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# THRESHOLD FOR BLOWUP FOR THE SUPERCRITICAL CUBIC WAVE EQUATION

IRFAN GLOGIĆ, MACIEJ MALIBORSKI, AND BIRGIT SCHÖRKHUBER

ABSTRACT. In this paper, we discuss singularity formation for the focusing cubic wave equation in the energy supercritical regime. For this equation an *explicit* nontrivial self-similar blowup solution was recently found by the first and third author in [27]. In the seven dimensional case it was proven to be stable along a co-dimension one manifold of initial data. Here, we provide numerical evidence that this solution is in fact a critical solution at the threshold between finite-time blowup and dispersion. Furthermore, we discuss the spectral problem arising in the stability analysis in general dimensions  $d \geq 5$ .

## 1. INTRODUCTION

We consider the focusing cubic wave equation

$$(\partial_t^2 - \Delta)u(t, x) = u(t, x)^3 \tag{1.1}$$

$(t, x) \in I \times \mathbb{R}^d$ , where  $I \subset \mathbb{R}$  is an interval containing zero. The model is invariant under rescaling  $u \mapsto u_\lambda$

$$u_\lambda(t, x) = \lambda^{-1}u(t/\lambda, x/\lambda), \quad \lambda > 0.$$

The corresponding scale invariant Sobolev space for  $(u(t, \cdot), \partial_t u(t, \cdot))$  is  $\dot{H}^{s_c} \times \dot{H}^{s_c-1}(\mathbb{R}^d)$ ,  $s_c = \frac{d}{2} - 1$ . Note that  $s_c = 1$  for  $d = 4$  and in this case, Eq. (1.1) is referred to as energy critical. The supercritical case thus corresponds to  $d \geq 5$ .

In the recent work [27] by the first and the third author it was found that

$$u_T^*(t, x) = \frac{1}{T-t} U^* \left( \frac{|x|}{T-t} \right), \quad U^*(\rho) = \frac{2\sqrt{2(d-1)(d-4)}}{d-4+3\rho^2} \tag{1.2}$$

solves the cubic wave equation (1.1) for  $d \geq 5$ . In  $d = 7$ , the solution is proven to be co-dimension one stable. More precisely, there exists a co-dimension one Lipschitz manifold of initial data in a small neighbourhood of  $u_T^*$ , whose solutions blow up in finite time and converge asymptotically to  $u_T^*$  (modulo space-time shifts and Lorentz boosts) in the backward lightcone

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of the blowup point. We conjecture that this manifold is a threshold between dispersion and finite-time blowup along a stable mechanism.

In this paper, we investigate this problem numerically in the radial case. The findings support our conjecture and show that  $u_T^*$  is indeed a critical solution sitting at the threshold for blowup. Thus, it provides the first explicit example of that kind for a supercritical wave equation.

The paper is organized as follows: In Section 1.1 we give a short overview on known blowup dynamics for Eq. (1.1). The results of [27] for  $d = 7$  are reviewed in Section 2, where we restrict the discussion to the radial case. In Section 2.1 we address in more detail the spectral problem corresponding to the linearization around  $U^*$  for general  $d \geq 5$ . In Section 3 we present numerical experiments in different space dimensions that establish the role of  $U^*$  as a threshold for blowup. The numerical technique that is used here has been developed by Biernat, Bizoń and the second author in [1].

**1.1. Known results on blowup dynamics.** It is well-known that the focusing wave equation admits solutions that blow up in finite time  $T < \infty$  from smooth, compactly supported initial data.

One is therefore naturally interested in the details of the blowup dynamics. Numerical experiments performed by Bizoń, Chmaj and Tabor in [9] suggest that locally the behavior of *generic* blowup solutions is driven by the so-called ODE blowup, which for the cubic wave equation is given by

$$u_T(t, x) = \frac{\sqrt{2}}{T-t}, \quad T > 0. \quad (1.3)$$

The stability of this solution has been established in the radial setting by Donninger and the third author [20], [21] for all odd  $d \geq 3$ , see also Donninger and Chatzikaleas [13] for a generalization to the non-radial case in  $d = 5$ .

These results prove the existence of an open set of initial data in a suitable Sobolev space with regularity  $k > s_c$  such that the corresponding solutions blow up in finite time and converge to  $u_T$  (modulo symmetries) in the backward lightcone of the blowup point

$$\mathcal{C}_{T,x_0} := \{(t, x) \in \mathbb{R}^d : |x - x_0| \leq T - t, t \in [0, T]\}. \quad (1.4)$$

