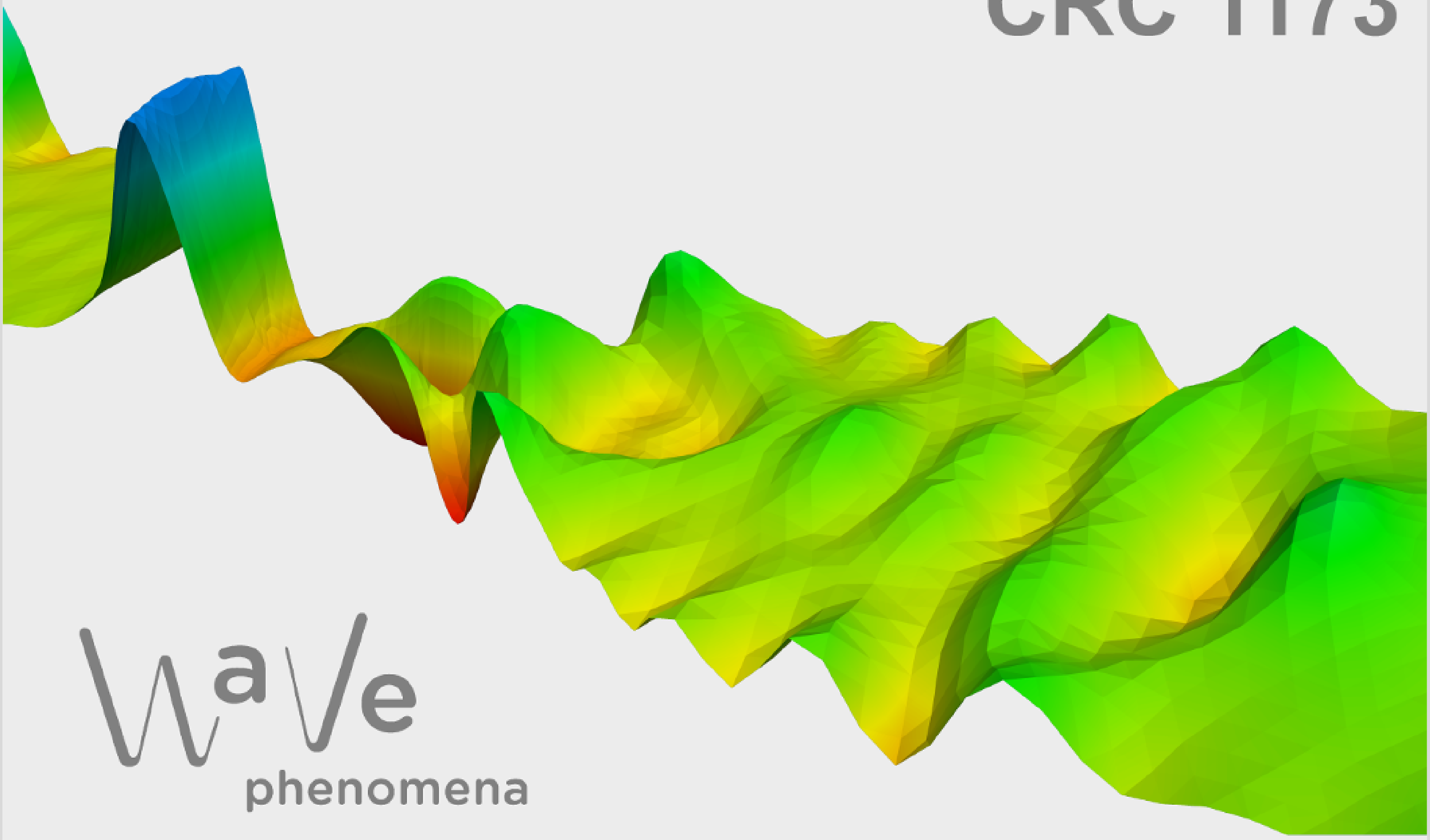
Local well-posedness for the nonlinear Schrödinger equation in the intersection of modulation spaces $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$

Leonid Chaichenets, Dirk Hundertmark,
Peer Kunstmann, Nikolaos Pattakos

CRC Preprint 2019/27, December 2019

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Participating universities

Universität Stuttgart

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

TU WIEN

Funded by

DFG

ISSN 2365-662X

LOCAL WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION IN THE INTERSECTION OF MODULATION SPACES $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$

L. CHAICHENETS, D. HUNDERTMARK, P. KUNSTMANN, AND N. PATTAKOS

Abstract. We introduce a Littlewood-Paley characterization of modulation spaces and use it to give an alternative proof of the algebra property, implicitly contained in [STW11], of the intersection $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$ for $d \in \mathbb{N}$, $p, q \in [1,\infty]$ and $s \geq 0$. We employ this algebra property to show the local well-posedness of the Cauchy problem for the cubic nonlinear Schrödinger equation in the above intersection. This improves [BO09, Theorem 1.1] by Bényi and Okoudjou, where only the case $q = 1$ is considered, and closes a gap in the literature. If $q > 1$ and $s > d\left(1 - \frac{1}{q}\right)$ or if $q = 1$ and $s \geq 0$ then $M^s_{p,q}(\mathbb{R}^d) \hookrightarrow M_{\infty,1}(\mathbb{R}^d)$ and the above intersection is superfluous. For this case we also obtain a new Hölder-type inequality for modulation spaces.

1. Introduction

In this paper we contribute to the general theory of modulation spaces. Modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ were introduced by Feichtinger in [Fei83]. Here, we only briefly recall their definition and refer to Section 2 and the literature mentioned there for more information. Fix a so-called *window function* $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The *short-time Fourier transform* $V_g f$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is defined by

$$(V_g f)(x,\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \overline{\langle f, M_\xi S_x g \rangle} \qquad \forall x, \xi \in \mathbb{R}^d, \tag{1}$$

where $S_x g(y) = g(y - x)$ denotes the *right-shift* by $x \in \mathbb{R}^d$, $(M_\xi g)(y) = e^{\mathrm{i}k \cdot y} g(y)$ the *modulation* by $\xi \in \mathbb{R}^d$ and $\langle f, g \rangle = \int_{\mathbb{R}^d} \overline{f}(x) g(x) \mathrm{d}x$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R})^d$. We define

$$\begin{aligned} M^s_{p,q}(\mathbb{R}^d) &= \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \middle| \, \|f\|_{M^s_{p,q}(\mathbb{R}^d)} < \infty \right\}, \text{ where} \\ \|f\|_{M^s_{p,q}(\mathbb{R}^d)} &= \left\| \xi \mapsto \langle \xi \rangle^s \left\| V_g f(\cdot, \xi) \right\|_p \right\|_q \end{aligned}$$

for $s \in \mathbb{R}$, $p, q \in [1,\infty]$. As usual in the literature, we set $M_{p,q}(\mathbb{R}^d) := M^0_{p,q}(\mathbb{R}^d)$ and often shorten the notation for $M^s_{p,q}(\mathbb{R}^d)$ to $M^s_{p,q}$. It can be shown, that the $M^s_{p,q}(\mathbb{R}^d)$ are Banach spaces and that different choices of the window function g lead to equivalent norms.

To state our first result, let us recall the definition of the Littlewood-Paley decomposition. Consider a smooth radial function $\phi_0 \in C^\infty_c(\mathbb{R}^d)$ with $\phi_0(\xi) = 1$ for

Date: April 30, 2019.

