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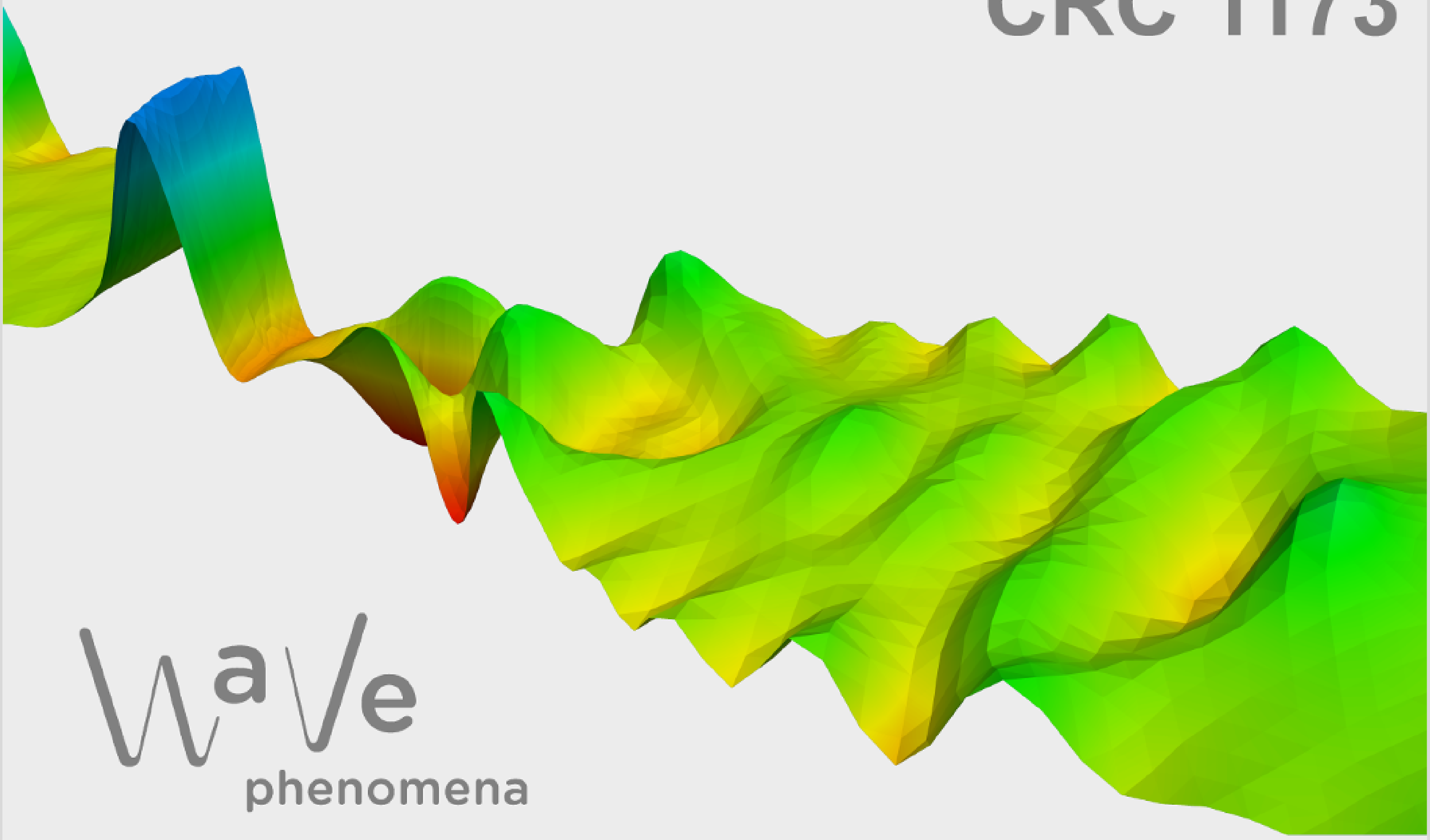

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ON THE GLOBAL WELL-POSEDNESS OF THE QUADRATIC NLS ON $L^2(\mathbb{R}) + H^1(\mathbb{T})$

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Abstract. We study the one dimensional nonlinear Schrödinger equation with power nonlinearity $|u|^{\alpha-1}u$ for $\alpha \in [1,5]$ and initial data $u_0 \in L^2(\mathbb{R}) + H^1(\mathbb{T})$. We show via Strichartz estimates that the Cauchy problem is locally well-posed. In the case of the quadratic nonlinearity ($\alpha = 2$) we obtain *global* well-posedness in the space $C(\mathbb{R}, L^2(\mathbb{R}) + H^1(\mathbb{T}))$ via Gronwall's inequality.

1. Introduction and main results

We are interested in the Cauchy problem for the nonlinear Schrödinger equation (NLS) with power nonlinearity on the space $L^2(\mathbb{R}) + H^1(\mathbb{T})$, i.e.

$$\begin{cases} \mathrm{i}u_t(x,t) + \partial_x^2 u(x,t) \pm |u|^{\alpha-1}u = 0 & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(\cdot,0) = u_0, \end{cases} \tag{1}$$

where $u_0 = v_0 + w_0 \in L^2(\mathbb{R}) + H^1(\mathbb{T})$ and $\alpha \in [1,5]$. By $\mathbb{T}$ we denote the one-dimensional torus, i.e. $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, where we consider functions on $\mathbb{T}$ to be 2π-periodic functions on $\mathbb{R}$. Before we state our main results, let us mention that the NLS (1) is globally well-posed in $L^2(\mathbb{R})$ via Strichartz estimates and mass conservation (see [Tsu87]) and it is globally well-posed in $L^2(\mathbb{T})$ via the Fourier restriction norm method and mass conservation (see [Bou93a]). Motivation for the investigation of hybrid initial values $u_0 \in L^2(\mathbb{R}) + H^1(\mathbb{T})$ comes from high–speed optical fiber communications, where in a certain approximation the behavior of pulses in glass–fiber cables is described by a NLS equation. The NLS (1) with initial data in $H^s(\mathbb{R}) + H^s(\mathbb{T})$ was referred to in [CHKP19] as the tooth problem. A tooth is, for example, w_0 restricted to one period. We think of the addition of v_0 to w_0 as eliminating finitely many of these teeth in the underlying periodic signal. A periodic signal is the simplest type of a non-decaying signal, encoding, for example, an infinite string of ones if there is exactly one tooth per period. However, such a purely periodic signal carries no information. One would like to be able to change it, at least locally. This leads necessarily to a hybrid formulation of the NLS where the signal is the sum of a periodic and a localized part, the localized part being able to remove one or more of the teeth in the underlying periodic signal. This way one can model, for example, a signal consisting of two infinite blocks of ones which are separated by a single zero, or even far more complicated patterns. In the optics literature the phenomenon of ghost pulses (see [MM99] and [ZM99]) occurs which in our terminology corresponds to the regrowth of missing teeth of the solution to the NLS (1).

The case of the cubic nonlinearity ($\alpha = 3$) and the initial data $u_0 \in H^s(\mathbb{R}) + H^s(\mathbb{T})$, where $s \geq 0$, was studied by the authors in [CHKP19], where the existence of weak solutions in the extended sense was established. Moreover, under some

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further assumptions, unconditional uniqueness was obtained. In this paper, due to the non-algebraic structure of the nonlinearity in (1) (for $\alpha \neq 3, 5$) we have to use different methods. For the relation between the solutions of [CHKP19] and the solutions of Theorem 2 we refer to Remark 4.

To state the main results of this paper we need some preparation. Let $u = v + w \in C([0,T], L^2(\mathbb{R}) + H^1(\mathbb{T}))$ where w satisfies the periodic NLS on the torus with initial data w_0. The following is known about w (the case $\alpha \geq 2$ has been treated in [LRS88, Theorem 2.1] while the remaining case $\alpha \in [1,2)$ is presented in Theorem 30).

Theorem 1. *The Cauchy problem for the periodic NLS*

$$\begin{cases} \mathrm{i}w_t(x,t) + \partial_x^2 w(x,t) \pm |w|^{\alpha-1} w = 0 & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ w(\cdot,0) = w_0. \end{cases} \tag{2}$$

is locally well-posed in $H^1(\mathbb{T})$ for $\alpha \geq 1$. That means that for any $w_0 \in H^1(\mathbb{T})$ there is a unique $w \in C([0,T], H^1(\mathbb{T}))$ satisfying (2) *in the mild sense. The guaranteed time of existence T depends only on $\|w_0\|_{H^1(\mathbb{T})}$.*

A solution w to the periodic NLS at hand dictates that the local part v has to be a solution of the Cauchy problem for the *modified NLS*

$$\begin{cases} \mathrm{i}v_t(x,t) + \partial_x^2 v(x,t) \pm G_\alpha(w,v) = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ v(\cdot,0) = v_0, \end{cases} \tag{3}$$

where

$$G_\alpha(w,v) := |v+w|^{\alpha-1}(v+w) - |w|^{\alpha-1} w. \tag{4}$$

The main results of the paper are the following two theorems on local and global wellposedness of NLS (3) and consequently NLS (1).

Theorem 2 (Local well-posedness of the NLS (1))**.** *For $\alpha \in [1,5]$ the Cauchy problem* (3) *is locally well-posed in $C([0,T], L^2(\mathbb{R})) \cap L^{\frac{4(\alpha+1)}{\alpha-1}}([0,T], L^{\alpha+1}(\mathbb{R}))$ for any $v_0 \in L^2(\mathbb{R})$.*

Hence, the original Cauchy problem (1) *is locally well-posed.*

In the case $\alpha \in [1,5)$, the guaranteed time of existence T depends only on $\|v_0\|_2$ and $\|w_0\|_{H^1(\mathbb{T})}$, whereas, for $\alpha = 5$, T depends on the profile of v_0 and $\|w_0\|_{H^1(\mathbb{T})}$.

Remark 3. *In the case $\alpha \in [1,2]$, the intersection in Theorem 2 is not needed, i.e. one has unconditional well-posedness for the perturbation v. However, it is not clear whether the Cauchy problem* (1) *is unconditionally well-posed, since the wellposedness we obtain for the periodic part w is only conditional (see the proof of Theorem 26).*

Remark 4. *Notice that the weak solution in the extended sense $\tilde{u}$ constructed in [CHKP19] and the solution u from Theorem 2 coincide. This can be seen as follows: u is a weak solution in the extended sense, which follows by the definition, Plancherel's theorem and the dominated convergence theorem. Moreover, in the aforementioned paper it was observed that $\tilde{u}$ is unique among those solutions, which can be approximated by smooth solutions. This is true for u and hence $\tilde{u} = u$ follows.*

For $\alpha = 2$, we need the Cauchy problem for the periodic NLS (2) to be globally well-posed in $H^1(\mathbb{T})$. Although this is claimed to be well-known in the community, we could not find a suitable reference. Several people refer to [Bou93a] for this, however in [Bou93a, Proposition 5.73] $\alpha \geq 3$ is required (in our notation). Moreover, in part ii) of the remark on page 152 in [Bou93a], Bourgain mentions that one could get existence of a solution for the quadratic nonlinearity using Schauder's fixed point theorem, but one would loose uniqueness. Hence, we provide a proof in the

Appendix (Theorem 30). This global existence and uniqueness result on the torus, together with a close inspection of the mass $\int |v|^2 \, \mathrm{d}x$ are essential ingredients in our proof of global well-posedness of (1) on the "tooth problem space" $L^2(\mathbb{R}) + H^1(\mathbb{T})$.

Theorem 5 (Global well-posedness of the quadratic NLS)**.** *For $\alpha = 2$ and $v_0 \in L^2(\mathbb{R})$ the unique solution v of (3) from Theorem 2 extends globally and obeys the bound*

$$\|v(\cdot,t)\|_2 \leq \|v_0\|_2 \exp\left[\|w\|_{L_t^\infty L_x^\infty} t\right] \qquad \forall t \in [0,\infty). \tag{5}$$

Hence, the original Cauchy problem (1) for $\alpha = 2$ is globally well-posed.

Although the local well-posedness result of Theorem 2 covers the whole range $\alpha \in [1,5]$, the methods of the proof of Theorem 5 only work for $\alpha = 2$. A more precise explanation is given in Remark 16.

Of course, one can consider hybrid problems for other dispersive equations. Here we confine ourselves to a remark on the KdV.

