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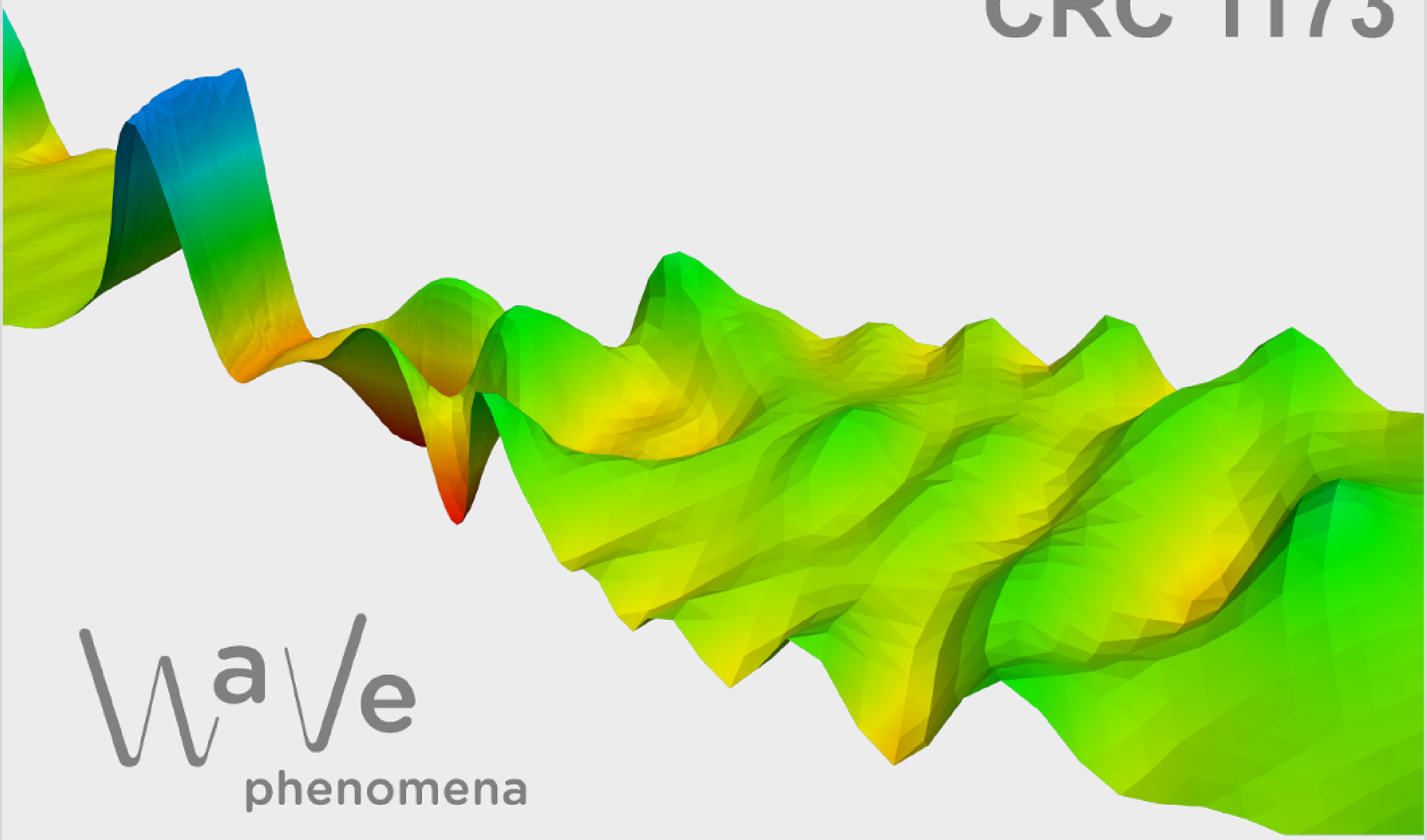

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DISPERSIVE ESTIMATES, BLOW-UP AND FAILURE OF STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION WITH SLOWLY DECAYING INITIAL DATA

RAINER MANDEL

Abstract. The initial value problem for the homogeneous Schrödinger equation is investigated for radially symmetric initial data with slow decay rates and not too wild oscillations. Our global wellposedness results apply to initial data for which Strichartz estimates fail.