2010 *Mathematics Subject Classification.* 35A01, 35A02, 35Q55, 42B25.

Key words and phrases. Nonlinear Schrödinger equation, modulation spaces, local well-posedness, Littlewood-Paley characterization, Hölder-type inequality.

all $|\xi| \leq \frac{1}{2}$ and $\operatorname{supp}(\phi_0) \subseteq B_1(0)$. Set $\phi_1 = \phi_0\left(\frac{\cdot}{2}\right) - \phi_0$ and $\phi_l = \phi_1\left(\frac{\cdot}{2^{l-1}}\right)$ for all $l \in \mathbb{N}$. The multiplier operators defined by

$$\Delta_l f \coloneqq \frac{1}{(2\pi)^{\frac{d}{2}}} \check{\phi}_l * f = \mathcal{F}^{(-1)} \phi_l \mathcal{F} f \qquad \forall \in \mathbb{N}_0 \, \forall f \in \mathcal{S}'(\mathbb{R}^d)$$

are called *dyadic decomposition operators* and the sequence $(\Delta_l f)_{l \in \mathbb{N}_0}$ is called the *Littlewood-Paley decomposition* of $f \in \mathcal{S}'(\mathbb{R}^d)$. Above, $\mathcal{F}$ denotes the usual *Fourier transform* and $\mathcal{F}^{(-1)}$ its inverse.

Our first result is

Theorem 1 (Littlewood-Paley characterization). *Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Then*

$$\|f\| \coloneqq \left\| \left(2^{ls} \|\Delta_l f\|_{M_{p,q}(\mathbb{R}^d)} \right)_{l \in \mathbb{N}_0} \right\|_q \qquad \forall f \in \mathcal{S}'(\mathbb{R}^d)$$

defines an equivalent norm on $M^s_{p,q}(\mathbb{R}^d)$. The constants of the norm equivalence depend only on d and s.

The above characterization of modulation spaces is new and we shall use it to prove that the intersections $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$ are *Banach *-algebras*[1]. To state this second result, let us denote by $C_{\mathrm{b}}(\mathbb{R}^d)$ the space of bounded complex-valued continuous functions on $\mathbb{R}^d$, where $d \in \mathbb{N}$. We then have

Theorem 2 (Algebra property). *Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \geq 0$. Then $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$ is a Banach *-algebra with respect to pointwise multiplication and complex conjugation. These operations are well-defined due to the embedding $M_{\infty,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$ Furthermore, if $q > 1$ and $s > d\left(1 - \frac{1}{q}\right)$ or if $q = 1$, then $M^s_{p,q}(\mathbb{R}^d) \hookrightarrow M_{\infty,1}(\mathbb{R}^d)$, so in particular $M^s_{p,q}(\mathbb{R}^d)$ is a Banach *-algebra, in that case.*

The latter case of Theorem 2 had been observed already in 1983 by Feichtinger in his aforementioned pioneering work on modulation spaces (cf. [Fei83, Proposition 6.9]), where he proves it using a rather abstract approach via Banach convolution triples. The case $q > 1$ and $s \in \left[0, d\left(1 - \frac{1}{q}\right)\right]$ seems to be new, at least as a statement. A different proof of Theorem 2 can be given following the idea of proof of [STW11, Proposition 3.2], see [Cha18, Proposition 4.2].

Our third result is a Hölder-type inequality for modulation spaces, which is stated in

Theorem 3 (Hölder-type inequality). *Let $d \in \mathbb{N}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For $q > 1$ let $s > d\left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Then there is a $C > 0$ such that for any $f \in M^s_{p_1,q}(\mathbb{R}^d)$ and any $g \in M^s_{p_2,q}(\mathbb{R}^d)$ one has $fg \in M^s_{p,q}(\mathbb{R}^d)$ and*

$$\|fg\|_{M^s_{p,q}(\mathbb{R}^d)} \leq C \|f\|_{M^s_{p_1,q}(\mathbb{R}^d)} \|g\|_{M^s_{p_2,q}(\mathbb{R}^d)} . \tag{2}$$

The above pointwise multiplication fg is well-defined due to the embedding formulated in Theorem 2. The constant C does *not* depend on p, p_1 or p_2.

Theorem 3 easily generalizes to $m \in \mathbb{N}$ factors and $p, p_1, \ldots, p_m \in (0, \infty]$. Hence, it extends the multilinear estimate from [BO09, Equation 2.4] to the case $q_0 = \ldots = q_m > 1$.

---

[1]For us, a Banach *-algebra X is a Banach algebra over $\mathbb{C}$ on which a continuous *involution* $*$ is defined, i.e. $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda} x^*$, $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$. We neither require X to have a unit nor $C = 1$ in the estimates $\|x \cdot y\| \leq C \|x\| \|y\|$, $\|x^*\| \leq C \|x\|$.

Here we present a direct proof of Theorem 3, close to the approach found in [WZG06, Corollary 4.2] and involving an application of Theorem 2. For a proof avoiding the Littewood-Paley characterization see the proof of [Cha18, Theorem 4.3]. A yet another and more abstract proof could be given by invoking [Fei80, Theorem 3] for a specific choice of Banach convolution triples.

Lastly, we employ Theorem 2 to study the Cauchy problem for the cubic nonlinear Schrödinger equation (*NLS*)

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t}(x,t) + \Delta u(x,t) \pm |u|^2 u(x,t) = 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d, \end{cases} \tag{3}$$

where the initial data u_0 is in an intersection of modulation spaces $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$. We are interested in *mild solutions* u of (3), i.e.

$$u \in C\left([0,T), M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)\right)$$

for some $T > 0$ which satisfy the corresponding integral equation

$$u(\cdot,t) = e^{\mathrm{i}t\Delta}u_0 \pm \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(|u|^2 u(\cdot,\tau)\right)\mathrm{d}\tau \qquad \forall t \in [0,T). \tag{4}$$

Our last result is stated in

Theorem 4 (Local well-posedness)**.** *Let $d \in \mathbb{N}$, $p \in [1,\infty]$, $q \in [1,\infty)$ and $s \geq 0$. Set $X = M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$ and $X(T) = C([0,T], X)$, $X_*(T) = C([0,T), X)$ for any $T > 0$. Assume that $u_0 \in X$. Then, there exists a unique maximal mild solution $u \in X_*(T_*)$ of* (3) *and the* blow-up alternative

$$T_* < \infty \qquad \Rightarrow \qquad \limsup_{t \to T_*-} \|u(\cdot,t)\|_X = \infty$$

holds. Moreover, for any $T' \in (0, T_*)$ there exists a neighborhood V of u_0 in X, such that the initial-data-to-solution-map $V \to X(T')$, $v_0 \mapsto v$ is Lipschitz continuous.

As already stated in Theorem 2 one has that, if $q > 1$ and $s > d\left(1 - \frac{1}{q}\right)$ or if $q = 1$, then $M^s_{p,q}(\mathbb{R}^d) \hookrightarrow M_{\infty,1}(\mathbb{R}^d)$ and so $X = M^s_{p,q}(\mathbb{R}^d)$, in that case.

In the case $q = \infty$ excluded in Theorem 4, the situation is more subtle. Following our proof, one obtains local well-posedness in the larger space

$$L^\infty([0,T), M^s_{p,\infty}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)).$$

The missing continuity in time is due to the properties of the free Schrödinger evolution and we refer to the remarks after Theorem 10.

