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The global Cauchy problem for the NLS with higher order anisotropic dispersion

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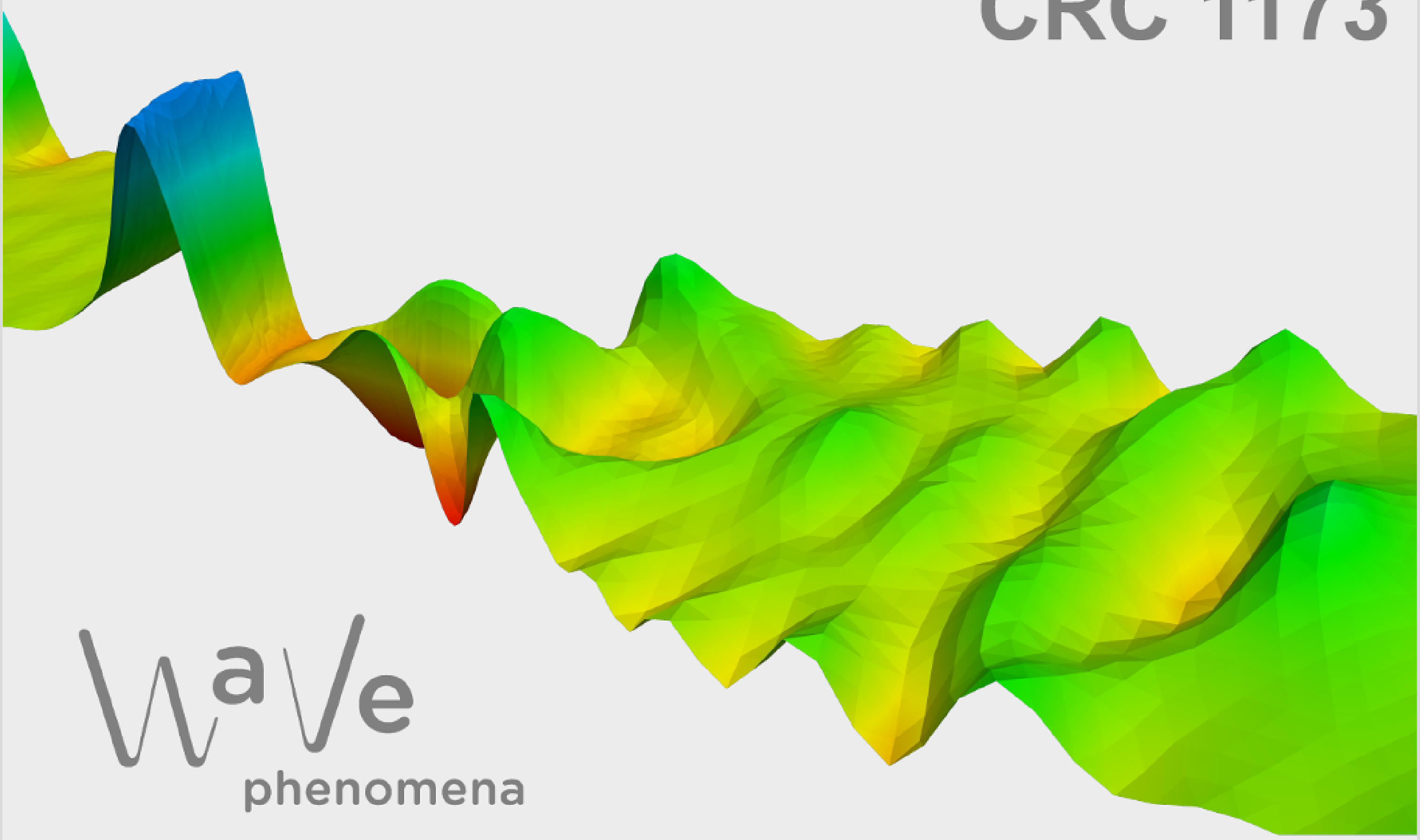

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THE GLOBAL CAUCHY PROBLEM FOR THE NLS WITH HIGHER ORDER ANISOTROPIC DISPERSION.

