Local well-posedness for the nonlinear Schrödinger equation in the intersection of modulation spaces $M_{p,q}^s(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$

Leonid Chaichenets, Dirk Hundertmark, Peer Kunstmann, Nikolaos Pattakos

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LOCAL WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION IN THE INTERSECTION OF MODULATION SPACES $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$

L. CHAICHENETS, D. HUNDERTMARK, P. KUNSTMANN, AND N. PATTAKOS

Abstract. We introduce a Littlewood-Paley characterization of modulation spaces and use it to give an alternative proof of the algebra property, implicitly contained in [STWi], of the intersection $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$ for $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \geq 0$. We employ this algebra property to show the local well-posedness of the Cauchy problem for the cubic nonlinear Schrödinger equation in the above intersection. This improves [BO09, Theorem 1.1] by Bényi and Okoudjou, where only the case $q = 1$ is considered, and closes a gap in the literature. If $q > 1$ and $s > d \left(1 - \frac{1}{q}\right)$ or if $q = 1$ and $s \geq 0$ then $M^s_{p,q}(\mathbb{R}^d) \hookrightarrow M_{\infty,1}(\mathbb{R}^d)$ and the above intersection is superfluous. For this case we also obtain a new Hölder-type inequality for modulation spaces.

1. Introduction

In this paper we contribute to the general theory of modulation spaces. Modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ were introduced by Feichtinger in [Fei83]. Here, we only briefly recall their definition and refer to Section 2 and the literature mentioned there for more information. Fix a so-called window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The short-time Fourier transform $V_g f$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window $g$ is defined by

$$
(V_g f)(x, \xi) = \frac{1}{(2\pi)^d} \langle f, M_{\xi} S_x g \rangle \quad \forall x, \xi \in \mathbb{R}^d,
$$

where $S_x g(y) = g(y - x)$ denotes the right-shift by $x \in \mathbb{R}^d$, $(M_{\xi} g)(y) = e^{ik \cdot y} g(y)$ the modulation by $\xi \in \mathbb{R}^d$ and $(f, g) = \int_{\mathbb{R}^d} f(x) g(x) dx$ for $f \in L^1_{\text{loc}}(\mathbb{R}^d), g \in \mathcal{S}(\mathbb{R}^d)$. We define

$$
M^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \left| \| f \|_{M^s_{p,q}(\mathbb{R}^d)} < \infty \right. \right\},
$$

where

$$
\| f \|_{M^s_{p,q}(\mathbb{R}^d)} = \left\| \xi \mapsto \langle \xi \rangle^s \| V_g f(\cdot, \xi) \|_p \right\|_q
$$

for $s \in \mathbb{R}$, $p, q \in [1, \infty]$. As usual in the literature, we set $M_{p,q}(\mathbb{R}^d) := M^0_{p,q}(\mathbb{R}^d)$ and often shorten the notation for $M^s_{p,q}(\mathbb{R}^d)$ to $M^s_{p,q}$. It can be shown, that the $M^s_{p,q}(\mathbb{R}^d)$ are Banach spaces and that different choices of the window function $g$ lead to equivalent norms.

To state our first result, let us recall the definition of the Littlewood-Paley decomposition. Consider a smooth radial function $\phi_0 \in C^\infty_c(\mathbb{R}^d)$ with $\phi_0(\xi) = 1$ for
all $|\xi| \leq \frac{1}{2}$ and supp$(\phi_0) \subseteq B_1(0)$. Set $\phi_1 = \phi_0 \left(\frac{1}{2}\right) - \phi_0$ and $\phi_l = \phi_1 \left(\frac{1}{2^{l-1}}\right)$ for all $l \in \mathbb{N}$. The multiplier operators defined by

$$\Delta_l f := \frac{1}{(2\pi)^2} \hat{\phi}_l \ast f = F(\phi_l F f) \quad \forall \in N_0 \forall f \in S'([\mathbb{R}^d])$$

are called dyadic decomposition operators and the sequence $(\Delta_l f)_{l \in N_0}$ is called the Littlewood–Paley decomposition of $f \in S'([\mathbb{R}^d])$. Above, $F$ denotes the usual Fourier transform and $F^{-1}$ its inverse.

Our first result is

**Theorem 1** (Littlewood–Paley characterization). Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Then

$$\|f\| := \left(\sum_{l \in \mathbb{N}_0} \| \Delta_l f \|_{M^{p,q}_s([\mathbb{R}^d])}^q\right)^{\frac{1}{q}} \quad \forall f \in S'([\mathbb{R}^d])$$

defines an equivalent norm on $M^{s,q}_p([\mathbb{R}^d])$. The constants of the norm equivalence depend only on $d$ and $s$.

The above characterization of modulation spaces is new and we shall use it to prove that the intersections $M^{s,q}_p([\mathbb{R}^d]) \cap M^{0,1}_{\infty}([\mathbb{R}^d])$ are Banach $*$-algebras. To state this second result, let us denote by $C_0([\mathbb{R}^d])$ the space of bounded complex-valued continuous functions on $[\mathbb{R}^d]$, where $d \in \mathbb{N}$. We then have

**Theorem 2** (Algebra property). Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \geq 0$. Then $M^{s,q}_p([\mathbb{R}^d]) \cap M^{0,1}_{\infty}([\mathbb{R}^d])$ is a Banach $*$-algebra with respect to pointwise multiplication and complex conjugation. These operations are well-defined due to the embedding $M^{0,1}_{\infty}([\mathbb{R}^d]) \hookrightarrow C_0([\mathbb{R}^d])$. Furthermore, if $q > 1$ and $s > d \left(1 - \frac{1}{q}\right)$ or if $q = 1$, then $M^{s,q}_p([\mathbb{R}^d]) \hookrightarrow M^{0,1}_{\infty}([\mathbb{R}^d])$, so in particular $M^{s,q}_p([\mathbb{R}^d])$ is a Banach $*$-algebra, in that case.