The precursors of Theorem 4 are [WZG06, Theorem 1.1] by Wang, Zhao and Guo for the space $M^0_{2,1}(\mathbb{R}^d)$ and [BO09, Theorem 1.1] due to Bényi and Okoudjou for the space $M^s_{p,1}(\mathbb{R}^d)$ with $p \in [1,\infty]$ and $s \geq 0$. In fact, Theorem 4 generalizes [BO09, Theorem 1.1] to $q \geq 1$: Although our theorem is stated for the cubic nonlinearity, this is for simplicity of the presentation only. The proof allows for an easy generalization to *algebraic nonlinearities* considered in [BO09], which are of the form

$$f(u) = g(|u|^2)u = \sum_{k=0}^{\infty} c_k |u|^{2k} u, \tag{5}$$

where g is an entire function. Also, [BO09, Theorems 1.2 and 1.3], which concern the nonlinear wave and the nonlinear Klein-Gordon equation respectively, can be generalized in the same spirit. The reason for this is that the proof of these results is based on the well-known Banach's contraction principle, on the fact that the free propagator is a C_0-group, and on the algebra property of the spaces under consideration. Although the ingredients seem to be known in the community, the results to be found in the literature (e.g. [WHHG11, Theorem 6.2]) are not as general as Theorem 4. In this sense, it closes a gap in the literature.

Let us remark that local well-posedness results in the case of modulation spaces that are not Banach *-algebras are [Guo16, Theorem 1.4] for $u_0 \in M_{2,q}(\mathbb{R})$ with $q \in [2, \infty)$ and [CHKP19, Theorem 6] with $u_0 \in M_{p,q}^s(\mathbb{R})$ with either $p \in [2, 3]$, $q \in \left[1, \frac{3}{2}\right]$ and $s \geq 0$ or $p \in [2, 3]$, $q \in \left(\frac{3}{2}, \frac{18}{11}\right]$ and $s > \frac{2}{3} - \frac{1}{q}$ or $q \in \left(\frac{18}{11}, 2\right]$, $p \in \left[2, \frac{10q}{7q-6}\right)$ and $s > \frac{2}{3} - \frac{1}{q}$ (see also [Pat18, Theorem 4]).

The remainder of our paper is structured as follows. We start with Section 2 providing an overview over modulation spaces and the free Schrödinger propagator on them. In Section 3 we apply methods from the Littlewood-Paley theory to prove Theorem 1. In the subsequent Section 4 we prove the algebra property from Theorem 2, notice the resulting similar property for weighted sequence spaces in Lemma 12 and deduce the Hölder-type inequality stated in Theorem 3. Finally, we prove Theorem 4 on the local well-posedness in Section 5.

Notation. We denote generic constants by C. To emphasize on which quantities a constant depends we write e.g. $C = C(d)$ or $C = C(d, s)$. Sometimes we omit a positive constant from an inequality by writing "$\lesssim$", e.g. $A \lesssim_d B$ instead of $A \leq C(d)B$. By $A \approx B$ we mean $A \lesssim B$ and $B \lesssim A$. Special constants are $d \in \mathbb{N}$ for the *dimension*, $p, q \in [1, \infty]$ for the *Lebesgue* exponents and $s \in \mathbb{R}$ for the *regularity* exponent. By p' we mean the *dual* exponent of p, that is the number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of *Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^d)$ the space of *tempered distributions*. Furthermore, we denote the *Bessel potential spaces* or simply L^2-based *Sobolev spaces* by $H^s = H^s(\mathbb{R}^d)$. For the space of smooth functions with compact support we write C_c^∞. The letters f, g, h denote either generic functions $\mathbb{R}^d \to \mathbb{C}$ or generic tempered distributions and $(a_k)_{k \in \mathbb{Z}^d} = (a_k)_k = (a_k)$, $(b_k)_{k \in \mathbb{Z}^d} = (b_k)_k = (b_k)$ denote generic complex-valued sequences. By $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$ we mean the *Japanese bracket*.

For a Banach space X we write X^* for its dual and $\|\cdot\|_X$ for the norm it is canonically equipped with. By $\mathscr{L}(X, Y)$ we denote the space of all bounded linear maps from X to Y, where Y is another Banach space, and set $\mathscr{L}(X) = \mathscr{L}(X, X)$. By $[X, Y]_\theta$ we mean complex interpolation between X and Y, if (X, Y) is an interpolation couple. For brevity we write $\|\cdot\|_p$ for the p-norm on the *Lebesgue space* $L^p = L^p(\mathbb{R}^d)$, the *sequence space* $l^p = l^p(\mathbb{Z}^d)$ or $l^p = l^p(\mathbb{N}_0)$ and $\|(a_k)\|_{q,s} \coloneqq \|(\langle k \rangle^s a_k)\|_q$ for the norm on $\langle \cdot \rangle^s$-weighted sequence spaces $l_s^q = l_s^q(\mathbb{Z}^d)$. If the norm is apparent from the context, we write $B_r(x)$ for a ball of radius r around $x \in X$.

We use the symmetric choice of constants for the Fourier transform and also write

$$\begin{aligned}
\hat{f}(\xi) &\coloneqq (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi \cdot x} f(x) \mathrm{d}x, \\
\check{g}(x) &\coloneqq \left(\mathcal{F}^{(-1)} g\right)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} g(\xi) \mathrm{d}\xi.
\end{aligned}$$

2. Preliminaries

As already mentioned in the introduction, modulation spaces were introduced by Feichtinger in [Fei83] in the setting of locally compact Abelian groups. A thorough introduction is given in the textbook [Grö01] by Gröchenig. A presentation incorporating the characterization of modulation spaces via *isometric decomposition operators*, which we are going to use below, is contained in the paper [WH07, Section 2, 3] by Wang and Hudzik. A survey on modulation spaces and nonlinear evolution equations is given in [RSW12].

A convenient equivalent norm on modulation spaces which we are going to use is constructed as follows (cf. [WH07, Propostition 2.1]): Set $Q_0 := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ and $Q_k := Q_0 + k$ for all $k \in \mathbb{Z}^d$. Consider a smooth partition of unity $(\sigma_k)_{k\in\mathbb{Z}^d} \in \left(C_c^\infty(\mathbb{R}^d)\right)^{\mathbb{Z}^d}$ satisfying

- $\exists c > 0 : \forall k \in \mathbb{Z}^d : \forall \eta \in Q_k : |\sigma_k(\eta)| \geq c$,
- $\forall k \in \mathbb{Z}^d : \operatorname{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k)$,
- $\sum_{k\in\mathbb{Z}^d} \sigma_k = 1$,
- $\forall m \in \mathbb{N}_0 : \exists C_m > 0 : \forall k \in \mathbb{Z}^d : \forall \alpha \in \mathbb{N}_0^d : |\alpha| \leq m \Rightarrow \|D^\alpha \sigma_k\|_\infty \leq C_m$

and define the *isometric decomposition operators* $\square_k := \mathcal{F}^{(-1)}\sigma_k\mathcal{F}$. We need the following often used (cf. [WH07, Proposition 1.9])

Lemma 5 (Bernstein multiplier estimate)**.** *Let $d \in \mathbb{N}$, $\sigma \in \mathcal{S}(\mathbb{R}^d)$ and $r, p_1, p_2 \in [1,\infty]$ such that $1 + \frac{1}{p_2} = \frac{1}{r} + \frac{1}{p_1}$. Consider the* multiplier operator $T_\sigma : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *with* symbol σ *defined by*

$$T_\sigma f = \mathcal{F}^{(-1)}\sigma\mathcal{F}f = \frac{1}{(2\pi)^{\frac{d}{2}}}\check{\sigma} * f \qquad \forall f \in \mathcal{S}'(\mathbb{R}^d).$$

Then, for any $f \in \mathcal{S}'(\mathbb{R}^d)$, every derivative of $T_\sigma f \in C^\infty(\mathbb{R}^d)$ (including $T_\sigma f$) has at most polynomial growth. Furthermore $\|T_\sigma f\|_{p_2} \leq \frac{\|\check{\sigma}\|_r}{(2\pi)^{\frac{d}{2}}} \|f\|_{p_1}$ for any $f \in L^{p_1}(\mathbb{R}^d)$.