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ABSTRACT. We use a method developed by Strauss to obtain global wellposedness results in the mild sense for the small data Cauchy problem in modulation spaces $M^s_{p,q}(\mathbb{R}^d)$, where $q=1$ and $s \geq 0$ or $q \in (1,\infty]$ and $s > \frac{d}{q'}$ for a nonlinear Schrödinger equation with higher order anisotropic dispersion and algebraic nonlinearities.

1. INTRODUCTION AND MAIN RESULTS

We are interested in the following Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t u(t,x) + \alpha\Delta u(t,x) + \mathrm{i}\beta\dfrac{\partial^3}{\partial x_1^3}u(t,x) + \gamma\dfrac{\partial^4}{\partial x_1^4}u(t,x) + f(u(t,x)) = 0, \\ \hfill u(0,\cdot) = u_0(\cdot), \end{cases} \tag{1}$$

where $(t,x) = (t,x_1,x') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{d-1}$, $d \geq 2$, $\alpha \in \mathbb{R}\setminus\{0\}$, and $(\beta,\gamma) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Such PDE arise in the context of high-speed soliton transmission in long-haul optical communication system, see [5]. The case where the coefficiets α,β,γ are time dependent has been studied in [2] in one dimension for the cubic nonlinearity, $f(u) = |u|^2u$, with initial data in $L^2(\mathbb{R})$-based Sobolev spaces. In [1] it is proved that (1) with nonlinearity $f(u) = |u|^pu$ where

$$p < \begin{cases} \frac{4}{d-\frac{1}{2}} &, \ \gamma \neq 0, \\ \frac{4}{d-\frac{1}{3}} &, \ \gamma = 0, \end{cases}$$

is globally wellposed in $L^2(\mathbb{R}^d)$ via Strichartz estimates and the charge conservation equation

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}, \qquad \forall t \in \mathbb{R}.$$

In the same paper the case of initial data $u_0 \in H^1_a(\mathbb{R}^d)$ is studied where

$$H^1_a(\mathbb{R}^d) := \left\{u \in L^2(\mathbb{R}^d) \middle| \nabla u, \partial^2_{x_1}u \in L^2(\mathbb{R}^d)\right\}$$

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is equipped with the norm

$$\|u\|_{H^1_a(\mathbb{R}^d)} := \left(\|u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \|\partial^2_{x_1} u\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}}.$$

In this paper we consider the Cauchy problem (1) with initial data u_0 in modulation spaces $M^s_{p,q}(\mathbb{R}^d)$. Modulation spaces were introduced by Feichtinger in [6] and since then, they have become canonical for both time-frequency and phase-space analysis. They provide an excellent substitute for estimates that are known to fail on Lebesgue spaces. To state the definition of a modulation space we need to fix some notation. We will denote by $S'(\mathbb{R}^d)$ the space of tempered distributions. Let Q_0 be the unit cube with center the origin in $\mathbb{R}^d$ and its translations $Q_k := Q_0 + k$ for all $k \in \mathbb{Z}^d$. Consider a partition of unity $\{\sigma_k = \sigma_0(\cdot - k)\}_{k\in\mathbb{Z}^d} \subset C^\infty(\mathbb{R}^d)$ satisfying

- $\exists c > 0 : \forall \eta \in Q_0 : |\sigma_0(\eta)| \geq c$,
- $\operatorname{supp}(\sigma_0) \subseteq \{\xi \in \mathbb{R}^d : |\xi| < \sqrt{d}\} =: B(0, \sqrt{d})$,

and define the isometric decomposition operators

$$\Box_k := \mathcal{F}^{(-1)} \sigma_k \mathcal{F}, \qquad \forall k \in \mathbb{Z}^d, \tag{2}$$

where $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^d$. Then the norm of a tempered distribution $f \in S'(\mathbb{R}^d)$ in the modulation space $M^s_{p,q}(\mathbb{R}^d)$, where $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, is given by