Remark 6. *Observe that the tooth problem for the KdV reduces to a known setting. More precisely, consider real solutions of*

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) + u_x u = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ & u(\cdot,0) = u_0 = v_0 + w_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T}). \end{cases} \tag{6}$$

Let $u = v + w \in C([0,T], H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T}))$, where $s_2 \in \mathbb{N}$ and w is a global solution of the periodic KdV for the initial data w_0 (see [Bou93b, Theorem 5]). Then v solves

$$v_t + v_{xxx} + v_x v + (wv)_x = 0$$

with the initial data v_0, which is the KdV with the potential w. This problem has been studied in e.g. [ET16, Section 3.1] using parabolic regularization. There it has been shown that v satisfies an exponential bound similar to (5). Combining both results we obtain:

> *For $s_1, s_2 \in \mathbb{N}$ satisfying $s_1 \geq 2$ and $s_2 \geq s_1 + 1$ the KdV tooth problem, i.e., the Cauchy problem (6), is globally well-posed in $H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$.*

The paper is organized as follows: In Section 2 we state the required prerequisites for the proofs of the main theorems. In Section 3 we present the proof of Theorem 2 and in Section 4 we present the proof of Theorem 5. Finally, in the Appendix we justify that the quadratic and subquadratic periodic NLS (2) is globally well-posed in $H^1(\mathbb{T})$.

2. Prerequisites

Let us fix the notation and state some results necessary for the proof of our main theorems. For the purpose of smoothing we will use the heat kernel $(\phi_\varepsilon)_{\varepsilon \geq 0}$. Recall, that $\phi_\varepsilon = \delta_0$, if $\varepsilon = 0$, and

$$\phi_\varepsilon(x) = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{|x|^2}{4\varepsilon}} \qquad \forall x \in \mathbb{R},$$

if $\varepsilon > 0$. We shall denote the convolution (in the space variable x) by e.g. $u * \phi_\varepsilon$.

For $s \in \mathbb{R}$ and $\Omega \in \{\mathbb{R}, \mathbb{T}\}$ we shall denote by $H^s(\Omega)$ the Sobolev spaces on Ω. Also, we set $H^\infty(\Omega) := \cap_{s\in\mathbb{R}} H^s(\Omega)$. By $\mathcal{F}$ we will denote the Fourier transform on $\mathbb{R}$.

We will use the following simple lemma, which can be found e.g. in [Cha18, Lemma 3.9].

Lemma 7 (Size estimate)**.** *Let* $\alpha \geq 1$*. Then the following* size estimate

$$\left| |v_1 + w|^{(\alpha-1)} (v_1 + w) - |v_2 + w|^{(\alpha-1)} (v_2 + w) \right| \tag{7}$$

$$\leq \alpha \max\left\{1, 2^{\alpha-1}\right\} \left(|v_1|^{\alpha-1} + |v_2|^{\alpha-1} + |w|^{\alpha-1} \right) |v_1 - v_2| \tag{8}$$

holds for any $v_1, v_2, w \in \mathbb{C}$*.*

A pair of exponents $(r, q) \in [2, \infty]^2$ is called *admissible* (in one dimension), if

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}. \tag{9}$$

Let us denote by $q_{\mathrm{a}}(r)$ the solution of (9) for any $r \in [2, \infty]$. Another pair of exponents $(\rho, \gamma) \in [1, 2]$ shall be called *dually admissible*, if $(\rho', \gamma') \in [2, \infty]$ is admissible, i.e. if

$$\frac{2}{\gamma} + \frac{1}{\rho} = \frac{5}{2}. \tag{10}$$

For any $\rho \in [1, 2]$ we denote by $\gamma_{\mathrm{a}}(\rho)$ the solution of (10).

Proposition 8 (Strichartz estimates)**.** *(Cf. [KT98, Theorem 1.2]) Let* $(r, q_a(r))$ *be admissible and* $(\rho, \gamma_a(\rho))$ *be dually admissible. Then there is a constant* $C = C(r, \rho) > 0$ *such that for any* $T > 0$*, any* $v_0 \in L^2(\mathbb{R})$ *and any* $F \in L^{\gamma}_{a}(\rho)([0, T], L^{\rho}(\mathbb{R}))$ *the* homogeneous *and* inhomogeneous *Strichartz estimates*

$$\left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L^{q_a(r)}([0,T],L^r(\mathbb{R}))} \leq C \|v_0\|_{L^2(\mathbb{R})}, \tag{11}$$

$$\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} F(\cdot, \tau) \right\|_{L^{q_a(r)}([0,T],L^r(\mathbb{R}))} \leq C \|F\|_{L^{\gamma_a(\rho)}([0,T],L^{\rho}(\mathbb{R}))}. \tag{12}$$

hold.

Lemma 9 (Gronwall, integral form)**.** *(See [Tao06, Theorem 1.10].) Let* $A, T \geq 0$ *and* $u, B \in C([0, T], \mathbb{R}_0^+)$ *be such that*

$$u(t) \leq A + \int_0^t B(s)u(s)\mathrm{d}s \qquad \forall t \in [0, T].$$

Then

$$u(t) \leq A \exp\left(\int_0^t B(s)\mathrm{d}s \right) \qquad \forall t \in [0, T].$$

Lemma 10 (Gronwall, differential form)**.** *(See [Tao06, Theorem 1.12].) Let* $T > 0$*,* $u : [0, T] \to \mathbb{R}_0^+$ *be absolutely continuous and* $B \in C([0, T], \mathbb{R}_0^+)$ *such that*

$$u'(t) \leq B(t)u(t) \qquad \text{for almost every } t \in [0, T].$$

Then

$$u(t) \leq u(0) \exp\left(\int_0^t B(s)\mathrm{d}s \right) \qquad \forall t \in [0, T].$$

Lemma 11. *(See [CHKP19, Equation (18)]). Let* $s \geq 0$*. Then there is a constant* $C = C(s) > 0$ *such that for any* $v \in H^s(\mathbb{R})$ *and any* $w \in H^{s+1}(\mathbb{T})$ *one has* $v \cdot w \in H^s(\mathbb{R})$ *and*

$$\|vw\|_{H^s(\mathbb{R})} \leq C \|v\|_{H^s(\mathbb{R})} \|w\|_{H^{s+1}(\mathbb{T})}.$$

The above estimate is not optimal w.r.t. the assumed regularity of w. However, we do not need a stronger version and the proof is straight forward.

3. Proof of Theorem 2

Consider first the case $\alpha \in [2,5)$. Let us define the space

$$X := C([0,T],L^2(\mathbb{R})) \cap L^{q_{\mathrm{a}}(\alpha+1)}([0,T],L^{\alpha+1}(\mathbb{R})) \tag{13}$$

equipped with the norm

$$\|v\|_X := \|v\|_{L_t^\infty L_x^2} + \|v\|_{L_t^{q_{\mathrm{a}}(\alpha+1)} L^{\alpha+1}} \qquad \forall v \in X,$$

where T will be fixed later in the proof. The integral formulation of (3) reads as

$$v = e^{\mathrm{i}t\partial_x^2} v_0 \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_\alpha(w,v)\mathrm{d}\tau =: \mathcal{T}(v). \tag{14}$$

By Banach's fixed-point theorem, it suffices to show that there are $R,T>0$ such that $\mathcal{T}$ is a contractive self-mapping of

$$M(R,T) := \left\{ v \in X \middle| \; \|v\|_X \le R \right\}.$$

Consider first the self-mapping property. For $r \in \{2, \alpha+1\}$ we have

$$\|\mathcal{T}v\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r} \le \left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r} + \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_\alpha(w,v)\mathrm{d}\tau \right\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r}.$$

By the homogeneous Strichartz estimate (11), we have

$$\left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r} \lesssim \|v_0\|_2$$

for the first summand. This suggests the choice $R \approx \|v_0\|_2$. For the second summand, whose norm also needs to be comparable with R, we will split the integral term. We proceed with the estimates for the contraction property of $\mathcal{T}$, because the self-mapping property follows from them by setting $v=v_1$ and $v_2=0$. To that end, let us define $G_\alpha(w,v_1,v_2) := G_\alpha(w,v_1) - G_\alpha(w,v_2)$, set

$$A := \{x \in \mathbb{R} \mid |w| \le (|v_1|+|v_2|)\}, \tag{15}$$

and introduce

$$G_{\alpha,1}(w,v_1,v_2) := \mathbb{1}_A \left(|w+v_1|^{\alpha-1}(w+v_1) - |w+v_2|^{\alpha-1}(w+v_2) \right)$$

and

$$G_{\alpha,2}(w,v_1,v_2) := \mathbb{1}_{A^c} \left(|w+v_1|^{\alpha-1}(w+v_1) - |w+v_2|^{\alpha-1}(w+v_2) \right).$$

By the triangle inequality one obtains for $r \in \{2,\alpha+1\}$ the estimate

$$\begin{aligned} &\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_\alpha(w,v_1,v_2)\mathrm{d}\tau \right\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r} \\ \le \; &\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_{\alpha,1}(w,v_1,v_2)\mathrm{d}\tau \right\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r} \\ &+ \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_{\alpha,2}(w,v_1,v_2)\mathrm{d}\tau \right\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r}. \end{aligned} \tag{16}$$

We use the inhomogeneous Strichartz inequality and the size estimate (7) to bound the first summand of (16) by

$$\begin{aligned} &\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_{\alpha,1}(w,v_1,v_2)\mathrm{d}\tau \right\|_{L_t^{q_{\mathrm{a}}(r)} L_x^r} \\ \le \; &\|G_{\alpha,1}(w,v_1,v_2)\|_{L_t^{q_{\mathrm{a}}((\alpha+1)')} L_x^{(\alpha+1)'}} \\ \lesssim \; &\left\| \mathbb{1}_A \left(|v_1|^{\alpha-1} + |v_2|^{\alpha-1} + |w|^{\alpha-1} \right) |v_1 - v_2| \right\|_{L_t^{q_{\mathrm{a}}((\alpha+1)')} L_x^{(\alpha+1)'}}. \end{aligned} \tag{17}$$

Using the definition of the set A and Hölder's inequality for the space and time norms we arrive at the upper bound

$$\left\| (|v_1|^{\alpha-1} + |v_2|^{\alpha-1}) \, |v_1 - v_2| \right\|_{L^{q_a((\alpha+1)')} L^{(\alpha+1)'}} \lesssim T^{1-\frac{\alpha-1}{4}} R^{\alpha-1} \|v_1 - v_2\|_X .$$

For the second summand of (16) we obtain by the same methods the bound

$$\begin{aligned} &\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_{\alpha,2}(w, v_1, v_2) \mathrm{d}\tau \right\|_{L_t^{q_a(r)} L_x^r} \\ \lesssim\ & \left\| |w|^{\alpha-1} |v_1 - v_2| \right\|_{L^1(L^2)} \lesssim T \|v_1 - v_2\|_X . \end{aligned}$$

Choosing T small enough shows the contraction property of $\mathcal{T}$ and the proof, in the case $\alpha \in (2, 5)$, concludes.

The case $\alpha \in [1, 2]$ is treated in the same way, but instead of setting $\rho = (\alpha+1)'$ one chooses $\rho = \frac{2}{\alpha}$ for the Strichartz exponent in (17). Applying Hölder's inequality subsequently leads to the $L_t^\infty L_x^2$-norm and hence no intersection in (13) is required, i.e. we indeed have unconditional uniqueness.