Putting $r = 1$ and $p_1 = p_2 = p$ in Lemma 5, shows that $\square_k f \in C^\infty(\mathbb{R}^d)$ for $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\|\square_k\|_{\mathscr{L}(L^p(\mathbb{R}^d))}$ is bounded independently of k and p. The aforementioned equivalent norm for the modulation space $M^s_{p,q}(\mathbb{R}^d)$ is given by (see [WH07, Proposition 2.1])

$$\|f\|_{M^s_{p,q}} \approx \left\| \left(\langle k \rangle^s \|\square_k f\|_p \right)_{k\in\mathbb{Z}^d} \right\|_q . \tag{6}$$

Choosing a different partition of unity (σ_k) yields yet another equivalent norm.

Lemma 6 (Continuous embeddings)**.** *Let $s_1 \geq s_2$, $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$, $q > 1$ and $s > \frac{d}{q'}$. Then*

1. *$M^{s_1}_{p_1,q_1}(\mathbb{R}^d) \subseteq M^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ and the embedding is continuous,*
2. *$M^s_{p_1,q}(\mathbb{R}^d) \subseteq M_{p_1,1}(\mathbb{R}^d)$ and the embedding is continuous,*
3. *$M_{p_1,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$.*

Lemma 6 is well-known (cf. [WH07, Proposition 2.5, 2.7]), but for convenience we sketch a

Proof. (1) One can change indices one by one. The inclusion for "s" is by monotonicity and the inclusion for "q" is by the embeddings of the l^q spaces. For the "p"-embedding consider $\tau \in C_c^\infty(\mathbb{R}^d)$ such that $\tau|_{B_{\sqrt{d}}} \equiv 1$ and $\operatorname{supp}(\tau) \subseteq B_d$. For every $k \in \mathbb{Z}^d$, consider the shifted symbol $\tau_k = S_k\tau$, define the corresponding multiplier operator $\tilde{\square}_k = \mathcal{F}^{(-1)}\tau_k\mathcal{F}$ and observe, that $\hat{\tau}_k = M_k\hat{\tau}$. Hence, by Lemma 5, the family $\left(\tilde{\square}_k\right)_{k\in\mathbb{Z}^d}$ is bounded in $\mathscr{L}(L^{p_1}(\mathbb{R}^d), L^{p_2}(\mathbb{R}^d))$. So, $\|\square_k f\|_{p_2} = \left\|\tilde{\square}_k \square_k f\right\|_{p_2} \lesssim_d \|\square_k f\|_{p_1}$ for any $k \in \mathbb{Z}^d$. Recalling (6) completes the argument.

(2) By Hölder's inequality we immediately have

$$\begin{aligned} \|f\|_{p_1,1} &\approx \sum_{k\in\mathbb{Z}^d} \|\square_k f\|_{p_1} \leq \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{-sq'}\right)^{\frac{1}{q'}} \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{sq} \|\square_k f\|_p^q\right)^{\frac{1}{q}} \\ &\approx \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{-sq'}\right)^{\frac{1}{q'}} \|f\|_{M^s_{p_1,q}} \end{aligned}$$

and the first factor is finite for $s > \frac{d}{q'}$ by comparison with the integral $\int_{\mathbb{R}^d} \langle x\rangle^{-sq'}\mathrm{d}x$.

(3) By part (1) it is enough to show that $M_{\infty,1} \hookrightarrow C_b$. For any $f \in M_{\infty,1}$ we have $\underbrace{\sum_{|k|\leq N} \square_k f}_{\in C^\infty} \to f$ in $\mathcal{S}'$ as $N \to \infty$. But simultaneously, the series $\sum_{k\in\mathbb{Z}^d} \square_k f$ is absolutely convergent in L^∞ to, say, $g \in C_b$. As $M_{\infty,1} \hookrightarrow S'$ (see [Fei83, Thm. 6.1 (B)]), we have $f = g$.

$\square$

For the proof of Theorem 2 we will need the following (cf. [BO09, eqn. (2.4)])

Lemma 7 (Bilinear estimate). *Let $d \in \mathbb{N}$ and $1 \leq p \leq \infty$. Assume $f \in M_{p,q}(\mathbb{R}^d)$ and $g \in M_{\infty,1}(\mathbb{R}^d)$. Then*

$$\|fg\|_{M_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{M_{p,q}(\mathbb{R}^d)} \|g\|_{M_{\infty,1}(\mathbb{R}^d)},$$

where the implicit constant does not *depend on p or q.*

For convenience, and because we will generalize Lemma 7 to Theorem 3, we present a proof close to the one of [WZG06, Corollary 4.2].

Proof. We use (6) to estimate the modulation space norm of the left-hand side. Fix a $k \in \mathbb{Z}^d$. By the definition of the operator $\square_k$ we have

$$\square_k(fg) = \frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}^{(-1)}\left(\sigma_k(\hat{f} * \hat{g})\right) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{l,m\in\mathbb{Z}^d} \mathcal{F}^{(-1)}\left(\sigma_k((\sigma_l\hat{f}) * (\sigma_m\hat{g}))\right).$$

As the supports of the partition of unity are compact, many summands vanish. Indeed, for any $k, l, m \in \mathbb{Z}^d$

$$\begin{aligned} \operatorname{supp}\left(\sigma_k\left((\sigma_l\hat{f}) * (\sigma_m\hat{g})\right)\right) &\subseteq \operatorname{supp}(\sigma_k) \cap (\operatorname{supp}(\sigma_l) + \operatorname{supp}(\sigma_m)) \\ &\subseteq B_{\sqrt{d}}(k) \cap B_{2\sqrt{d}}(l+m) \end{aligned}$$

and so $\sigma_k\left((\sigma_l\hat{f}) * (\sigma_m\hat{g})\right) \equiv 0$ if $|(k-l)-m| > 3\sqrt{d}$. Hence, the double series over $l, m \in \mathbb{Z}^d$ boils down to a finite sum of discrete convolutions

$$\begin{aligned} \square_k(fg) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\mathcal{F}^{(-1)}\left(\sigma_k \sum_{m\in M}\sum_{l\in\mathbb{Z}^d} (\sigma_l\hat{f}) * (\sigma_{k-l+m}\hat{g})\right) \\ &= \square_k \sum_{m\in M}\sum_{l\in\mathbb{Z}^d} (\square_l f)\cdot(\square_{k+m-l} g), \end{aligned}$$

where $M = \left\{m \in \mathbb{Z}^d \,\middle|\, |m| \leq 3\sqrt{d}\right\}$ and $\#M \leq \left(6\sqrt{d}+1\right)^d < \infty$. That was the job of $\square_k$ and we now get rid of it,

$$\|\square_k(fg)\|_p \lesssim \sum_{m\in M}\sum_{l\in\mathbb{Z}^d} \|(\square_l f)\cdot(\square_{k+m-l} g)\|_p,$$

using the Bernstein multiplier estimate from Lemma 5.