$$\|f\|_{M^s_{p,q}} := \left\| \left\{ \langle k\rangle^s \|\Box_k f\|_p \right\}_{k\in\mathbb{Z}^d} \right\|_{l^q(\mathbb{Z}^d)}, \tag{3}$$

where we denote by $\langle k\rangle = 1 + |k|$ the Japanese bracket. It can be proved that different choices of the function σ_0 lead to equivalent norms in $M^s_{p,q}(\mathbb{R}^d)$ (see e.g. [3, Proposition 2.9]). When $s = 0$ we denote the space $M^0_{p,q}(\mathbb{R}^d)$ by $M_{p,q}(\mathbb{R}^d)$. In the special case where $p = q = 2$ we have $M^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ the usual Sobolev spaces.

For $\alpha \in \mathbb{R}$ we define the weighted mixed-norm space

$$L^\infty_\alpha(\mathbb{R}, M^s_{p,q}(\mathbb{R}^d)) := \left\{ u \in L^\infty(\mathbb{R}, M^s_{p,q}(\mathbb{R}^d)) \middle| \, \|u\|_{L^\infty_\alpha(\mathbb{R}, M^s_{p,q}(\mathbb{R}^d))} < \infty \right\},$$

where

$$\|u\|_{L^\infty_\alpha(\mathbb{R}, M^s_{p,q}(\mathbb{R}^d))} := \sup_{t\in\mathbb{R}} \langle t\rangle^\alpha \|u(t,\cdot)\|_{M^s_{p,q}}.$$

Let us denote by $\pi(u^{m+1})$ any $(m+1)$-time product of u and $\bar{u}$, where $m \in \mathbb{Z}_+$. Define also the quantity

$$\frac{2}{\gamma_{m,d}} = \begin{cases} (d - \frac{1}{2})(\frac{m}{2(m+2)}) & , \gamma \neq 0, \\ (d - \frac{1}{3})(\frac{m}{2(m+2)}) & , \gamma = 0. \end{cases} \tag{4}$$

Futhermore, let m_0 denote the positive root of

$$\begin{cases} (2d-1)x^2 + (2d-5)x - 8 = 0 & , \gamma \neq 0, \\ (3d-1)x^2 + (3d-7)x - 12 = 0 & , \gamma = 0. \end{cases} \tag{5}$$

The main results are the following theorems.

Theorem 1. *Suppose that* $d \geq 1$, $f(u) = \pm\pi(u^{m+1})$, $m \in \mathbb{Z}_+$ *with* $m > m_0$ *and* $q \in [1, \infty]$. *For* $q = 1$, *let* $s \geq 0$ *and for* $q > 1$, *let* $s > \frac{d}{q'}$. *Then there exists a* $\delta > 0$ *such that for any* $u_0 \in M^s_{\frac{m+2}{m+1},q}(\mathbb{R}^d)$ *with* $\|u_0\|_{M^s_{\frac{m+2}{m+1},q}} \leq \delta$ *the Cauchy problem* (1) *admits a unique global solution*

$$u \in L^{\infty}_{\frac{2}{\gamma_{m,d}}}(\mathbb{R}, M^s_{2+m,q}(\mathbb{R}^d)). \tag{6}$$

The restriction on the power of the nonlinearity described in Theorem 1 is explained in remark 9.

Theorem 2. *Suppose that* $d \geq 2$, $f(u) = \lambda(e^{\rho|u|^2} - 1)u$, $\lambda \in \mathbb{C}$ *and* $\rho > 0$. *In addition, let* $s \geq 0$ *if* $q = 1$ *and let* $s > \frac{d}{q'}$ *if* $q \in (1, \infty]$. *There exists* $\delta > 0$ *such that for any* $u_0 \in M^s_{\frac{4}{3},q}(\mathbb{R}^d)$ *with* $\|u_0\|_{M^s_{\frac{4}{3},q}} \leq \delta$ *the Cauchy problem* (1) *admits a unique global solution* u *in the space* $L^{\infty}_{\frac{2}{\gamma_{2,d}}}(\mathbb{R}, M^s_{4,q}(\mathbb{R}^d))$.

