2}v\|_{L^2}^2 \le \tfrac{C(\alpha)}{|\lambda|}\|(\Delta + z^2a)v\|_{L^2}^2.$$

For any first-order operator $b(x,\partial)$ with bounded coefficients, the above inequality implies

$$\begin{aligned} \|\nabla v\|_{L^2}^2 + \|zv\|_{L^2}^2 &\le C(\alpha, \lambda_0)\|(\Delta + z^2a + b(x,\partial))v\|_{L^2}^2, \\ \|v\|_X^2 + \|\nabla v\|_Y^2 + \|zv\|_Y^2 &\le C(\alpha, \lambda_0)\|(\Delta + z^2a + b(x,\partial))v\|_{Y^*}^2 \end{aligned} \quad (3.38)$$

for all $\Re z^2 \le -\lambda_0(a,b)$, where $\lambda_0(a,b) > 0$ depends only on $\alpha = \inf a(x)$ and the supremum of the coefficients of $b(x,\partial)$. In the second line we employ (3.18) from [5] and (2.3).

We thus focus on the case $\lambda > 0$, starting with the main part $\Delta + az^2$. We use the notations

$$r = |x|, \qquad \widehat{x} = \tfrac{x}{|x|}, \qquad \partial_r = \widehat{x}\cdot\nabla, \qquad \Omega = r\nabla - x\partial_r, \qquad \widetilde{\Omega} = \Omega - 2\widehat{x}.$$

As above, we denote the radial derivative of a radially symmetric function with an apex, i.e., $\phi'(r) = \partial_r\phi$. The vector fields Ω and $\widetilde{\Omega}$ satisfy the relations

$$\widehat{x}\cdot\Omega = 0, \qquad \textstyle\int_{\mathbb{S}^2} \widetilde{\Omega} f dS = 0$$

and we have

$$\Omega^2 = \Delta_{\mathbb{S}^2}, \qquad \Delta = \partial_r^2 + \tfrac{2}{r}\partial_r + \tfrac{1}{r^2}\Omega^2, \qquad |\nabla v|^2 = |\partial_r v|^2 + \tfrac{1}{r^2}|\Omega v|^2.$$

Fix two radially symmetric, real valued functions ϕ and γ. We introduce the transformed operator

$$Q = re^{\phi}(\Delta + z^2 a(x))e^{-\phi}r^{-1}, \qquad z^2 = \lambda + i\eta, \quad \lambda,\eta\in\mathbb{R},$$

or more explicitly

$$Q = \partial_r^2 + \tfrac{1}{r^2}\Omega^2 + z^2a(x) + \phi'^2 - \phi'' - 2\phi'\partial_r.$$

It is straighforward to check

$$\begin{aligned}\partial_r\{\gamma A_0\} + \widetilde{\Omega}\cdot\{\gamma Z_0\} = 2\gamma\cdot\Re[Qv\cdot\overline{v_r}] + 2\gamma\eta a(x)\Im(v\cdot\overline{v_r}) +\\ + (\gamma' + 4\phi'\gamma)|v_r|^2 - (\tfrac{\gamma}{r^2})'|\Omega v|^2 + [(\lambda a(x) + \phi'^2 - \phi'')\gamma]'|v|^2,\end{aligned} \tag{3.39}$$

where $\lambda = \Re z^2$, $\eta = \Im z^2$ and

$$A_0 = |\partial_r v|^2 - \tfrac{1}{r^2}|\Omega v|^2 + (\lambda a(x) + \phi'^2 - \phi'')|v|^2, \qquad Z_0 = 2\Re(r^{-2}\Omega v\cdot\overline{v_r}).$$

Lemma 3.3. *Assume $a(x)$ satisfies*

$$0 < a(x) \le M, \qquad (\nu + r)a'_- \le 2a - \nu, \tag{3.40}$$

for some $M > 0$ and $\nu\in(0,1]$. Let $\lambda = \Re z^2$, $\eta = \Im z^2$, $\nu\lambda \ge 2\eta^2$, and $\tau \ge M^2 + 4$. Then we have the estimate

$$\|e^{\phi}\langle x\rangle^{-1/2}\nabla u\|_{L^2}^2 + (\Re z^2 + \tau^2)\|e^{\phi}u\|_{L^2}^2 \le 10\nu^{-4}\tau^{-1}\|e^{\phi}(\nu + r)(\Delta + za)u\|^2 \tag{3.41}$$

where $\phi(r) = \tau(r^2 + r)$.

Proof. Identity (3.39) implies

$$\begin{aligned}&\partial_r\{\gamma A_0\} + \widetilde{\Omega}\cdot\{\gamma Z_0\} + \tau^{-1}\gamma|Qv|^2\\ &\quad\ge (\gamma' + (4\phi' - \tau)\gamma - M^2\gamma^{\frac{3}{2}})|v_r|^2 - (\tfrac{\gamma}{r^2})'|\Omega v|^2 + [(\lambda a(x) + \phi'^2 - \phi'')\gamma]'|v|^2 - \eta^2\gamma^{\frac{1}{2}}|v|^2.\end{aligned}$$

We make the choices

$$\gamma(r) = (\nu + r)^2, \qquad \phi(r) = \tau(r^2 + r)$$

with the parameters $\tau \ge M^2 + 4$ and $\nu\in(0,1]$. We obtain

$$\begin{aligned}\ell := \partial_r\{\gamma A_0\} + \widetilde{\Omega}\cdot\{\gamma Z_0\} + \tau^{-1}\gamma|Qv|^2 \ge 2\tau(r+1)\gamma|v_r|^2 + \tfrac{2\nu(\nu+r)}{r^3}|\Omega v|^2\\ + (\lambda(a\gamma)' + \tau^2(\nu + r)^3 - \eta^2\gamma^{\frac{1}{2}})|v|^2.\end{aligned}$$

Condition (3.40) yields $(a\gamma)' \ge \nu\gamma^{1/2}$, and $\nu\lambda - \eta^2 \ge \frac{1}{2}\nu\lambda$ follows from the assumption on z. We can thus continue the previous inequality as

$$\ell \ge 2\tau(r+1)\gamma|v_r|^2 + \tfrac{2\nu(\nu+r)}{r^3}|\Omega v|^2 + (\tfrac{1}{2}\nu\lambda(\nu + r) + \tau^2(\nu + r)^3)|v|^2.$$

Now we integrate over the cylinder $\Pi = [0,+\infty)\times\mathbb{S}^2$ and use the notation

$$\|v\|_{\Pi}^2 := \textstyle\int_0^{+\infty}\int_{\mathbb{S}^2}|v|^2 dSdr.$$

So the above lower bound leads to

$$\tau\|v_r\|_\Pi^2 + \|\tfrac{1}{r^{3/2}}\Omega v\|_\Pi^2 + \lambda\|v\|_\Pi^2 + 2\tau^2\|(\nu+r)v\|_\Pi^2 \le 2\tau^{-1}\nu^{-2}\|(\nu+r)Qv\|_\Pi^2.$$

Setting $v = re^\phi u$, we have

$$\|v\|_\Pi = \|e^\phi u\|_{L^2(\mathbb{R}^3)}, \qquad \|r^{-3/2}\Omega v\|_\Pi = \|r^{-3/2}e^\phi \Omega u\|_{L^2(\mathbb{R}^3)},$$
$$\|(\nu+r)Qv\|_\Pi = \|e^\phi(\nu+r)(\Delta + z^2 a(x))u\|_{L^2(\mathbb{R}^3)},$$

which implies the first partial estimate

$$\begin{aligned}\tau\|v_r\|_\Pi^2 + \|\tfrac{e^\phi}{r^{3/2}}\Omega u\|_{L^2}^2 + \lambda\|e^\phi u\|_{L^2} + 2\tau^2\|(\nu+r)e^\phi u\|_{L^2}^2 \\ \le 2\nu^{-2}\tau^{-1}\|e^\phi(\nu+r)(\Delta+z^2a(x))u\|_{L^2}^2.\end{aligned} \tag{3.42}$$

In order to handle the v_r term, we first define $v = rw$, i.e., $w = e^\phi u$. Observe that

$$\|v_r\|_\Pi^2 = \int |w_r + \tfrac{w}{r}|^2 dx = \int (|w_r|^2 + \tfrac{|w|^2}{r^2} + 2\Re w_r \tfrac{\overline{w}}{r})dx,$$
$$\int 2\Re w_r \tfrac{\overline{w}}{r} dx = \int \tfrac{1}{r}\widehat{x}\cdot\nabla|w|^2 dx = -\int |w|^2 \nabla\cdot(\tfrac{\widehat{x}}{r})dx = -\int \tfrac{|w|^2}{r^2}dx,$$

and hence

$$\|v_r\|_\Pi = \|w_r\|_{L^2(\mathbb{R}^3)} = \|e^\phi(u_r + \phi' u)\|_{L^2(\mathbb{R}^3)}, \qquad \phi'(r) = \tau(2r+1).$$

We deduce

$$\|e^\phi u_r\|_{L^2(\mathbb{R}^3)}^2 \le 2\|v_r\|_\Pi^2 + 8\nu^{-2}\tau^2\|e^\phi(\nu+r)u\|_{L^2(\mathbb{R}^3)}^2.$$

So estimate (3.42) gives

$$\|e^\phi u_r\|_{L^2}^2 + \|\tfrac{e^\phi}{r^{3/2}}\Omega u\|_{L^2}^2 + (\lambda+\tau^2)\|e^\phi u\|_{L^2}^2 \le 10\nu^{-4}\tau^{-1}\|e^\phi(\nu+r)(\Delta+z^2a)u\|_{L^2}^2.$$

Inserting $|\nabla u|^2 = |u_r|^2 + \frac{1}{r^2}|\Omega u|^2$, the assertion (3.41) follows. □

We now take a first-order operator $b(x,\partial)$ and let $L = \Delta + z^2 a + b$. Note that

$$\|e^\phi(\nu+r)(\Delta+z^2a)u\|_{L^2} \le \|e^\phi(\nu+r)Lu\|_{L^2} + \|e^\phi(\nu+r)bu\|_{L^2}.$$

Assume that u has support in the ball $|x| \le K$ for some $K \ge 1$ and that $b(x,\partial)$ satisfies

$$|b(x,\partial)v| \le N(|v| + \langle x\rangle^{-1/2}|\nabla v|). \tag{3.43}$$

We can then estimate

$$\|e^\phi(\nu+r)bu\|_{L^2}^2 \le 2N^2(K+1)^2(\|e^\phi u\|_{L^2}^2 + \|e^\phi\langle x\rangle^{-1/2}\nabla u\|_{L^2}^2).$$

Taking a large parameter $\tau \ge 1$, the lower order terms on the RHS of (3.41) can be absorbed yielding our Carleman estimate.