In the subcritical case  $d = 3$ , the seminal work of Merle and Zaag [45] proves that the *blowup rate* of generic blowup solutions is described by (1.3). This result however does not provide information on the generic *blowup profile*. In fact, from [2] it is known that there are infinitely many self-similar blowup solutions of the form

$$u(t, x) = (T - t)^{-1} U_n \left( \frac{|x|}{T-t} \right) \quad (1.5)$$

with radial profiles  $\{U_n\}_{n \in \mathbb{N}_0}$ , where  $U_0 = \sqrt{2}$  corresponds to the ODE blowup. For  $n \geq 1$ , these profiles are smooth within a ball of radius one and become singular somewhere outside. Nevertheless, one can exploit the finite speed of propagation and use cut-off functions to produce smooth and

compactly supported initial data that lead to blow up at a self-similar rate with a profile different from the trivial one inside the backward lightcone of the blowup point. We note that in three dimensions, Eq. (1.1) is invariant under conformal inversion  $u \mapsto \tilde{u}$ ,

$$\tilde{u}(t, x) = (t^2 - |x|^2)^{-\frac{d-1}{2}} u \left( \frac{t}{t^2 - |x|^2}, \frac{x}{t^2 - |x|^2} \right). \quad (1.6)$$

Numerical experiments by Bizoń and Zenginoglu [11] indicate that in this case the excited self-similar profiles do not seem to play a role in the generic time-evolution. Instead, a conformally transformed version of the ODE blowup solution (1.3) governs the dynamics and also describes the borderline between blowup and dispersion.

In  $d = 4$ , non-trivial self-similar solutions can be excluded, see [31], [9], [43]. Instead, so-called type II blowup solutions occur. These solutions blow up by concentrating the *static* ground state solution  $Q$ , which is the unique radial, positive solution to the elliptic problem and known explicitly in the energy critical case. For the focusing wave equation such solutions have been constructed first by Krieger, Schlag and Tataru [42] for a quintic nonlinearity in  $d = 3$ , using their methods developed for wave maps [34], see also [41], [19], [18] and in particular [35], [12] for the stability of these solutions.

For the cubic wave equation, following the different approach of [47], [46], Hillairet and Raphaël constructed in [29] *smooth* type II solutions to Eq. (1.1) that behave like

$$u(t, x) \sim \lambda(t)^{-1} Q \left( \frac{|x|}{\lambda(t)} \right), \quad (1.7)$$

with  $\lambda(t) \sim (T - t)e^{-\sqrt{|\log(T-t)|}}$  as  $t \rightarrow T^-$  and

$$Q(r) = \left(1 + \frac{r^2}{8}\right)^{-1}. \quad (1.8)$$

Furthermore, these solutions are stable along a co-dimension one manifold of initial data. It is well-known that for energy critical wave equations, the ground state solution  $Q$  gives rise to a threshold for blowup. For the focusing wave equation, this was first observed numerically in [9]. Rigorously, this has been investigated in a large body of works following different lines of research, see e.g. [32], [26], [30], [33], [37], [38], [39], [40], [36]. We also refer to [22], [24], [23], [25], see also [17], for a characterization of possible blowup dynamics in the critical case.

In contrast, the energy supercritical case is much less explored. For the supercritical focusing wave equation in three space dimensions the existence of a countable family of globally *smooth* self-similar profiles of the form (1.5) has been shown rigorously by Bizoń, Maison and Wasserman [10]. Analogous results exist for wave maps and the Yang-Mills equation, see Bizoń [3], [4], but only the corresponding (non-trivial) stable ground states are known in closed form [6]. For the cubic wave equation this has not been established so far in the supercritical case. However, the existence of infinitely many

self-similar solutions for Eq. (1.1) is conjectured in dimensions  $5 \leq d < 13$  based on numerical experiments performed by Kycia [44]. In higher space dimensions  $d \geq 13$ , a result by Collot [14] implies the existence of smooth type II blowup solutions of the form (1.7) with blowup rates  $\lambda(t) \sim (T-t)^{\frac{\ell}{\alpha}}$  for  $\ell > \alpha(d) > 2$ . In this case, the corresponding ground state  $Q$  is not known explicitly and the constructed solutions are co-dimension  $\ell - 1$  stable.

In contrast to the energy critical and subcritical case, the threshold for blowup seems to be described by a *non-trivial self-similar* profile in the supercritical case. This has been observed in numerical experiments performed in [9] for a supercritical focusing wave equation, as well as for wave maps and the Yang-Mills equation in [4], [8] and [1]. These observations are especially interesting due to the striking similarity with critical phenomena in gravitational collapse, where threshold solutions are typically self-similar [28]. From an analytic point of view, the description of threshold phenomena for supercritical wave equations in terms of self-similar solutions is completely open.

To the best of our knowledge, the solution (1.2) found in [27] is the first known example of a non-trivial self-similar solution given in closed form for the wave equation with focusing power nonlinearity. The aim of the present work is to numerically explore the role of  $u_T^*$  in the generic time-evolution and to demonstrate that it is a critical solution at the threshold for blowup. Its explicit character makes it accessible to rigorous analysis and thus it can provide a starting point for the investigation of threshold phenomena in supercritical problems.