Remark 3. For $q < \infty$, the solution from Theorem 1 and 2 is a continuous function with values in the corresponding modulation space, i.e. indeed a mild solution. For the more delicate situation $q = \infty$ see [9].

The idea of studying the Cauchy problem (1) with such time-decay norm is inspired by [13], where the authors considered the NLS and the NLKG equations. As mentioned there, this idea goes back to the work of Strauss, see [11]. Their results were improved in [7] and [8] where the author considered the nonlinear higher order Schrödinger equation

$$\mathrm{i}\partial_t u + \phi(\sqrt{-\Delta})u = f(u), \tag{7}$$

where $\phi(\sqrt{-\Delta}) = \mathcal{F}^{-1}\phi(|\xi|)\mathcal{F}$ and ϕ is a polynomial, with initial data u_0 in a modulation space.

Remark 4. Notice that Theorem 1 does not include the cubic nonlinearity in dimension $d = 1$ since m has to be strictly bigger than m_0 which is the positive root of the quadratics in (5), that is $m_0 = \frac{3+\sqrt{41}}{2}$, if $\gamma \neq 0$ and $m_0 = \frac{4+\sqrt{110}}{4}$, if $\gamma = 0$. In both cases $m_0 > 3$.

Remark 5. In [13, Theorem 1.1 and Theorem 1.2], the authors only considered modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ with $q = 1$. But, by Theorem 6, their crucial estimate (6.6) also holds for $q \in (1, \infty]$ and $s > \frac{d}{q'}$. Hence, the statements of their theorems is true in this case too.

1.1. **Preliminaries.** It is known that for $s > d/q'$ (where q' is the conjugate exponent of q) and $p, q \in [1, \infty]$, the embedding

$$M^s_{p,q}(\mathbb{R}^d) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \,\middle|\, f \text{ continuous and bounded} \right\}, \tag{8}$$

is continuous. The same is true for the embedding

$$M^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{s_2}_{p_2,q_2}(\mathbb{R}^d), \tag{9}$$

which holds for any $s_1, s_2 \in \mathbb{R}$ and any $p_1, p_2, q_1, q_2 \in [1, \infty]$ satisfying $p_1 \le p_2$ and either

$$q_1 \le q_2 \quad \text{and} \quad s_1 \ge s_2$$

$$\text{or}$$

$$q_2 < q_1 \quad \text{and} \quad s_1 > s_2 + \frac{d}{q_2} - \frac{d}{q_1}$$

(see [6, Proposition 6.8 and Proposition 6.5]).

We are going to use the following Hölder type inequality for modulation spaces which appeared in [3, Theorem 4.3] (see also [4]).

Theorem 6. *Let $d \ge 1$ and $1 \le p, p_1, p_2, q \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For $q = 1$ let $s \ge 0$ and for $q \in (1, \infty]$ let $s > \frac{d}{q'}$. Then there exists a constant $C = C(d, s, q) > 0$ such that*

$$\|fg\|_{M^s_{p,q}} \le C \|f\|_{M^s_{p_1,q}} \|g\|_{M^s_{p_2,q}},$$

for all $f \in M^s_{p_1,q}(\mathbb{R}^d)$ and $g \in M^s_{p_2,q}(\mathbb{R}^d)$.

The propagator of the homogeneous Schrödigner equation with higher order anisotropic dispersion is given by

$$W(t) = \mathcal{F}^{(-1)} e^{\mathrm{i}(\alpha|\xi|^2 + \beta\xi_1^3 + \gamma\xi_1^4)t} \mathcal{F}, \tag{10}$$

where $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1}$. For the rest of the paper, $A \lesssim B$ shall mean that there is a constant $C > 0$ such that $A \le CB$. The next dispersive estimate is from [1, Theorem 1.1]:

Theorem 7. *Consider $p \in [2, \infty]$ and $f \in L^{p'}(\mathbb{R}^d)$. Then*

$$\|W(t)f\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{-\mu} \|f\|_{L^{p'}(\mathbb{R}^d)} \tag{11}$$

where

$$\mu = \mu(d, \gamma, p) := \begin{cases} \left(d - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{p}\right) & \gamma \ne 0, \\ \left(d - \frac{1}{3}\right)\left(\frac{1}{2} - \frac{1}{p}\right) & \gamma = 0, \end{cases} \tag{12}$$

and the implicit constant is independent of the function f and the time t.