Invoking Hölder's inequality we further estimate

$$\|\Box_k(fg)\|_p \lesssim \sum_{m\in M} \left(\left(\|\Box_l(f)\|_p \right)_l * \left(\|\Box_{n+m}(g)\|_\infty \right)_n \right)(k) \tag{7}$$

pointwise in $k \in \mathbb{Z}^d$, where $*$ denotes the convolution of sequences, and hence obtain

$$\|fg\|_{M_{p,q}} \lesssim \left\| \left(\|\Box_l f\|_p \right)_l \right\|_q \left\| \left(\|\Box_n g\|_\infty \right)_n \right\|_1$$

by Young's inequality. □

Lemma 8 (Dual space)**.** *For $s \in \mathbb{R}$, $p, q \in [1, \infty)$ we have*

$$\left(M^s_{p,q}(\mathbb{R}^d) \right)^* = M^{-s}_{p',q'}(\mathbb{R}^d)$$

(see [WH07, Theorem 3.1]*).*

Theorem 9 (Complex interpolation)**.** *For $p_1, q_1 \in [1, \infty)$, $p_2, q_2 \in [1, \infty]$, $s_1, s_2 \in \mathbb{R}$ and $\theta \in (0, 1)$ one has*

$$\left[M^{s_1}_{p_1,q_1}(\mathbb{R}^d), M^{s_2}_{p_2,q_2}(\mathbb{R}^d) \right]_\theta = M^s_{p,q}(\mathbb{R}^d),$$

with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2$$

(see [Fei83, Theorem 6.1 (D)]*).*

We are now ready to state and prove the following

Theorem 10 (Schrödinger propagator bound)**.** *There is a constant $C > 0$ such that for any $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$ the inequality*

$$\left\| e^{\mathrm{i}t\Delta} \right\|_{\mathscr{L}(M^s_{p,q}(\mathbb{R}^d))} \le C^d (1 + |t|)^{d\left|\frac{1}{2} - \frac{1}{p}\right|} \tag{8}$$

holds for all $t \in \mathbb{R}$. Furthermore, the exponent of the time dependence is sharp.

The boundedness has been obtained e.g. in [BGOR07, Theorem 1] whereas the sharpness was proven in [CN09, Proposition 4.1]. If $q < \infty$, then $(e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$ is a C_0-group on $M^s_{p,q}$, i.e.

$$\lim_{t\to 0} \left\| e^{\mathrm{i}t\Delta} f - f \right\|_{M^s_{p,q}} = 0 \qquad \forall f \in M^s_{p,q}$$

(see e.g. [Cha18, Proposition 3.5]). This is not true for $q = \infty$ and we refer to [Kun19] for this more subtle case.

Theorem 10. By definition, we have

$$(V_g e^{\mathrm{i}t\Delta} f)(x, \xi) = e^{-\mathrm{i}t|\xi|^2} (V_{e^{\mathrm{i}t\Delta} g} f)(x + 2t\xi, \xi)$$

for any $f \in \mathcal{S}'(\mathbb{R}^d)$, any $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, and any $t \in \mathbb{R}$, i.e. the Schrödinger time evolution of the initial data can be interpreted as the time evolution of the window function. The price for changing from window g_0 to window g_1 is $\|V_{g_0} g_1\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$ by [Grö01, Proposition 11.3.2 (c)]. For $g(x) = e^{-|x|^2}$ one explicitly calculates

$$\left\| V_{e^{-\mathrm{i}t\Delta} g} g \right\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = C^d \left(1 + |t|\right)^{\frac{d}{2}},$$

which proves the claimed bound for $p \in \{1, \infty\}$. Conservation for $p = 2$ is easily seen from (6). Complex interpolation between the cases $p = 2$ and $p = \infty$ yields (8) for $p \in [2, \infty]$. The remaining case $p \in (1, 2)$ is covered by duality.

Optimality in the case $p \in [1, 2]$ is proven by choosing the window g and the argument f to be a Gaussian and explicitly calculating $\left\| e^{\mathrm{i}t\Delta} f \right\|_{M^s_{p,q}} \approx (1 + |t|)^{d\left(\frac{1}{p} - \frac{1}{2}\right)}$. This implies the optimality for $p \in (2, \infty]$ by duality. □

3. Littlewood-Paley theory

In this section we extend some ideas of the Littlewood-Paley decomposition from Sobolev spaces $H^s(\mathbb{R}^d)$ to modulation spaces $M^s_{p,q}(\mathbb{R}^d)$. The inspiration for this was [AG07, Chapter II].

Observe, that for any $\xi \in \mathbb{R}^d$ one has

$$\sum_{l=0}^{\infty} \phi_l(\xi) = \phi_0(\xi) + \lim_{N\to\infty} \sum_{l=1}^{N} \left[\phi_1\left(\frac{\xi}{2^l}\right) - \phi_1\left(\frac{\xi}{2^{l-1}}\right)\right] = \lim_{N\to\infty} \phi_0\left(\frac{\xi}{2^N}\right) = 1,$$

i.e. $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is a smooth partition of unity. Moreover, $\operatorname{supp}(\phi_l) \subseteq A_l$ for any $l \in \mathbb{N}_0$, where

$$A_0 := \left\{\xi \in \mathbb{R}^d \middle|\, |\xi| \leq 1\right\} \quad \text{and} \quad A_l := \left\{\xi \in \mathbb{R}^d \middle|\, 2^{l-2} \leq |\xi| \leq 2^l\right\} \qquad \forall l \in \mathbb{N}.$$

The symbols of the dyadic decomposition operators satisfy

$$\left\|\hat{\phi}_l\right\|_1 = \left\|\mathcal{F}\left[\phi_1\left(\frac{\cdot}{2^{l-1}}\right)\right]\right\|_1 = \left\|2^{l-1}\hat{\phi}_1(2^{l-1}\cdot)\right\|_1 = \left\|\hat{\phi}_1\right\|_1 \leq 2\left\|\hat{\phi}_0\right\|_1$$

for all $l \in \mathbb{N}$. Applying Lemma 5 shows that for any $l \in \mathbb{N}_0$ and any $f \in \mathcal{S}'(\mathbb{R}^d)$ one has that $\Delta_l f \in C^\infty$ and any of its derivates has at most polynomial growth. Furthermore, $\|\Delta_l\|_{\mathscr{L}(L^p(\mathbb{R}^d))}$ is bounded independently of $l \in \mathbb{N}_0$ and $p \in [1, \infty]$.

Theorem 1. We start by gathering some useful facts. Fix $l \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$. Recall, that $\operatorname{supp}(\phi_l) \subseteq A_l$ and $\operatorname{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k)$. Hence,

$$\Box_k \Delta_l \not\equiv 0 \Rightarrow k \in A'_l := \left\{k' \in \mathbb{Z}^d \middle|\, 2^{l-2} - \sqrt{d} \leq |k'| \leq 2^l + \sqrt{d}\right\}. \tag{9}$$

On A'_l the Japanese bracket can be controlled. In fact, for all $t \in \mathbb{R}$ we have

$$\langle k\rangle^t \approx 2^{lt}, \tag{10}$$

where the implicit constant does not depend on l.

Finally, observe that $k \in A'_l$ is satisfied for only finitely many $l \in \mathbb{N}_0$, whose number is independent of $k \in \mathbb{Z}^d$, i.e.

$$\sum_{l=0}^{\infty} \mathbb{1}_{A'_l}(k) \lesssim 1, \tag{11}$$

where the implicit constant depends on d only.