Proposition 3.4. *Assume $a(x)$ satisfies* (3.40) *and $b(x,\partial)$ satisfies* (3.43). *Take $z \in \mathbb{C}$ with $\lambda = \Re z^2$, $\eta = \Im z^2$ and $\nu\lambda \ge 2\eta^2$. Let $\phi(r) = \tau(r^2+r)$, $u \in H^2$ have support in $|x| \le K$ for some $K \ge 1$, and $\tau \ge \max\{4+M^2, 80\nu^{-4}N^2(K+1)^2\}$. Then the following estimate holds*

$$\|e^\phi\langle x\rangle^{-1/2}\nabla u\|_{L^2}^2 + (\Re z^2 + \tau^2)\|e^\phi u\|_{L^2}^2 \le 40\nu^{-4}\tau^{-1}\|e^\phi Lu\|_{L^2}^2. \tag{3.44}$$

Since $\langle x\rangle^{-1} \ge (2K)^{-1}$ on the support of u, choosing τ sufficiently large we deduce from (3.44) the estimate

$$\|u\|_X + \|zu\|_Y + \|\nabla u\|_Y \le c(K,M,N,\nu)\|(\Delta + z^2a(x)+b)u\|_{Y^*} \tag{3.45}$$

provided u is supported in $|x| \le K$ and $z^2 = \lambda + i\eta$ lies in the parabolic region $\nu\lambda \ge 2\eta^2$.

4. The complete resolvent estimate

We are ready to patch the previous estimates and deduce a global one valid for all frequencies $z^2 = \lambda + i\eta$ in a region of the form

$$\Omega = \Omega(\nu, \lambda_1) = \{\lambda \leq -\lambda_1/2\} \cup \{\lambda^2 + \eta^2 \leq \lambda_1^2\} \cup \{\nu\lambda \geq 2\eta^2\} \tag{4.1}$$

for suitable $\nu, \lambda_1 > 0$. Recall that

- if $\Re z^2 \leq -\lambda_0$ for a sufficiently large $\lambda_0 > 0$, we can use the elliptic estimate (3.38);
- if z^2 belongs to an arbitrarily large (but fixed) ball $|z^2| \leq \lambda_1$, we can use Proposition 2.5.

Thus to cover the entire region $\Omega(\nu, \lambda_1)$ it remains to consider frequencies $z^2 = \lambda + i\eta$ in the parabolic region given by $\lambda \geq \lambda_0'$ and $\nu\lambda \geq 2\eta^2$ for a sufficiently large $\lambda_0' > 0$.

To this aim, we combine estimates (3.37) and (3.44) for functions vanishing inside, resp. outside, balls. The assumptions on $a(x)$ are

$$0 < \alpha \leq a(x) \leq M, \qquad \|\langle x\rangle a^{-1} a'_-\|_{\ell^1 L^\infty} \leq \tfrac{1}{4}, \qquad (\nu + r)a'_- \leq 2a - \nu \tag{4.2}$$

for some $\nu \in (0, 1]$. For $b(x, \partial)$ we require

$$|b(x, \partial)v| \leq C_b(\langle x\rangle^{-2-\delta}|v| + \langle x\rangle^{-1-\delta}|\nabla v|) \tag{4.3}$$

for some $C_b, \delta > 0$, which is the same as (2.16) in Proposition 2.5. Note that (4.2) contains both (3.26) and (3.40), and (4.3) implies (3.43) (after possibly increasing N). On the other hand, if we take $S_0 > 1$ sufficiently large (and possibly decrease δ), we see that (4.3) implies (3.36) for $|x| \geq S$ for any $S \geq S_0$. From now on, S_0 is fixed. Thus the assumptions of both Propositions 3.2 and 3.4 are verified.

Fix a radial cutoff function χ_0 such that $\chi_0 = 0$ for $|x| \leq 1$ and $\chi_0 = 1$ for $|x| \geq 2$. Set $S = 2S_0$ and $\chi(x) = \chi_0(S^{-1}x)$. We then decompose

$$\|u\|_X + \|zu\|_Y + \|\nabla u\|_Y \leq I + II \tag{4.4}$$

with

$$I = \|\chi u\|_X + \|\chi z u\|_Y + \|\nabla(\chi u)\|_Y,$$
$$II = \|(1-\chi)u\|_X + \|(1-\chi)zu\|_Y + \|\nabla((1-\chi)u)\|_Y.$$

Writing $L = L(z) = \Delta + z^2 a + b$, we can apply (3.45) to II since $(1-\chi)u$ is compactly supported in $|x| \leq 2S$, obtaining

$$II \lesssim \|L((1-\chi)u)\|_{Y^*} \lesssim \|Lu\|_{Y^*} + \||u| + |\nabla u|\|_{L^2(S \leq |x| \leq 2S)}$$

The last term at the right is supported in $|x| \geq 2S_0$. It can thus be estimated via (3.37) in Proposition 3.2 with S_0 instead of S, and hence

$$II \leq C\|Lu\|_{Y^*} + \rho(1 + |z|)\|u\|_Y + C\|(|u| + |\nabla u|)u\|^{1/2}_{L^1(|x| \sim S_0)}$$

where $\rho > 0$ is arbitrarily small and $C = C(\alpha, M, \sigma, \delta, \rho, S_0)$. We next treat I again using (3.37) with S_0 instead of S (recall that we have $S = 2S_0$), which yields

$$I \leq C\|Lu\|_{Y^*} + \rho(1 + |z|)\|u\|_Y + C\|(|u| + |\nabla u|)u\|^{1/2}_{L^1(|x| \sim S_0)}.$$

Summing up, we get

$$I + II \leq C\|Lu\|_{Y^*} + \rho(1 + |z|)\|u\|_Y + C\|(|u| + |\nabla u|)u\|^{1/2}_{L^1(|x| \sim S_0)}$$

For every $\rho > 0$, the last summand is bounded by

$$\|(|u| + |\nabla u|)u\|^{1/2}_{L^1(|x| \sim S_0)} \leq \rho\|\nabla u\|_Y + C(S_0, \rho)\|u\|_Y, \tag{4.5}$$

leading to

$$\|u\|_X + \|zu\|_Y + \|\nabla u\|_Y \le C\|Lu\|_{Y^*} + \rho(\|zu\|_Y + \|\nabla u\|_Y) + C\|u\|_Y.$$

Here $\rho > 0$ is arbitrary and $C = C(\alpha, M, \sigma, \delta, \rho, S_0)$. Taking $\rho = 1/2$ and absorbing two terms by the LHS, we infer

$$\|u\|_X + \|zu\|_Y + \|\nabla u\|_Y \le C\|Lu\|_{Y^*} + C\|u\|_Y.$$

If we assume $|z| \ge 2C$, we can also absorb the last summand and we obtain

$$\|u\|_X + \|zu\|_Y + \|\nabla u\|_Y \le C\|Lu\|_{Y^*}$$

for all z in the region $|z| \ge 2C(\alpha, M, \sigma, \delta, S_0)$ such that $\nu\lambda \ge 2\eta^2$. We now choose a sufficiently large $\lambda_1 > 0$ in the definition (4.1) of Ω and employ (3.38) and Proposition 2.5 as indicated after (4.1). In this way, the following main resolvent estimate is proved.

Proposition 4.1. *Assume $a(x)$ and $b(x, \partial)$ satisfy (4.2), (4.3), $|x|^2\langle x\rangle^\delta(a-1) \in L^\infty$, and the spectral assumption (S). Then we can find $\lambda_1 > 0$ such that for all $z^2 = \lambda + i\eta \in \mathbb{C}$ in the region $\Omega = \Omega(\nu, \lambda_1)$ defined in (4.1), the operator $L(z) = \Delta + z^2 a(x) + b(x, \partial)$ satisfies the estimate*

$$\|u\|_X + \|zu\|_Y + \|\nabla u\|_Y \lesssim \|L(z)u\|_{Y^*} \tag{4.6}$$

with a constant uniform in z.

The same proof applies to a matrix operator of the special form

$$L(z) = I_3\Delta + I_3 a(x)z^2 + b(x, \partial).$$

Remark 4.2. The last condition in (4.2) is implied by

$$\frac{a'_-}{a} \le \frac{1}{\nu + r}$$

(provided ν is small enough). Thus we see that the following assumption

$$a'_-(x) \le \nu_0 a(x)\langle x\rangle^{-1-\delta} \tag{4.7}$$

implies the last two conditions in (4.2), provided ν_0 is small enough.

5. Smoothing estimates

We shall now convert estimate (4.6) into a smoothing estimate for the wave equation. First, we repackage (4.6) in a weaker form in terms of weighted L^2 norms, in order to apply the Laplace transform. Recall from Propositions 2.6–2.8 that hypothesis (S) is valid for our Maxwell system, under mild extra decay conditions.

Corollary 5.1. *Let $L(z) = I_3\Delta + I_3 a(x)z^2 + b(x, \partial)$ be a matrix operator such that*

(1) $\alpha = \inf a(x) > 0$, $\langle x\rangle^{2+\delta}(a-1) \in L^\infty$, and $a'_- \le \frac{1}{4}(1-2^{-\delta})^{-1}a\langle x\rangle^{-1-\delta}$,
(2) $|b(x, \partial)v| \lesssim \langle x\rangle^{-2-\delta}|v| + \langle x\rangle^{-1-\delta}|\nabla v|$,
(3) the spectral assumption (S) holds

for some $\delta > 0$. Then there exists $\lambda_1 > 0$ such that for any z with $z^2 \in \Omega(1 \wedge \alpha, \lambda_1)$ we have

$$\|v\|_{L^2_{-3/2-}} + \|zv\|_{L^2_{-1/2-}} + \|\nabla v\|_{L^2_{-1/2-}} \lesssim \|L(z)v\|_{L^2_{1/2+}} \tag{5.1}$$

where we use the notation $\|u\|_{L^2_s} = \|\langle x\rangle^s u\|_{L^2(\mathbb{R}^3)}$.