## 2. CO-DIMENSION ONE STABLE SELF-SIMILAR BLOWUP

In the following, we consider the cubic wave equation in the radial case

$$\partial_t^2 u(t, r) - \partial_r^2 u(t, r) - \frac{d-1}{r} \partial_r u(t, r) = u(t, r)^3 \quad (2.1)$$

for  $d \geq 5$  and review the results of [27] on the stability of  $u_T^*$ . For simplicity, we restrict the discussion to the radial case. Since we are interested in the blow up behavior of solutions near the origin, we consider Eq. (2.1) in backward lightcones  $\mathcal{C}_T := \mathcal{C}_{T,0}$  as defined in (1.4). Measured in homogeneous Sobolev norms on balls  $\mathbb{B}_{T-t}^d$ , the blowup solution behaves according to

$$\|u_T^*(t, \cdot)\|_{\dot{H}^k(\mathbb{B}_{T-t}^d)} \simeq (T-t)^{s_c - k}, \quad s_c = \frac{d}{2} - 1. \quad (2.2)$$

In particular, these norms blow up only if  $k > s_c$ . For the stability analysis, we consider Eq. (2.1) for suitable small perturbations of the blowup initial data

$$u(0, \cdot) = u_1^*(0, \cdot) + f, \quad \partial_t u(0, \cdot) = \partial_t u_1^*(0, \cdot) + g \quad (2.3)$$

and look for solutions that can be written as

$$u(t, r) = u_T^*(t, r) + (T-t)^{-1} \varphi(-\log(T-t), \frac{r}{T-t}) \quad (2.4)$$

for some  $T > 0$ , such that the perturbation vanishes in a suitable sense as  $t \rightarrow T^-$ . In [27] it is shown that in  $d = 7$  this is possible under a co-dimension one condition on the initial data. To formulate the result, we define  $(h_1, h_2)$  by

$$h_1(r) = (1 + r^2)^{-2}, \quad h_2(r) = 4(1 + r^2)^{-3}. \quad (2.5)$$

**Theorem 2.1** ([27], radial version). *Let  $d = 7$ . There are constants  $\omega, \delta, c > 0$  such that for all smooth, radial  $(f, g)$  with*

$$\|(f, g)\|_{H^4 \times H^3(\mathbb{B}_2^7)} \leq \frac{\delta}{c}$$

*the following holds: There is an  $\alpha \in [-\delta, \delta]$  and a  $T \in [1 - \delta, 1 + \delta]$  depending Lipschitz continuously on  $(f, g)$  such that for initial data*

$$u(0, \cdot) = u_1^*(0, \cdot) + f + \alpha h_1, \quad \partial_t u(0, \cdot) = \partial_t u_1^*(0, \cdot) + g + \alpha h_2 \quad (2.6)$$

*there is a unique solution  $u$  in the backward lightcone  $\mathcal{C}_T$  blowing up at  $t = T$  and converging to  $u_T^*$  according to*

$$\begin{aligned} (T - t)^{k - s_c} \|u(t, \cdot) - u_T^*(t, \cdot)\|_{\dot{H}^k(\mathbb{B}_{T-t}^7)} &\lesssim (T - t)^\omega \\ (T - t)^{k - s_c} \|\partial_t u(t, \cdot) - \partial_t u_T^*(t, \cdot)\|_{\dot{H}^{k-1}(\mathbb{B}_{T-t}^7)} &\lesssim (T - t)^\omega \end{aligned} \quad (2.7)$$

for  $k = 1, 2, 3$ .

*Remark 1.* We note that the right-hand side of Eq. (2.7) is normalized to the behavior of  $u_T^*$  in the respective norm. Furthermore, by a solution, we mean a solution to the corresponding operator equation in adapted coordinates, see [27].

The proof of this result relies on the analysis of the time evolution for the perturbation  $\varphi$  in adapted self-similar coordinates

$$\tau = -\log(T - t), \quad \rho = \frac{r}{T - t}. \quad (2.8)$$

By setting

$$\psi(\tau, \rho) = e^{-\tau} u(T - e^{-\tau}, e^{-\tau} \rho), \quad (2.9)$$

the ansatz (2.4) yields

$$\left( \partial_\tau^2 + 3\partial_\tau + 2\rho\partial_\rho\partial_\tau - \Delta_\rho + \rho^2\partial_\rho^2 + 4\rho\partial_\rho + 2 - V(\rho) \right) \varphi(\tau, \rho) = N(\varphi(\tau, \rho)) \quad (2.10)$$

with  $V(\rho) = 3U^*(\rho)^2$  and  $N(\varphi) = (U^* + \varphi)^3 - 3U^{*2}$ . It turns out that the dynamics are in fact governed by the left-hand side corresponding to the linearized problem and the remaining nonlinearity  $N$  can be treated perturbatively under suitable smallness assumptions. In [27], the time-evolution for the perturbation is studied as a first order system and analyzed in a suitable function space by using semigroup theory, spectral analysis and fixed point arguments. In this framework, it turns out that the enemies to stability are *mode solutions*

$$\varphi(\tau, \rho) = e^{\lambda\tau} f(\rho) \quad (2.11)$$

with  $\lambda \in \mathbb{C}$ ,  $\operatorname{Re}\lambda \geq 0$  and *smooth* profiles  $f \in C^\infty[0, 1]$ , that satisfy the linearized equation

$$\left(\partial_\tau^2 + 3\partial_\tau + 2\rho\partial_\rho\partial_\tau - \Delta_\rho + \rho^2\partial_\rho^2 + 4\rho\partial_\rho + 2 - V(\rho)\right) \varphi(\tau, \rho) = 0. \quad (2.12)$$