- $\gtrsim$: Consider $q < \infty$ first. By (6), (9), Bernstein multiplier estimate, (10) and (11) we have

$$\begin{aligned}
&\left\|\left(2^{ls}\|\Delta_l f\|_{M_{p,q}}\right)_l\right\|_q \\
\approx\ & \left(\sum_{l=0}^{\infty} 2^{lsq} \sum_{k\in\mathbb{Z}^d} \|\Box_k\Delta_l f\|_p^q\right)^{\frac{1}{q}} \lesssim \left(\sum_{l=0}^{\infty}\sum_{k\in A'_l} 2^{lsq}\|\Box_k f\|_p^q\right)^{\frac{1}{q}} \\
\approx\ & \left(\sum_{l=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \mathbb{1}_{A'_l}(k)\langle k\rangle^{qs}\|\Box_k f\|_p^q\right)^{\frac{1}{q}} \lesssim \|f\|_{M^s_{p,q}}.
\end{aligned}$$

  Similarly, for $q = \infty$, we have

$$\begin{aligned}
\left\|\left(2^{ls}\|\Delta_l f\|_{M_{p,\infty}}\right)_l\right\|_\infty &= \sup_{l\in\mathbb{N}_0} 2^{ls} \sup_{k\in\mathbb{Z}^d} \|\Box_k\Delta_l f\|_p \\
&\lesssim \sup_{l\in\mathbb{N}_0}\sup_{k\in A'_l} \langle k\rangle^s \|\Box_k f\|_p \approx \|f\|_{M^s_{p,\infty}}.
\end{aligned}$$

- $\lesssim$: Again, consider $q < \infty$ first. By (6), $f = \sum_{l=0}^{\infty} \Delta_l f$ in $\mathcal{S}'$ and (9) we have

$$\|f\|_{M^s_{p,q}} \lesssim \left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \left(\sum_{l=0}^{\infty} \|\Box_k \Delta_l f\|_p \right)^q \right)^{\frac{1}{q}}$$
$$\lesssim \left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \left(\sum_{l=0}^{\infty} \mathbb{1}_{A'_l}(k) \|\Box_k \Delta_l f\|_p \right)^q \right)^{\frac{1}{q}}.$$

  For each $k \in \mathbb{Z}^d$ the sum over l contains only finitely many non-vanishing summands and their number is independent of k by (11). Hölder's inequality estimates the last term against

$$\left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \sum_{l=0}^{\infty} \mathbb{1}_{A'_l}(k) \|\Box_k \Delta_l f\|_p^q \right)^{\frac{1}{q}} \approx \left(\sum_{l=0}^{\infty} 2^{lsq} \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{A'_l}(k) \|\Delta_l \Box_k f\|_p^q \right)^{\frac{1}{q}}$$
$$\leq \left\| \left(2^{ls} \|\Delta_l f\|_{M_{p,q}} \right)_l \right\|_q,$$

  where we additionally used (10). The proof for $q = \infty$ is along the same lines.

□

The individual parts of the Littlewood-Paley decomposition had their Fourier transform supported in almost disjoint dyadic annuli. Theorem 1 characterized elements of modulation spaces by the decay of these parts. The following lemma provides a sufficient condition for the case of non-disjoint balls.

Lemma 11 (Sufficient condition). *Let $1 \leq q \leq \infty$ and $s > 0$. For $m \in \mathbb{N}_0$ let $f_m \in \mathcal{S}'(\mathbb{R}^d)$ be such that*

$$\operatorname{supp}(\hat{f}_m) \subseteq B_m := \left\{ \xi \in \mathbb{R}^d \middle| |\xi| \leq 2^m \right\} \qquad \forall m \in \mathbb{N}_0.$$

Set $f := \sum_{m=0}^{\infty} f_m$ in $\mathcal{S}'(\mathbb{R}^d)$. Then

$$\|f\|_{M^s_{p,q}(\mathbb{R}^d)} \lesssim \left\| \left(2^{ms} \|f_m\|_{M_{p,q}(\mathbb{R}^d)} \right)_{m \in \mathbb{N}_0} \right\|_q,$$

where the implicit constant depends on d and s only.

Proof. Observe, that $A_l \cap B_m = \emptyset$ if $l > m + 2$. Hence, we have

$$\|f\|_{M^s_{p,q}} \approx \left\| \left(2^{ls} \|\Delta_l f\|_{M_{p,q}} \right)_l \right\|_q \lesssim \left\| \left(2^{ls} \sum_{m=l}^{\infty} \|\Delta_l f_m\|_{M_{p,q}} \right)_l \right\|_q$$
$$\lesssim \left\| \left(2^{ls} \sum_{m=l}^{\infty} \|f_m\|_{M_{p,q}} \right)_l \right\|_q,$$

where we additionally used Theorem 1 and Bernstein multiplier estimate. From now on, we assume $q \in (1, \infty)$, as the proof for the other cases is easier and follows the same lines. Hölder's inequality and geometric sum formula estimates the last term against

$$\begin{aligned}
&\left(\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty}2^{ls}\|f_m\|_{M_{p,q}}\right)^q\right)^{\frac{1}{q}}\\
=\;&\left(\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty}2^{\frac{(l-m)s}{q'}}\times 2^{\frac{(l-m)s}{q}}2^{ms}\|f_m\|_{M_{p,q}}\right)^q\right)^{\frac{1}{q}}\\
\leq\;&\left(\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty}2^{(l-m)s}\right)^{\frac{q}{q'}}\left(\sum_{m=l}^{\infty}2^{(l-m)s}2^{msq}\|f_m\|_{M_{p,q}}^q\right)\right)^{\frac{1}{q}}\\
\approx\;&\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}2^{(l-m)s}2^{msq}\|f_m\|_{M_{p,q}}^q\right)^{\frac{1}{q}}\\
\approx\;&\left\|\left(2^{ms}\|f_m\|_{M_{p,q}}\right)_m\right\|_q,
\end{aligned}$$

finishing the proof. □

4. Algebra property and Hölder-type inequality

Main goal of this section is to prove Theorem 2, which was inspired by the fact that $H^s(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ is a Banach *-algebra with respect to pointwise multiplication for $s\geq 0$.

Theorem 2. Parts 2 and 3 of Lemma 6 prove the claimed embedding. Continuity of complex conjugation is obvious from (6). Continuity of multiplication follows by the paraproduct argument

$$fg=\left(\sum_{l=0}^{\infty}\Delta_l f\right)\left(\sum_{m=0}^{\infty}\Delta_m g\right)=\sum_{l=0}^{\infty}\underbrace{\left(\Delta_l f\sum_{m=0}^{l}\Delta_m g\right)}_{=:u_l}+\sum_{m=1}^{\infty}\underbrace{\left(\Delta_m g\sum_{l=0}^{m-1}\Delta_l f\right)}_{=:v_m}.$$

Observe, that for any $l,m\in\mathbb{N}_0$ we have $\operatorname{supp}(\hat{u}_l)\subseteq B_{l+1}$ and $\operatorname{supp}(\hat{v}_m)\subseteq B_m$ by the properties of convolution. Hence, Lemma 11 could be applied. Bilinear estimate from Lemma 7 and Theorem 1 show

$$\left\|\left(2^{ls}\|u_l\|_{M_{p,q}}\right)_l\right\|_q\;\leq\;\left\|\left(2^{ls}\|\Delta_l f\|_{M_{p,q}}\right)_l\right\|_q\sum_{m=0}^{\infty}\|\Delta_m g\|_{M_{\infty,1}}\approx\|f\|_{M_{p,q}^s}\|g\|_{M_{\infty,1}}.$$