Proof. It is easy to check that assumption (1) implies (4.2), with $\nu = 1 \wedge \alpha$. In view of (2.4), estimate (4.6) implies (5.1). ☐

Let $u:\mathbb{R}_t\times\mathbb{R}^3_x\to\mathbb{C}^3$ be a function with $u(t,x)=0$ for $t<0$ and such that the maps $\partial_t^k u:\mathbb{R}\to H^{2-k}(\mathbb{R}^3)$ are continuous and grow sub-exponentially for $k=0,1,2$. For $z=\alpha+i\beta$ in the upper half plane $\Im z>0$, then the 'damped' Fourier transform

$$v(z,\cdot):=\int_{-\infty}^{+\infty}e^{itz}u(t,\cdot)dt$$

is defined in $L^2(\mathbb{R}^3)$. It satisfies

$$-z^2v(z,\cdot)=\int_{-\infty}^{+\infty}e^{itz}\partial_t^2u(t,\cdot)dt,\qquad(\Delta+b)v(z,\cdot)=\int_{-\infty}^{+\infty}e^{itz}(\Delta+b)u(t,\cdot)dt$$

so that

$$(\Delta+az^2+b)v(z,x)=\int_{-\infty}^{+\infty}e^{itz}(\Delta+b-a\partial_t^2)u(t,x)dt$$

for a.e. $x\in\mathbb{R}^3$. Plancherel's formula thus yields

$$\int|(\Delta+(\alpha+i\beta)^2a+b)v(\alpha+i\beta,x)|^2d\alpha=2\pi\int e^{-2\beta t}|(\Delta+b-a\partial_t^2)u(t,x)|^2dt.$$

We multiply by the weight $\langle x\rangle^{2s}$ and integrate also in x, obtaining

$$\|(\Delta+(\cdot+i\beta)^2a+b)v\|_{L^2(d\alpha)L^2_s(\mathbb{R}^3)}\eqsim\|e^{-\beta t}(\Delta+b-a\partial_t^2)u\|_{L^2(dt)L^2_s(\mathbb{R}^3)}$$

for any $s\in\mathbb{R}$, though the norms could be infinite. In a similar way we deduce

$$\begin{aligned}\|v(\cdot+i\beta)\|_{L^2(d\alpha)L^2_s(\mathbb{R}^3)}&\eqsim\|e^{-\beta t}u\|_{L^2(dt)L^2_s(\mathbb{R}^3)},\\ \|\nabla v(\cdot+i\beta)\|_{L^2(d\alpha)L^2_s(\mathbb{R}^3)}&\eqsim\|e^{-\beta t}\nabla u\|_{L^2(dt)L^2_s(\mathbb{R}^3)},\\ |z|\|v(\cdot+i\beta)\|_{L^2(d\alpha)L^2_s(\mathbb{R}^3)}&\eqsim\|e^{-\beta t}\partial_t u\|_{L^2(dt)L^2_s(\mathbb{R}^3)}.\end{aligned}$$

Note that if $z=\alpha+i\beta$ with $\beta>0$ sufficiently small, then $z^2=\lambda+i\eta$ lies in the parabolic region Ω. We assume that $G=(\Delta+b-a\partial_t^2)u$ belongs to $L^2L^2_{1/2+}$. Estimate (5.1) thus implies that

$$\begin{aligned}\|e^{-\beta t}u\|_{L^2(dt)L^2_{-3/2-}}+\|e^{-\beta t}\partial_t u\|_{L^2(dt)L^2_{-1/2-}}+\|e^{-\beta t}\nabla u\|_{L^2(dt)L^2_{-1/2-}}&\\ \lesssim\|e^{-\beta t}(\Delta+b-a\partial_t^2)u\|_{L^2(dt)L^2_{1/2+}}&.\end{aligned}\tag{5.2}$$

for sufficiently small $\beta>0$. (In particular, the involved norms are finite.) Here the implicit constant does not depend on β, so that one can let $\beta\to0$ by Fatou's lemma. As usual, no modification is necessary in the matrix case.

We apply (5.2) to a solution of the problem

$$(a\partial_t^2-\Delta-b(x,\partial))U=G(t,x),\qquad U(0,x)=\partial_tU(0,x)=0.\tag{5.3}$$

Proposition 5.2. *Let $U(t,x):\mathbb{R}_t\times\mathbb{R}^3_x\to\mathbb{C}^3$ be a solution of the Cauchy problem* (5.3) *subject to the above growth conditions, where $a(x)$ and $b(x,\partial)$ are as in Corollary 5.1 and $\langle x\rangle^{1/2+}G\in L^2L^2$. Then the following estimate holds:*

$$\|U\|_{L^2L^2_{-3/2-}}+\|\partial_tU\|_{L^2L^2_{-1/2-}}+\|\nabla U\|_{L^2L^2_{-1/2-}}\lesssim\|G\|_{L^2L^2_{1/2+}}.\tag{5.4}$$

Proof. Assume $G=0$ for $t\le0$, so that $U=0$ for $t\le0$ and we can apply (5.2). Letting $\beta\downarrow0$ we obtain (5.4). The same estimate is valid if $G=0$ for $t\ge0$ (just by time reversal $t\to-t$). By linearity, estimate (5.4) holds for arbitrary G. $\square$

We next focus on the actual Maxwell equations

$$\partial_t^2\mathbf{E}+\tfrac{1}{\epsilon}\nabla\times\tfrac{1}{\mu}\nabla\times\mathbf{E}=0,\qquad\nabla\cdot(\epsilon\mathbf{E})=0,$$

or equivalently (with $\mathbf{D}=\epsilon\mathbf{E}$)

$$\partial_t^2\mathbf{D}+\nabla\times\tfrac{1}{\mu}\nabla\times\tfrac{1}{\epsilon}\mathbf{D}=0,\qquad\nabla\cdot\mathbf{D}=0.$$

Let $\mathcal{H}$ be the Hilbert space

$$\mathcal{H}=\{u\in L^2(\mathbb{R}^3;\mathbb{C}^3):\nabla\cdot u=0\}\tag{5.5}$$

endowed with the scalar product $(u,v)_{\mathcal{H}} = \int \epsilon^{-1} u \cdot \overline{v} dx$ and the corresponding norm $\|u\|_{\mathcal{H}} = (u,u)_{\mathcal{H}}^{1/2}$, and let $H = H(x,\partial)$ be the operator

$$H(x,\partial)U = \nabla \times \tfrac{1}{\mu} \nabla \times \tfrac{1}{\epsilon} U \tag{5.6}$$

which is selfadjoint and non negative on $\mathcal{H}$. The spectral theorem implies that the flow $e^{it\sqrt{H}}$ is well defined, bounded and continuous on $\mathcal{H}$. Let $U(t,x)$ be the solution to

$$\partial_t^2 U + HU = F(t,x), \qquad U(0,x) = 0, \qquad \partial_t U(0,x) = 0, \tag{5.7}$$

where F is $\mathcal{H}$–valued and hence

$$\nabla \cdot F = 0.$$

By Duhamel's formula U is given by

$$U(t,x) = \textstyle\int_0^t H^{-1/2} \sin((t-s)\sqrt{H}) F(s) ds.$$

We thus have

$$\sqrt{H} U = \textstyle\int_0^t \sin((t-s)\sqrt{H}) F(s) ds, \quad \partial_t U = \int_0^t \cos((t-s)\sqrt{H}) F(s) ds. \tag{5.8}$$

Note that also U is divergence free and hence HU is given by

$$HU = -\tfrac{1}{\epsilon\mu} \Delta U + \nabla \times \tfrac{1}{\mu} \nabla \times \tfrac{1}{\epsilon} U - \tfrac{1}{\epsilon\mu} \nabla \times \nabla \times U.$$

Therefore problem (5.7) can be written in the form (5.3) with the choices

$$a = \epsilon\mu, \qquad b(x,\partial)U = \nabla \times \nabla \times U - \epsilon\mu \nabla \times \tfrac{1}{\mu} \nabla \times \tfrac{1}{\epsilon} U, \qquad G = aF. \tag{5.9}$$

We collect in the next lemma some basic estimates involving $\sqrt{H}$ and H. Observe that a map $u \in \dot{H}^2$ satisfies $\nabla u \in L^6$ and $|u| \lesssim \langle x \rangle^{1/2}$. Hence, $H : \dot{H}^2 \to L^2$ is bounded if

$$|\nabla \epsilon| + |\nabla \mu| \lesssim \langle x \rangle^{-1-\delta}, \qquad |D^2 \epsilon| \lesssim \langle x \rangle^{-2-\delta}. \tag{5.10}$$

Lemma 5.3. *Let H be the operator in* (5.6) *and $c, \delta > 0$. Assume that the coefficients ϵ, μ and their first and second derivatives are bounded and that $\epsilon, \mu \geq c$. We take divergence free functions f from $\dot{H}^1$ in (i) and (ii), from H^2 in (iii), and from $\dot{H}^2$ in (iv). Then the following estimates hold.*

(i) If $|\nabla \epsilon| \lesssim \langle x \rangle^{-1-\delta}$, then

$$\|\sqrt{H} f\|_{L^2} \lesssim \|\nabla f\|_{L^2}. \tag{5.11}$$

(ii) If

$$|\nabla \epsilon| + |\nabla \mu| \lesssim \langle x \rangle^{-\frac{3}{2}-\delta}, \qquad |D^2 \epsilon| \lesssim \langle x \rangle^{-\frac{5}{2}-\delta}, \tag{5.12}$$

then

$$\|\sqrt{H} f\|_{L^2} \eqsim \|\nabla f\|_{L^2}. \tag{5.13}$$

(iii) We have

$$\|\langle x \rangle^{-\frac{1}{2}-\delta} \Delta f\|_{L^2} \lesssim \|\langle x \rangle^{-\frac{1}{2}-\delta} H f\|_{L^2} + \|\langle x \rangle^{-\frac{1}{2}-\delta} f\|_{L^2}, \tag{5.14}$$

$$\|\langle x \rangle^{-\frac{1}{2}-\delta} H f\|_{L^2} \lesssim \|\langle x \rangle^{-\frac{1}{2}-\delta} \Delta f\|_{L^2} + \|\langle x \rangle^{-\frac{1}{2}-\delta} f\|_{L^2}. \tag{5.15}$$

(iv) If (5.12) *is true, then for $\sigma \in (0,\delta)$*

$$\|\langle x \rangle^{-\frac{1}{2}-\sigma} \Delta f\|_{L^2} \lesssim \|\langle x \rangle^{-\frac{1}{2}-\sigma} H f\|_{L^2}. \tag{5.16}$$

Proof. PROOF OF (5.11). Integrating by parts we have

$$\|\sqrt{H}f\|_{L^2}^2 \approx \|\sqrt{H}f\|_{\mathcal{H}}^2 = (Hf,f)_{\mathcal{H}} = \int \tfrac{1}{\mu}|\nabla\times\tfrac{1}{\epsilon}f|^2dx \lesssim \|\nabla\times f\|_{L^2}^2 + \|\nabla\epsilon\times f\|_{L^2}^2 \qquad (5.17)$$
$$\lesssim \|\nabla f\|_{L^2}^2 + \|\langle x\rangle^{-1-\delta}f\|_{L^2}^2 \lesssim \|\nabla f\|_{L^2}^2 + \|f\|_{L^6}^2 \lesssim \|\nabla f\|_{L^2}^2.$$

This computation is valid for $f\in H^2\cap\mathcal{H}$, and extends to $f\in\dot{H}^1\cap\mathcal{H}$ by approximation.