For the remaining critical case $\alpha = 5$, consider the complete metric space

$$M(R, T) := \left\{ v \in X \;\middle|\; \left\| v - e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^\infty L_x^2} + \|v\|_{L_t^6 L_x^6} \le R \right\}.$$

We have to show again that $\mathcal{T}$ is a contractive self-mapping of $M(R, T)$ for some $R, T > 0$. Candidates for R and T are determined from the first term of (16), corresponding to the effective power $|v|^5$, exactly as in the treatment of the usual mass critical NLS (see e.g. [LP15, Theorem 5.3]). Subsequently, the remaining terms corresponding to the effective power $|v|^1$ are treated via the Strichartz estimates as in the case $\alpha \in (2, 5)$ enforcing a possibly smaller choice of T. We omit the details. □

4. Proof of Theorem 5

The proof of Theorem 5 will be done by looking at the mass $\frac{1}{2} \|v(t)\|_2^2$ of the solution. In order to make this rigorous we have to work with solutions which are differentiable in time. We will get time regularity from regularity in space. Hence we replace G_2 in (3) by its smooth version G^ε. We obtain

$$\begin{cases} \mathrm{i}v_t(x,t) + \partial_x^2 v(x,t) \pm G^\varepsilon(w, v) = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \qquad\qquad v(\cdot, 0) = v_0, \end{cases} \tag{18}$$

where

$$G^\varepsilon(w, v) := [|v + w| * \phi_\varepsilon](v + w) - [|w| * \phi_\varepsilon]w. \tag{19}$$

Theorem 12 (Local well-posedness of the smoothened modified NLS)**.** *Let $\varepsilon \ge 0$. Then there is a constant $C > 0$ such that for any $v_0 \in L^2$ and any $w \in C(\mathbb{R}, L_x^\infty)$ the Cauchy problem (18) has a unique solution in $C([0,T], L^2(\mathbb{R}))$, provided*

$$T \le C \min \left\{ \|v_0\|_2^{-\frac{4}{3}}, \|w\|_{L_t^\infty, L_x^\infty}^{-1} \right\}. \tag{20}$$

Proof. Consider the integral formulation of (18), i.e.

$$v = e^{\mathrm{i}t\partial_x^2} v_0 \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w, v) \mathrm{d}\tau =: \mathcal{T}^\varepsilon(v) \tag{21}$$

and notice that

$$G^\varepsilon(w, v) = \underbrace{([|v + w| - |w|] * \phi_\varepsilon)\, v}_{=:G_1^\varepsilon(w,v)} + \underbrace{([|v + w| - |w|] * \phi_\varepsilon)\, w + [|w| * \phi_\varepsilon] v}_{=:G_2^\varepsilon(w,v)} .$$

By Banach's fixed-point theorem, it suffices to show that there are $R,T>0$ such that $\mathcal{T}^\varepsilon$ is a contractive self-mapping of

$$M(R,T) := \left\{ v \in C([0,T],L^2(\mathbb{R})) \,\Big|\, \|v\| \le R \right\}.$$

Consider first the self-mapping property. We have

$$\|\mathcal{T}^\varepsilon v\|_{L^\infty_t L^2_x} \le \left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L^\infty_t L^2_x} + \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L^\infty_t L^2_x}.$$

Since the operator $e^{it\partial_x^2}$ is an isometry on L^2 we have

$$\left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L^\infty_t L^2_x} = \|v_0\|_2$$

for the first summand. This suggests the choice $R \approx \|v_0\|_2$. For the second summand, whose norm needs to also be comparable with R, we split the integral term and obtain

$$\begin{aligned} &\left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L^\infty_t L^2_x} \\ \le\ & \left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_1^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L^\infty_t L^2_x} + \left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_2^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L^\infty_t L^2_x}. \end{aligned}$$

Now, both summands are treated via the inhomogeneous Strichartz estimate as in the proof of Theorem 2. More precisely, one has

$$\begin{aligned} \left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_1^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L^\infty_t L^2_x} &\lesssim \|([|v+w|-|w|]*\phi_\epsilon)v)\|_{L^\gamma(L^\rho)} \\ &\le \left\| \|[|v+w|-|w|]*\phi_\epsilon\|_{L^{2\rho}_x} \|v\|_{L^{2\rho}_x} \right\|_{L^\gamma_t} \\ &\le \left\| \|v\|^2_{L^{2\rho}_x} \right\|_{L^\gamma_t} = \|v\|^2_{L^{2\gamma}(L^{2\rho})}. \end{aligned}$$

Above, we used the Cauchy-Schwartz inequality to arrive at the second line and Young's inequality (if $\varepsilon \neq 0$) and a size estimate to pass to the last line (all in the space variable).

As we want to arrive at the norm in $C([0,T],L^2(\mathbb{R}))$, we put $2\rho = 2$, i.e. $\rho = 1$. Then, from the admissibility condition (9) for (ρ',γ'), one obtains $\gamma = \frac{4}{3}$. As $2\gamma = \frac{8}{3} < \infty = q_{\mathrm{a}}(2)$, one can raise the time exponent to ∞ by Hölder's inequality for the time variable, i.e.

$$\|v\|^2_{L^{2\gamma}(L^{2\rho})} \le T^{\frac{3}{4}} \|v\|^2_{L^\infty(L^2)} \le T^{\frac{3}{4}} R^2 \overset{!}{\lesssim} R. \tag{22}$$

This inequality holds under the condition

$$T \lesssim \|v_0\|_2^{-\frac{4}{3}},$$

which is satisfied by (20).

For G_2^ε we similarly obtain

$$\begin{aligned} &\left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_2^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L^\infty_t L^2_x} \\ \lesssim\ & \|([|v+w|-|w|]*\phi_\varepsilon])w\|_{L^{\tilde\gamma}(L^{\tilde\rho})} + \|[|w|*\phi_\varepsilon]v\|_{L^{\tilde\gamma}(L^{\tilde\rho})} \\ \le\ & \|w\|_{L^\infty(L^\infty)} \|[|v+w|-|w|]*\phi_\varepsilon]\|_{L^{\tilde\gamma}(L^{\tilde\rho})} + \|[|w|*\phi_\varepsilon]\|_{L^\infty(L^\infty)} \|v\|_{L^{\tilde\gamma}(L^{\tilde\rho})} \\ \lesssim\ & \|w\|_{L^\infty(L^\infty)} \|v\|_{L^{\tilde\gamma}(L^{\tilde\rho})}, \end{aligned}$$

where we employed Young's inequality and a size estimate to obtain the last line. In contrast to the G_1-case, we choose $\tilde{\rho} = 2$ to arrive at the norm in $C([0,T], L^2(\mathbb{R}))$. Then, by the admissibility condition (9), $\tilde{\gamma} = 1 < \infty = q_{\mathrm{a}}(2)$. Hence, by exploiting again the Hölder's inequality for the time variable, we get

$$\begin{aligned} \|w\|_{L^\infty(L^\infty)} \|v\|_{L^{\tilde{\gamma}}(L^{\tilde{\rho}})} &= \|w\|_{L^\infty(L^\infty)} \|v\|_{L^1(L^2)} \\ &\leq \|w\|_{L^\infty(L^\infty)} T \|v\|_{L^\infty(L^2)} \\ &\leq \|w\|_{L^\infty(L^\infty)} RT \\ &\overset{!}{\lesssim}_1 R. \end{aligned}$$

From this we obtain the additional condition

$$T \lesssim \|w\|_{L^\infty(L^\infty)}^{-1},$$

which is also satisfied by (20).

For the contraction property, consider the splitting

$$\begin{aligned} G^\varepsilon(w, v_1, v_2) &:= G^\varepsilon(w, v_1) - G^\varepsilon(w, v_2) \\ &= [|v_1 + w| * \phi_\varepsilon](v_1 + w) - [|v_2 + w| * \phi_\varepsilon](v_2 + w) \\ &= \underbrace{([|v_1 + w| - |w|] * \phi_\varepsilon)(v_1 - v_2) + ([|v_1 + w| - |v_2 + w|] * \phi_\varepsilon)v_2}_{=:G_1^\varepsilon(w,v_1,v_2)} \\ &\quad + \underbrace{([|v_1 + w| - |v_2 + w|] * \phi_\varepsilon)w + [|w| * \phi_\varepsilon](v_1 - v_2)}_{=:G_2^\varepsilon(w,v_1,v_2)}. \end{aligned}$$

Arguments similar to those used in the proof of the self-mapping property shown above yield the contraction property of $\mathcal{T}^\varepsilon$, possibly requiring an even smaller implicit constant in (20). $\square$

Lemma 13 (Convergence of the solutions for vanishing smoothing)**.** *Fix $v_0 \in L^2$ and $w \in C(\mathbb{R}, C(\mathbb{T}))$, and for all $\varepsilon \geq 0$ denote by $v^\varepsilon \in C([0,T], L^2(\mathbb{R}))$ the unique solution of the Cauchy problem* (18) *from Theorem 12. Then,*

$$\left\|v^\varepsilon - v^0\right\|_{L_t^\infty L_x^2} \xrightarrow{\varepsilon \to 0+} 0.$$

Proof. Recall, that by construction v^ε and v^0 are fixed points of $\mathcal{T}^\varepsilon$ and $\mathcal{T}^0$ respectively and hence

$$\begin{aligned} \left\|v^\varepsilon - v^0\right\|_{L_t^\infty L_x^2} &\leq \left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w, v^\varepsilon) - G^0(w, v^0)\right) \mathrm{d}\tau\right\|_{L_t^\infty L_x^2} \\ &\leq \left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w, v^\varepsilon) - G^\varepsilon(w, v^0)\right) \mathrm{d}\tau\right\|_{L_t^\infty L_x^2} \\ &\quad + \left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w, v^0) - G^0(w, v^0)\right) \mathrm{d}\tau\right\|_{L_t^\infty L_x^2}. \end{aligned}$$

Due to the fact that $\mathcal{T}^\varepsilon$ is contractive, the first summand is controlled by

$$\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w, v^\varepsilon) - G^\varepsilon(w, v^0)\right) \mathrm{d}\tau\right\|_{L_t^\infty L_x^2} \leq C \left\|v^0 - v^\varepsilon\right\|_{L_t^\infty L_x^2},$$

where $C < 1$ is the contraction constant. Thus, it suffices to show that the second summand converges to zero. To that end we first gather terms with the same effective powers of v^0 and w, i.e.

$$\begin{aligned}
&\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left(G^\varepsilon(w,v^0)-G^0(w,v^0)\right)\mathrm{d}\tau\\
=&\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left(\left[\left|w+v^0\right|*\phi_\varepsilon\right](v^0+w)-\left[|w|*\phi_\varepsilon\right]w\right.\\
&\left.-\left|w+v^0\right|(v^0+w)+|w|\,w\right)\mathrm{d}\tau\\
=&\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left(\left[\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right]v^0\right)\mathrm{d}\tau \qquad (23)\\
&+\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left(\left[\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right]w\right. \qquad (24)\\
&\left.+\left(|w|*\phi_\varepsilon-|w|\right)v^0\right)\mathrm{d}\tau.
\end{aligned}$$

The first summand corresponding to $\left|v^0\right|^2$ is treated in the same way as the G_1^ε-term in the proof of Theorem 12, i.e. via a Strichartz estimate and Hölder's inequality. We arrive at

$$\begin{aligned}
&\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left(\left[\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right]v^0\right)\mathrm{d}\tau\right\|_{L_t^\infty L_x^2}\\
\leq\;&\left\|\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right\|_{L_t^{\frac{4}{3}}L_x^2}\cdot\left\|v^0\right\|_{L_t^\infty L_x^2}.
\end{aligned}$$

It suffices to show that the first factor above tends to zero, as ε tends to zero. For almost every $t\in[0,T]$ we have that $\left(\left|w+v^0\right|-|w|\right)\in L^2$, which implies, due to the fact that $(\phi_\varepsilon)_{\varepsilon>0}$ is an approximation to the identity, that

$$\left\|\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right\|_{L_x^2}\xrightarrow{\varepsilon\to0+}0.$$

Furthermore, by Young's inequality,

$$\left\|\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right\|_{L_x^2}^{\frac{4}{3}}\lesssim\left\|v^0\right\|_{L_x^2}^{\frac{4}{3}}$$

for every $\varepsilon>0$ and almost every $t\in[0,T]$. Also,

$$\int_0^T\left\|v^0(\cdot,\tau)\right\|_{L_x^2}^{\frac{4}{3}}\mathrm{d}\tau=\left\|v^0\right\|_{L_t^{\frac{4}{3}}L_x^2}^{\frac{4}{3}}\lesssim_T\left\|v^0\right\|_{L_t^\infty L_x^2}^{\frac{4}{3}}$$

and hence the claim follows by the dominated convergence theorem.

The second summand (Equation (24)), corresponding to $\left|v^0w\right|$, is treated like the G_2^ε-term and we arrive at

$$\begin{aligned}
&\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[\left(\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right)w\right]\mathrm{d}\tau\right\|_{L_t^\infty L_x^2}\\
&+\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[\left(|w|*\phi_\varepsilon-|w|\right)v^0\right]\mathrm{d}\tau\right\|_{L_t^\infty L_x^2}\\
\leq\;&\left\|\left(\left|w+v^0\right|-|w|\right)*\phi_\varepsilon-\left(\left|w+v^0\right|-|w|\right)\right\|_{L_t^1L_x^2}\cdot\|w\|_{L_t^\infty L_x^\infty}\\
&+\left\|v^0\right\|_{L_t^\infty L_x^2}\left\||w|*\phi_\varepsilon-|w|\right\|_{L_t^1L_x^\infty}.
\end{aligned}$$

Observe, that $|w|$ is uniformly continuous in the x-variable on the whole of $\mathbb{R}$. Hence, as for (23), the fact that $(\phi_\varepsilon)_{\varepsilon>0}$ is an approximation to the identity implies the convergence to zero of (24). □

Lemma 14 (Smooth solutions for smooth initial data)**.** *(Cf. [Tao06, Proposition 3.11].) Let $\varepsilon>0$, $w\in C([0,T],H^\infty(\mathbb{T}))$ and $v_0\in\mathcal{S}$ and let v denote the unique* *solution of* (18). *Then* $v \in C^1([0,T], H^\infty(\mathbb{R}))$ *and for any* $s > \frac{1}{2}$ *one has*

$$\|v\|_{L_t^\infty H_x^s} \leq C \|v_0\|_{H^s} \exp\left(\|v\|_{L_t^1 L_x^\infty} + T \|w\|_{C(H^{s+1}(\mathbb{T}))}\right) \tag{25}$$

for some $C = C(\varepsilon, s) > 0$.