The same argument yields $\|\sum_{m=1}^{\infty}v_m\|_{M_{p,q}^s}\lesssim\|f\|_{M_{\infty,1}}\|g\|_{M_{p,q}^s}$ and finishes the proof. □

The analogon of Theorem 2 for sequence spaces is stated in

Lemma 12 (Algebra property)**.** *Let $1\leq q\leq\infty$ and $s\geq 0$. Then $l_s^q(\mathbb{Z}^d)\cap l^1(\mathbb{Z}^d)$ is a Banach algebra with respect to convolution*

$$(a_l)*(b_m)=\left(\sum_{m\in\mathbb{Z}^d}a_{k-m}b_m\right)_{k\in\mathbb{Z}^d},\tag{12}$$

which is well-defined, as the series above always converge absolutely.

Furthermore, if $q>1$ and $s>d\left(1-\frac{1}{q}\right)$ or $q=1$, then $l_s^q(\mathbb{Z}^d)\hookrightarrow l^1(\mathbb{Z}^d)$, so in particular $l_s^q(\mathbb{Z}^d)$ is a Banach algebra, in that case.

Although this result is certainly not new, we could not find a suitable reference. A proof can be given using the same techniques as for the proof of Theorem 2, i.e. by proving analoga of Theorem 1 and Lemma 11 for the weighted sequence spaces. Another approach is to notice that by definition

$$\left\| \sum_{k\in\mathbb{Z}^d} a_k e^{\mathrm{i}kx} \right\|_{M^s_{\infty,q}} \approx \|(a_k)\|_{l^q_s}$$

and hence, by Theorem 2, one has

$$\begin{aligned}
& \|(a_k) * (b_k)\|_{l^q_s} \\
\approx\ & \left\| \left(\sum_{k\in\mathbb{Z}^d} a_k e^{\mathrm{i}kx} \right) \cdot \left(\sum_{k\in\mathbb{Z}^d} b_k e^{\mathrm{i}kx} \right) \right\|_{M^s_{\infty,q}} \\
\lesssim\ & \left\| \sum_{k\in\mathbb{Z}^d} a_k e^{\mathrm{i}kx} \right\|_{M^s_{\infty,q}} \left\| \sum_{k\in\mathbb{Z}^d} b_k e^{\mathrm{i}kx} \right\|_{M_{\infty,1}} + \left\| \sum_{k\in\mathbb{Z}^d} a_k e^{\mathrm{i}kx} \right\|_{M_{\infty,1}} \left\| \sum_{k\in\mathbb{Z}^d} b_k e^{\mathrm{i}kx} \right\|_{M^s_{\infty,q}} \\
\approx\ & \|(a_k)\|_{l^q_s} \|(b_k)\|_{l^1} + \|(a_k)\|_{l^1} \|(b_k)\|_{l^q_s} .
\end{aligned}$$

We are now ready to give a

Theorem 3. We arrive, as for equation (7) in the proof of Lemma 7, at

$$\|\square_k(fg)\|_p \lesssim \sum_{m\in M} \left(\left(\|\square_l(f)\|_{p_1} \right)_l * \left(\|\square_{n+m}(g)\|_{p_2} \right)_n \right)(k)$$

pointwise in $k \in \mathbb{Z}^d$. By the algebra property from Lemma 12, it follows that

$$\|fg\|_{M^s_{p,q}} \lesssim \left\| \left(\|\square_l f\|_{p_1} \right)_l \right\|_{q,s} \left(\sum_{m\in M} \left\| \left(\|\square_{n+m} g\|_{p_2} \right)_n \right\|_{q,s} \right)$$

and the first factor is already $\|f\|_{M^s_{p,q}}$. Finally, we remove the sum over m in the second factor

$$\sum_{m\in M} \left\| \left(\|\square_{n+m} g\|_{p_2} \right)_n \right\|_{q,s} \lesssim \|g\|_{M^s_{p_2,q}}$$

applying Peetre's inequality $\langle k+l \rangle^s \leq 2^{|s|} \langle k \rangle^s \langle l \rangle^{|s|}$ (see e.g. [RT10, Proposition 3.3.31]).

Let us finish the proof remarking that the only estimate involving "p"s we used was Hölder's inequality and thus the implicit constant indeed does not depend on p, p_1 or p_2. $\square$

5. Proof of the local well-posedness, Theorem 4.

Theorem 2 immediately implies that $X(T)$ is a Banach *-algebra, i.e.

$$\begin{aligned}
\|uv\|_{X(T)} &= \sup_{0\leq t\leq T} \|uv(\cdot,t)\|_X \lesssim \left(\sup_{0\leq s\leq T} \|u(\cdot,s)\|_X \right) \left(\sup_{0\leq t\leq T} \|v(\cdot,t)\|_X \right) \\
&= \|u\|_{X(T)} \|v\|_{X(T)} .
\end{aligned}$$

For $R > 0$ we denote by $M(R,T) = \left\{ u \in X(T) \middle| \|u\|_{X(T)} \leq R \right\}$ the closed ball of radius R in $X(T)$ centered at the origin. We show that for some $R, T > 0$ the right-hand side of (4),

$$(\mathcal{T}u)(\cdot,t) := e^{\mathrm{i}t\Delta} u_0 \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\Delta} \left(|u|^2 u(\cdot,\tau) \right) \mathrm{d}\tau \qquad (\forall t \in [0,T]), \tag{13}$$

defines a contractive self-mapping $\mathcal{T} = \mathcal{T}(u_0) : M_{R,T} \to M_{R,T}$.

To that end, let us observe that Theorem 10 implies the *homogeneous estimate*

$$\left\|t \mapsto e^{\mathrm{i}t\Delta}v\right\|_X \leq C_0(1+T)^{\frac{d}{2}} \left\|v\right\|_X \qquad (\forall v \in X),$$

which, together with the algebra property of $X(T)$, proves the *inhomogeneous estimate*

$$\begin{aligned} &\left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(\left|u\right|^2 u(\cdot,\tau)\right)\mathrm{d}\tau\right\|_X \\ \leq\ & C_0(1+T)^{\frac{d}{2}}\int_0^t \left\|\left|u\right|^2 u(\cdot,\tau)\right\|_X \mathrm{d}\tau \leq C_0C_1T(1+T)^{\frac{d}{2}}\left\|u\right\|_X^3, \end{aligned}$$

holding for $0 \leq t \leq T$ and $u \in X(T)$.

Applying the triangle inequality in (13) yields

$$\left\|\mathcal{T}u\right\|_X \leq C_0(1+T)^{\frac{d}{2}}(\left\|u_0\right\|_X + C_1TR^3)$$

for any $u \in M(R,T)$. Thus, $\mathcal{T}$ maps $M(R,T)$ into itself for $R = 2C_0C_1\left\|u_0\right\|_X$ and T small enough. Furthermore,

$$\left|u\right|^2 u - \left|v\right|^2 v = (u-v)\left|u\right|^2 + (\overline{u}u - \overline{v}v)v = (u-v)(\left|u\right|^2 + \overline{u}v) + (\overline{u}-\overline{v})v^2$$

and hence

$$\left\|\mathcal{T}u - \mathcal{T}v\right\|_{X(T)} \lesssim T(1+T)^{\frac{d}{2}}R^2\left\|u-v\right\|_{X(T)}$$

for $u, v \in M(R,T)$, where we additionally used the algebra property of $X(T)$ and the homogeneous estimate. Taking T sufficiently small makes $\mathcal{T}$ a contraction.