In a rigorous formulation, the values  $\lambda$  correspond to eigenvalues of a suitably defined differential operator  $\mathbf{L}_0$ , see [27]. The resulting ordinary differential equation for  $f$  arises as an eigenvalue equation and contains the crucial information on the stability properties of  $u_T^*$ . A fundamental observation is that the time translation invariance of the blowup solution induces a mode solution for  $\lambda_0 = 1$ . More precisely,

$$\phi_0(t, r) := \frac{d}{d\varepsilon} u_{T+\varepsilon}^*(t, r)|_{\varepsilon=0} = (T-t)^{-2} f_0\left(\frac{r}{T-t}\right) \quad (2.13)$$

with

$$f_0(\rho) = \frac{(d-4)(d-4-3\rho^2)}{(d-4+3\rho^2)^2}, \quad (2.14)$$

satisfies the linearized equation in physical variables

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r - 3(T-t)^{-2}U^*\left(\frac{r}{T-t}\right)^2\right) \phi(t, r) = 0. \quad (2.15)$$

Hence,  $\varphi_0(\tau, \rho) = e^{-\tau}\phi_0(T-e^{-\tau}, e^{-\tau}\rho) = e^\tau f_0(\rho)$  yields an unstable solution to (2.12). In the nonlinear time-evolution this instability can be controlled by suitably adjusting the blowup time  $T > 0$ . However, as will be discussed in the following, there is additional genuine instability that yields the co-dimension one condition on the data.

**2.1. The spectral ODE.** By inserting the mode ansatz (2.11) into Eq. (2.12) we obtain the ordinary differential equation

$$(1 - \rho^2)f''(\rho) + \left(\frac{d-1}{\rho} - 2(\lambda+2)\rho\right) f'(\rho) - ((\lambda+1)(\lambda+2) - V(\rho)) f(\rho) = 0, \quad (2.16)$$

where

$$V(\rho) = \frac{24(d-1)(d-4)}{(d-4+3\rho^2)^2}.$$

Note that the backward lightcone  $\mathcal{C}_T$ , see (1.4), is mapped onto the cylinder  $(\tau, \rho) \in [0, \infty) \times [0, 1]$  via self-similar coordinates (2.8). We are therefore interested in the values of  $\lambda$  that yield solutions to Eq. (2.16) that are smooth on  $[0, 1]$ . Furthermore, in the sequel we refer to such values of  $\lambda$  as *eigenvalues* and the corresponding solutions as *eigenfunctions*. To numerically compute eigenvalues we follow the approach in [7]. First, we introduce the change of variables

$$x = \rho^2, \quad f(\rho) = \frac{\sqrt{x}}{(d-4+3x)^2} y(x). \quad (2.17)$$



This yields a Heun equation (in its canonical form)

$$y''(x) + \left( \frac{d}{2x} + \frac{2\lambda + 5 - d}{2(x-1)} - \frac{12}{3x + d - 4} \right) y'(x) + \frac{3(\lambda^2 - 5\lambda + 6)x + (d-4)\lambda^2 + 3(d-4)\lambda - 2(5d+8)}{4x(x-1)(3x+d-4)} y(x) = 0. \quad (2.18)$$

Note that (2.17) preserves the smoothness of solutions on the interval  $[0, 1]$ , i.e. Eq. (2.16) and Eq. (2.18) have the same “spectrum”. For that reason, by focusing on Eq. (2.18) we reduce the analysis from an equation that has six to the one that has four singular points.

The sets of Frobenius indices for Eq. (2.18) at  $x = 0$  and  $x = 1$  are  $\{0, -\frac{d-1}{2}\}$  and  $\{0, \frac{d-3}{2} - \lambda\}$  respectively. Taking the first of the pairs leads to analytic solutions  $y_0(x)$  and  $y_1(x)$ .<sup>1</sup> A scenario that yields an eigenvalue is precisely when  $y_0$  and  $y_1$  are constant multiples of each other, which is in turn equivalent to the Wronskian

$$W[y_0, y_1](x) := y_0'(x)y_1(x) - y_0(x)y_1'(x) \quad (2.19)$$

being identically equal to zero. The eigenvalues are therefore given by the zeros of  $W[y_0, y_1](x)$  for an arbitrary choice of  $x \in (0, 1)$ . Since Eq. (2.18) has four singular points there is unfortunately no closed form expression for the Wronskian, unlike for example for the hypergeometric equation where (2.19) is given in terms of Gamma functions. However, functions  $y_0$  and  $y_1$  are built into Maple, and we can numerically compute the zeros of  $W[y_0, y_1](x)$  with high precision. Our findings are displayed in Table 1. Furthermore, we obtained the same results by using shooting method and the method of continued fractions as in [5].

First of all, all eigenvalues appear to be real, even though there is no a priori reason for that.<sup>2</sup> Furthermore, in each dimension, in addition to the *gauge* eigenvalue  $\lambda_0 = 1$ , there is exactly one more non-negative eigenvalue. It is unfortunately very difficult to rigorously prove this observation. However, for  $d = 7$  this eigenvalue happens to be  $\lambda_1 = 3$  with an explicit corresponding eigenfunction; we were able to exploit this fact and prove the following result.