Proof. We begin by showing that $v \in C([0,T], H^s(\mathbb{R}))$ for any $s \in \mathbb{N}$. It suffices to prove that the operator $\mathcal{T}^\varepsilon$ from Theorem 12 is a self mapping in $M(R, T') \subseteq H^s$, for a possibly smaller $T' \leq T$. To that end, observe that

$$\begin{aligned} \|\mathcal{T}^\varepsilon v\|_{H^s} &\leq \left\|e^{\mathrm{i}t\partial_x^2} v_0\right\|_{H^s} + \left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w,v) \mathrm{d}\tau\right\|_{H^s} \\ &\leq \|v_0\|_{H^s} + \int_0^t \|G^\varepsilon(w,v)\|_{H^s} \mathrm{d}\tau. \end{aligned}$$

The first summand fixes $R \approx \|v_0\|_{H^s}$. For the integrand in the second summand we have (the variable τ is omitted in the notation)

$$\begin{aligned} &\|G^\varepsilon(w,v)\|_{H^s} \\ \leq\ &\underbrace{\|([|w+v| - |w|] * \phi_\varepsilon)\, v\|_{H^s}}_{=:I} + \underbrace{\|(|w| * \phi_\varepsilon)\, v\|_{H^s}}_{=:II} + \underbrace{\|([|w+v| - |w|] * \phi_\varepsilon)\, w\|_{H^s}}_{=:III}. \end{aligned} \tag{26}$$

As $H^s(\mathbb{R})$ is an algebra with respect to point-wise multiplication, the first summand is estimated against

$$\|([|w+v| - |w|] * \phi_\varepsilon)\, v\|_{H^s} \lesssim \|[|w+v| - |w|] * \phi_\varepsilon\|_{H^s} \|v\|_{H^s}.$$

The first product above is further estimated via the definition of the H^s-norm as

$$\|[|w+v| - |w|] * \phi_\varepsilon\|_{H^s} \lesssim \|\langle\cdot\rangle^s \mathcal{F}\phi_\varepsilon\|_{L^\infty} \||w+v| - |w|\|_2. \tag{27}$$

Further estimating $\|v\|_2 \leq \|v\|_{H^s} \leq R$ and recalling the integral concludes the discussion of this term. The second summand (II) is treated via Lemma 11:

$$\|(|w| * \phi_\varepsilon)\, v\|_{H^s} \lesssim_s \||w| * \phi_\varepsilon\|_{H^{s+1}(\mathbb{T})} \|v\|_{H^s}.$$

We again estimate $\|v\|_{H^s} \leq R$ and observe for the other factor that

$$\begin{aligned} \||w| * \phi_\varepsilon\|_{H^{s+1}(\mathbb{T})} &\approx \sum_{|\alpha| \leq \lceil s+1 \rceil} \||w| * [D^\alpha \phi_\varepsilon]\|_{L^2(\mathbb{T})} \\ &\leq \|w\|_\infty \sum_{|\alpha| \leq \lceil s+1 \rceil} \|D^\alpha \phi_\varepsilon\|_{L^1(\mathbb{R})} \\ &\lesssim_{\varepsilon,s} \|w\|_{H^{s+1}(\mathbb{T})}. \end{aligned}$$

The last summand (III) is estimated via

$$\|([|w+v| - |w|] * \phi_\varepsilon)\, w\|_{H^s} \lesssim_{\varepsilon,s} \|v\|_{H^s} \|w\|_{H^{s+1}(\mathbb{T})}.$$

The proof of the above requires no new techniques and is omitted. All in all this shows the local well-posedness of (18) in $C([0,T'], H^s)$, where the guaranteed time of existence is

$$T' \approx_{\varepsilon,s} \left\{\|w\|_{H^{s+1}(\mathbb{T})}^{-1}, \|v_0\|_{H^s(\mathbb{R})}^{-1}\right\}.$$

To prove the estimate (25), we will employ Lemma 9 (Gronwall's inequality). To that end, let T' be now the maximal time of existence of the solution $v \in C([0,T'), H^s)$. Observe that

$$\|v(\cdot,t)\|_{H^s} = \|(\mathcal{T}^\varepsilon v)(\cdot,t)\|_{H^s} \leq \|v_0\|_{H^s} + \int_0^t \|G^\varepsilon(w,v)(\cdot,\tau)\|_{H^s} \mathrm{d}\tau \qquad \forall t \in [0,T').$$

The integrand above is estimated as in inequality (26). The first term (I), however, needs retreatment, as it is quadratic in $\|v\|_{H^s}$. The algebra property of $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ implies

$$I \leq \|([|w+v| - |w|] * \phi_\varepsilon)\|_{H^s} \|v\|_\infty + \|([|w+v| - |w|] * \phi_\varepsilon)\|_\infty \|v\|_{H^s}.$$

We estimate the first factor in the first summand by (27). For the first factor of the second summand we have

$$\|([|w+v| - |w|] * \phi_\varepsilon)\|_\infty \leq \|[|w+v| - |w|]\|_\infty \|\phi_\varepsilon\|_1 \leq \|v\|_\infty$$

by Young's inequality. Reinserting the estimates for the terms (II) and (III) yields

$$\|v(\cdot,t)\|_{H^s} \lesssim_{s,\varepsilon} \|v_0\|_{H^s} + \int_0^t \left(\|v(\cdot,\tau)\|_\infty + \|w(\cdot,\tau)\|_{H^{s+1}(\mathbb{T})}\right) \|v(\cdot,\tau)\|_{H^s} \,\mathrm{d}\tau.$$

Gronwall's inequality now implies

$$\begin{aligned} \|v(\cdot,t)\|_{H^s} &\lesssim_{\varepsilon,s} \|v_0\|_{H^s} \exp\left(\int_0^t \left(\|v(\cdot,\tau)\|_\infty + \|w(\cdot,\tau)\|_{H^{s+1}(\mathbb{T})}\right) \mathrm{d}\tau\right) \\ &\leq \|v_0\|_{H^s} \exp\left(\|v\|_{L^1_t L^\infty_x} + T' \|w\|_{C(H^{s+1}(\mathbb{T}))}\right) \qquad \forall t \in [0,T'). \end{aligned}$$

Thus we see that a blowup cannot occur for any $T' < T$ and so $T' = T$.

This indeed shows that $v \in C([0,T], H^s)$. As $v_0 \in \mathcal{S}$ and $w \in C([0,T], H^\infty(\mathbb{T}))$ are smooth, a classical result from semi-group theory (see [Paz92, Theorem 4.2.4]) implies that $v \in C^1([0,T], H^s)$. Since $s > \frac{1}{2}$ was arbitrary, the proof is complete. □

Proposition 15. *The unique solution v of* (18) *from Theorem 12 satisfies*

$$\|v(\cdot,t)\|_2 \leq \|v_0\|_2 \exp\left[\|w\|_{L^\infty_t L^\infty_x} t\right] \qquad \forall t \in [0,T]. \tag{28}$$

Proof. Let $w^n \in C([0,T], H^\infty(\mathbb{T}))$ be functions with the property

$$\|w^n - w\|_{C([0,T],H^1(\mathbb{T}))} \xrightarrow{n\to\infty} 0$$

and let $v_n \xrightarrow{n\to\infty} v_0$ in the L^2-norm where $v_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. Moreover, let $v^{\varepsilon,n} \in C^1([0,T], L^2))$ be the solution of (18) with initial data v_n and nonlinearity $G^\varepsilon(w^n, v^{\varepsilon,n})$ (the smoothness of $v^{\varepsilon,n}$ follows from Lemma 14). We have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\varepsilon,n}(\cdot,t)\|_2^2 &= \mathrm{Re}\,\langle \dot{v}^{\varepsilon,n}(\cdot,t), v^{\varepsilon,n}(\cdot,t)\rangle = \mathrm{Re}\left\langle \mathrm{i}\partial_x^2 v^{\varepsilon,n} \pm \mathrm{i}G^\varepsilon(w^n, v^{\varepsilon,n}), v^{\varepsilon,n}\right\rangle \\ &= \underbrace{-\,\mathrm{Re}\,\mathrm{i}\,\langle \nabla v^{\varepsilon,n}, \nabla v^{\varepsilon,n}\rangle}_{=0} \\ &\quad \pm \mathrm{Re}\,\mathrm{i}\,\langle (|v^{\varepsilon,n} + w^n| * \phi_\varepsilon)(v^{\varepsilon,n} + w^n) - (|w^n| * \phi_\varepsilon) w^n, v^{\varepsilon,n}\rangle \\ &= \pm \underbrace{\mathrm{Re}\,\mathrm{i}\,\langle (|v^{\varepsilon,n} + w^n| * \phi_\varepsilon) v^{\varepsilon,n}, v^{\varepsilon,n}\rangle}_{=0} \\ &\quad \pm \mathrm{Re}\,\mathrm{i}\,\langle ([|v^{\varepsilon,n} + w^n| - |w^n|] * \phi_\varepsilon) w^n, v^{\varepsilon,n}\rangle \end{aligned} \tag{29}$$

and hence

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\varepsilon,n}(\cdot,t)\|_2^2 &\leq |\langle ([|v^{\varepsilon,n} + w^n| - |w^n|] * \phi_\varepsilon) w^n, v^{\varepsilon,n}\rangle| \\ &\leq \|[|v^{\varepsilon,n} + w^n| - |w^n|] * \phi_\varepsilon) w^n\|_{L^2_x} \|v^{\varepsilon,n}\|_{L^2_x} \\ &\leq \|w^n\|_{L^\infty_t L^\infty_x} \|v^{\varepsilon,n}\|_{L^2_x}^2 \end{aligned} \tag{30}$$

for all $t \in [0,T]$. Above, we obtained the first estimate by the Cauchy-Schwarz inequality and the second one by Hölder's inequality, Young's inequality and the size estimate. By the differential form of the Gronwall's inequality from Lemma 10, we obtain

$$\|v^{\varepsilon,n}(\cdot,t)\|_2 \le \|v_n\|_2 \exp\left[\|w^n\|_{L_t^\infty L_x^\infty} t\right] \qquad \forall t \in [0,T].$$

In the limit $n \to \infty$, the right-hand side above converges to the right-hand side of (28). It remains to show

$$\|v^{\varepsilon,n} - v^\varepsilon\|_{L^\infty L^2} \xrightarrow{n\to\infty} 0, \tag{31}$$

because then the left-hand side converges to $\|v^\varepsilon\|_{L_t^\infty L_x^2}$ in the limit $n \to \infty$. Finally, Lemma 13 yields

$$\|v^\varepsilon\|_{L_t^\infty L_x^2} \xrightarrow{\varepsilon\to 0} \|v^0\|_{L_t^\infty L_x^2}.$$

To prove (31), observe that the linear evolution poses no problems and hence it suffices to control the integral term

$$\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left[G^\varepsilon(w,v^\varepsilon) - G^\varepsilon(w^n, v^{\varepsilon,n})\right] \mathrm{d}\tau \right\|_{L^\infty L^2}.$$