The latter case of Theorem 2 had been observed already in 1983 by Feichtinger in his aforementioned pioneering work on modulation spaces (cf. [Fei83, Proposition 6.9]), where he proves it using a rather abstract approach via Banach convolution triples. The case $q > 1$ and $s \in \left[0, d \left(1 - \frac{1}{q}\right)^{-1}\right]$ seems to be new, at least as a statement. A different proof of Theorem 2 can be given following the idea of proof of [STW11, Proposition 3.2], see [Cha18, Proposition 4.2].

Our third result is a Hölder-type inequality for modulation spaces, which is stated in

**Theorem 3** (Hölder-type inequality). Let $d \in \mathbb{N}$ and $p, p_1, p_2, q, q \in [1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For $q > 1$ let $s > d \left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Then there is a $C > 0$ such that for any $f \in M^{s,q}_{p_1}([\mathbb{R}^d])$ and any $g \in M^{s,q}_{p_2}([\mathbb{R}^d])$ one has

$$\|fg\|_{M^{s,q}_{p_3}([\mathbb{R}^d])} \leq C \|f\|_{M^{s,q}_{p_1}([\mathbb{R}^d])} \|g\|_{M^{s,q}_{p_2}([\mathbb{R}^d])}.$$  

The above pointwise multiplication $fg$ is well-defined due to the embedding formulated in Theorem 2. The constant $C$ does not depend on $p, p_1$ or $p_2$.

Theorem 3 easily generalizes to $m \in \mathbb{N}$ factors and $p, p_1, \ldots, p_m \in (0, \infty]$. Hence, it extends the multilinear estimate from [BO09, Equation 2.4] to the case $q_0 = \ldots = q_m > 1$.

\footnote{For us, a Banach $*$-algebra $X$ is a Banach algebra over $\mathbb{C}$ on which a continuous involution $*$ is defined, i.e. $(x + y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda} x^*$ and $(x^*)^* = x$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$. We neither require $X$ to have a unit nor $C = 1$ in the estimates $\|x \cdot y\| \leq C \|x\| \|y\|$, $\|x^*\| \leq C \|x\|$.}
Here we present a direct proof of Theorem 4 close to the approach found in [WZG06, Corollary 4.2] and involving an application of Theorem 2. For a proof avoiding the Littlewood-Paley characterization see the proof of [Cha13, Theorem 4.3]. A yet another and more abstract proof could be given by invoking [Fei80, Theorem 1.1] for a specific choice of Banach convolution triples.

Lastly, we employ Theorem 2 to study the Cauchy problem for the cubic nonlinear Schrödinger equation (NLS)

\[
\begin{cases}
\frac{\partial u}{\partial t}(x,t) + \Delta u(x,t) \pm |u|^2 u(x,t) = 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\
u(x,0) = u_0(x) & x \in \mathbb{R}^d,
\end{cases}
\]

where the initial data \(u_0\) is in an intersection of modulation spaces \(M^s_{p,q}(\mathbb{R}^d) \cap M^1_{\infty,1}(\mathbb{R}^d)\). We are interested in mild solutions \(u\) of (3), i.e.

\[u \in C([0,T), M^s_{p,q}(\mathbb{R}^d) \cap M^1_{\infty,1}(\mathbb{R}^d))\]

for some \(T > 0\) which satisfy the corresponding integral equation

\[u(\cdot,t) = e^{i\Delta}u_0 \pm i \int_0^t e^{i(t-\tau)\Delta} \left(|u|^2 u(\cdot,\tau)\right) d\tau \quad \forall t \in [0,T].\]

Our last result is stated in

**Theorem 4** (Local well-posedness). Let \(d \in \mathbb{N}, p \in [1,\infty], q \in [1,\infty)\) and \(s \geq 0\). Set \(X = M^s_{p,q}(\mathbb{R}^d) \cap M^1_{\infty,1}(\mathbb{R}^d)\) and \(X(T) = C([0,T], X), X_s(T) = C([0,T], X)\) for any \(T > 0\). Assume that \(u_0 \in X\). Then, there exists a unique maximal mild solution \(u \in X_s(T_*)\) of (3) and the blow-up alternative

\[T_* < \infty \quad \Rightarrow \quad \limsup_{t \to T_*} \|u(\cdot,t)\|_X = \infty\]

holds. Moreover, for any \(T' \in (0,T_*)\) there exists a neighborhood \(V\) of \(u_0\) in \(X\), such that the initial-data-to-solution-map \(V \to X(T')\), \(v_0 \mapsto v\) is Lipschitz continuous.

As already stated in Theorem 2 one has that, if \(q > 1\) and \(s > d \left(1 - \frac{1}{q}\right)\) or if \(q = 1\), then \(M^s_{p,q}(\mathbb{R}^d) \hookrightarrow M^1_{\infty,1}(\mathbb{R}^d)\) and so \(X = M^s_{p,q}(\mathbb{R}^d)\), in that case.

In the case \(q = \infty\) excluded in Theorem 4 the situation is more subtle. Following our proof, one obtains local well-posedness in the larger space

\[L^\infty([0,T), M^s_{p,\infty}(\mathbb{R}^d) \cap M^1_{\infty,1}(\mathbb{R}^d)).\]

The missing continuity in time is due to the properties of the free Schrödinger evolution and we refer to the remarks after Theorem 10.