**Proposition 2.2** ([27], case  $\ell = 0$ ). *For  $\text{Re}\lambda \geq 0$ , the only solutions  $f(\rho; \lambda) \in C^\infty[0, 1]$  to Eq. (2.16) are  $f(\rho; 1) = f_0(\rho)$  and  $f(\rho; 3) = f_1(\rho)$  with*

$$f_1(\rho) = \frac{1}{(1 + \rho^2)^2}. \quad (2.20)$$

<sup>1</sup>Of course, this is in the case when  $\frac{d-3}{2} - \lambda \notin \mathbb{N}$ , because otherwise the analytic solution at  $x = 1$  has a zero of power  $\frac{d-3}{2} - \lambda$ , but this complementary case is treated similarly.

<sup>2</sup>More precisely, following the reasoning in [1], one can prove that the eigenvalues for which  $\text{Re}\lambda > \frac{d-3}{2}$  are necessarily real. Note, however, that already for  $d = 9$  we observe no eigenvalues in this region.

$d$	$\lambda_1$	$\lambda_0$	$\lambda_{-1}$	$\lambda_{-2}$	$\lambda_{-3}$
5	4.3721349687	1	-0.5372149224	-1.8885795302	-3.1761090510
6	3.3952423458	1	-0.5489567604	-1.9623530271	-3.3260178298
7	3.0000000000	1	-0.5524221092	-2.0000000000	-3.4107729035
8	2.7819974065	1	-0.5538793302	-2.0235597344	-3.4669858823
9	2.6429561127	1	-0.5546167803	-2.0398648867	-3.5074416442
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\infty$	2.0000000000	1	-0.5559264834	-2.1334365196	-3.7697439469

TABLE 1. All eigenvalues of Eq. (2.18) appear to be real. In addition to two non-negative eigenvalues, there seem to be infinitely many negative ones, the largest three of them being shown.

On the other hand, it seems that there are infinitely many negative eigenvalues; in Table 1 we listed the largest three of them. Here we should point out that only the eigenvalues for which  $\operatorname{Re}\lambda > -\frac{1}{2}$ , correspond to the isolated eigenvalues of the operator  $\mathbf{L}_0$  in [27], see Lemma 5.2. However, the spectral cutoff value  $-\frac{1}{2}$  is dictated by the choice of the space on which  $\mathbf{L}_0$  is studied. So by imposing more regularity on this space (in terms of Sobolev space order) the spectral cutoff is pushed to the left and in this way more and more negative eigenvalues get uncovered.

A particularly interesting feature of our numerical results in Table 1 is that eigenvalues seem to decrease as the dimension increases. It is therefore natural to ask as to whether they have limiting values as  $d$  goes to infinity. To further investigate this we do the following. Namely, by rescaling the independent variable

$$x = (d - 1)z \tag{2.21}$$

in Eq. (2.18) and letting  $d$  go to infinity we obtain the following equation

$$zy''(z) + \left( \frac{1}{2z} + \frac{4}{3z+1} + \lambda - \frac{3}{2} \right) y'(z) + \frac{\lambda - 2}{4} \frac{3(\lambda - 3)z + \lambda + 5}{z(3z+1)} y(z) = 0. \tag{2.22}$$

In this process the interval  $[0, 1]$  contracts into a single point  $z = 0$ . We therefore study solutions of Eq. (2.22) that are analytic at  $z = 0$ . Since  $z = 0$  is an irregular singular point for Eq. (2.22) there is a formal series solution centered at  $z = 0$  which is asymptotic to an actual solution but generically not convergent. By requiring convergence of such series we impose a quantization condition on  $\lambda$ . Furthermore, by a method analogous to the one in [5], we numerically calculate such values of  $\lambda$ , the largest five of which are given in the last row of Table 1.

Interestingly, Table 1 suggests that one eigenvalue of Eq. (2.22) is  $\lambda_1 = 2$ . In fact, this is obvious from Eq. (2.22) since any constant function is a solution for  $\lambda = 2$ . Furthermore, since both (numerically observed) non-negative eigenvalues of Eq. (2.22) are known explicitly together with their corresponding eigenfunctions<sup>3</sup> it is likely that an adjustment of spectral techniques developed in [16, 15, 27] would yield a rigorous proof of this observation. Subsequently, by using some kind of perturbative argument to prove that for large values of  $d$  the spectrum of Eq. (2.18) is close to the one of Eq. (2.22) one would obtain an analog of Theorem 2.1 for all large enough dimensions  $d$ .