To that end, we will split the difference of the nonlinear terms according to their effective power up to one exception. We begin by observing that

$$\begin{aligned} & G^\varepsilon(w,v^\varepsilon) - G^\varepsilon(w^n,v^{\varepsilon,n}) \\ = \;& (|w+v^\varepsilon| * \phi_\varepsilon)v^\varepsilon - (|w^n + v^{\varepsilon,n}| * \phi_\varepsilon)v^{\varepsilon,n} \\ & + ([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)w - ([|w^n+v^{\varepsilon,n}| - |w^n|] * \phi_\varepsilon)w^n \end{aligned}$$

and gather the first and the second summand, as well as the third and the last summand. In the first sum we have

$$\begin{aligned} & (|w+v^\varepsilon| * \phi_\varepsilon)v^\varepsilon - (|w^n+v^{\varepsilon,n}| * \phi_\varepsilon)v^{\varepsilon,n} \\ = \;& \underbrace{(|w+v^\varepsilon| * \phi_\varepsilon)v^\varepsilon - (|w+v^\varepsilon| * \phi_\varepsilon)v^{\varepsilon,n}}_{=:I} \\ & + \underbrace{(|w+v^\varepsilon| * \phi_\varepsilon)v^{\varepsilon,n} - (|w^n+v^{\varepsilon,n}| * \phi_\varepsilon)v^{\varepsilon,n}}_{=:II}, \end{aligned}$$

whereas for the second sum

$$\begin{aligned} & ([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)w - ([|w^n+v^{\varepsilon,n}| - |w^n|] * \phi_\varepsilon)w^n \\ = \;& \underbrace{([|w^n+v^\varepsilon| - |w^n|] * \phi_\varepsilon)w^n - ([|w^n+v^{\varepsilon,n}| - |w^n|] * \phi_\varepsilon)w^n}_{=:III} \\ & + \underbrace{([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)w - ([|w^n+v^\varepsilon| - |w^n|] * \phi_\varepsilon)w^n}_{=:IV} \end{aligned}$$

holds. We now complete the splitting of $G^\varepsilon(w,v^{\varepsilon,n}) - G^\varepsilon(w^n,v^{\varepsilon,n})$ into terms of the same effective powers. We have

$$\begin{aligned} I \;&=\; (|w+v^\varepsilon| * \phi_\varepsilon)(v^\varepsilon - v^{\varepsilon,n}) \\ &=\; ([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)(v^\varepsilon - v^{\varepsilon,n}) + (|w| * \phi_\varepsilon)(v^\varepsilon - v^{\varepsilon,n}), \\ II \;&=\; ([|w+v^\varepsilon| - |w^n+v^{\varepsilon,n}|] * \phi_\varepsilon)v^{\varepsilon,n} \\ &=\; ([|w+v^\varepsilon| - |w+v^{\varepsilon,n}|] * \phi_\varepsilon)v^{\varepsilon,n} + ([|w+v^{\varepsilon,n}| - |w^n+v^{\varepsilon,n}|] * \phi_\varepsilon)v^{\varepsilon,n}, \\ III \;&=\; ([|w^n+v^\varepsilon| - |w^n+v^{\varepsilon,n}|] * \phi_\varepsilon)w^n \text{ and} \\ IV \;&=\; ([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)w - ([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)w^n \\ & \quad - ([|w^n+v^\varepsilon| - |w^n|] * \phi_\varepsilon)w^n + ([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)w^n \\ &=\; ([|w+v^\varepsilon| - |w|] * \phi_\varepsilon)(w - w^n) \\ & \quad + ([|w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n|] * \phi_\varepsilon)w^n, \end{aligned}$$

from which the effective powers are obvious, and put

$$\begin{aligned}\tilde{G}_1^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n}) &:= ([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})\\ &\quad +([|w+v^\varepsilon|-|w+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n},\\ \tilde{G}_2^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n}) &:= (|w|*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})+([|w^n+v^\varepsilon|-|w^n+v^{\varepsilon,n}|]*\phi_\varepsilon)w^n\\ &\quad +([|w+v^{\varepsilon,n}|-|w^n+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n}\\ &\quad +([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(w-w^n)\\ &\quad +([|w+v^\varepsilon|-|w|-|w^n+v^\varepsilon|+|w^n|]*\phi_\varepsilon)w^n.\end{aligned}$$

Now, by the triangle inequality and the inhomogeneous Strichartz estimate, one has

$$\begin{aligned}&\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[G^\varepsilon(w,v^{\varepsilon,n})-G^\varepsilon(w^n,v^{\varepsilon,n})\right]\mathrm{d}\tau\right\|_{L^\infty L^2}\\ \leq\ &\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\tilde{G}_1^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\mathrm{d}\tau\right\|_{L^\infty L^2}\\ &+\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\tilde{G}_2^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\mathrm{d}\tau\right\|_{L^\infty L^2}\\ \lesssim\ &\left\|\tilde{G}_1^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\right\|_{L_t^{\frac{4}{3}}L_x^1}+\left\|\tilde{G}_2^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\right\|_{L_t^1L_x^2}.\end{aligned}$$

We begin by estimating the first summand above. In fact, we have

$$\begin{aligned}&\|([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})\|_{L_t^{\frac{4}{3}}L_x^1}\\ \leq\ &\left\|t\mapsto\|[|w+v^\varepsilon|-|w|]*\phi_\varepsilon\|_{L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_x^2}\right\|_{\frac{4}{3}}\\ \leq\ &\left\|t\mapsto\|v^\varepsilon\|_{L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_x^2}\right\|_{\frac{4}{3}}\\ \leq\ &T^{\frac{3}{4}}\|v^\varepsilon\|_{L_t^\infty L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.\end{aligned}$$

by the Cauchy-Schwarz, Young's and the inverse triangle inequalities for the space variable and Hölder's inequality for the time variable. Choosing T sufficiently small shows that

$$\|([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})\|_{L_t^{\frac{4}{3}}L_x^1}\leq\frac{1}{5}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.$$

For the second term in the definition of $\tilde{G}_1^\varepsilon$ the same techniques are applied which yield the bound

$$\|([|w+v^\varepsilon|-|w+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n}\|_{L_t^{\frac{4}{3}}L_x^1}\leq T^{\frac{3}{4}}\|v^{\varepsilon,n}\|_{L_t^\infty L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.$$

By the proof of Theorem 12, one has

$$\|v^{\varepsilon,n}\|_{L_t^\infty L_x^2}\lesssim\|v_n\|_2\approx\|v_0\|_2 \tag{32}$$

and thus choosing T sufficiently small again implies

$$\|([|w+v^\varepsilon|-|w+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n}\|_{L_t^{\frac{4}{3}}L_x^1}\leq\frac{1}{5}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.$$

The first term in the definition of $\tilde{G}_2^\varepsilon$ is treated similarly to the above. The same is true for the second term, where we additionally observe that

$$\sup_{n\in\mathbb{N}}\|w^n\|_{C([0,T],H^1(\mathbb{T}))}<\infty. \tag{33}$$

For the third term, we have

$$\begin{aligned}
&\left\|\left(\left[|w+v^{\varepsilon,n}|-|w^n+v^{\varepsilon,n}|\right]*\phi_\varepsilon\right)v^{\varepsilon,n}\right\|_{L^1_tL^2_x}\\
\le\ &\left\|\left[|w+v^{\varepsilon,n}|-|w^n+v^{\varepsilon,n}|\right]*\phi_\varepsilon\right\|_{L^\infty_tL^\infty_x}\left\|v^{\varepsilon,n}\right\|_{L^\infty_tL^2_x}\\
\le\ &\left\|w-w^n\right\|_{L^\infty_tL^\infty_x}\left\|v^{\varepsilon,n}\right\|_{L^\infty_tL^2_x}\\
\lesssim\ &\left\|w-w^n\right\|_{L^\infty_tH^1_x(\mathbb{T})}\xrightarrow{n\to\infty}0,
\end{aligned}$$

where the Cauchy-Schwarz inequality was used for the first estimate, the embedding $L^\infty_t\hookrightarrow L^1_t$, Young's inequality and the inverse triangle inequality for the second estimate and the embedding $C([0,T],H^1(\mathbb{T}))\hookrightarrow L^\infty_tL^\infty_x$ together with (32) for the last estimate. By the same techniques, one obtains the convergence of the fourth term to zero.

Finally, for the last term in the definition of $\tilde{G}^\varepsilon_2$, one has

$$\begin{aligned}
&\left\|\left(\left[|w+v^\varepsilon|-|w|-|w^n+v^\varepsilon|+|w^n|\right]*\phi_\varepsilon\right)w^n\right\|_{L^1_tL^2_x}\\
\le\ &\left\|\left|w+v^\varepsilon\right|-|w|-|w^n+v^\varepsilon|+|w^n|\right\|_{L^1_tL^2_x}\left\|w^n\right\|_{L^\infty H^1_x(\mathbb{T})}\\
\lesssim\ &\left\|\left|w+v^\varepsilon\right|-|w|-|w^n+v^\varepsilon|+|w^n|\right\|_{L^1_tL^2_x},
\end{aligned}$$

where Hölder's inequality, the embedding $C([0,T],H^1(\mathbb{T}))\hookrightarrow L^\infty_tL^\infty_x$ and Young's inequality were used for the first estimate and (33) for the second estimate. Observe that by the inverse triangle inequality, the bound

$$\left||w+v^\varepsilon|-|w|-|w^n+v^\varepsilon|+|w^n|\right|\le 2\min\left\{|w-w^n|,|v^\varepsilon|\right\}\le 2|v^\varepsilon|$$

holds pointwise (in t and x). This implies that

$$|w+v^\varepsilon|-|w|-|w^n+v^\varepsilon|+|w^n|\xrightarrow{n\to\infty}0$$

and hence, by the theorem of dominated convergence for the space variable,

$$g_n(t):=\left\|\left|w+v^\varepsilon\right|-|w|-|w^n+v^\varepsilon|+|w^n|\right\|_{L^2_x}\xrightarrow{n\to\infty}0\qquad\forall t\in[0,T].$$

Moreover, for all $t\in[0,T]$, we have $g_n(t)\le 2\left\|v^\varepsilon(\cdot,t)\right\|_2$ and $\|v^\varepsilon\|_{L^1_tL^2_x}\lesssim\|v^\varepsilon\|_{L^\infty_tL^2_x}<\infty$. Hence, reapplying the theorem of dominated convergence for the time variable yields

$$\left\|\left(\left[|w+v^\varepsilon|-|w|-|w^n+v^\varepsilon|+|w^n|\right]*\phi_\varepsilon\right)w^n\right\|_{L^1_tL^2_x}\xrightarrow{n\to\infty}0$$

as claimed. □

Notice that (28) together with the local well-posedness of NLS (3) from Theorem 12 imply that NLS (3) is globally well-posed, i.e. Theorem 5 is proved.

Remark 16. *Observe, that in the case $\alpha\neq 2$, the proof would proceed roughly unchanged up to Equation* (29). *However, we could replace the differential inequality* (30) *by*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|v^{\varepsilon,n}(\cdot,t)\right\|_2^2\lesssim\|w^n\|^{\alpha-1}_{L^\infty_tL^\infty_x}\left\|v^{\varepsilon,n}\right\|^2_{L^2_x}+\|w^n\|_{L^\infty_tL^\infty_x}\left\|v^{\varepsilon,n}\right\|^\alpha_{L^\alpha_x}$$

and this bound is not sufficient to exclude a blow-up of the L^2-norm.

Appendix A. Quadratic and Subquadratic NLS on the torus

To prove global existence of solutions to the Cauchy problem of the quadratic and subquadratic nonlinear Schrödinger equation on $\mathbb{T}$ (that is (2) with $\alpha\in[1,2]$), we will employ the mass and energy conservation laws. The justification of conservation laws requires solutions which are differentiable in time. Again, time regularity will be obtained from regularity in space. To that end we will smoothen out the rough quadratic nonlinearity in such a way that the solutions of the resulting equation still admit suitable conservation laws. The regularization is slightly different from the one used in the proof of Theorem 5. Let us mention that the ideas presented here are borrowed from [GV79] where the same problem was studied on $\mathbb{R}^d$, using a contraction argument and conservation laws. Since our setting is based on the torus, we have to work with Bourgain spaces. For the convenience of the reader, we present some of the arguments in detail.

Observe that, if w is a sufficiently nice 2π-periodic function and $\varepsilon > 0$, then

$$\begin{aligned}(w * \phi_\varepsilon)(x) &= \int_{-\infty}^{\infty} w(y)\phi_\varepsilon(x-y)\mathrm{d}y = \sum_{n\in\mathbb{Z}} \int_{(2n-1)\pi}^{(2n+1)\pi} w(y)\phi_\varepsilon(x-y)\mathrm{d}y \\ &= \int_{-\pi}^{\pi} w(y) \sum_{n\in\mathbb{Z}} \phi_\varepsilon(x-y-2n\pi)\mathrm{d}y.\end{aligned}$$

Hence, convolution of w with ϕ_ε on $\mathbb{R}$ corresponds to convolution of w with the periodization of ϕ_ε on $\mathbb{T}$. For the rest of the paper we will slightly abuse the notation and denote this periodization also by ϕ_ε. In the same spirit we will use from now on $*$ to denote the convolution on $\mathbb{T}$.