PROOF OF (5.14) AND (5.15). Integration by parts yields (all norms are L^2)

$$\|\langle x\rangle^{-s}\nabla f\|^2 \le \|\langle x\rangle^{-s}\Delta f\|\|\langle x\rangle^{-s}f\| + 2s\|\langle x\rangle^{-s}\nabla f\|\|\langle x\rangle^{-s-1}f\|$$

for every $s>0$. By the Cauchy–Schwarz inequality and absorbing one term at the right, we obtain

$$\|\langle x\rangle^{-s}\nabla f\|^2 \le 2\|\langle x\rangle^{-s}\Delta f\|\|\langle x\rangle^{-s}f\| + 2s^2\|\langle x\rangle^{-s-1}f\|^2$$

and then, for arbitrary $\rho>0$,

$$\|\langle x\rangle^{-s}\nabla f\| \le \rho\|\langle x\rangle^{-s}\Delta f\| + C(\rho,s)\|\langle x\rangle^{-s}f\|.$$

Since $\nabla\cdot f=0$, we can write Hf in the form

$$\epsilon\mu Hf = \Delta f + b_1(x)\cdot\nabla f + b_0(x)f$$

for suitable bounded matrices b_j. Taking ρ small in the previous estimate, we deduce

$$\|\langle x\rangle^{-s}Hf\|_{L^2} \gtrsim \|\langle x\rangle^{-s}\Delta f\|_{L^2} - \|\langle x\rangle^{-s}\nabla f\|_{L^2} - \|\langle x\rangle^{-s}f\|_{L^2}$$
$$\gtrsim \|\langle x\rangle^{-s}\Delta f\|_{L^2} - C\|\langle x\rangle^{-s}f\|_{L^2}.$$

The proof of (5.15) is similar.

PROOF OF (5.16). 1) Assume by contradiction the existence of a sequence (f_n) such that $\nabla\cdot f_n=0$, $\|\langle x\rangle^{-1/2-\sigma}\Delta f_n\|_{L^2}=1$ and $\|\langle x\rangle^{-1/2-\sigma}Hf_n\|_{L^2}\to 0$. (We may assume that f_n is a Schwartz function vanishing at 0 together with its derivatives.) By compact embedding we can extract a subsequence (again denoted by f_n) which converges in H^1_{loc} to a limit function f such that $\nabla\cdot f=0$, $\|\langle x\rangle^{-1/2-\sigma}\Delta f\|_{L^2}\le 1$ and $\|\langle x\rangle^{-1/2-\sigma}Hf\|_{L^2}=0$.

We first prove that $f\neq 0$. Note that for this step it is enough to assume (5.10). Recalling (2.33), for a sufficiently regular v we have

$$|\nabla\times\nabla\times v| \lesssim |Hv| + |b(x,\partial)v|, \quad |b(x,\partial)v| \lesssim (|\nabla\mu|^2+|\nabla\epsilon|^2+|D^2\epsilon|)|v| + (|\nabla\epsilon|+|\nabla\mu|)|\nabla v|.$$

The decay of the coefficients thus implies (all norms are L^2)

$$\|\langle x\rangle^{-s}\nabla\times\nabla\times v\| \lesssim \|\langle x\rangle^{-s}Hv\| + \|\langle x\rangle^{-s-1-\delta}\nabla v\| + \|\langle x\rangle^{-s-2-\delta}v\|.$$

for $s=\frac{1}{2}+\sigma$. As in the proof of (5.14), we integrate by parts and get

$$\|\langle x\rangle^{-s-1-\delta}\nabla v\|^2 \le \|\langle x\rangle^{-s-\delta}\Delta v\|\|\langle x\rangle^{-s-2-\delta}v\| + C(s,\delta)\|\langle x\rangle^{-s-1-\delta}\nabla v\|\|\langle x\rangle^{-s-2-\delta}v\|.$$

Cauchy–Schwarz allows us to absorb a term at the left and hence

$$\|\langle x\rangle^{-s-1-\delta}\nabla v\| \le \rho\|\langle x\rangle^{-s-\delta}\Delta v\| + C(\rho)\|\langle x\rangle^{-s-2-\delta}v\|$$

where $\rho>0$ can be taken arbitrarily small. In conclusion we obtain

$$\|\langle x\rangle^{-s}\nabla\times\nabla\times v\| \le \rho\|\langle x\rangle^{-s-\delta}\Delta v\| + C\|\langle x\rangle^{-s}Hv\| + C\|\langle x\rangle^{-s-2-\delta}v\|.$$

Now take $\chi_R(x)=\chi(x/R)$ as above and suppose $v=(1-\chi_R)w$, with w divergence free. Then we have

$$\|\langle x\rangle^{-s}\Delta((1-\chi_R)w)\| \lesssim \|\langle x\rangle^{-s}\nabla\times\nabla\times((1-\chi_R)w)\| + \||w|+|\nabla w|\|_{L^2(R\le|x|\le 2R)}.$$

We combine this inequality with the previous one. For sufficiently small ρ we can absorb a term at the left and derive, for $v=(1-\chi_R)w$ with $\nabla\cdot w=0$,

$$\|\langle x\rangle^{-s}\Delta v\| \lesssim C\|\langle x\rangle^{-s}Hv\| + C\|\langle x\rangle^{-s-2-\delta}v\| + \||w|+|\nabla w|\|_{L^2(R\le|x|\le 2R)}. \qquad (5.18)$$

Now we split

$$1 = \|\langle x\rangle^{-s}\Delta f_n\| \le \|\langle x\rangle^{-s}\Delta(\chi_R f_n)\| + \|\langle x\rangle^{-s}\Delta((1-\chi_R)f_n)\|.$$

For the first term we write

$$\begin{aligned}\|\langle x\rangle^{-s}\Delta(\chi_R f_n)\| &\le \|\langle x\rangle^{-s}H(\chi_R f_n)\| + \|\langle x\rangle^{-s}b(x,\partial)(\chi_R f_n)\| \\ &\lesssim \|\langle x\rangle^{-s}Hf_n\| + \||f_n| + |\nabla f_n|\|_{L^2(|x|\le 2R)}.\end{aligned}$$

For the second one we use (5.18) with $w = f_n$. Summing up, we infer

$$1 \lesssim \|\langle x\rangle^{-s}Hf_n\| + \||f_n| + |\nabla f_n|\|_{L^2(|x|\le 2R)} + R^{-\delta}\|\langle x\rangle^{-s-2}(1-\chi_R)f_n\|. \tag{5.19}$$

We recall the Allegretto–Rellich inequality

$$\||x|^{-a-2}u\|_{L^2} \lesssim \||x|^{-a}\Delta u\|_{L^2}, \tag{5.20}$$

if $\inf_{k\in\mathbb{N}_0}|4k(k+1)-(2a+3)(2a+1)| > 0$ which can be applied to functions that vanish in a neighborhood of 0 and decay fast enough at ∞. (See Theorem 6.4.1 and Remark 6.4.2 in [2].) Then the last term can be estimated by

$$CR^{-\delta}\|\langle x\rangle^{-s}\Delta((1-\chi_R)f_n)\|_{L^2} \lesssim R^{-\delta}\|\langle x\rangle^{-s}\Delta f_n\|_{L^2} + R^{-\delta}\||f_n| + |\nabla f_n|\|_{L^2(|x|\le 2R)}.$$

Using $\|\langle x\rangle^{-s}\Delta f_n\|_{L^2} = 1$ and fixing a large R, we arrive at

$$\begin{aligned}1 &\lesssim \|\langle x\rangle^{-s}Hf_n\| + \||f_n| + |\nabla f_n|\|_{L^2(|x|\le 2R)} + R^{-\delta}, \\ 1 &\lesssim \|\langle x\rangle^{-s}Hf_n\| + \||f_n| + |\nabla f_n|\|_{L^2(|x|\le 2R)}.\end{aligned}$$

Since $\|\langle x\rangle^{-s}Hf_n\| \to 0$ and $f_n \to f$ in H^1_{loc} as $n\to\infty$, we conclude that $f \ne 0$. Moreover, Δf belongs to $L^2_{-1/2-\sigma}$.

2) We finally prove that $f = 0$ and so deduce the required contradiction. To this aim we use that $\Delta f = -b(x,\partial)f =: F$ since $\nabla\cdot f = 0$ and $Hf = 0$. We want to proceed as in Proposition 2.8 for which we will need to establish $F \in L^2_{3/2+\delta}$. The above argument shows that the functions $(1-\chi_1)f_n$ are uniformly bounded in $L^2_{-5/2-\sigma}$, and so f belongs to $L^2_{-5/2-\sigma}$. Interpolation yields $\nabla f \in L^2_{-3/2-\sigma}$. We now use the additional decay (5.12) of the coefficients to deduce that

$$\langle x\rangle^{\delta-\sigma}|\Delta f| = \langle x\rangle^{\delta-\sigma}|F| \lesssim \langle x\rangle^{-\frac{5}{2}-\sigma}|f| + \langle x\rangle^{-\frac{3}{2}-\sigma}|\nabla f| \ \in L^2. \tag{5.21}$$

We take numbers $\delta > \gamma' > \gamma > \sigma$, where we may assume that $\delta \le \frac{1}{6}$. Setting

$$\frac{1}{p_1} = \frac{\gamma'-\sigma}{3} + \frac{1}{2} < 1,$$

Hölder's inequality implies that $\Delta f \in L^{p_1}$. From Sobolev's embedding we then infer

$$\nabla f \in L^{q_1}, \qquad \frac{1}{q_1} = \frac{1}{p_1} - \frac{1}{3} = \frac{\gamma'-\sigma}{3} + \frac{1}{6}.$$

as well as

$$\langle x\rangle^{-\alpha}f \in L^\infty, \qquad \alpha = 2 - \frac{3}{p_1} = \frac{1}{2} - \gamma' + \sigma > 0.$$

The functions $\langle x\rangle^{\gamma-\sigma-2}f$ and $\langle x\rangle^{\gamma-\sigma-1}\nabla f$ are thus contained in L^2 due to Hölder's inequality. Assumption (5.12) then yields

$$\langle x\rangle^{\delta+\gamma-\sigma+\frac{1}{2}}|\Delta f| \lesssim \langle x\rangle^{\gamma-\sigma-2}|f| + \langle x\rangle^{\gamma-\sigma-1}|\nabla f| \ \in L^2.$$

As a result, Δf is an element of L^{p_2} with

$$\frac{1}{p_2} = \frac{\gamma'+\gamma-\sigma+\frac{1}{2}}{3} + \frac{1}{2} < 1.$$

Employing Sobolev's embedding again, we obtain

$$\nabla f \in L^{q_2} \text{ with } \frac{1}{q_2} = \frac{\gamma' + \gamma - \sigma}{3} + \frac{1}{3} \in \Big(\frac{1}{3}, \frac{1}{2}\Big), \qquad f \in L^{r_2} \text{ with } \frac{1}{r_2} = \frac{\gamma' + \gamma - \sigma}{3}.$$

Hence $\langle x\rangle^{2\gamma-\sigma-\frac{3}{2}} f \in L^2$ and $\langle x\rangle^{2\gamma-\sigma-\frac{1}{2}}\nabla f \in L^2$ so that $\langle x\rangle^{\delta+2\gamma-\sigma+1}\Delta f \in L^2$ by (5.12).