Finally, we remark that in the case  $d = 7$  the genuine instability  $\lambda_1 = 3$  can be related to the conformal invariance of Eq. (2.15) which is due to the self-similar character of the potential. More specifically, we adapt the transformation defined in Eq. (1.6) to the backward lightcone  $\mathcal{C}_T$  and define

$$\tilde{\phi}(t, r) = ((T - t)^2 - r^2)^{-\frac{d-1}{2}} \phi \left( \frac{t-T}{(T-t)^2 - r^2} + T, \frac{r}{(T-t)^2 - r^2} \right), \quad (2.23)$$

which leaves Eq. (2.15) invariant. In particular,

$$\tilde{\phi}_0(t, r) = (T - t)^{3-d} \left(1 - \frac{r}{T-t}\right)^{\frac{5-d}{2}} f_0\left(\frac{r}{T-t}\right) \quad (2.24)$$

gives rise to a solution of Eq. (2.12) given by

$$\tilde{\varphi}_0(\tau, \rho) = e^{(d-4)\tau} (1 - \rho^2)^{\frac{5-d}{2}} f_0(\rho). \quad (2.25)$$

For  $d = 5$ , this is just an identity. In higher space dimensions, this transformation induces a singularity at the lightcone, since  $f_0(\rho) = \mathcal{O}(1)$  around  $\rho = 1$  in general. However, in  $d = 7$ ,  $f_0$  vanishes at the lightcone such that

$$\tilde{\varphi}_0(\tau, \rho) = e^{3\tau} (1 - \rho^2)^{-1} f_0(\rho) \quad (2.26)$$

corresponds to the mode solution for  $\lambda_1 = 3$ . This effect can be observed also in other supercritical wave equations, see e.g. [1] for the wave maps problem.

### 3. THRESHOLD BEHAVIOR

**3.1. Numerical approach.** To effectively cope with the blowup phenomena numerically we follow closely the scheme introduced in [1] and used in the context of self-similar blowup of equivariant wave maps. Thus we introduce new (computational) coordinates  $(s, y)$  through

$$t = \int e^{-s} h(s) ds, \quad r = e^{-s} y, \quad (3.1)$$

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<sup>3</sup>The solution corresponding to the gauge eigenvalue  $\lambda_0 = 1$  is  $y(z; 1) = 1 - 3z$ , as can be found in advance by calculating the limit of (2.14) as  $d \rightarrow \infty$ , having in mind (2.17) and (2.21).

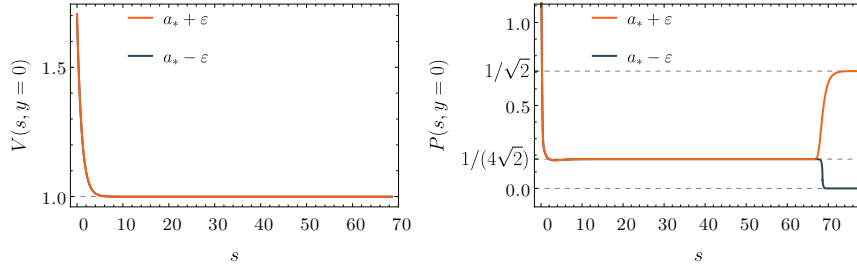


FIGURE 1. The evolution of marginally sub- (blue line) and supercritical (orange line) evolutions in  $d = 5$  in computational variables (3.1)-(3.2).

The arbitrary choice of  $h(s)$  is used to make the coordinate transformation adapt to the blowing up solution; note that  $(s, y)$  are the self-similar coordinates  $(\tau, \rho)$  defined in (2.8) when  $h(s) \equiv 1$ . We also define new dependent variables

$$V(s, y) = e^{-s}u(t, r), \quad P(s, y) = e^{-2s}\partial_t u(t, r). \quad (3.2)$$

The time evolution of the new variables (3.2) follows from (3.1) and the equation (2.1), explicitly we have

$$\begin{aligned} \partial_s V(s, y) &= h(s)P(s, y) - V(s, y) - y\partial_y V(s, y), \\ \partial_s P(s, y) &= h(s) \left( \partial_y^2 V(s, y) + \frac{d-1}{y} \partial_y V(s, y) + V(s, y)^3 \right) \\ &\quad - 2P(s, y) - y\partial_y P(s, y). \end{aligned} \quad (3.3)$$

The main advantage of this rescaling and the redefinition of dependent variables is that both  $V(s, y)$  and  $P(s, y)$  stay finite and the blowup is shifted to infinity ( $s \rightarrow \infty$ ) if we set  $h(s) = 1/P(s, 0)$ . This choice of  $h$  leads to the following long time asymptotics of the new variables at  $y = 0$ :  $V(s, 0) = 1 + Ce^{-s}$ , for some constant  $C \in \mathbb{R}$  and in the case of dispersion  $P(s, 0) \rightarrow 0$ , while in the case of self-similar blowup  $P(s, 0) \rightarrow 1/U(0)$ , where  $U(\rho)$  is a self-similar profile. This behavior is in stark contrast to the original variable  $u(t, r)$  which blows up in finite time.