The smooth version of (2) for $\alpha \in [1,2]$ reads as

$$\begin{cases} \mathrm{i}w_t(x,t) + \partial_x^2 w(x,t) \pm \left(|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon)\right)*\phi_\varepsilon = 0 \qquad (x,t)\in\mathbb{T}\times\mathbb{R}, \\ \qquad\qquad w(\cdot,0) = w_0 * \phi_\varepsilon \end{cases} \tag{34}$$

and the corresponding Duhamel's formula is (cf. [GV79, Equations (2.14), (2.13), (2.11) and (1.15)])

$$w(\cdot,t) = e^{\mathrm{i}t\partial_x^2}(w_0*\phi_\varepsilon) \pm \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[\left(|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon)\right)*\phi_\varepsilon(\cdot,\tau)\right]\mathrm{d}\tau. \tag{35}$$

From now on, we denote by $\mathcal{F}$ and $\mathcal{F}^{(-1)}$ the *Fourier transform* and the inverse Fourier transform, on the torus, respectively. We use the symmetric choice of constants and write also

$$\begin{aligned}\hat{f}(\xi) &:= (\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x, \\ \check{g}(x) &:= \left(\mathcal{F}^{(-1)}g\right)(x) = \frac{1}{\sqrt{2\pi}}\sum_{\xi\in\mathbb{Z}} e^{\mathrm{i}\xi\cdot x} g(\xi).\end{aligned}$$

One has $\mathcal{F}(f*g) = \sqrt{2\pi}\hat{f}\hat{g}$. Furthermore, let $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ for any $\xi\in\mathbb{R}$ and $J^s w := \mathcal{F}^{(-1)}\langle\cdot\rangle^s\mathcal{F}w$ for any $w \in (C^\infty(\mathbb{T}))'$.

A.1. **Prerequisites.** In this subsection, we present some technical results from the literature, needed for treatment of the quadratic nonlinearity.

Lemma 17. *Let $p\in[1,\infty]$ and $\varepsilon \geq 0$. Then for any $w\in L^p(\mathbb{T})$ one has*

$$\|w*\phi_\varepsilon\|_{L^p(\mathbb{T})} \leq \|w\|_{L^p(\mathbb{T})}.$$

Lemma 18. *Let $s\in\mathbb{R}$ and $w\in H^s(\mathbb{T})$. Then*

$$\|w*\phi_\varepsilon\|_{H^s(\mathbb{T})} \leq \|w\|_{H^s(\mathbb{T})} \qquad \text{and} \qquad \|w*\phi_\varepsilon\|_{\dot{H}^s(\mathbb{T})} \leq \|w\|_{\dot{H}^s(\mathbb{T})} \qquad \forall\varepsilon\geq 0$$

where we denote by $\dot{H}^s(\mathbb{T})$ the homogeneous Sobolev norm on the torus. Furthermore, if $\varepsilon>0$, then

$$\|w*\phi_\varepsilon\|_{H^s(\mathbb{T})} \lesssim_{\varepsilon,s} \|w\|_{L^2(\mathbb{T})}.$$

Lemma 19. *(Cf. [Bre11, Theorem 3.16].) Let $w^n \xrightarrow{n\to\infty} w$ in $L^2(\mathbb{T})$ and assume that $\sup_{n\in\mathbb{N}}\|w^n\|_{H^1(\mathbb{T})} < \infty$. Then $w\in H^1(\mathbb{T})$ and*

$$\|w\|_{H^1(\mathbb{T})} \leq \liminf_{n\to\infty}\|w^n\|_{H^1(\mathbb{T})}, \qquad \|w\|_{\dot{H}^1(\mathbb{T})} \leq \liminf_{n\to\infty}\|w^n\|_{\dot{H}^1(\mathbb{T})}, \tag{36}$$

and $w^n \rightharpoonup w$ *in* $H^1(\mathbb{T})$*, i.e. for any* $u \in H^1(\mathbb{T})$ *one has*

$$\lim_{n\to\infty} \langle w^n, u\rangle_{H^1(\mathbb{T})} = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \langle k\rangle^2 \overline{\hat{w}_k^n} \hat{u}_k = \langle w, u\rangle_{H^1(\mathbb{T})} . \tag{37}$$

If additionally $\|w^n\|_{H^1} \xrightarrow{n\to\infty} \|w\|_{H^1}$*, then* $w^n \xrightarrow{n\to\infty} w$ *in* $H^1(\mathbb{T})$*.*

In the following we are going to use the $X^{s,b}$ spaces on the torus where $s, b \in \mathbb{R}$. They are defined via the norm (see equation (3.49) in [ET16])

$$\|w\|_{X^{s,b}} = \|\langle k\rangle^s \langle \tau + k^2\rangle^b \hat{w}(\tau, k)\|_{L^2_\tau l^2_k}. \tag{38}$$

Lemma 20 ($X^{0,\frac{3}{8}} \hookrightarrow L^4(\mathbb{T}\times\mathbb{R})$)**.** *(See [Tao06, Proposition 2.13].) We have*

$$\|w\|_{L^4(\mathbb{T}\times\mathbb{R})} \lesssim \|w\|_{X^{0,\frac{3}{8}}}$$

for any $w \in \mathcal{S}(\mathbb{R}, C^\infty(\mathbb{T}))$*.*

Lemma 21 ($X^{s,b}_\delta \hookrightarrow C(H^s)$)**.** *(Cf. [ET16, Lemma 3.9].) Let* $b > \frac{1}{2}$ *and* $s \in \mathbb{R}$*. Then*

$$\|w\|_{C([0,\delta],H^s(\mathbb{T}))} \lesssim \|w\|_{X^{s,b}_\delta} .$$

Lemma 22 (Linear Schrödinger evolution in $X^{s,b}_\delta$)**.** *(Cf. [ET16, Lemma 3.10].) Let* $b, s \in \mathbb{R}$*,* $\delta \in (0, 1]$ *and* η *a smooth cut-off in time. Then*

$$\left\|\eta(t) e^{\mathrm{i}t\partial_x^2} w_0\right\|_{X^{s,b}_\delta} \lesssim \|w_0\|_{H^s(\mathbb{T})} \qquad \forall w_0 \in H^s(\mathbb{T}).$$

Lemma 23 (Treating the integral term in $X^{s,b}_\delta$)**.** *(Cf. [ET16, Lemma 3.12].) Let* $b \in \left(\frac{1}{2}, 1\right]$*,* $s \in \mathbb{R}$ *and* $\delta \le 1$*. Set* $b' \coloneqq b - 1$*. Then*

$$\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} F(\tau) \mathrm{d}\tau\right\|_{X^{s,b}_\delta} \lesssim_b \|F\|_{X^{s,b'}_\delta} \qquad \forall F \in X^{s,b'}_\delta .$$

Lemma 24 (Changing b in $X^{s,b}_\delta$)**.** *(Cf. [ET16, Lemma 3.11].) Let* $b, b' \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ *with* $b' < b$*,* $s \in \mathbb{R}$ *and* $\delta \in (0, 1]$*. Then*

$$\|w\|_{X^{s,b'}_\delta} \lesssim \delta^{b-b'} \|w\|_{X^{s,b}_\delta} \qquad \forall w.$$

The next proposition appears in [ET16] for the case of the cubic nonlinearity and $\varepsilon = 0$. Since we need the corresponding result for (sub)quadratic nonlinearities which are more complicated than the algebraic cubic nonlinearity, we present the proof, too.

Proposition 25 (Control of the nonlinearity in $X^{s,b}_\delta$)**.** *(Cf. [ET16, Proposition 3.26].) Let* $s \ge 0$ *and* $\varepsilon > 0$ *or* $\varepsilon = s = 0$*. Then, for all* w_1, w_2 *we have*

$$\begin{aligned} &\left\|\left(|w_1 * \phi_\varepsilon|^{\alpha-1} (w_1 * \phi_\varepsilon)\right) * \phi_\varepsilon - \left(|w_2 * \phi_\varepsilon|^{\alpha-1} (w_2 * \phi_\varepsilon)\right) * \phi_\varepsilon\right\|_{X^{s,-\frac{3}{8}}_\delta} \\ \lesssim_{\varepsilon,s} &\left(\|w_1\|^{\alpha-1}_{X^{0,\frac{3}{8}}_\delta} + \|w_2\|^{\alpha-1}_{X^{0,\frac{3}{8}}_\delta}\right)\left(\|w_1 - w_2\|_{X^{0,\frac{3}{8}}_\delta}\right). \end{aligned}$$

Proof. Fix w_1, w_2. Then, by Plancherel theorem and duality in $L^2(\mathbb{R}\times\mathbb{T})$, one has

$$\begin{aligned} &\left\|\left(|w_1 * \phi_\varepsilon|^{\alpha-1} (w_1 * \phi_\varepsilon)\right) * \phi_\varepsilon - \left(|w_2 * \phi_\varepsilon|^{\alpha-1} (w_2 * \phi_\varepsilon)\right) * \phi_\varepsilon\right\|_{X^{s,-\frac{3}{8}}_\delta} \\ = &\sup_{\|w\|_{X^{-s,\frac{3}{8}}_\delta}=1} \left|\left\langle\left(|w_1 * \phi_\varepsilon|^{\alpha-1} (w_1 * \phi_\varepsilon) - |w_2 * \phi_\varepsilon|^{\alpha-1} (w_2 * \phi_\varepsilon)\right) * \phi_\varepsilon, w\right\rangle_{L^2(\mathbb{R}\times\mathbb{T})}\right| . \end{aligned}$$

Fix any $w \in X_\delta^{-s,\frac{3}{8}}$ with $\|w\|_{X_\delta^{-s,\frac{3}{8}}} = 1$. Then

$$\begin{aligned}
& \left|\left\langle\left(|w_1 * \phi_\varepsilon|^{\alpha-1}(w_1 * \phi_\varepsilon) - |w_2 * \phi_\varepsilon|^{\alpha-1}(w_2 * \phi_\varepsilon)\right) * \phi_\varepsilon, w\right\rangle_{L^2(\mathbb{R}\times\mathbb{T})}\right| \\
= & \left|\left\langle J^s\left[\left(|w_1 * \phi_\varepsilon|^{\alpha-1}(w_1 * \phi_\varepsilon) - |w_2 * \phi_\varepsilon|^{\alpha-1}(w_2 * \phi_\varepsilon)\right) * \phi_\varepsilon\right], J^{-s}w\right\rangle_{L^2(\mathbb{R}\times\mathbb{T})}\right| \\
\leq & \left\|\left(|w_1 * \phi_\varepsilon|^{\alpha-1}(w_1 * \phi_\varepsilon) - |w_2 * \phi_\varepsilon|^{\alpha-1}(w_2 * \phi_\varepsilon)\right) * (J^s\phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \\
& \cdot \left\|J^{-s}w\right\|_{L^4(\mathbb{R}\times\mathbb{T})} \\
\lesssim_{\varepsilon,s} & \left\||w_1 * \phi_\varepsilon|^{\alpha-1}(w_1 * \phi_\varepsilon) - |w_2 * \phi_\varepsilon|^{\alpha-1}(w_2 * \phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \underbrace{\left\|J^{-s}w\right\|_{X_\delta^{0,\frac{3}{8}}}}_{=\|w\|_{X_\delta^{-s,\frac{3}{8}}}=1} \\
\leq & \left\||w_1 * \phi_\varepsilon|^{\alpha-1}(w_1 * \phi_\varepsilon) - |w_2 * \phi_\varepsilon|^{\alpha-1}(w_2 * \phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \\
\leq & \left\|\left(|w_1 * \phi_\varepsilon|^{\alpha-1} + |w_1 * \phi_\varepsilon|^{\alpha-1}\right)((w_1 - w_2) * \phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})},
\end{aligned}$$

where, for the first estimate, we used Hölder's inequality and Young's inequality, Lemma 20 for the second and the size estimate (7) for the last inequality. Applying Hölder's inequality again yields the upper bound

$$\left(\left\||w_1 * \phi_\varepsilon|^{\alpha-1}\right\|_{L^4(\mathbb{R}\times\mathbb{T})} + \left\||w_1 * \phi_\varepsilon|^{\alpha-1}\right\|_{L^4(\mathbb{R}\times\mathbb{T})}\right)\|(w_1 - w_2) * \phi_\varepsilon\|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

For the first factor, we apply Hölder's and Young's inequalities as well as the embedding from Lemma 20 and arrive at the upper bound of

$$\|w_1\|_{L^4(\mathbb{R}\times\mathbb{T})}^{\alpha-1} + \|w_2\|_{L^4(\mathbb{R}\times\mathbb{T})}^{\alpha-1} \lesssim \|w_1\|_{X^{0,\frac{3}{8}}}^{\alpha-1} + \|w_2\|_{X^{0,\frac{3}{8}}}^{\alpha-1}.$$

For the second factor we use Young's inequality and the definition of the norm in $X^{0,\frac{3}{8}}$ to arrive at the final estimate

$$\left(\|w_1\|_{X^{0,\frac{3}{8}}}^{\alpha-1} + \|w_2\|_{X^{0,\frac{3}{8}}}^{\alpha-1}\right)\|w_1 - w_2\|_{X^{0,\frac{3}{8}}}.$$

□

A.2. **Results.** First, we consider local wellposedness:

Theorem 26. *(Cf. [ET16, Theorem 3.27] for the cubic NLS.) Let $\varepsilon > 0$ and $s \geq 0$ or $\varepsilon = s = 0$. Then the (smoothened) (sub)quadratic NLS* (34) *is* locally *well-posed in $H^s(\mathbb{T})$.*

Proof. It suffices to show that the right-hand side of (35) defines a contractive self-mapping $\mathcal{T} : M(R, \delta) \to M(R, \delta)$ for some $R, \delta > 0$, where

$$M(R, \delta) := \left\{w \in Y \middle| \|w\|_Y \leq R\right\}$$

and Y is a suitable subspace of $C([0, \delta], H^s(\mathbb{T}))$.