Banach's fixed-point theorem implies the existence and uniqueness of a mild solution up to the *guaranteed time of existence* $T_0 = T_0(\left\|u_0\right\|_X) \approx \left\|u_0\right\|_X^{-2} > 0$. Uniqueness of the maximal solution and the blow-up alternative now follow easily by the usual contradiction argument.

For the proof of the Lipschitz continuity, let us notice that for any $r > \left\|u_0\right\|_X$, $v_0 \in B_r(0)$ and $0 < T \leq T_0(r)$ we have

$$\begin{aligned} \left\|u-v\right\|_{X(T)} &= \left\|\mathcal{T}(u_0)u - \mathcal{T}(v_0)v\right\|_{X(T)} \\ &\lesssim (1+T)^{\frac{d}{2}}\left\|u_0-v_0\right\|_X + T(1+T)^{\frac{d}{2}}R^2\left\|u-v\right\|_{X(T)}, \end{aligned}$$

where v is the mild solution corresponding to the initial data v_0 and $R = 2Cr$, similar to the above. Collecting terms containing $\left\|u-v\right\|_{X(T)}$ shows Lipschitz continuity with constant $L = L(r)$ for sufficiently small T, say $T_l = T_l(r)$. For arbitrary $0 < T' < T_*$ put $r = 2\left\|u\right\|_{X(T')}$ and divide $[0,T']$ into n subintervals of length $\leq T_l$. The claim follows for $V = B_\delta(u_0)$ where $\delta = \frac{\left\|u_0\right\|_X}{L^n}$ by iteration. This concludes the proof. □

Acknowledgments

We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

References

[AG07] Alinhac, Serge and Patrick Gérard: *Pseudo-differential Operators and the Nash-Moser Theorem*, volume 82 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2007, ISBN 978-0-8218-3454-1.

[BGOR07] Bényi, Árpád, Karlheinz Gröchenig, Kasso Akochayé Okoudjou, and Luke Gervase Rogers: *Unimodular Fourier multipliers for modulation spaces*. Journal of Functional Analysis, 246(2):366–384, 2007, ISSN 0022-1236. https://doi.org/10.1016/j.jfa.2006.12.019.

[BO09] Bényi, Árpád and Kasso Akochayé Okoudjou: *Local well-posedness of nonlinear dispersive equations on modulation spaces*. Bulletin of the London Mathematical Society, 41(3):549–558, 2009, ISSN 0024-6093. https://doi.org/10.1112/blms/bdp027.

[Cha18] Chaichenets, Leonid: *Modulation spaces and nonlinear Schrödinger equations.* PhD thesis, Karlsruhe Institute of Technology (KIT), 2018. https://doi.org/10.5445/IR/1000088173.

[CHKP19] Chaichenets, Leonid, Dirk Hundertmark, Peer Christian Kunstmann, and Nikolaos Pattakos: *Nonlinear Schrödinger equation, differentiation by parts and modulation spaces.* Journal of Evolution Equations, 2019, ISSN 1424-3202. https://doi.org/10.1007/s00028-019-00501-z.

[CN09] Cordero, Elena and Fabio Nicola: *Sharpness of some properties of Wiener amalgam and modulation spaces.* Bulletin of the Australian Mathematical Society, 80(1):105 – 116, 2009, ISSN 0004-9727. https://doi.org/10.1017/S0004972709000070.

[Fei80] Feichtinger, Hans Georg: *Banach convolution algebras of Wiener type.* Functions, Series, Operators, 35:509 – 524, 1980. https://www.univie.ac.at/nuhag-php/bibtex/open_files/fe83_wientyp1.pdf.

[Fei83] Feichtinger, Hans Georg: *Modulation spaces on locally compact abelian groups.* University Vienna, 1983. https://www.univie.ac.at/nuhag-php/bibtex/open_files/120_ModICWA.pdf.

[Grö01] Gröchenig, Karlheinz: *Foundations of time-frequency analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, 2001, ISBN 978-0-8176-4022-4. https://doi.org/10.1007/978-1-4612-0003-1.

[Guo16] Guo, Shaoming: *On the 1D cubic NLS in an almost critical space.* Journal of Fourier Analysis and Applications, 23(1):91 – 124, 2016, ISSN 1531-5851. https://doi.org/10.1007/s00041-016-9464-z.

[Kun19] Kunstmann, Peer Christian: *Modulation type spaces for generators of polynomially bounded groups and Schrödinger equations.* Semigroup Forum, 2019, ISSN 1432-2137. https://doi.org/10.1007/s00233-019-10016-1.

[Pat18] Pattakos, Nikolaos: *NLS in the modulation space $M_{2,q}(\mathbb{R})$.* Journal of Fourier Analysis and Applications, 2018, ISSN 1531-5851. https://doi.org/10.1007/s00041-018-09655-9.

[RSW12] Ruzhansky, Michael, Mitsuru Sugimoto, and Baoxiang Wang: *Modulation spaces and nonlinear evolution equations.* In *Evolution equations of hyperbolic and Schrödinger type*, volume 301, pages 267–283. Springer, Basel, 2012, ISBN 978-3-0348-0453-0. https://doi.org/10.1007/978-3-0348-0454-7_14.

[RT10] Ruzhansky, Michael Vladimirovich and Ville Turunen: *Pseudo-Differential Operators and Symmetries.* Number 2 in *Pseudo-Differential Operators*. Birkhäuser, Basel, 2010, ISBN 978-3-7643-8513-2. https://doi.org/10.1007/978-3-7643-8514-9.

[STW11] Sugimoto, Mitsuru, Naohito Tomita, and Baoxiang Wang: *Remarks on nonlinear operations on modulation spaces.* Integral Transforms and Special Functions, 22(4 – 5):351 – 358, 2011, ISSN 1065-2469. https://doi.org/10.1080/10652469.2010.541054.

[WH07] Wang, Baoxiang and Henryk Hudzik: *The global Cauchy problem for the NLS and NLKG with small rough data.* Journal of Differential Equations, 232(1):36–73, 2007, ISSN 0022-0396. https://doi.org/10.1016/j.jde.2006.09.004.

[WHHG11] Wang, Baoxiang, Zhaohui Huo, Chengchun Hao, and Zinhua Guo: *Harmonic Analysis Method for Nonlinear Evolution Equations, I.* World Scientific, 2011, ISBN 978-981-4360-73-9. https://doi.org/10.1142/8209.

[WZG06] Wang, Baoxiang, Lifeng Zhao, and Boling Guo: *Isometric decomposition operators, function spaces $E^{\lambda}_{p,q}$ and applications to nonlinear evolution equations.* Journal of Functional Analysis, 233(1):1–39, 2006, ISSN 0022-1236. https://doi.org/10.1016/j.jfa.2005.06.018.

Leonid Chaichenets, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
Email address: `leonid.chaichenets@kit.edu`

Dirk Hundertmark, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
Email address: `dirk.hundertmark@kit.edu`

Peer Christian Kunstmann, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
Email address: `peer.kunstmann@kit.edu`

Nikolaos Pattakos, Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
Email address: `nikolaos.pattakos@kit.edu`