As pointed out in [1] the transformation (3.1) with a proper choice of  $h$  introduces self-adapting coordinates that accurately resolve both spatial and temporal scales of blowing up solution. In this way we avoid using an adaptive spatial mesh and a time rescaling techniques. In fact to solve (3.3) we use a standard method of lines with 6th order finite difference approximation in space and a 6th order Runge-Kutta method with fixed time step as a time stepping algorithm. Staggered spatial grid with fixed mesh size is used to deal with the  $y = 0$  singularity in (3.3). In addition, a symmetry of the variables (3.2) at the origin is used to construct finite difference stencils close to the coordinate singularity. We add a standard dissipation term to suppress high frequency noise in the data introduced by spatial

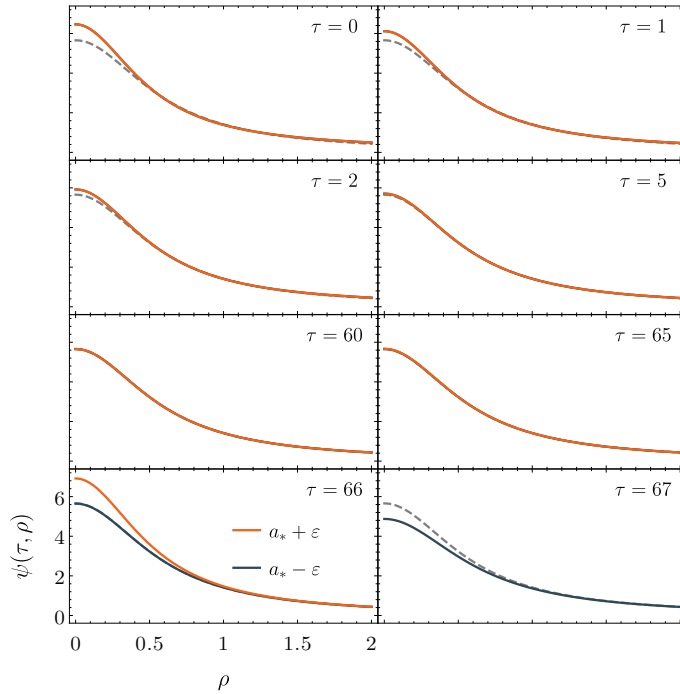
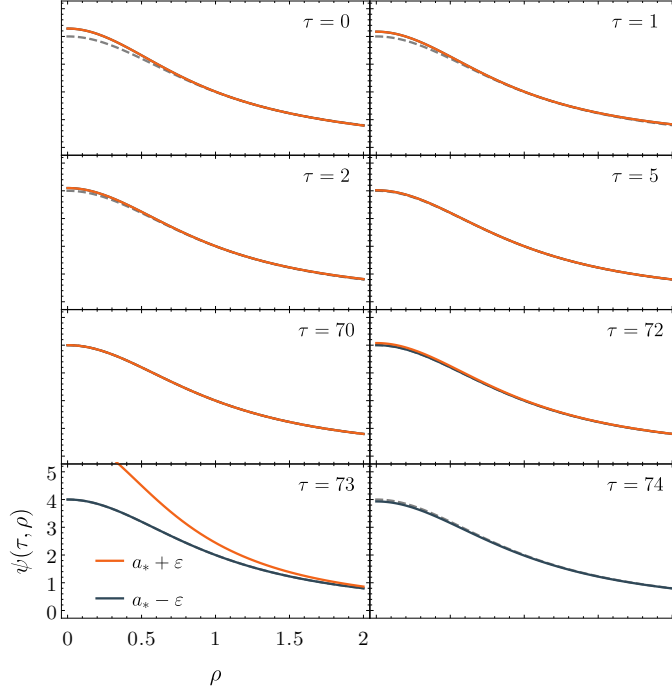


FIGURE 2. The evolution of marginally sub- (blue line) and supercritical (orange line) evolutions for  $d = 5$  in self-similar coordinates adapted to the threshold solution. Both solutions approach the intermediate attractor  $U^*(\rho)$  (dashed line) along the stable eigenmode  $f_{-1}(\rho)$  (see discussion in the text). After some time solutions depart from  $U^*(\rho)$  along the genuine unstable eigenmode  $f_1(\rho)$  in opposite directions. In the last frame, the supercritical solution is out of range of the plot as the coordinates used differ from the ones in which we would see the approach to  $U_0(\rho)$  (the generic blowup).

discretisation. The code was written in Mathematica whose flexibility and functionality allowed us to use arbitrary precision arithmetics seamlessly.

It turned out that higher precision was crucial to get close enough to the critical solution and obtain a detailed description of near critical evolutions. To speed up computations we parallelized the bisection search (discussed in the following section) probing the search interval using multiple (typically 64) cores simultaneously.

FIGURE 3. Same as Fig. 2 but for  $d = 7$ .

**3.2. Results.** With fixed family of initial data<sup>4</sup>

$$V(0, y) = P(0, y) = \frac{a}{\cosh y}, \quad (3.4)$$

we integrate (3.3) forward in  $s$  with  $a$  as the only free parameter. For small initial amplitude  $a$  the solution disperses and  $P(s, 0)$  goes to zero, whereas for large values of  $a$  we observe the solution approaching a homogeneous profile (in particular  $P(s, 0)$  goes to  $1/U_0(0) = 1/\sqrt{2}$ ). We perform a bisection search in amplitude  $a$  based on the criteria just outlined; in this way, we find the threshold evolution which is analyzed in detail below. Sample results of such bisection search are shown in Fig. 1. We present results for  $d = 5$  and  $d = 7$  cases only but we expect analogous behavior for other dimensions  $d \geq 5$ .