We consider the case $s \geq 1$ first. Put $Y = C([0, \delta], H^s(\mathbb{T}))$. Due to $e^{\mathrm{i}t\partial_x^2}$ being an isometry on $H^s(\mathbb{T})$, for any $t \in \mathbb{R}$, and Lemma 18 we have

$$\begin{aligned}
& \|\mathcal{T}w\|_Y \\
\leq & \left\|e^{\mathrm{i}t\partial_x^2}(w_0 * \phi_\varepsilon)\right\|_{H^s(\mathbb{T})} + \left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[\left(|w * \phi_\varepsilon|^{\alpha-1}(w * \phi_\varepsilon)\right) * \phi_\varepsilon\right] \mathrm{d}\tau\right\|_Y \\
\leq & \|w_0\|_{H^s(\mathbb{T})} + \delta\left\|\left(|w * \phi_\varepsilon|^{\alpha-1}(w * \phi_\varepsilon)\right) * \phi_\varepsilon\right\|_Y.
\end{aligned}$$

This suggests the choice $R \approx \|w_0\|_{H^s}$. Fix $\tau \in [0,\delta]$. Then, due to Lemma 18 and the embedding $H^s(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, we have that

$$\begin{aligned}
& \left\| ((|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon))*\phi_\varepsilon)(\cdot,\tau) \right\|_{H^s(\mathbb{T})} \\
\lesssim_{\varepsilon,s} \ & \left\| (|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon))(\cdot,\tau) \right\|_{L^2(\mathbb{T})} \\
\leq \ & \|(w*\phi_\varepsilon)(\cdot,\tau)\|_{L^\infty(\mathbb{T})}^{\alpha-1} \|(w*\phi_\varepsilon)(\cdot,\tau)\|_{L^2(\mathbb{T})} \\
\lesssim \ & R^\alpha.
\end{aligned}$$

By the above, the condition $\|\mathcal{T}w\|_Y \leq R$ is satisfied, if $\delta \lesssim_{\varepsilon,s} R^{1-\alpha}$. The contraction property of $\mathcal{T}$ is shown in the same way, possibly requiring a smaller implicit constant in the last inequality.

In the case $s \in [0,1)$ and $\varepsilon > 0$, consider any $b \in \left(\frac{1}{2}, \frac{5}{8}\right)$ and put $Y = X_\delta^{s,b}$ (by Lemma 21 one indeed has $Y \hookrightarrow C([0,\delta], H^s(\mathbb{T}))$). Then, by the triangle inequality and Lemmata 22 and 23 we have

$$\begin{aligned}
& \|\mathcal{T}w\|_{X_\delta^{s,b}} \\
\leq \ & \left\| e^{\mathrm{i}t\partial_x^2}(w_0*\phi_\varepsilon) \right\|_{X_\delta^{s,b}} + \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left[\left(|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon) \right) * \phi_\varepsilon \right] \mathrm{d}\tau \right\|_{X_\delta^{s,b}} \\
\lesssim \ & \|w_0\|_{H^s(\mathbb{T})} + \left\| \left(|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon) \right) * \phi_\varepsilon \right\|_{X_\delta^{s,b-1}}.
\end{aligned}$$

This estimate suggests $R \approx \|w_0\|_{H^s(\mathbb{T})}$. For the second summand, apply Lemma 24 and Proposition 25 (with $w = 0$) to obtain the upper bound

$$\begin{aligned}
\left\| \left(|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon) \right) * \phi_\varepsilon \right\|_{X_\delta^{s,b-1}} & \lesssim \delta^{1-b-\frac{3}{8}} \left\| \left(|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon) \right) * \phi_\varepsilon \right\|_{X_\delta^{s,-\frac{3}{8}}} \\
& \lesssim \delta^{1-b-\frac{3}{8}} \|w\|_{X_\delta^{0,\frac{3}{8}}}^{\alpha-1} \|w\|_{X_\delta^{s,\frac{3}{8}}} \qquad (39) \\
& \leq \delta^{1-b-\frac{3}{8}} \|w\|_{X_\delta^{s,\frac{3}{8}}}^{\alpha} \leq \delta^{1-b-\frac{3}{8}} R^\alpha.
\end{aligned}$$

As the exponent of δ is positive, we can choose δ small enough to make $\mathcal{T}$ a self-mapping of $M(R,\delta)$. The fact that $\mathcal{T}$ is contractive is proven similarly, possibly requiring a smaller δ.

The remaining case $\varepsilon = s = 0$ is treated exactly as the last case. □

In order to prove the conservation laws, we need to be able to approximate by smooth solutions.

Lemma 27 (Smooth solutions for smooth initial data)**.** *(Cf. [Tao06, Proposition 3.11].) Let $\varepsilon > 0$, and $w_0 \in L^2(\mathbb{T})$ and let w denote the unique solution of* (35)*. Then $w \in C([0,\delta], H^\infty(\mathbb{R}))$ and for any $s > \frac{1}{2}$ one has*

$$\|w\|_{L_t^\infty H_x^s} \leq C \|w_0\|_{L^2} e^{Ct\|w\|_{L_t^\infty H^1}^{\alpha-1}} \qquad (40)$$

for some $C = C(\varepsilon, s) > 0$.

Proof. As w is the solution to (35), one immediately has

$$\begin{aligned}
\|w(\cdot,t)\|_{H^s} &\leq \|w_0 * \phi_\varepsilon\|_{H^s} + \int_0^t \left\| \left(|w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon)\right) * \phi_\varepsilon \right\|_{H^s} \mathrm{d}\tau \\
&\lesssim_{\varepsilon,s} \|w_0\|_{L^2} + \int_0^t \left\| |w*\phi_\varepsilon|^{\alpha-1}(w*\phi_\varepsilon) \right\|_{L^2} \mathrm{d}\tau \\
&\leq \|w_0\|_{L^2} + \|w*\phi_\varepsilon\|_{L_t^\infty L_x^\infty}^{\alpha-1} \int_0^t \|w*\phi_\varepsilon\|_{L^2} \mathrm{d}\tau \\
&\lesssim \|w_0\|_{L^2} + \|w\|_{L_t^\infty H^1}^{\alpha-1} \int_0^t \|w\|_{H^s} \mathrm{d}\tau.
\end{aligned}$$

Now (40) follows from Lemma 9. □

Theorem 28. *Let $\varepsilon > 0$ and $s \in [1,\infty)$. Then the smoothened NLS* (34) *is* globally *well-posed in $H^s(\mathbb{T})$.*

Proof. Local well-posedness has already been shown in Theorem 26 and it remains to show that the solution w exists globally. By the blow-up alternative, it suffices to see that $\|w(\cdot,t)\|_{H^s(\mathbb{T})}$ cannot explode. Moreover, by Lemma 27 it suffices to consider $s = 1$. By the same lemma, one has that $w \in C([0,\delta], H^\infty(\mathbb{T}))$ and in particular, $w \in C^1([0,\delta], H^1(\mathbb{T}))$. Hence, the energy conservation (cf. [GV79, Equations (3.14) and (1.18)])

$$E_\varepsilon(w(\cdot,t)) := \int_{\mathbb{T}} \frac{1}{2}|\nabla w(x,t)|^2 \mp \frac{1}{\alpha+1}|(w*\phi_\varepsilon)(x,t)|^{\alpha+1} \mathrm{d}x = E_\varepsilon(w_0 * \phi_\varepsilon) \tag{41}$$

is applicable to w. But

$$\|w(\cdot,t)\|_{H^1(\mathbb{T})}^2 = \|w_0\|_2^2 + 2E_\varepsilon(w_0*\phi_\varepsilon) \pm \frac{2}{\alpha+1}\|w(\cdot,t)\|_{\alpha+1}^{\alpha+1} \tag{42}$$

and so $\|w(\cdot,t)\|_{\dot{H}^1(\mathbb{T})}$ is controlled by $E_\varepsilon(w_0 * \phi_\varepsilon)$ in the defocusing case. In the focusing case we can assume w.l.o.g. that $\|w(\cdot,t)\|_{\dot{H}^1(\mathbb{T})}^2$ is an unbounded function of t, (otherwise, there is nothing to show) and say that $\|w(\cdot,t)\|_{\dot{H}^1(\mathbb{T})}^2$ is large. Then, by the Gagliardo-Nirenberg inequality from [Bre11, Chapter 8, Eqn. (42)], we have

$$\|w(\cdot,t)\|_{\alpha+1}^{\alpha+1} \lesssim \|w(\cdot,t)\|_2^{\frac{\alpha+3}{2}} \|w(\cdot,t)\|_{\dot{H}^1(\mathbb{T})}^{\frac{\alpha-1}{2}} \leq \frac{1}{2}\|w(\cdot,t)\|_{\dot{H}^1(\mathbb{T})}^2, \tag{43}$$

where above we additionally used the mass conservation

$$\|w(\cdot,t)\|_{L^2(\mathbb{T})} = \|w(\cdot,0)\|_{L^2(\mathbb{T})}.$$

Hence, inserting (43) into (42) and rearranging the inequality shows that the quantity $\|w(\cdot,t)\|_{\dot{H}^1(\mathbb{T})}^2$ is bounded, in contradiction to the assumption. This completes the proof. □

Theorem 29. *(Cf. [ET16, Theorem 3.28] for the cubic NLS.) The Cauchy problem for the (sub)quadratic periodic NLS ((2) with $\alpha \in [1,2]$) is* globally *well-posed in $L^2(\mathbb{T})$ and the solution w enjoys mass conservation $\|w(\cdot,t)\|_{L^2(\mathbb{T})} = \|w_0\|_{L^2(\mathbb{T})}$.*

Proof. Local well-posedness has already been shown in Theorem 26. Let w denote this local solution. By the blow-up alternative, it suffices to show mass conservation. To that end, let us denote by w^ε the global solution of (34) for $\varepsilon > 0$ from Theorem 28. We will show that for any $b \in \left(\frac{1}{2}, \frac{5}{8}\right)$ one has $\|w^\varepsilon - w\|_{X_\delta^{0,b}} \to 0$ as $\varepsilon \to 0+$. To that end, notice that

$$\begin{aligned}
&\left\| w^{\varepsilon} - w \right\|_{X_\delta^{0,b}} \qquad (44)\\
\leq\ & \left\| e^{\mathrm{i}t\partial_x^2} (w_0 * \phi_\varepsilon - w_0) \right\|_{X_\delta^{0,b}}\\
&+ \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left[\left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} (w^\varepsilon * \phi_\varepsilon) \right) * \phi_\varepsilon - |w|^{\alpha-1} w \right] \mathrm{d}\tau \right\|_{X_\delta^{0,b}} \qquad (45)\\
\lesssim\ & \left\| w_0 * \phi_\varepsilon - w_0 \right\|_{L^2(\mathbb{T})} + \left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} (w^\varepsilon * \phi_\varepsilon) \right) * \phi_\varepsilon - |w|^{\alpha-1} w \right\|_{X_\delta^{0,b-1}}\\
\lesssim\ & \left\| w_0 * \phi_\varepsilon - w_0 \right\|_{L^2(\mathbb{T})} + \delta^{1-b} \left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} (w^\varepsilon * \phi_\varepsilon) \right) * \phi_\varepsilon - |w|^{\alpha-1} w \right\|_{X_\delta^{0,0}},
\end{aligned}$$

where we used the fact that w and w^ε solve the corresponding fixed-point equations and Lemmata 22, 23 and 24.