The precursors of Theorem 4 are [WZG06, Theorem 1.1] by Wang, Zhao and Guo for the space \(M^s_{p,1}(\mathbb{R}^d)\) and [BO09, Theorem 1.1] due to Bényi and Okoudjou for the space \(M^s_{p,1}(\mathbb{R}^d)\) with \(p \in [1,\infty]\) and \(s \geq 0\). In fact, Theorem 4 generalizes [BO09, Theorem 1.1] to \(q \geq 1\): Although our theorem is stated for the cubic nonlinearity, this is for simplicity of the presentation only. The proof allows for an easy generalization to algebraic nonlinearities considered in [BO09], which are of the form

\[f(u) = g(|u|^2)u = \sum_{k=0}^{\infty} c_k |u|^{2k} u,\]

where \(g\) is an entire function. Also, [BO09, Theorems 1.2 and 1.3], which concern the nonlinear wave and the nonlinear Klein-Gordon equation respectively, can be generalized in the same spirit. The reason for this is that the proof of these results is based on the well-known Banach’s contraction principle, on the fact that the free propagator is a \(C_0\)-group, and on the algebra property of the spaces under
with compact support we write
\[C\]

tempered distributions

Notation. We denote generic constants by \(C\). To emphasize on which quantities a constant depends we write e.g. \(C = C(d)\) or \(C = C(d,s)\). Sometimes we omit a positive constant from an inequality by writing “\(\lesssim\)”, e.g. \(A \lesssim d B\) instead of \(A \leq C(d)B\). By \(A \approx B\) we mean \(A \leq B\) and \(B \leq A\). Special constants are \(d \in \mathbb{N}\) for the dimension, \(p, q \in [1,\infty]\) for the Lebesgue exponents and \(s \in \mathbb{R}\) for the regularity exponent. By \(p'\) we mean the dual exponent of \(p\), that is the number satisfying \(\frac{1}{p} + \frac{1}{p'} = 1\).

We denote by \(\mathcal{S}(\mathbb{R}^d)\) the set of Schwartz functions and by \(\mathcal{S}'(\mathbb{R}^d)\) the space of tempered distributions. Furthermore, we denote the Bessel potential spaces or simply \(L^2\)-based Sobolev spaces by \(H^s = H^s(\mathbb{R}^d)\). For the space of smooth functions with compact support we write \(C^\infty_c\). The letters \(f, g, h\) denote either generic functions \(\mathbb{R}^d \to \mathbb{C}\) or generic tempered distributions and \((a_k)_{k \in \mathbb{Z}^d} = (a_k)_k = (a_k), (b_k)_{k \in \mathbb{Z}^d} = (b_k)_k = (b_k)\) denote generic complex-valued sequences. By \(\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}\) we mean the Japanese bracket.

For a Banach space \(X\) we write \(X^*\) for its dual and \(\| \cdot \|_X\) for the norm it is canonically equipped with. By \(\mathcal{L}(X,Y)\) we denote the space of all bounded linear maps from \(X\) to \(Y\), where \(Y\) is another Banach space, and set \(\mathcal{L}_b(X,Y) = \mathcal{L}(X,Y,\mathbb{C})\). By \([X,Y]_\theta\) we mean complex interpolation between \(X\) and \(Y\), if \((X,Y)\) is an interpolation couple. For brevity we write \(\| \cdot \|_p\) for the \(p\)-norm on the Lebesgue space \(L^p = L^p(\mathbb{R}^d)\), the sequence space \(l^p = l^p(\mathbb{Z}^d)\) or \(l^p = l^p(\mathbb{N})\) and \(\| (a_k) \|_{q,s} := \|(k)^s a_k\|_q\) for the norm on \(\langle \cdot \rangle^s\)-weighted sequence spaces \(l^p_s = l^p_s(\mathbb{Z}^d)\). If the norm is apparent from the context, we write \(B_r(x)\) for a ball of radius \(r\) around \(x \in X\).

We use the symmetric choice of constants for the Fourier transform and also write
\[
\hat{f}(\xi) := (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx,
\]
\[
g(x) := (\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} g(\xi) d\xi.
\]

2. Preliminaries

As already mentioned in the introduction, modulation spaces were introduced by Feichtinger in [F83] in the setting of locally compact Abelian groups. A thorough introduction is given in the textbook [G01] by Gröchenig. A presentation
Proof. (1) One can change indices one by one. The inclusion for “\(q\) is by
monotonicity and the inclusion for “\(q\) is by the embeddings of the \(l^p\)
spaces. For the \(p\)-embedding consider \(\tau \in C_0^\infty(\mathbb{R}^d)\) such that \(\tau|_{B_1} \equiv 1\) and
\(\text{supp}(\tau) \subseteq B_d\). For every \(k \in \mathbb{Z}^d\), consider the shifted symbol \(\tau_k = S_k\tau\),
define the corresponding multiplier operator \(\tilde{\tau}_k = F(-1)\tau_k F\) and observe,
that \(\tilde{\tau}_k = M_k \tau\). Hence, by Lemma 5 the family \((\tilde{\tau}_k)_{k \in \mathbb{Z}^d}\) is bounded
in \(\mathcal{L}(L^p(\mathbb{R}^d), L^{p_2}(\mathbb{R}^d))\). So, \(\|\tilde{\tau}_k f\|_{L^{p_2}} \leq \|\tilde{\tau}_k\|_{L^{p_1}(\mathbb{R}^d)} \|\tau\|_{L^{p_2}(\mathbb{R}^d)}\) for any
\(k \in \mathbb{Z}^d\). Recalling (6) completes the argument.