Using this, we claim the following

Theorem 8. *Consider $s \in \mathbb{R}, p \in [2, \infty]$ and $q \in [1, \infty]$. Then*

$$\|W(t)f\|_{M^s_{p,q}(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\mu} \|f\|_{M^s_{p',q}(\mathbb{R}^d)}, \tag{13}$$

where $\mu = \mu(d, \gamma, p)$ is as in Equation (12) and the implicit constant is independent of the function f and the time t.

Proof. The operators $\Box_k$ and $W(t)$ commute and hence we immediately arrive at

$$\|\Box_k W(t) f\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{-\mu} \|\Box_{k+l} f\|_{L^{p'}(\mathbb{R}^d)} \qquad \forall k \in \mathbb{Z}^d \, \forall t \in \mathbb{R} \setminus \{0\} \tag{14}$$

by invoking Theorem 7. Moreover, as $p \in [2, \infty]$, we have

$$\|\Box_k W(t) f\|_{L^p(\mathbb{R}^d)} \lesssim \left\| \sigma_k e^{\mathrm{i}(\alpha|\xi|^2 + \beta\xi_1^3 + \gamma\xi_1^4)t} \hat{f} \right\|_{L^{p'}(\mathbb{R}^d)} \lesssim \left\| \sigma_k \hat{f} \right\|_{L^p(\mathbb{R}^d)} \lesssim \|\Box_k f\|_{L^{p'}(\mathbb{R}^d)} \tag{15}$$

for any $k \in \mathbb{Z}^d$ and any $t \in \mathbb{R}$. Above, we used the Hausdoff-Young inequality for the first and last estimate and the fact that $\operatorname{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k)$ for the second inequality. Taking the minimum of the right-hand sides of (14) and (15) shows

$$\|\Box_k W(t) f\|_{L^p(\mathbb{R}^d)} \lesssim \langle t\rangle^{-\mu} \|\Box_k f\|_{L^{p'}(\mathbb{R}^d)} \qquad \forall k \in \mathbb{Z}^d \, \forall t \in \mathbb{R}.$$

Multiplying by the weight $\langle k\rangle^s$ and taking the $l^q(\mathbb{Z}^d)$-norm on both sides we arrive at the desired estimate. □

2. Proofs of the main theorems

In this section we present the proofs of the main theorems.

Proof of Theorem 1. For the sake of brevity, let us shorten the notation by setting

$$\|u\| := \|u\|_{L^\infty_{\frac{2}{\gamma_{m,d}}}(\mathbb{R}, M^s_{m+2,q}(\mathbb{R}^d))}. \tag{16}$$

By the Banach fixed-point theorem, it suffices to show that the operator defined by

$$\mathcal{T}u := W(t)u_0 \pm i \int_0^t W(t-\tau)\left(\pi(u^{m+1})\right) \mathrm{d}\tau \tag{17}$$

is a contractive self-mapping of the complete metric space

$$M(R) = \left\{ u \in L^\infty_{\frac{2}{\gamma_{m,d}}}(\mathbb{R}, M^s_{m+2,q}(\mathbb{R}^d)) \,\middle|\, \|u\| \le R \right\} \tag{18}$$

for some $R \in \mathbb{R}_+$. We begin with the self-mapping property and observe that

$$\|\mathcal{T}u\| \le \|W(t)u_0\| + \left\| \int_0^t W(t-\tau)\left(\pi(u^{m+1})\right) \mathrm{d}\tau \right\|. \tag{19}$$