For the first summand, observe that

$$\left\| w_0 * \phi_\varepsilon - w_0 \right\|_{L^2(\mathbb{T})} = \left\| \left(\langle k \rangle^s \hat{w}_0(k) (\sqrt{2\pi} \hat{\phi}_\varepsilon(k) - 1) \right)_k \right\|_{l^2(\mathbb{Z})}$$

and the right-hand side above converges to 0 as $\varepsilon \to 0+$ by the dominated convergence theorem and the definition of ϕ_ε.

For the second summand, note that $X_\delta^{0,0} = L^2([0,\delta] \times \mathbb{T})$ and hence

$$\begin{aligned}
&\left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} (w^\varepsilon * \phi_\varepsilon) \right) * \phi_\varepsilon - |w|^{\alpha-1} w \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
\leq\ & \left\| \left(|w|^{\alpha-1} w \right) * \phi_\varepsilon - |w|^{\alpha-1} w \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
&+ \left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} (w^\varepsilon * \phi_\varepsilon) - |w|^{\alpha-1} w \right) * \phi_\varepsilon \right\|_{L^2([0,\delta]\times\mathbb{T})}.
\end{aligned}$$

The first summand above goes to zero due to $(\phi_\varepsilon)_\varepsilon$ being an approximation to the identity on $L^{2\alpha}(\mathbb{T})$. The other summand is further estimated by

$$\begin{aligned}
&\left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} (w^\varepsilon * \phi_\varepsilon) - |w|^{\alpha-1} w \right) * \phi_\varepsilon \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
\leq\ & \left\| |w^\varepsilon * \phi_\varepsilon|^{\alpha-1} (w^\varepsilon * \phi_\varepsilon) - |w|^{\alpha-1} w \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
\lesssim\ & \left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} + |w|^{\alpha-1} \right) (w^\varepsilon * \phi_\varepsilon - w) \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
\leq\ & \left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} + |w|^{\alpha-1} \right) [(w^\varepsilon - w) * \phi_\varepsilon] \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
&+ \left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} + |w|^{\alpha-1} \right) (w * \phi_\varepsilon - w) \right\|_{L^2([0,\delta]\times\mathbb{T})}. \qquad (46)
\end{aligned}$$

Let us introduce the set $A^\varepsilon = \{|w^\varepsilon| * \phi_\varepsilon \leq 1\}$. Then the first summand above is further estimated by

$$\begin{aligned}
&\left\| \left(|w^\varepsilon * \phi_\varepsilon|^{\alpha-1} + |w|^{\alpha-1} \right) [(w^\varepsilon - w) * \phi_\varepsilon] \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
\leq\ & \left\| \left(\mathbb{1}_{A^\varepsilon} |w^\varepsilon * \phi_\varepsilon|^{\alpha-1} + \mathbb{1}_{A^0} |w|^{\alpha-1} \right) [(w^\varepsilon - w) * \phi_\varepsilon] \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
&+ \left\| \left(\mathbb{1}_{(A^\varepsilon)^c} |w^\varepsilon * \phi_\varepsilon|^{\alpha-1} + \mathbb{1}_{(A^0)^c} |w|^{\alpha-1} \right) [(w^\varepsilon - w) * \phi_\varepsilon] \right\|_{L^2([0,\delta]\times\mathbb{T})}\\
\leq\ & \left\| w^\varepsilon - w \right\|_{L^2([0,\delta]\times\mathbb{T})} + \|w\|_{L^4([0,\delta]\times\mathbb{T})} \left\| w^\varepsilon - w \right\|_{L^4([0,\delta]\times\mathbb{T})}\\
\lesssim\ & \left(1 + \|w\|_{X_\delta^{0,b}} \right) \left\| w^\varepsilon - w \right\|_{X_\delta^{0,b}},
\end{aligned}$$

where we used Hölder's and Young's inequalities for the penultimate estimate and Lemma 20 for the last step. Recall that in front of this term is δ^{1-b} and, w.l.o.g.,

$\delta \ll 1$. Hence we can just move it to the left-hand side of (44). The treatment of the last remaining term (46) does not require any new techniques.

By the above, $\|w^\varepsilon - w\|_{X_\delta^{0,b}} \to 0$ as $\varepsilon \to 0+$. Applying Lemma 21, we see that

$$\begin{aligned}\|w\|_{C([0,T],L^2(\mathbb{T}))} &\leq \limsup_{\varepsilon\to 0+} \left[\|w^\varepsilon - w\|_{X_\delta^{0,b}} + \|w^\varepsilon\|_{C([0,T],L^2(\mathbb{T}))}\right] \\ &\leq \limsup_{\varepsilon\to 0+} \left[\|w_0 * \phi_\varepsilon\|_{L^2(\mathbb{T})}\right] = \|w_0\|_{L^2(\mathbb{T})}\end{aligned}$$

and hence the solution w indeed enjoys mass conservation. This finishes the proof. □

In addition to mass conservation, we also have conservation of the energy.

Theorem 30. *(Cf. [GV79, Theorem 3.1] and [LRS88, Theorem 2.1].) The Cauchy problem for the (sub)quadratic periodic NLS ((2) with $\alpha \in [1,2]$) is* globally *well-posed in $H^1(\mathbb{T})$ and the solution w enjoys energy conservation $E(w(\cdot,t)) = E(w_0)$.*

Remark 31. *In [LRS88] it is claimed that the quadratic NLS is globally well-posed on the torus. They refer to [GV79], where it is done on the real line. While our proof of Theorem 30 borrows some ideas from [GV79], we believe that in order to be able to do the torus case, one needs the result of Bourgain [Bou93a], in particular, the Bourgain spaces, which appeared 5 years after [LRS88].*

Proof. Let $w_0 \in H^1(\mathbb{T}) \subseteq L^2(\mathbb{T})$. By Theorem 29, the (sub)quadratic periodic NLS has the unique global solution $w \in C_\mathrm{b}(\mathbb{R}, L^2(\mathbb{T}))$. It remains to show that $w \in C_\mathrm{b}(\mathbb{R}, H^1(\mathbb{T}))$. To show that for any $t \in \mathbb{R}$ one has $w(\cdot,t) \in H^1(\mathbb{T})$ we first prove that

$$\sup_{\varepsilon>0} \|w^\varepsilon\|_{C(\mathbb{R},H^1(\mathbb{T}))} < \infty. \tag{47}$$

By calculations similar to those in the proof of Theorem 29, it suffices to prove the corresponding bound for the energy $E_\varepsilon(w^\varepsilon(\cdot,t))$.

To that end let w^ε be the unique global solution of the modified NLS (34) for $\varepsilon > 0$ from Theorem 28. The energy conservation from Equation (41) implies

$$E_\varepsilon(w^\varepsilon(\cdot,t)) = E_\varepsilon(w_0 * \phi_\varepsilon) = \frac{1}{2}\|w_0 * \phi_\varepsilon\|^2_{\dot{H}^1(\mathbb{T})} \mp \frac{1}{\alpha+1}\|w_0 * \phi_\varepsilon\|^{\alpha+1}_{L^3(\mathbb{T})}.$$

Observe that by Lemma 18 the first summand above satisfies

$$\|w_0 * \phi_\varepsilon\|^2_{\dot{H}^1(\mathbb{T})} \leq \|w_0\|^2_{\dot{H}^1(\mathbb{T})}.$$

If the sign of the second summand is negative (focusing case), there is nothing left to do. If the sign is positive (defocusing case), one has

$$\|w_0 * \phi_\varepsilon\|^{\alpha+1}_{\alpha+1} \leq \|w_0\|^{\alpha+1}_{\alpha+1} \leq \|w_0\|^{\alpha-1}_{L^\infty(\mathbb{T})}\|w_0\|^2_{L^2(\mathbb{T})} \leq \|w_0\|^{\alpha+1}_{H^1(\mathbb{T})}$$

by Lemma 17. Therefore, the bound (47) holds.

Assume for now that $t \in [0,\delta]$, where δ is the guaranteed time of existence of w in $L^2(\mathbb{T})$. From the proof of Theorem 29, one has that

$$\lim_{\varepsilon\to 0+} \|w^\varepsilon - w\|_{C([0,T],L^2(\mathbb{T}))} = 0. \tag{48}$$

Hence, from Equations (47) and (48) and Lemma 19 it follows that

$$\|w(\cdot,t)\|_{H^1(\mathbb{T})} \leq \liminf_{\varepsilon\to 0+} \|w^\varepsilon(\cdot,t)\|_{H^1(\mathbb{T})} < \infty.$$

Observe, that by the above we have

$$\begin{aligned}
&\|w^\varepsilon(\cdot,t)*\phi_\varepsilon - w\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1} \\
\lesssim\; &\|(w^\varepsilon(\cdot,t)-w(\cdot,t))*\phi_\varepsilon\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1} + \|(w(\cdot,t)*\phi_\varepsilon - w(\cdot,t)\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1} \\
\leq\; &\left(\|w^\varepsilon(\cdot,t)\|_{L^\infty}^{\alpha-1} + \|w(\cdot,t)\|_{L^\infty}^{\alpha-1}\right)\|w^\varepsilon(\cdot,t)-w(\cdot,t)\|_{L^2(\mathbb{T})}^2 \\
&+ \|(w(\cdot,t)*\phi_\varepsilon - w(\cdot,t)\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1} \xrightarrow{\varepsilon\to 0+} 0
\end{aligned}$$

and hence

$$E_0(w(\cdot,t)) \leq \liminf_{\varepsilon\to 0+} E_\varepsilon(w^\varepsilon(\cdot,t)) \leq E_0(w_0).$$

Interchanging 0 and t shows the reverse inequality and proves the energy conservation $E_0(w_0) = E_0(w(\cdot,t))$.

Reiterating the argument proves that $w \in L^\infty(\mathbb{R}, H^1(\mathbb{T}))$. It remains to show that $w \in C(\mathbb{R}, H^1(\mathbb{T}))$. To that end, observe that $t \mapsto w(\cdot,t)$ is weakly continuous in $L^2(\mathbb{T})$. But, by the above, $\sup_{t\in\mathbb{R}} \|w(\cdot,t)\|_{H^1(\mathbb{T})} < \infty$ and hence $t \mapsto w(\cdot,t)$ is weakly continuous in $H^1(\mathbb{T})$. By the observation

$$\|w(\cdot,t)-w(\cdot,s)\|_{H^1(\mathbb{T})}^2 = \|w(\cdot,t)\|_{H^1(\mathbb{T})}^2 + \|w(\cdot,s)\|_{H^1(\mathbb{T})}^2 - 2\operatorname{Re}\langle w(\cdot,t), w(\cdot,s)\rangle_{H^1(\mathbb{T})},$$

it is enough to show that $t \mapsto \|w(\cdot,t)\|_{H^1(\mathbb{T})}$ is continuous. (See [Bre11, Proposition 3.32] for this result in a more general setting.)

To that end, observe that by the mass and energy conservation we have

$$\begin{aligned}
\|w(\cdot,t)\|_{H^1(\mathbb{T})}^2 &= 2E(w(\cdot,t)) \pm \frac{2}{\alpha+1}\|w(\cdot,t)\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1} + \|w(\cdot,t)\|_{L^2(\mathbb{T})}^2 \\
&= 2E_0(w_0) \pm \frac{2}{\alpha+1}\|w(\cdot,t)\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1} + \|w_0\|_{L^2(\mathbb{T})}^2.
\end{aligned}$$

Moreover, for any $t, s \in \mathbb{R}$ we have

$$\begin{aligned}
&\left|\|w(\cdot,t)\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1} - \|w(\cdot,s)\|_{L^{\alpha+1}(\mathbb{T})}^{\alpha+1}\right| \\
\lesssim\; &\int_{\mathbb{T}} |w(x,t)-w(x,s)|\left(|w(x,t)|^\alpha + |w(x,s)|^\alpha\right)\mathrm{d}x \\
\lesssim\; &\|w\|_{L^\infty(\mathbb{R},H^1(\mathbb{T}))}^\alpha \|w(\cdot,t)-w(\cdot,s)\|_{L^2(\mathbb{T})}.
\end{aligned}$$

The fact that $w \in C_{\mathrm{b}}(\mathbb{R}, L^2(\mathbb{T}))$ concludes the argument. □

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