(2) \(M_{p_1,q_1}(\mathbb{R}^d) \subseteq M_{p_2,q_2}(\mathbb{R}^d)\) and the embedding is continuous,
and \(M_{p_1,q}(\mathbb{R}^d) \subseteq M_{p_1,1}(\mathbb{R}^d)\) and the embedding is continuous,
and \(M_{p_1,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)\).

Lemma 6 is well-known (cf. WH07 Proposition 2.5, 2.7), but for convenience
we sketch a

Proof. (1) One can change indices one by one. The inclusion for “\(s\) is by
monotonicity and the inclusion for “\(q\) is by the embeddings of the \(l^p\)
spaces. For the \(p\)-embedding consider \(\tau \in C_0^\infty(\mathbb{R}^d)\) such that \(\tau|_{B_1} \equiv 1\) and
\(\text{supp}(\tau) \subseteq B_d\). For every \(k \in \mathbb{Z}^d\), consider the shifted symbol \(\tau_k = S_k\tau\),
define the corresponding multiplier operator \(\tilde{\tau}_k = F(-1)\tau_k F\) and observe,
that \(\tilde{\tau}_k = M_k \tau\). Hence, by Lemma 5 the family \((\tilde{\tau}_k)_{k \in \mathbb{Z}^d}\) is bounded
in \(\mathcal{L}(L^p(\mathbb{R}^d), L^{p_2}(\mathbb{R}^d))\). So, \(\|\tilde{\tau}_k f\|_{L^{p_2}} \leq \|\tilde{\tau}_k\|_{L^{p_1}(\mathbb{R}^d)} \|\tau\|_{L^{p_2}(\mathbb{R}^d)}\) for any
\(k \in \mathbb{Z}^d\). Recalling (6) completes the argument.
Indeed, for any $A$ as the supports of the partition of unity are compact, many summands vanish.

Proof. We use (6) to estimate the modulation space norm of the left-hand side. Fix $k$ we have

$$\|\sigma_k f\|_{p,1} \approx \sum_{k \in \mathbb{Z}^d} \|\Delta_k f\|_{p,1} \leq \left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-sq'} \right)^{\frac{1}{p'}} \left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{sq} \|\Delta_k f\|_p^q \right)^{\frac{1}{q}}$$

and the first factor is finite for $s > \frac{d}{q}$ by comparison with the integral $\int_{\mathbb{R}^d} (x)^{-sq} \, dx$.

(3) By H"older’s inequality we immediately have

$$\|f\|_{p,1} \approx \|\sigma_k f\|_{p,1} \leq \left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-sq'} \right)^{\frac{1}{p'}} \|f\|_{M^{p,q}}.$$ 

For the proof of Theorem 2 we will need the following (cf. [BO09, eqn. (2.4)])

Lemma 7 (Bilinear estimate). Let $d \in \mathbb{N}$ and $1 \leq p \leq \infty$. Assume $f \in M_{p,q}(\mathbb{R}^d)$ and $g \in M_{\infty,1}(\mathbb{R}^d)$. Then

$$\|fg\|_{M_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{M_{p,q}(\mathbb{R}^d)} \|g\|_{M_{\infty,1}(\mathbb{R}^d)},$$

where the implicit constant does not depend on $p$ or $q$.

For convenience, and because we will generalize Lemma 7 to Theorem 3, we present a proof close to the one of [WZ06, Corollary 4.2].

Proof. We use (6) to estimate the modulation space norm of the left-hand side. Fix $k \in \mathbb{Z}^d$. By the definition of the operator $\Delta_k$ we have

$$\Delta_k(fg) = \frac{1}{(2\pi)^d} \mathcal{F}^{(-1)} \left( \sigma_k (f \ast \hat{g}) \right) = \frac{1}{(2\pi)^d} \sum_{l,m \in \mathbb{Z}^d} \mathcal{F}^{(-1)} \left( \sigma_k ((\sigma_l f) \ast (\sigma_m \hat{g})) \right).$$

As the supports of the partition of unity are compact, many summands vanish. Indeed, for any $k,l,m \in \mathbb{Z}^d$

$$\text{supp} \left( \sigma_k ((\sigma_l f) \ast (\sigma_m \hat{g})) \right) \subseteq \text{supp}(\sigma_k) \cap (\text{supp}(\sigma_l) + \text{supp}(\sigma_m)) \subseteq B_{\sqrt{d}}(k) \cap B_{2\sqrt{d}}(l+m)$$

and so $\sigma_k ((\sigma_l f) \ast (\sigma_m \hat{g})) \equiv 0$ if $|l-m| > 3\sqrt{d}$. Hence, the double series over $l,m \in \mathbb{Z}^d$ boils down to a finite sum of discrete convolutions

$$\Delta_k(fg) = \frac{1}{(2\pi)^d} \mathcal{F}^{(-1)} \left( \sigma_k \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} (\sigma_l f) \ast (\sigma_{k-l+m} \hat{g}) \right)$$

$$= \Delta_k \left( \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} (\mathcal{F} f) \cdot (\mathcal{F} \Delta_{k+m-l} g) \right),$$

where $M = \left\{ m \in \mathbb{Z}^d \mid |m| \leq 3\sqrt{d} \right\}$ and $\#M \leq \left( 6\sqrt{d} + 1 \right)^d < \infty$. That was the job of $\Delta_k$ and we now get rid of it,

$$\|\Delta_k(fg)\|_p \lesssim \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} \|\mathcal{F} f \cdot (\mathcal{F} \Delta_{k+m-l} g)\|_p.$$
using the Bernstein multiplier estimate from Lemma [4].