Notice, that $\mu(d, \gamma, m+2) = \frac{2}{\gamma_{m,d}}$ and hence, by the dispersive estimate (13), one obtains

$$\|W(t)u_0\| = \sup_{t\in\mathbb{R}} \left[\langle t\rangle^{\frac{2}{\gamma_{m,d}}} \|W(t)u_0\|_{M^s_{m+2,q}} \right] \lesssim \|u_0\|_{M^s_{\frac{m+2}{m+1},q}}. \tag{20}$$

Introducing the smallness condition

$$\|u_0\|_{M^s_{\frac{m+2}{m+1},q}(\mathbb{R}^d)} \lesssim \frac{R}{2}$$

leads to $\|W(t)u_0\| \le \frac{R}{2}$.

For the integral term we have the upper bound

$$\sup_{t\in\mathbb{R}} \left[\langle t\rangle^{\frac{2}{\gamma_{m,d}}} \int_0^t \langle t-\tau\rangle^{-\frac{2}{\gamma_{m,d}}} \|\pi(u^{m+1})\|_{M^s_{\frac{m+2}{m+1},q}} \mathrm{d}\tau \right]. \tag{21}$$

Hölder's inequality for modulation spaces from Theorem 6 is applicable (due to the assumptions on s, q) and yields

$$\|\pi(u^{m+1})\|_{M^s_{\frac{m+2}{m+1},q}} \lesssim \|u\|^{m+1}_{M^s_{m+2,q}}. \tag{22}$$

Furthermore, as $u \in M(R)$, one has

$$\|u(\tau,\cdot)\|_{M^s_{m+2,q}} \leq \langle\tau\rangle^{-\frac{2}{\gamma_{m,d}}}\|u\| \leq \langle\tau\rangle^{-\frac{2}{\gamma_{m,d}}}R \qquad \forall\tau\in\mathbb{R} \tag{23}$$

and we obtain the upper bound for the integral term

$$R^{m+1}\sup_{t\in\mathbb{R}}\left[\langle t\rangle^{\frac{2}{\gamma_{m,d}}}\int_0^t \langle t-\tau\rangle^{-\frac{2}{\gamma_{m,d}}}\langle\tau\rangle^{-\frac{2(m+1)}{\gamma_{m,d}}}\mathrm{d}\tau\right]. \tag{24}$$

To be able to control the individual factors of the integral, we split it into $\int_0^t = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t$. For the first summand we have

$$\begin{aligned}\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{2}{\gamma_{m,d}}}\langle\tau\rangle^{-\frac{2(m+1)}{\gamma_{m,d}}}d\tau &\lesssim \left\langle\frac{t}{2}\right\rangle^{-\frac{2}{\gamma_{m,d}}}\frac{1}{1-\frac{2}{\gamma_{m,d}}(m+1)}\left(\left\langle\frac{t}{2}\right\rangle^{1-\frac{2}{\gamma_{m,d}}(m+1)}-1\right)\\ &\lesssim \langle t\rangle^{-\frac{2}{\gamma_{m,d}}},\end{aligned} \tag{25}$$

where we used the monotonicity of $\langle\cdot\rangle$ and the assumption $m > m_0$, which implies $\frac{2(m+1)}{\gamma_{m,d}} > 1$. We similarly estimate the second summand by

$$\int_{\frac{t}{2}}^t\langle t-\tau\rangle^{-\frac{2}{\gamma_{m,d}}}\langle\tau\rangle^{-\frac{2(m+1)}{\gamma_{m,d}}}d\tau \lesssim \langle t\rangle^{-\frac{2(m+1)}{\gamma_{m,d}}}\int_{\frac{t}{2}}^t\langle t-\tau\rangle^{-\frac{2}{\gamma_{m,d}}} \lesssim \langle t\rangle^{-\frac{2}{\gamma_{m,d}}}. \tag{26}$$

Putting everything together we arrive at the condition

$$\|\mathcal{T}u\| \lesssim \frac{R}{2} + R^{m+1} \overset{!}{\leq} R, \tag{27}$$

which is satisfied for suffieciently small R. Similarly we obtain

$$\|\mathcal{T}u - \mathcal{T}v\| \lesssim (\|u\|^m + \|v\|^m)\|u-v\| \leq 2R^m\|u-v\|. \tag{28}$$

Hence, under a possibly smaller choice of R, the operator $\mathcal{T}$ is a contraction and the proof is complete. □

Remark 9. Observe that the restriction $m > m_0$ corresponds to the boundedness of the terms in (25) and (26).