Repeating these steps, we gain another factor $\langle x\rangle^{\gamma+\frac{1}{2}}$. We thus obtain $f \in L^2_{-1}$, $\nabla f \in L^2$, and $\Delta f = F \in L^2_{3/2+\delta}$, also interpolating with $\nabla f \in L^{q_2}$. Moreover, we have $\nabla f \in L^6$ since $\Delta f \in L^2$, as well as

$$|\nabla(\langle x\rangle^{-1} f)| \lesssim \langle x\rangle^{-1}|\nabla f| + \langle x\rangle^{-2}|f| \ \in L^2$$

so that $\langle x\rangle^{-1} f \in L^6$. Using (5.12) once more, we derive

$$\langle x\rangle^{\frac{3}{2}+\delta}|\Delta f| \lesssim \langle x\rangle^{-1}|f| + |\nabla f| \ \in L^6.$$

The argument used in Proposition 2.8 now leads to $f = 0$.

Proof of (5.13). The converse inequality is proved by interpolation with the inequality

$$\|\Delta f\|_{L^2} \lesssim \|Hf\|_{L^2}. \tag{5.22}$$

for divergence free $f \in \dot H^2$. One shows (5.22) by contradiction, assuming the existence of (f_n) with $\nabla \cdot f_n = 0$, $\|\Delta f_n\|_{L^2} = 1$ and $\|Hf_n\|_{L^2} \to 0$. Here we may assume that f_n is a Schwartz function vanishing at 0 together with its derivatives.

We proceed as the proof of (5.16) above. As in step 1) of this proof we deduce that f_n tends in H^1_{loc} to a function $f \neq 0$ with $\nabla \cdot f = 0$, $Hf = 0$, and $\|\Delta f\|_{L^2} \le 1$. One only has to modify the last summand in (5.19) with $s = 0$ to $R^{-\frac{\delta}{2}}\|\langle x\rangle^{-\frac{\delta}{2}-2}(1-\chi_R)f_n\|$, in order to use the Rellich-Allegretto inequality (5.20) with $a = \delta/2$. As in step 2) of the proof of (5.16), one also sees that f belongs to $L^2_{-2-\sigma}$ and ∇f to $L^2_{-1-\sigma}$ for some $\sigma > 0$. Just using assumption (5.10) we deduce (5.21) and can proceed as before to conclude the contradiction $f = 0$. □

We are now in position to apply Corollary 5.1, combined with Propositions 2.6–2.8 concerning hypothesis (S).

Proposition 5.4. *Let $\epsilon, \mu : \mathbb{R}^3 \to \mathbb{R}$ and $b(x, \partial)$ as in (5.9). For a $\delta > 0$ assume that*

(1) $\inf \epsilon\mu > 0$ and $(\epsilon\mu)'_- \le \frac{1}{4}(1-2^{-\delta})^{-1}\epsilon\mu\langle x\rangle^{-1-\delta}$,

(2) $|\nabla\epsilon| + |\nabla\mu| \lesssim \langle x\rangle^{-1-\delta}$ and $|\epsilon - 1| + |\mu - 1| + |D^2\epsilon| + |D^2\mu| \lesssim \langle x\rangle^{-2-\delta}$,

(3) either 0 is not a resonance (i.e., if $(\Delta + b)u = 0$ with $u = \Delta^{-1} f$ for some $\langle x\rangle^{\frac{1}{2}+\delta} f \in L^2$ then $u = 0$) or we strengthen (2) by

$$|\nabla\epsilon| + |\nabla\mu| \lesssim \langle x\rangle^{-\frac{3}{2}-\delta}. \tag{5.23}$$

Then for any divergence free forcing term $F \in L^2L^2_{1/2+}$ we have

$$\|\langle x\rangle^{-\frac{1}{2}-} \textstyle\int_0^t \cos((t-s)\sqrt{H})F(s)ds\|_{L^2(dt)L^2} \lesssim \|\langle x\rangle^{1/2+}F\|_{L^2L^2}, \tag{5.24}$$

$$\|\langle x\rangle^{-\frac{1}{2}-} \textstyle\int_{\mathbb{R}} \cos((t-s)\sqrt{H})F(s)ds\|_{L^2(dt)L^2} \lesssim \|\langle x\rangle^{1/2+}F\|_{L^2L^2}. \tag{5.25}$$

Proof. We check the hypotheses in Corollary 5.1 on the coefficients. Assumption (1) in the corollary follows from conditions (1) and (2) here, and assumption (2) in the corollary is an easy consequence of (2) here (compare with (2.25)). The spectral assumption (S) reduces to (3) here in view of Propositions 2.6–2.8.

We can approximate F in $L^2L^2_{1/2+}$ by divergence free functions $F_n \in C^1L^2_{1/2+}$ with compact support in time. The corresponding solutions U_n to (5.7) then satisfy the conditions of Proposition 5.2. By density we can thus assume that U has the required regularity and growth. Since for divergence free solutions the wave equations $\epsilon\mu U_{tt} = (\Delta + b)U + \epsilon\mu F$ and $U_{tt} = HU + F$ coincide, we can apply estimate (5.4). Hence the solution U of problem (5.7) satisfies

$$\|\langle x\rangle^{-\frac{3}{2}-}U\|_{L^2L^2} + \|\langle x\rangle^{-\frac{1}{2}-}\nabla U\|_{L^2L^2} + \|\langle x\rangle^{-\frac{1}{2}-}\partial_t U\|_{L^2L^2} \lesssim \|\langle x\rangle^{\frac{1}{2}+}F\|_{L^2L^2}.$$

Because of (5.8), we obtain

$$\|\langle x\rangle^{-\frac{1}{2}-}\int_0^t \cos((t-s)\sqrt{H})F(s)ds\|_{L^2L^2} \lesssim \|\langle x\rangle^{\frac{1}{2}+}F\|_{L^2L^2}$$

i.e., (5.24). By time translation invariance, we can replace the integral $\int_0^t$ with $\int_T^t$ for an arbitrary $T < 0$, where the implicit constant does not dependent of T, and hence with $\int_{-\infty}^t$ by letting $T \to -\infty$. By time reversal, the estimate is then valid also with the integral $\int_{-\infty}^t$ replaced by $\int_t^{+\infty}$. Summing the two, we arrive at (5.25). □

By a modification of the standard TT^* method we obtain the corresponding homogeneous estimates.

Proposition 5.5. *Under the assumptions of Proposition 5.4 we have for any divergence free data f in the respective spaces*

$$\|\langle x\rangle^{-1/2-}e^{it\sqrt{H}}f\|_{L^2(dt)L^2} \lesssim \|f\|_{L^2}, \quad \|\langle x\rangle^{-1/2-}\partial_t e^{it\sqrt{H}}f\|_{L^2(dt)L^2} \lesssim \|f\|_{\dot{H}^1}, \tag{5.26}$$

$$\|\langle x\rangle^{-1/2-}\nabla e^{it\sqrt{H}}f\|_{L^2(dt)L^2} \lesssim \|f\|_{H^1}, \quad \|\langle x\rangle^{-1/2-}\Delta e^{it\sqrt{H}}f\|_{L^2(dt)L^2} \lesssim \|f\|_{H^2}. \tag{5.27}$$

Proof. 1) By the change of variable $s \to -s$, estimate (5.25) implies

$$\|\langle x\rangle^{-1/2-}\int_{\mathbb{R}} \cos((t+s)\sqrt{H})F(s)ds\|_{L^2(dt)L^2} \lesssim \|\langle x\rangle^{1/2+}F\|_{L^2L^2}.$$

Summing the two inequalities, we get

$$\|\langle x\rangle^{-1/2-}\int_{\mathbb{R}} \cos(t\sqrt{H})\cos(s\sqrt{H})F(s)ds\|_{L^2(dt)L^2} \lesssim \|\langle x\rangle^{1/2+}F\|_{L^2L^2}. \tag{5.28}$$

By subtraction we obtain this estimate with $\sin(t\sqrt{H})\sin(s\sqrt{H})$ instead of $\cos(t\sqrt{H})\cos(s\sqrt{H})$.

To exploit the above bounds, we consider the duality

$$((F,G)) = \int_{\mathbb{R}}(F(t),G(t))_{\mathcal{H}}dt = \int_{\mathbb{R}}\int_{\mathbb{R}^3} F(t,x)\overline{G}(t,x)\epsilon^{-1}dt$$

and the weighted space Z^* of divergence free F with finite norm

$$\|F\|_{Z^*} = \|\langle x\rangle^{1/2+}F\|_{L^2L^2}.$$

Define $(Tf)(t) = \cos(t\sqrt{H})f$ for $f \in \mathcal{H}$ and $t \in \mathbb{R}$, as well as $T^*F = \int_{\mathbb{R}}\cos(s\sqrt{H})F(s)\,ds \in \mathcal{H}$ at first for $F \in Z^*$ with compact support in time. (Recall H is selfadjoint for the ϵ–product.) We obtain

$$TT^*F = \int \cos(t\sqrt{H})\cos(s\sqrt{H})F(s)ds,$$

as well as $((F,Tf)) = (T^*F,f)_{\mathcal{H}}$ and $((TT^*F,G)) = (T^*F,T^*G)_{\mathcal{H}}$ for such f, F and G. Estimate (5.28) yields

$$|((T^*F,T^*G))| = |((\langle x\rangle^{-1/2-}TT^*F, \langle x\rangle^{1/2+}G))| \lesssim \|F\|_{Z^*}\|G\|_{Z^*}.$$

Taking $F = G$ we deduce

$$\|T^*F\|_{\mathcal{H}}^2 \lesssim \|F\|_{Z^*}^2 = \|\langle x\rangle^{1/2+}F\|_{L^2L^2}^2.$$

By density, the operator $T^* : Z^* \to \mathcal{H}$ is bounded. Duality implies

$$\|\langle x\rangle^{-1/2-}\cos(t\sqrt{H})f\|_{L^2L^2} = \|Tf\|_Z = \sup_{\|F\|_{Z^*}\le 1}|((F,Tf))| = \sup_{\|F\|_{Z^*}\le 1}(T^*F,f)_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}$$

where the suprema are taken over F with compact support in time. A similar argument gives

$$\|\langle x\rangle^{-1/2-}\sin(t\sqrt{H})f\|_{L^2L^2} \lesssim \|f\|_{\mathcal{H}}.$$

Combining the two estimates we get the first of (5.26). By the estimate already proved and (5.11) we have

$$\|\langle x\rangle^{-1/2-}\partial_t e^{it\sqrt{H}}f\|_{L^2L^2} = \|\langle x\rangle^{-1/2-}e^{it\sqrt{H}}\sqrt{H}f\|_{L^2L^2} \lesssim \|\sqrt{H}f\|_{L^2} \lesssim \|\nabla f\|_{L^2}$$

and this concludes the proof of (5.26).