Invoking Hölder’s inequality we further estimate
\[
\|\square_k(fg)\|_p \leq \sum_{m \in M} \left( \left( \|\square_l(f)\|_p \right)^* \left( \|\square_{n+m}(g)\|_\infty \right)_n \right) (k)
\]
pointwise in \( k \in \mathbb{Z}^d \), where * denotes the convolution of sequences, and hence obtain
\[
\|fg\|_{M_{p,q}} \leq \left\| \left( \|\square_l(f)\|_p \right)^* \left( \|\square_{n}(g)\|_\infty \right)_n \right\|_1
\]
by Young’s inequality. \( \square \)

Lemma 8 (Dual space). For \( s \in \mathbb{R} \), \( p, q \in [1, \infty) \) we have
\[
(M_{p,q}^s(\mathbb{R}^d))^* = M_{p,q}^s(\mathbb{R}^d)
\]
(see [WH07] Theorem 3.1).

Theorem 9 (Complex interpolation). For \( p_1, q_1 \in [1, \infty) \), \( p_2, q_2 \in [1, \infty) \), \( s_1, s_2 \in \mathbb{R} \) and \( \theta \in (0, 1) \) one has
\[
[M_{p_1,q_1}^{s_1}(\mathbb{R}^d), M_{p_2,q_2}^{s_2}(\mathbb{R}^d)]_{\theta} = M_{p,q}^s(\mathbb{R}^d),
\]
with
\[
\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2
\]
(see [Fei83] Theorem 6.1(D)).

We are now ready to state and prove the following

Theorem 10 (Schrödinger propagator bound). There is a constant \( C > 0 \) such that for any \( d \in \mathbb{N} \), \( p, q \in [1, \infty] \) and \( s \in \mathbb{R} \) the inequality
\[
\|e^{it\Delta}\|_{L^p(M_{p,q}^s(\mathbb{R}^d))} \leq C_d(1 + |t|)^{\frac{d}{2} - \frac{s}{2}}
\]
holds for all \( t \in \mathbb{R} \). Furthermore, the exponent of the time dependence is sharp.

The boundedness has been obtained e.g. in [BGOR07] Theorem 1 whereas the sharpness was proven in [CN09] Proposition 4.1. If \( q < \infty \), then \( (e^{it\Delta})_{t \in \mathbb{R}} \) is a \( C_0 \)-group on \( M_{p,q}^s \), i.e.
\[
\lim_{t \to 0} \|e^{it\Delta}f - f\|_{M_{p,q}^s} = 0 \quad \forall f \in M_{p,q}^s
\]
(see e.g. [Cha18] Proposition 3.5). This is not true for \( q = \infty \) and we refer to [Kum19] for this more subtle case.

Theorem 11. By definition, we have
\[
\|V_0 e^{it\Delta}f(x, \xi)\|_p = e^{-|t|\xi^2} \|V_0 \Delta g(x + 2it\xi, \xi)\|
\]
for any \( f \in \mathcal{S}(\mathbb{R}^d) \), any \( (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \), and any \( t \in \mathbb{R} \), i.e., the Schrödinger time evolution of the initial data can be interpreted as the time evolution of the window function. The price for changing from window \( g_0 \) to window \( g_1 \) is \( \|V_0 g_1\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} \) by [Grö01] Proposition 11.3.2 (c)). For \( g(x) = e^{-|x|^2} \) one explicitly calculates
\[
\|V_0 e^{-|t|\Delta}g\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = C_d (1 + |t|)^{\frac{d}{2}},
\]
which proves the claimed bound for \( p \in [1, \infty) \). Conservation for \( p = 2 \) is easily seen from [4]. Complex interpolation between the cases \( p = 2 \) and \( p = \infty \) yields \( [5] \) for \( p \in [2, \infty] \). The remaining case \( p \in (1, 2] \) is covered by duality.

Optimality in the case \( p \in [1, 2] \) is proven by choosing the window \( g \) and the argument \( f \) to be a Gaussian and explicitly calculating \( \|e^{it\Delta}f\|_{M_{p,q}^s} \approx (1 + |t|)^d \left( \frac{1}{2} - \frac{s}{2} \right) \). This implies the optimality for \( p \in (2, \infty] \) by duality. \( \square \)
3. Littlewood-Paley theory

In this section we extend some ideas of the Littlewood-Paley decomposition from Sobolev spaces $H^s(\mathbb{R}^d)$ to modulation spaces $M^s_{p,q}(\mathbb{R}^d)$. The inspiration for this was Chapter II.