For  $a \approx a_*$  (explicitly  $a_* \approx 1.710572581$  in  $d = 5$  and  $a_* \approx 2.335609125$  in  $d = 7$ ) the numerical solution approaches, for intermediate times, a static profile which matches  $U^*(\rho)$ , cf. (1.2). This is illustrated in Figs. 2 and 3. As outlined in the previous section we expect the solution which is close to the critical solution to behave as (here we look at solution at  $\rho = 0$ )

$$\psi(\tau, \rho = 0) = c + a_1 e^{\lambda_1 \tau} + a_0 e^\tau + a_{-1} e^{\lambda_{-1} \tau} + \dots, \quad (3.5)$$

<sup>4</sup>The result is universal with respect to the choice of initial data family interpolating between dispersion and blowup.

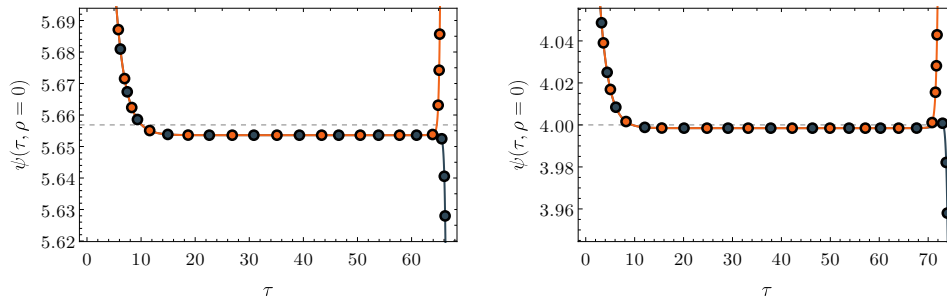


FIGURE 4. The convergence of nearly critical evolutions (blue line for subcritical case, orange line for supercritical case) to the self-similar profile  $U^*$ . Value of  $\psi(\tau, 0)$  converges to  $U^*(0)$  while the rate of convergence is determined by the stable eigenvalue  $\lambda_{-1}$ . At a later time, the unstable mode takes over and we see the divergence with the rate given by unstable eigenmode  $\lambda_1$ . The gauge mode is successively removed by a proper choice of the blowup time  $T$ . Results of the fit of theoretical prediction (3.5) are summarized in Tab. 2. Both  $d = 5$  (left panel) and  $d = 7$  (right panel) are presented.

in adapted self-similar coordinates (2.8)-(2.9), where dots stand for even faster decaying modes. The bisection procedure ensures the coefficient of the unstable mode  $a_1 \sim a - a_* \equiv \varepsilon$  is small<sup>5</sup> so that for long enough time  $s$  (equivalently for large  $\tau$ ) we see the convergence to  $U^*(\rho)$ , thus  $c \approx U^*(0)$ .

To perform a qualitative comparison of the numerical data with the analytical prediction first we have to change variables from the numerical  $(s, y)$  to the adapted self-similar coordinates  $(\tau, \rho)$  (this step requires finding an approximate blowup time  $T$ ; the procedure is described in more details in [1]). Then we do the fit to the transformed data. Comparison of the theoretical prediction (3.5) with the numerical results is presented in Fig. 4, while the results of the fits are collected in Tab. 2.

<sup>5</sup>Using 128 digits of precision for  $d = 5$  we were able to fine-tune  $a$  up to  $\varepsilon \approx 10^{-129}$  whereas for  $d = 7$  and 96 digits we obtain  $\varepsilon \approx 10^{-96}$ .

$d = 5$	subcritical	supercritical
$c$	5.653595752	5.653595793
$a_1$	$-5.423965718 \cdot 10^{-128}$	$5.002789061 \cdot 10^{-126}$
$a_0$	$2.184193927 \cdot 10^{-32}$	$2.144231964 \cdot 10^{-32}$
$a_{-1}$	0.7585737069	0.7585801575
$\lambda_{-1}$	-0.5387977976	-0.5387995060

$d = 7$	subcritical	supercritical
$c$	3.998463960	3.998463963
$a_1$	$-2.707332535 \cdot 10^{-98}$	$1.138265563 \cdot 10^{-95}$
$a_0$	$1.107517572 \cdot 10^{-34}$	$1.106153203 \cdot 10^{-34}$
$a_{-1}$	0.3001361956	0.3001362363
$\lambda_{-1}$	-0.5535087559	-0.5535088248

TABLE 2. Results of the fit of theoretical prediction to the sub- and supercritical evolutions ( $d = 5$  and  $d = 7$  cases) cf. (3.5). In both cases, we fix values of exponents of (genuine) unstable modes (the exponent of the gauge mode was also fixed) using values as found from the linear stability analysis. Listed values are results of the fit. Quality of the fit can be assessed by the agreement with results of the linear stability analysis ( $U^*(0) = 4\sqrt{2} \approx 5.65685$  in  $d = 5$ ,  $U^*(0) = 4$  in  $d = 7$ ), the magnitude of the gauge mode, and the magnitude of the unstable mode (in super- and subcritical evolutions  $a_1$  should be small, of the order of precision used, and of opposite sign).

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