2) Applying (5.14) to $u = e^{it\sqrt{H}}f$ and using the first inequality in (5.26) we have

$$\begin{aligned}\|\langle x\rangle^{-1/2-}\Delta u\|_{L^2L^2} \lesssim&\|\langle x\rangle^{-1/2-}Hu\|_{L^2L^2} + \|\langle x\rangle^{-1/2-}u\|_{L^2L^2}\\ =&\|\langle x\rangle^{-1/2-}e^{it\sqrt{H}}Hf\|_{L^2L^2} + \|\langle x\rangle^{-1/2-}e^{it\sqrt{H}}f\|_{L^2L^2}\\ \lesssim&\|Hf\|_{L^2} + \|f\|_{L^2}\end{aligned}$$

and by (5.15) we obtain the second part of (5.27). The first estimate in (5.27) then follows by complex interpolation of weighted Sobolev spaces with the first inequality in (5.26). □

Under more restrictive decay conditions on the coefficients, we can prove a variant of (5.27) in homogeneous norms that is needed below.

Proposition 5.6. *Let $\epsilon, \mu : \mathbb{R}^3 \to \mathbb{R}$ and $b(x, \partial)$ as in* (5.9)*. For some $\delta > 0$ assume that*

(1) $\inf \epsilon\mu > 0$ *and* $(\epsilon\mu)'_- \le \frac{1}{4}(1-2^{-\delta})^{-1}\epsilon\mu\langle x\rangle^{-1-\delta}$,

(2) $|\epsilon-1| + |\mu-1| + |D^2\mu| \lesssim \langle x\rangle^{-2-\delta}$, $|\nabla\epsilon| + |\nabla\mu| \lesssim \langle x\rangle^{-\frac{3}{2}-\delta}$, *and* $|D^2\epsilon| \lesssim \langle x\rangle^{-\frac{5}{2}-\delta}$.

Let f be divergence free. Then in addition to (5.26) *and* (5.27)*, we have the estimates*

$$\|\langle x\rangle^{-1/2-}\nabla e^{it\sqrt{H}}f\|_{L^2(dt)L^2} \lesssim \|\nabla f\|_{L^2}, \quad \|\langle x\rangle^{-1/2-}\Delta e^{it\sqrt{H}}f\|_{L^2(dt)L^2} \lesssim \|Hf\|_{L^2}, \qquad (5.29)$$

Proof. Note that under these assumptions, the spectral condition (S) is satisfied due to Propositions 2.6–2.8. By (5.16) in Lemma 5.3 combined with (5.26) we have

$$\|\langle x\rangle^{-\frac{1}{2}-}\Delta e^{it\sqrt{H}}f\|_{L^2L^2} \lesssim \|\langle x\rangle^{-\frac{1}{2}-}He^{it\sqrt{H}}f\|_{L^2L^2} = \|\langle x\rangle^{-\frac{1}{2}-}e^{it\sqrt{H}}Hf\|_{L^2L^2} \lesssim \|Hf\|_{L^2}.$$

This proves the second estimate in (5.29). Complex interpolation with (5.26) then yields

$$\|\langle x\rangle^{-\frac{1}{2}-}\nabla e^{it\sqrt{H}}f\|_{L^2L^2} \lesssim \|\sqrt{H}f\|_{L^2},$$

and recalling (5.11) we obtain also the first estimate. □

6. Strichartz estimates

We first deduce from the results in [29] a conditional Strichartz estimate for the wave equation

$$(a\partial_t^2 - \Delta - b(x,\partial))U = F, \qquad U(0,\cdot) = U_0, \qquad \partial_t U(0,\cdot) = U_1. \qquad (6.1)$$

Recall that a couple $(p, q) \in [2,\infty]^2$ is *wave admissible* in dimension $n = 3$ if

$$\tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{2}, \qquad p \in [2,\infty], \qquad q \in [2,\infty).$$

We often use that multiplication by ϵ, ϵ^{-1}, μ or μ^{-1} is continuous on the Strichartz spaces, as shown in the next lemma.

Lemma 6.1. *Let $m \in W^{1,\infty}$ be positive with $\frac{1}{m} \in L^\infty$ and $|\nabla m| \lesssim \langle x\rangle^{-1-}$ and let (p, q) be wave admissible. Then the operator $f \mapsto mf$ is bounded on spaces $\dot H^{2/p}_{q'}$, $\dot H^{-2/p}_q$ and $\dot H^{1-2/p}_q$. In addition, assume that $|D^2m| \lesssim \langle x\rangle^{-2-}$. Then $f \mapsto mf$ is also bounded on $\dot H^{1+2/p}_{q'}$.*

Proof. Observe that $|\,|D|^{2/p}m| \lesssim \langle x\rangle^{-2/p-}$ by interpolation so that $|D|^{2/p}m$ belongs to $L^{3p/2}$. Let $\frac{1}{\kappa} = \frac{1}{q'} - \frac{2}{3p} \in [\frac{1}{2}, \frac{1}{q'}]$. Sobolev's inequality yields $\dot H^{2/p}_{q'} \hookrightarrow L^\kappa$. We thus obtain

$$\|\,|D|^{2/p}(mf)\|_{L^{q'}} \lesssim \|m\|_{L^\infty}\,\|\,|D|^{2/p}f\|_{L^{q'}} + \||D|^{2/p}m\|_{L^{3p/2}}\,\|f\|_{L^\kappa} \lesssim \|f\|_{\dot H^{2/p}_{q'}}\,.$$

Duality then implies the boundedness on $\dot H_q^{-2/p}$. We further have $|\,|D|^{1-2/p}m| \lesssim \langle x\rangle^{-2/q-}$ and $\dot H_q^{1-2/p} \hookrightarrow L^{3q}$. As above, one now shows that $f \mapsto mf$ is continuous on $\dot H_q^{1-2/p}$.

For the last claim, let $f \in \dot H^{1+2/p}_{q'}$. The first step allows us to bound $|D|^{2/p}(m\nabla f)$ in $L^{q'}$. For the term $|D|^{2/p}(\nabla m f)$, we note that $\dot H^{1+2/p}_{q'} \hookrightarrow L^{3q'}$ since $1 + \frac{2}{p} - \frac{3}{q'} = -\frac{1}{q'}$ and that $|D|^{2/p}\nabla m \in L^{3q'/2}$ since $(1+\frac{2}{p})\frac{3q'}{2} = 3$. Hence $|D|^{2/p}\nabla m f$ belongs to $L^{q'}$ by Hölder's inequality. The remaining summand $\nabla m|D|^{2/p}f$ is contained in $L^{q'}$ because $\nabla m \in L^3$ and $\dot H^1_{q'} \hookrightarrow L^{\frac{3q'}{3-q'}}$. □

Proposition 6.2. *Assume that the coefficients $a(x) > 0$ and $b(x,\partial)$ satisfy*

(1) $\langle x\rangle^{2+\delta}|D^2a| + \langle x\rangle^{1+\delta}|Da| + \langle x\rangle^{\delta}|a-1| \in L^\infty$,

(2) $|b(x,\partial)v| \lesssim \langle x\rangle^{-2-\delta}|v| + \langle x\rangle^{-1-\delta}|\nabla v|$

for some $\delta > 0$. Let (p,q) and (r,s) be wave admissible. Then there exists $R_0 > 0$ such that, for any $R \ge R_0$ and any solution U of problem (6.1), we have the estimate

$$\||D|^{-\frac{2}{p}}D_{t,x}U\|_{L^pL^q} \lesssim \|D_{t,x}U(0)\|_{L^2} + \||D|^{\frac{2}{r}}F\|_{L^{r'}L^{s'}} + \|D_{t,x}U\|_{L^2L^2(|x|\le R+1)} \tag{6.2}$$

with an implicit constant depending on R.

Proof. Let R be a large parameter to be chosen below and $\chi(x) = \chi_R(x)$ be a smooth cutoff equal to 1 on a ball $B(0,R)$ and vanishing outside $B(0,R+1)$, whose derivatives are bounded independently of R. We decompose $U = v + w$ with $v = \chi U$ and $w = (1-\chi)U$. Let (p,q) and (r,s) be wave admissible.

1) The piece w is supported in $|x| \ge R$ and solves the problem

$$(a\partial_t^2 - \Delta - b(x,\partial))w = G + (1-\chi)F, \qquad w(0,\cdot) = (1-\chi)U_0, \qquad w_t(0,\cdot) = (1-\chi)U_1,$$

where G is the commutator $G = [\chi, \Delta + b]U$. Note that G is supported in $R \le |x| \le R+1$ and satisfies, for some constant depending on the coefficients,

$$|G| \le C(|U| + |D_xU|)\mathbf{1}_{R\le|x|\le R+1}. \tag{6.3}$$

Moreover, by choosing R large enough, we see that in the region $|x| \ge R$ the coefficients fulfill assumptions (8), (9) and (10) of [29] (modified as in Remark 1 of that paper) with a constant $\varepsilon > 0$ which can be made arbitrarily small as $R \to \infty$.