Observe, that for any $\xi \in \mathbb{R}^d$ one has
\[ \sum_{l=0}^{\infty} \phi_l(\xi) = \phi_0(\xi) + \lim_{N \to \infty} \sum_{l=1}^{N} \left[ \phi_l \left( \frac{\xi}{2^l} \right) - \phi_1 \left( \frac{\xi}{2^{l-1}} \right) \right] = \lim_{N \to \infty} \phi_0 \left( \frac{\xi}{2^N} \right) = 1, \]
\text{i.e. } \{\phi_0, \phi_1, \phi_2, \ldots\} \text{ is a smooth partition of unity. Moreover, supp}(\phi_l) \subseteq A_l \text{ for any } l \in \mathbb{N}, \text{ where}
\[ A_0 := \{ \xi \in \mathbb{R}^d | |\xi| \leq 1 \} \quad \text{and} \quad A_l := \{ \xi \in \mathbb{R}^d | 2^{l-2} \leq |\xi| \leq 2^l \} \quad \forall l \in \mathbb{N}. \]
The symbols of the dyadic decomposition operators satisfy
\[ \| \hat{\phi}_l \|_1 = \left\| \mathcal{F} \left[ \phi_l \left( \frac{\xi}{2^{l-1}} \right) \right] \right\|_1 = \left\| 2^{l-1} \hat{\phi}_l \left( 2^{l-2} \xi \right) \right\|_1 = \left\| \hat{\phi}_1 \right\|_1 \leq 2 \left\| \hat{\phi}_0 \right\|_1 \]
for all $l \in \mathbb{N}$. Applying Lemma 3 shows that for any $l \in \mathbb{N}_0$ and any $f \in S'(\mathbb{R}^d)$ one has that $\Delta_l f \in C^\infty$ and any of its derivates has at most polynomial growth. Furthermore, $\| \Delta_l \|_{L^p(L^q(\mathbb{R}^d))}$ is bounded independently of $l \in \mathbb{N}_0$ and $p \in [1, \infty]$.

**Theorem 7** We start by gathering some useful facts. Fix $l \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$. Recall, that supp($\phi_l$) $\subseteq A_l$ and supp($\sigma_k$) $\subseteq B_{\sqrt{d}}(k)$. Hence,
\[ (9) \quad \square_k \Delta_l \neq 0 \Rightarrow k \in A_l^c := \left\{ k' \in \mathbb{Z}^d | 2^{l-2} - \sqrt{d} \leq |k'| \leq 2^l + \sqrt{d} \right\}. \]

On $A_l^c$ the Japanese bracket can be controlled. In fact, for all $t \in \mathbb{R}$ we have
\[ (10) \quad \langle k \rangle^t \approx 2^t, \]
where the implicit constant does not depend on $l$.

Finally, observe that $k \in A_l^c$ is satisfied for only finitely many $l \in \mathbb{N}_0$, whose number is independent of $k \in \mathbb{Z}^d$, i.e.
\[ (11) \quad \sum_{l=0}^{\infty} \mathbb{1}_{A_l^c}(k) \lesssim 1, \]
where the implicit constant depends on $d$ only.

- $\geq$: Consider $q < \infty$ first. By (6), (9), Bernstein multiplier estimate, (10) and (11) we have
\[ \left\| \left( 2^{ls} \| \Delta_l f \|_{M^s_{p,q}} \right) \right\|_q \approx \left( \sum_{l=0}^{\infty} 2^{lsq} \left\| \square_k \Delta_l f \right\|_q^q \right)^{\frac{1}{q}} \lesssim \left( \sum_{l=0}^{\infty} 2^{lsq} \left\| \square_k f \right\|_p^q \right)^{\frac{1}{q}} \approx \| f \|_{M^s_{p,q}}. \]

Similarly, for $q = \infty$, we have
\[ \left\| \left( 2^{ls} \| \Delta_l f \|_{M^s_{p,\infty}} \right) \right\|_\infty \approx \sup_{l \in \mathbb{N}_0} \left( \sup_{k \in \mathbb{Z}^d} 2^{ls} \| \square_k \Delta_l f \|_p \right) \lesssim \sup_{l \in \mathbb{N}_0} \left( \sup_{k \in A_l^c} \langle k \rangle^s \| \square_k f \|_p \right) \approx \| f \|_{M^s_{p,\infty}}. \]
\[ LWP\ \text{for the NLS in } M_{p,q}^* \cap M_{\infty,1} \]

- \[ \lesssim \]: Again, consider \( q < \infty \) first. By (6), \( f = \sum_{l=0}^{\infty} \Delta_l f \) in \( S' \) and (9) we have
  \[ \|f\|_{M_{p,q}^*} \lesssim \left( \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \left( \sum_{l=0}^{\infty} \left\| \square_k \Delta_l f \right\|_p^q \right)^{\frac{1}{q}} \right)^{\frac{1}{p'}} \]

  \[ \lesssim \left( \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \left( \sum_{l=0}^{\infty} \lambda_{A_l}(k) \left\| \square_k \Delta_l f \right\|_p^q \right)^{\frac{1}{q}} \right)^{\frac{1}{p'}} \]

  For each \( k \in \mathbb{Z}^d \) the sum over \( l \) contains only finitely many non-vanishing summands and their number is independent of \( k \) by (11). Hölder’s inequality estimates the last term against

  \[ \left( \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{qs} \left( \sum_{l=0}^{\infty} \lambda_{A_l}(k) \left\| \square_k \Delta_l f \right\|_p^q \right)^{\frac{1}{q}} \right)^{\frac{1}{p'}} \lesssim \left( \sum_{l=0}^{\infty} \| \Delta_l f \|_{M_{p,q}^*} \right)^{\frac{1}{q'}} \]

  where we additionally used (10). The proof for \( q = \infty \) is along the same lines.

\[ \square \]

The individual parts of the Littlewood-Paley decomposition had their Fourier transform supported in almost disjoint dyadic annuli. Theorem 1 characterized elements of modulation spaces by the decay of these parts. The following lemma provides a sufficient condition for the case of non-disjoint balls.

**Lemma 11** (Sufficient condition). Let \( 1 \leq q \leq \infty \) and \( s > 0 \). For \( m \in \mathbb{N}_0 \) let \( f_m \in S'([\mathbb{R}^d]) \) be such that

\[ \text{supp}(\hat{f}_m) \subseteq B_m := \{ \xi \in \mathbb{R}^d \mid |\xi| \leq 2^m \} \quad \forall m \in \mathbb{N}_0. \]

Set \( f := \sum_{m=0}^{\infty} f_m \) in \( S'([\mathbb{R}^d]) \). Then

\[ \|f\|_{M_{p,q}^*([\mathbb{R}^d])} \lesssim \left\| \left( 2^{ms} \| f_m \|_{M_{p,q}([\mathbb{R}^d])} \right)_{m \in \mathbb{N}_0} \right\|_q, \]

where the implicit constant depends on \( d \) and \( s \) only.