Proof of Theorem 2. As in the proof of Theorem 1, we shorten the notation of the norm by

$$\|u\| := \|u\|_{L^\infty_{\frac{2}{\gamma_{2,d}}}(\mathbb{R},M^s_{4,q})} \tag{29}$$

and introduce the operator

$$\mathcal{T}u = W(t)u_0 \pm \mathrm{i}\int_0^t W(t-\tau)(f(u))\mathrm{d}\tau, \tag{30}$$

which we want to be a contractive self-mapping of the complete metric space $M(R)$ for some $R \in \mathbb{R}_+$. We begin with the self-mapping property. By the definition of the nonlinearity $f(u) = \lambda(e^{\rho|u|^2} - 1)u$ we have

$$f(u) = \lambda \sum_{k=1}^{\infty} \frac{\rho^k}{k!} |u|^{2k} u. \tag{31}$$

Following the proof of Theorem 1, we arrive at

$$\|\mathcal{T}u\| \lesssim \|u_0\|_{M^s_{\frac{4}{3},q}} + \sum_{k=1}^{\infty} \sup_{t\in\mathbb{R}} \left[\langle t \rangle^{\frac{2}{\gamma_{2,d}}} \int_0^t \langle t-\tau \rangle^{-\frac{2}{\gamma_{2,d}}} \frac{\rho^k}{k!} \, \||u|^{2k}u\|_{M^s_{\frac{4}{3},q}} \mathrm{d}\tau \right]. \tag{32}$$

Hölder's inequality for modulation spaces from Theorem 6 is applicable (due to the assumptions on s, q) and yields the estimate

$$\||u|^{2k}u\|_{M^s_{\frac{4}{3},q}} \lesssim \|u\|^3_{M^s_{4,q}} \|u\|^{2k-2}_{M^s_{\infty,q}} \lesssim \|u\|^{2k+1}_{M^s_{4,q}}, \tag{33}$$

where in the second inequality we used (9), i.e. the embedding $M^s_{4,q}(\mathbb{R}^d) \hookrightarrow M^s_{\infty,q}(\mathbb{R}^d)$. Hence, by (23) for $m = 2$, we obtain

$$\|\mathcal{T}u\| \lesssim \|u_0\|_{M^s_{\frac{4}{3},q}} + \sum_{k=1}^{\infty} \frac{\rho^k}{k!} R^{2k+1} \sup_{t\in\mathbb{R}} \left[\langle t \rangle^{\frac{2}{\gamma_{2,d}}} \int_0^t \langle t-\tau \rangle^{-\frac{2}{\gamma_{2,d}}} \langle \tau \rangle^{-(2k+1)\frac{2}{\gamma_{2,d}}} \mathrm{d}\tau \right]. \tag{34}$$

The supremum above is finite by the same reasoning as in the proof of Theorem 1 and we therefore arrive at the condition

$$\|\mathcal{T}u\| \lesssim \|u_0\|_{M^s_{\frac{4}{3},q}} + \sum_{k=1}^{\infty} \frac{\rho^k}{k!} R^{2k+1} = \|u_0\|_{M^s_{\frac{4}{3},q}} + \left(R\, e^{\rho R^2} - 1 \right) \overset{!}{\leq} R. \tag{35}$$

Thus, if $\|u_0\|_{M^s_{\frac{4}{3},q}} \lesssim \frac{R}{2}$ and $R > 0$ is sufficiently small, the operator $\mathcal{T}$ is a self-mapping of the space $M(R)$. The contraction property is proved in a similar way. □

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