We want to apply Theorem 2 of [29] with $s = 0$ for $R \ge R_0$ and some sufficiently large $R_0 \ge 2$. However this theorem does directly apply to zero-order terms in our situation. So we use it with the modified inhomogeneity $G + (1-\chi)F + b_0w$, where $bw = b_1\cdot\nabla w + b_0w$. We combine Theorem 2 with estimates (12) and (16) of [29], all for $s = 0$. These estimates allow to bound the X^0 and Y^0 norms of [29] from below and above, respectively, by weighted L^2-based norms. Using also Lemma 6.1, it follows

$$\begin{aligned}
&\sup_{j\ge j_R}\|\langle x\rangle^{-\frac{1}{2}}\nabla w\|_{L^2L^2(A_j)} + \||D|^{\sigma}D_{t,x}w\|_{L^pL^q}\\
&\quad\lesssim \|D_{t,x}w(0)\|_{L^2} + \||D|^{\rho}((1-\chi)F)\|_{L^{r'}L^{s'}} + \sum_{j\ge j_R}\|\langle x\rangle^{\frac{1}{2}}(G + b_0w)\|_{L^2L^2(A_j)}\\
&\quad\lesssim \|D_{t,x}w(0)\|_{L^2} + \||D|^{\rho}F\|_{L^{r'}L^{s'}} + \|\langle x\rangle^{\frac{1}{2}+}G\|_{L^2L^2} + \sum_{j\ge j_R}\|\langle x\rangle^{-\frac{3}{2}-\delta}w\|_{L^2L^2(A_j)},
\end{aligned}$$

where $R \approx 2^{j_R}$, $\sigma = -\frac{2}{p}$, $\rho = \frac{2}{r}$ and $A_j = \{2^{j-1} \le |x| \le 2^j\}$. Hölder's and Sobolev's inequalities imply

$$\sum_{j\ge j_R} \|\langle x\rangle^{-\frac{3}{2}-\delta} w\|_{L^2L^2(A_j)} \lesssim \sum_{j\ge j_R} 2^{-j(1/2+\delta)}\|w\|_{L^2L^6(A_j)} \lesssim \sum_{j\ge j_R} 2^{-j(1/2+\delta)}\|\nabla w\|_{L^2L^2(A_j)}$$
$$\lesssim R_0^{-\delta} \sup_{j\ge j_R} \|\langle x\rangle^{-\frac{1}{2}}\nabla w\|_{L^2L^2(A_j)}.$$

Taking a large R_0, we infer

$$\||D|^\sigma D_{t,x} w\|_{L^pL^q} \lesssim \|D_{t,x}w(0)\|_{L^2} + \||D|^\rho F\|_{L^{r'}L^{s'}} + \|\langle x\rangle^{\frac{1}{2}+} G\|_{L^2L^2}.$$

Since G is compactly supported in $R \le |x| \le R+1$, we can bound

$$\|\langle x\rangle^{\frac{1}{2}+} G\|_{L^2L^2} \lesssim \|G\|_{L^2L^2}$$

with an implicit constant $\approx R^{1/2+}$, and Sobolev's embedding further yields

$$\|D_{t,x}w(0)\|_{L^2} \le C(R)\|D_{t,x}U(0)\|_{L^2}.$$

Recalling the definition of G, the previous estimate can thus be simplified to

$$\||D|^\sigma D_{t,x} w\|_{L^pL^q} \lesssim \|D_{t,x}U(0)\|_{L^2} + \||D|^\rho F\|_{L^{r'}L^{s'}} + \||U| + |D_xU|\|_{L^2L^2(R\le|x|\le R+1)}. \tag{6.4}$$

2) Next we consider the remaining piece v supported in $|x| \le R+1$, which solves

$$(a\partial_t^2 - \Delta - b(x,\partial))v = -G + \chi F, \qquad v(0,x) = \chi U_0, \qquad \partial_t v(0,x) = \chi U_1.$$

In the region $|x| \le R+1$ the coefficients satisfy assumptions (8), (9) and (10) of [29] (again modified as in Remark 1 of the paper) but with a possibly large constant ϵ there. We want to apply Theorem 3 of this paper. As above, we have to generalize this result to the case of potentials. Moreover, in this theorem only treats inhomogenities in Y^0 and not in $L^{r'}\dot H^{-2/r}_{s'} + Y^0$, as needed by us.

We first extend this result to a forcing term of the form $f + g \in L^{r'}\dot H^{-2/r}_{s'} + L^2L^2_{1/2+}$ using the parametrix K from Theorem 3 of [29]. (We note that $L^2L^2_{1/2+} \hookrightarrow Y^0$ by (16) of this paper). Let $P = a\partial_t^2 - \Delta - b_1\cdot\nabla$. We consider a function u with $Pu = f + g$. Set $\tilde u = u - Kf$ so that $P\tilde u = (I - PK)f + g$. (See the proof of Lemma 9 in [29].) We restrict the time interval to $t \in [0,\tau]$ for some $\tau \in (0,2]$ and let $L^p_\tau = L^p_{(0,\tau)}$. Estimates (24) and (25) of [29] then yield

$$\begin{aligned}
&\||D|^{-\rho} D_{t,x}u\|_{L^r_\tau L^s} + \||D|^\sigma D_{t,x}u\|_{L^p_\tau L^q}\\
&\le \||D|^{-\rho} D_{t,x}\tilde u\|_{L^r_\tau L^s} + \||D|^{-\rho} D_{t,x}Kf\|_{L^r_\tau L^s} + \||D|^\sigma D_{t,x}\tilde u\|_{L^p_\tau L^q} + \||D|^\sigma D_{t,x}Kf\|_{L^p_\tau L^q}\\
&\lesssim \|D_{t,x}(u - Kf)\|_{L^\infty_\tau L^2} + \|(I-PK)f + g\|_{Y^0} + \||D|^\rho f\|_{L^{r'}_\tau L^{s'}}\\
&\lesssim \|D_{t,x}u\|_{L^\infty_\tau L^2} + \||D|^\rho f\|_{L^{r'}_\tau L^{s'}} + \|g\|_{Y^0}.
\end{aligned}$$

Here we also use that, on $(0,\tau)$, the X^0 norm (modified as in Remark 1 of [29]) is controlled by the $L^\infty L^2$ norm. The implicit constant is uniform in $\tau \le 2$, but depends on R.

Let $t \in [0,\tau]$ and $D_{t,x}u(0) = 0$. A standard energy estimate yields

$$\begin{aligned}
\|D_{t,x}u(t)\|^2_{L^2} &\lesssim \int_0^t \int_{\mathbb{R}^3} (|f| + |g| + |\nabla u|)|\partial_t u|\,dx\,ds\\
&\lesssim \||D|^\rho f\|_{L^{r'}_\tau L^{s'}}\,\||D|^{-\rho}\partial_t u\|_{L^r_\tau L^s} + \|g\|_{L^1_\tau L^2}\,\|\partial_t u\|_{L^\infty_\tau L^2} + \int_0^t \|D_{t,x}u(s)\|^2_{L^2}ds.
\end{aligned}$$

By means of Gronwall's inquality we infer

$$\begin{aligned}
\|D_{t,x}u\|_{L^\infty_\tau L^2} \le \|D_{t,x}u(0)\|_{L^2} &+ \kappa\||D|^{-\rho}\partial_t u\|_{L^r_\tau L^s} + \kappa\|\partial_t u\|_{L^\infty_\tau L^2}\\
&+ c(\kappa)(\||D|^\rho f\|_{L^{r'}_\tau L^{s'}} + \|\langle x\rangle^{1/2+} g\|_{L^2_\tau L^2}).
\end{aligned}$$

We can absorb the second and third term on the right choosing a small $\kappa > 0$ so that

$$\||D|^\sigma D_{t,x} u\|_{L^p_\tau L^q} \lesssim \|D_{t,x} u(0)\|_{L^2} + \||D|^\rho f\|_{L^{r'}_\tau L^{s'}} + \|\langle x\rangle^{1/2+} g\|_{L^2_\tau L^2}. \tag{6.5}$$

3) We put the term $-b_0 v$ to the RHS as before, now applying (6.5) with $f = \chi F$ and $g = b_0 v - G$. To deal with the zero-order part, we also involve the trivial Strichartz pair $(\infty, 2)$, obtaining

$$\begin{aligned}\|\nabla v\|_{L^\infty_\tau L^2} + \||D|^\sigma D_{t,x} v\|_{L^p_\tau L^q} \lesssim \|D_{t,x} v(0)\|_{L^2} + \||D|^\rho (\chi F)\|_{L^{r'}_\tau L^{s'}} + \|\langle x\rangle^{\frac12+} G\|_{L^2_\tau L^2} \\ + \|\langle x\rangle^{-\frac32-} v\|_{L^2_\tau L^2}.\end{aligned}$$

Employing Hölder's and Sobolev's inequalities as in step 1), we control the last summand by

$$\|\langle x\rangle^{-\frac32-} v\|_{L^2_\tau L^2} \lesssim \|\nabla v\|_{L^2_\tau L^2} \le \tau^{\frac12} \|\nabla v\|_{L^\infty_\tau L^2}.$$

For a fixed small $\tau > 0$ we can absorb this term by the LHS. As before we then simplify the estimate to

$$\||D|^\sigma D_{t,x} v\|_{L^p_\tau L^q} \lesssim \|D_{t,x} v(0)\|_{L^2} + \||D|^\rho F\|_{L^{r'}_\tau L^{s'}} + \|G\|_{L^2_\tau L^2}.$$

This inequality is invariant under time translations. By a finite iteration, we conclude

$$\||D|^\sigma D_{t,x} v\|_{L^p_{[0,2]} L^q} \lesssim \|D_{t,x} v(0)\|_{L^2} + \||D|^\rho F\|_{L^{r'}_{[0,2]} L^{s'}} + \|G\|_{L^2_{[0,2]} L^2} \tag{6.6}$$

controlling the initial values by means of the Strichartz pair $(\infty, 2)$. Observe that this estimate is valid on any time interval of length 2.