**Proof.** Observe, that \( A_l \cap B_m = \emptyset \) if \( l > m + 2 \). Hence, we have

\[ \|f\|_{M_{p,q}^*} \approx \left\| \left( 2^{ls} \| \Delta_l f \|_{M_{p,q}^*} \right)_{l \in \mathbb{N}_0} \right\|_q \lesssim \left\| \left( 2^{ls} \sum_{m=l}^{\infty} \| \Delta_l f_m \|_{M_{p,q}^*} \right)_{l \in \mathbb{N}_0} \right\|_q \]

where we additionally used Theorem 1 and Bernstein multiplier estimate. From now on, we assume \( q \in (1, \infty) \), as the proof for the other cases is easier and follows the same lines. Hölder’s inequality and geometric sum formula estimates the last
term against
\[
\left( \sum_{l=0}^{\infty} \left( \sum_{m=l}^{\infty} 2^{ls} \|f_m\|_{M_{p,q}} \right)^q \right)^{\frac{1}{q}}
\]
\[
= \left( \sum_{l=0}^{\infty} \left( \sum_{m=l}^{\infty} 2^{l(l-m)s} \times 2^{(l-m)s} 2^{ms} \|f_m\|_{M_{p,q}} \right)^q \right)^{\frac{1}{q}}
\]
\[
\leq \left( \sum_{l=0}^{\infty} \left( \sum_{m=l}^{\infty} 2^{l(l-m)s} 2^{ms} \|f_m\|_{M_{p,q}} \right) \right)^{\frac{1}{q}}
\]
\[
\approx \left( \sum_{m=0}^{m} \sum_{l=0}^{m} 2^{l(l-m)s} 2^{ms} \|f_m\|_{M_{p,q}} \right)^{\frac{1}{q}}
\]
\[
\approx \left( ||2^{ms} \|f_m\|_{M_{p,q}}\|_m \right)\|_q ,
\]
finishing the proof. \[\square\]

4. Algebra property and Hölder-type inequality

Main goal of this section is to prove Theorem 2, which was inspired by the fact that $H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a Banach *-algebra with respect to pointwise multiplication for $s \geq 0$.

**Theorem 2** Parts 2 and 3 of Lemma 6 prove the claimed embedding. Continuity of complex conjugation is obvious from (6). Continuity of multiplication follows by the paraproduct argument

\[
fg = \left( \sum_{l=0}^{\infty} \Delta_l f \right) \left( \sum_{m=0}^{\infty} \Delta_m g \right) = \sum_{l=0}^{\infty} \Delta_l f \sum_{m=0}^{l} \Delta_m g + \sum_{m=1}^{\infty} \Delta_m g \sum_{l=0}^{m-1} \Delta_l f.
\]

Observe, that for any $l, m \in \mathbb{N}_0$ we have supp($\hat{u}_l$) \subseteq $B_{l+1}$ and supp($\hat{v}_m$) \subseteq $B_m$ by the properties of convolution. Hence, Lemma 11 could be applied. Bilinear estimate from Lemma 7 and Theorem 1 show

\[
\left\| \left( 2^{ls} \|u\|_{M_{p,q}} \right) \right\|_q \leq \left\| \left( 2^{ls} \|\Delta_l f\|_{M_{p,q}} \right) \right\|_q \sum_{l=0}^{\infty} ||\Delta_m g||_{M_{\infty,1}} \approx ||f||_{M_{p,q}} \|g\|_{M_{\infty,1}}.
\]

The same argument yields $||\sum_{m=1}^{\infty} v_m ||_{M_{p,q}} \leq ||f||_{M_{\infty,1}} \|g\|_{M_{p,q}}$ and finishes the proof. \[\square\]

The analogon of Theorem 2 for sequence spaces is stated in

**Lemma 12** (Algebra property). Let $1 \leq q \leq \infty$ and $s \geq 0$. Then $l^q_2(\mathbb{Z}^d) \cap l^1(\mathbb{Z}^d)$ is a Banach algebra with respect to convolution

\[
(a_l) * (b_m) = \left( \sum_{m \in \mathbb{Z}^d} a_{k-m} b_m \right),
\]

which is well-defined, as the series above always converge absolutely.

Furthermore, if $q > 1$ and $s > d \left( 1 - \frac{1}{q} \right)$ or $q = 1$, then $l^q_2(\mathbb{Z}^d) \hookrightarrow l^1(\mathbb{Z}^d)$, so in particular $l^1_2(\mathbb{Z}^d)$ is a Banach algebra, in that case.
Theorem 3. We arrive, as for equation (7) in the proof of Lemma 7, at

$$p \approx \| \sum_{k \in \mathbb{Z}^d} a_k e^{i k x} \| \approx \| a_k \|_{M_{\infty,q}^\ast}$$

and hence, by Theorem 2, one has

$$\| (a_k) * (b_k) \|_{M_{\infty,q}^\ast} \approx \left( \sum_{k \in \mathbb{Z}^d} a_k e^{i k x} \right) \cdot \left( \sum_{k \in \mathbb{Z}^d} b_k e^{i k x} \right)$$

and the first factor is already

$$\approx \| (a_k) \|_{M_{\infty,q}^\ast} \quad \text{by Peetre’s inequality}$$

Second factor

$$\sum_{k \in \mathbb{Z}^d} a_k e^{i k x} \sum_{k \in \mathbb{Z}^d} b_k e^{i k x} \sum_{k \in \mathbb{Z}^d} a_k e^{i k x} \sum_{k \in \mathbb{Z}^d} b_k e^{i k x}$$

$$\approx \| (a_k) \|_{M_{\infty,q}^\ast} \| (b_k) \|_1 + \| (a_k) \|_1 \| (b_k) \|_{M_{\infty,q}^\ast}.$$