We now use a reduction which (to our knowledge) originates in [4]. Let $\mathcal{I}$ be the sequence of intervals $I = [k, k+2]$ for $k \in \mathbb{Z}$ and $\{\phi_I\}_{I\in\mathcal{I}}$ be a smooth partition of unity adapted to $\mathcal{I}$. The cutoffed function $v_I = \phi_I(t) v(t,x)$ solves

$$(a\partial_t^2 - \Delta - b(x,\partial)) v_I = F_I + G_I, \qquad v_I(k,x) = \partial_t v_I(k,x) = 0,$$

where $F_I = \chi\phi_I F$ and $G_I = a[\partial_t^2, \phi_I]v - \phi_I G$ are supported in $\{|x| \le R+1, t \in I\}$. We have

$$|G_I(t,x)| \le C(|U| + |D_{t,x} U|)\mathbf{1}_{|x|\le R+1}(x)\mathbf{1}_I(t). \tag{6.7}$$

Estimate (6.6) on the time interval I yields

$$\||D|^\sigma D_{t,x} v_I\|_{L^p_I L^q} \lesssim \||D|^\rho F_I\|_{L^{r'}_I L^{s'}} + \|G_I\|_{L^2_I L^2}.$$

$\|D_{t,x}\mathbf{D}^h\|_{L^2 L^2_{-1/2-}} + \|D_{t,x}\mathbf{D}^h\|_{L^2 L^2_{-1/2-}}$. We now raise both sides to the power $p \ge 2$ and sum over $I \in \mathcal{I}$, obtaining

$$\||D|^\sigma D_{t,x} v\|^p_{L^p L^q} \lesssim \textstyle\sum_I \||D|^\sigma D_{t,x} v_I\|^p_{L^p_I L^q} \lesssim \sum_I \||D|^\rho F_I\|^p_{L^{r'}_I L^{s'}} + \sum_I \|G_I\|^p_{L^2_I L^2}.$$

Since $\sum c_I^p \le (\sum c_I^\kappa)^{p/\kappa}$ for $\kappa \in [1, p]$, we deduce

$$\||D|^\sigma D_{t,x} v\|^p_{L^p L^q} \lesssim (\textstyle\sum_I \||D|^\rho F_I\|^{r'}_{L^{r'}_I L^{s'}})^{p/r'} + (\sum_I \|G_I\|^2_{L^2_I L^2})^{p/2}.$$

Inquality (6.7) then leads to

$$\||D|^\sigma D_{t,x} v\|_{L^p L^q} \lesssim \||D|^\rho (\chi F)\|_{L^{r'} L^{s'}} + \||U| + |D_{t,x} U|\|_{L^2 L^2(|x|\le R+1)}.$$

We also use Sobolev's inequality to estimate U by $D_x U$ with a constant depending on R. Together with Lemma 6.1 and (6.4), the assertion follows. □

Using estimate (19) of [29] one checks easily that the results in this paper, and hence Proposition 6.2, are valid more generally for the system of wave equations

$$(aI_3\partial_t^2 - I_3\Delta - b(x,\partial))U = F, \qquad U(0,\cdot) = U_0, \qquad \partial_t U(0,\cdot) = U_1, \tag{6.8}$$

with diagonal principal part, where $b(x,\partial)$ is a matrix first-order operator which satisfies decay assumptions as in Proposition 6.2. We now apply (6.2) to the Maxwell system

$$\partial_t^2\mathbf{D}+\nabla\times\tfrac{1}{\mu}\nabla\times\tfrac{1}{\epsilon}\mathbf{D}=F,\qquad \mathbf{D}(0,\cdot)=\mathbf{D}_0,\ \ \partial_t\mathbf{D}(0,\cdot)=\mathbf{D}_1,\quad \nabla\cdot\mathbf{D}_0=\nabla\cdot\mathbf{D}_1=\nabla\cdot F=0. \tag{6.9}$$

Recall that the solution to the above problem is given by

$$\mathbf{D}(t)=\cos(t\sqrt{H})\mathbf{D}_0+\sin(t\sqrt{H})H^{-1/2}\mathbf{D}_1+H^{-1/2}\int_0^t\sin((t-s)\sqrt{H})F(s)ds. \tag{6.10}$$

We denote by $\mathbf{D}^h$ the solution to the problem with $F=0$ and by $\mathbf{D}^i$ that one with $\mathbf{D}_0=\mathbf{D}_1=0$.

For $F=0$, the conditional Strichartz estimate in Proposition 6.2 and the smoothing estimates in Propositions 5.5 and 5.6 easily yield the Strichartz inequality for $\mathbf{D}^h$. The usual TT^* argument then allows us to bound $\partial_t\mathbf{D}^i$ and $\sqrt{H}\mathbf{D}^i$ in $L^p\dot{H}_q^{-2/p}$. To replace here $\sqrt{H}$ by ∇, one would need a variant of Lemma 5.3 in $\dot{H}_q^{-2/p}$ which should require a substantial effort. We by-pass this difficulty by means of a modified TT^* argument that also uses ideas from Proposition 5.5. This is possible since we only have to control an error term in $L^2L^2_{-1/2-}$ arising from Proposition 6.2. Here it turns out to be enough to estimate $\nabla H^{-1/2}$ in $L^2_{-1/2-}$ just using Lemma 5.3.

Theorem 6.3. *Under the assumptions of Proposition 5.6, Then the solution $\mathbf{D}(t,x)$ to problem (6.9) satisfies for any wave admissible (p,q) and (r,s) the estimate*

$$\||D|^{-\frac{2}{p}}D_{t,x}\mathbf{D}\|_{L^pL^q}\lesssim\|\nabla\mathbf{D}_0\|_{L^2}+\|\mathbf{D}_1\|_{L^2}+\||D|^{\frac{2}{r}}F\|_{L^{r'}L^{s'}}. \tag{6.11}$$

Proof. As in the proof of Proposition 5.4 we can recast (6.9) in the form (6.8). Since the conditions in Proposition 6.2 are satisfied, estimate (6.2) yields

$$\||D|^{-\frac{2}{p}}D_{t,x}\mathbf{D}\|_{L^pL^q}\lesssim\|\nabla\mathbf{D}_0\|_{L^2}+\|\mathbf{D}_1\|_{L^2}+\||D|^{\frac{2}{r}}F\|_{L^{r'}L^{s'}}+\|D_{t,x}\mathbf{D}\|_{L^2L^2(|x|\le R_0+1)}.$$

for some fixed radius $R_0\ge 1$. The last term is bounded by a constant times

$$\|D_{t,x}\mathbf{D}^h\|_{L^2L^2_{-1/2-}}+\|D_{t,x}\mathbf{D}^i\|_{L^2L^2_{-1/2-}}.$$

In view of (6.10), Propositions 5.5 and 5.6 and (5.13) yield

$$\|D_{t,x}\mathbf{D}^h\|_{L^2L^2_{-1/2-}}\lesssim\|\nabla\mathbf{D}_0\|_{L^2}+\|\mathbf{D}_1\|_{L^2}.$$

We thus have shown (6.11) for $F=0$.

Consider now the case $F\ne 0$. To complete the proof it is sufficient to prove the estimate

$$\left\|D_{t,x}\int_0^t H^{-\frac{1}{2}}e^{i(t-s)\sqrt{H}}F(s)ds\right\|_Z\le C\|F\|_{E^*}$$

where F is divergence free and we set $E=L^r\dot{H}_s^{-2/r}$ and $Z=L^2L^2_{-1/2-}$. First, we notice that by the Christ–Kiselev Lemma this estimate follows from the analogous unretarded one (since $r>2$)

$$\left\|D_{t,x}\int H^{-\frac{1}{2}}e^{i(t-s)\sqrt{H}}F(s)ds\right\|_Z\le C\|F\|_{E^*}.$$

Next, we split $D_{t,x}H^{-\frac{1}{2}}\int e^{i(t-s)\sqrt{H}}F(s)ds = D_{t,x}e^{it\sqrt{H}}H^{-1/2}\int e^{-is\sqrt{H}}F(s)ds$ and we recall from Propositions 5.5 and 5.6 the inequalities

$$\|e^{it\sqrt{H}}f\|_Z\lesssim\|f\|_{L^2},\qquad\|\nabla e^{it\sqrt{H}}f\|_Z\lesssim\|\nabla f\|_{L^2}.$$

Combined with (5.13), these estimates yield

$$\left\|D_{t,x}\int H^{-\frac{1}{2}}e^{i(t-s)\sqrt{H}}F(s)ds\right\|_Z\lesssim\left\|\int e^{-is\sqrt{H}}F(s)ds\right\|_{L^2}+\left\|\nabla H^{-\frac{1}{2}}\int e^{-is\sqrt{H}}F(s)ds\right\|_{L^2}$$

$$\lesssim \left\| \int \cos(s\sqrt{H})F(s)ds \right\|_{L^2} + \left\| \int \sin(s\sqrt{H})F(s)ds \right\|_{L^2}. \quad (6.12)$$

In the first part of the proof we have seen

$$\|D_{t,x}H^{-\frac{1}{2}}\sin(t\sqrt{H})f\|_E \lesssim \|f\|_{L^2}, \qquad \|D_{t,x}\cos(t\sqrt{H})f\|_E \lesssim \|\nabla f\|_{L^2}.$$

Considering only D_t, the first inequality implies

$$\|\cos(t\sqrt{H})f\|_E \lesssim \|f\|_{L^2},$$

while the second one gives

$$\|\sin(t\sqrt{H})f\|_E = \|D_t\cos(t\sqrt{H})H^{-\frac{1}{2}}f\|_E \lesssim \|\nabla H^{-\frac{1}{2}}f\|_{L^2} \lesssim \|f\|_{L^2}$$

using again (5.13). Applying the dual estimates we see that both terms in (6.12) can be estimated by $\|F\|_{E^*}$, and this concludes the proof.

□

The above theorem now easily implies our main result.

Proof of Theorem 1.1. In view of (1.6), the main results for $\mathbf{D}$ is an immediate consequence of the above theorem. The magnetic field $\mathbf{B}$ solves (1.7) which is of the same form as equation (1.6) for $\mathbf{D}$ except that ϵ and μ are interchanged. So Theorem 6.3 is true with $\mathbf{B}$ instead of $\mathbf{D}$ if we replace the condition on second derivatives in (2) of Proposition 5.6 by

$$|D^2\epsilon| \lesssim \langle x\rangle^{-2-\delta}, \qquad |D^2\mu| \lesssim \langle x\rangle^{-\frac{5}{2}-\delta}.$$

Theorem 1.1 for $\mathbf{B}$ again follows easily taking into account (1.7) and Lemma 6.1.

By Lemma 6.1 we can replace $\mathbf{D}$ by $\mathbf{E}$ in Theorem 1.1, with the divergence conditions $\nabla\cdot(\epsilon\mathbf{E}_0) = \nabla\cdot(\epsilon\mathbf{E}_1) = 0$. One can pass from $\mathbf{B}$ to $\mathbf{H}$ in same way as from $\mathbf{D}$ to $\mathbf{E}$. □

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Piero D'Ancona: Dipartimento di Matematica, Sapienza Università di Roma, Piazzale A. Moro 2, 00185 Roma, Italy

E-mail address: dancona@mat.uniroma1.it

Roland Schnaubelt: Karlsruhe Institute of Technology, Department of Mathematics, 76128 Karlsruhe, Germany

E-mail address: schnaubelt@kit.edu