We are now ready to give a

**Theorem 3** We arrive, as for equation (7) in the proof of Lemma 7 at

$$\| \Box_k (f g) \| \approx \sum_{m \in M} \left( \left( \Box_l (f) \right) \right)_l + \left( \left( \Box_n (g) \right) \right)_n$$

pointwise in $k \in \mathbb{Z}^d$. By the algebra property from Lemma 12, it follows that

$$\| f g \|_{M_{p,q}^\ast} \lesssim \left( \left( \Box_l f \right) \right)_l \left( \sum_{m \in M} \left( \left( \Box_n (g) \right) \right)_n \right)$$

and the first factor is already $\| f \|_{M_{p,q}^\ast}$. Finally, we remove the sum over $m$ in the second factor

$$\sum_{m \in M} \left( \left( \Box_n (g) \right) \right)_n \lesssim \| g \|_{M_{p,q}^\ast}$$

applying Peetre’s inequality $\langle k + l \rangle \lesssim 2^{|s|} \langle k \rangle^s |t|^s$ (see e.g. [RT10, Proposition 3.3.31]).

Let us finish the proof remarking that the only estimate involving “$p$” we used was Hölder’s inequality and thus the implicit constant indeed does not depend on $p$, $p_1$ or $p_2$.

\[ \square \]

5. **Proof of the local well-posedness, Theorem 4**

Theorem 2 immediately implies that $X(T)$ is a Banach *-algebra, i.e.

$$\| u v \|_{X(T)} = \sup_{0 \leq t \leq T} \| u v (\cdot, t) \|_{X} \lesssim \left( \sup_{0 \leq s \leq T} \| u (\cdot, s) \|_{X} \right) \left( \sup_{0 \leq t \leq T} \| v (\cdot, t) \|_{X} \right)$$

and

$$= \| u \|_{X(T)} \| v \|_{X(T)}.$$
To that end, let us observe that Theorem 10 implies the homogeneous estimate
\[ \|t \mapsto e^{t\Delta} v\|_X \leq C_0 (1 + T)^{\frac{3}{2}} \|v\|_X \quad (\forall v \in X), \]
which, together with the algebra property of \(X(T)\), proves the inhomogeneous estimate
\[ \left\| \int_0^t e^{(t-\tau)\Delta} \left( |u|^2 u(\cdot, \tau) \right) \, d\tau \right\|_X \leq C_0 (1 + T)^{\frac{3}{2}} \|v\|_X, \]
holding for \(0 \leq t \leq T\) and \(u \in X(T)\).

Applying the triangle inequality in (13) yields
\[ \|Tu\|_X \leq C_0 (1 + T)^{\frac{3}{2}} (\|u_0\|_X + C_1 T R^3) \]
for any \(u \in M(R, T)\). Thus, \(T\) maps \(M(R, T)\) into itself for \(R = 2C_0C_1\|u_0\|_X\) and \(T\) small enough. Furthermore,
\[ |u|^2 - |v|^2 v = (u - v) |u|^2 + (\overline{u} - \overline{v}) v = (u - v) (|u|^2 + \overline{u} v) + (\overline{u} - \overline{v}) v^2 \]
and hence
\[ \|Tu - Tv\|_{X(T)} \leq T (1 + T)^{\frac{3}{2}} R^2 \|u - v\|_{X(T)} \]
for \(u, v \in M(R, T)\), where we additionally used the algebra property of \(X(T)\) and the homogeneous estimate. Taking \(T\) sufficiently small makes \(T\) a contraction.

Banach’s fixed-point theorem implies the existence and uniqueness of a mild solution up to the guaranteed time of existence \(T_0 = T_0 (\|u_0\|_X) \approx \|u_0\|_X^{-2} > 0\). Uniqueness of the maximal solution and the blow-up alternative now follow easily by the usual contradiction argument.

For the proof of the Lipschitz continuity, let us notice that for any \(r > \|u_0\|_X\), \(v_0 \in B_r(0)\) and \(0 < T \leq T_0(r)\) we have
\[ \|u - v\|_{X(T)} = \|T(u_0)u - T(v_0)v\|_{X(T)} \leq (1 + T)^{\frac{3}{2}} \|u_0 - v_0\|_X + T (1 + T)^{\frac{3}{2}} R^2 \|u - v\|_{X(T)}, \]
where \(v\) is the mild solution corresponding to the initial data \(v_0\) and \(R = 2Cr\), similar to the above. Collecting terms containing \(\|u - v\|_{X(T)}\) shows Lipschitz continuity with constant \(L = L(r)\) for sufficiently small \(T\), say \(T_1 = T_1 (r)\). For arbitrary \(0 < T' < T_1\), put \(r = 2 \|u\|_{X(T')}\) and divide \([0, T']\) into \(n\) subintervals of length \(\leq T_1\). The claim follows for \(V = B_3(u_0)\) where \(\delta = \frac{\|u_0\|_X}{2}\) by iteration. This concludes the proof.

\[ \square \]

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**References**