1. Introduction

In this paper we investigate the initial value problem for the Schrödinger equation

$$i\partial_t\psi + \Delta\psi = 0 \quad \text{in } \mathbb{R}^n, \qquad \psi(0) = \phi \tag{1}$$

for radial initial data ϕ with slow decay at infinity. In particular, we are interested in a solution theory for (1) without assuming ϕ to belong to one of the Lebesgue spaces $L^r(\mathbb{R}^n)$ with $r \in [1,2]$. In this case Strichartz estimates are not available and local or global wellposedness results for (1) are unknown. Surprisingly, we could not find a complete statement about Strichartz estimates for such initial data in the literature, so we clarify this point here.

Theorem 1. *Let $n \in \mathbb{N}, n \geq 2$ and $p,q \in [1,\infty], r > 2$. Then there is no Strichartz estimate*

$$\|e^{it\Delta}\phi\|_{L^p_t(\mathbb{R};L^q(\mathbb{R}^n))} \lesssim \|\phi\|_{L^r(\mathbb{R}^n)}. \tag{2}$$

This theorem partly generalizes the known fact that for any given $t > 0$ the Schrödinger propagator $e^{it\Delta}$ is unbounded as a map from $L^r(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $r > 2, q \in [1,\infty]$, cf. [6, p.63] for the case $q = r$. Theorem 1 may seem surprising in view of the fact that the optimal conditions for Strichartz estimates in the most important special case $r = 2$ do not provide any obvious reason why the estimates should break down completely for $r > 2$. Recall that these conditions are given by

$$p,q \geq 2, \qquad (p,q,n) \neq (2,\infty,2), \qquad \frac{2}{p} + \frac{n}{q} = \frac{n}{2},$$

see for instance [2, Theorem 2.3.3]. We refer to the papers [3, 5, 8] for three milestone contributions related to the discovery of these conditions. At least for $n \geq 3$, each of the above conditions has a counterpart in the range $r > 2$. The scaling invariance of the Schrödinger equation implies $\frac{2}{p} + \frac{n}{q} = \frac{n}{r}$ so that $q \geq r$ would be an immediate consequence that replaces the condition $q \geq 2$. As we discuss in the Appendix $p \geq 2$ generalizes to $p \geq \frac{2r}{(2n-r(n-1))_+}$. In particular, there is no evident reason for the necessity of $r \leq 2$ so that Theorem 1 seems to

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fill a gap in the literature. Its short proof relies on a thorough analysis of a counterexample due to Bona, Ponce, Saut and Sparber [1]. The main feature of their solution is that the corresponding initial datum oscillates quadratically with respect to the distance to the origin, which produces blow-up of the solution at some prescribable finite time, cf. [1, Lemma 2.1]. We reconsider this self-similar blow-up analysis for partly more general initial data and estimate the blow-up rate in $L^q(\mathbb{R}^n)$, which eventually leads to Theorem 1. Accordingly, our proof even reveals that local Strichartz estimates cannot hold either and that no improvement in the radial situation is possible.

Given that Strichartz estimates fail, the question arises how wellposedness results for the Schrödinger equation can be achieved if the initial datum lies in $L^r(\mathbb{R}^n)$ only for $r > 2$. In view of the above-mentioned counterexample it seems reasonable to impose a condition on the oscillations of the initial datum. In the following we present one possible approach in the radially symmetric case which relies on suitably weighted Sobolev norms of the initial data. For instance we identify a class of initial data lying in $L^r(\mathbb{R}^n)$ only for $r > \frac{2n}{n-1}$ with solutions that are bounded in time and uniformly localized in space, see Corollary 1. In that case dispersion need not occur because there are solutions of the form

$$\psi(x,t) = e^{-i\omega^2 t}\phi(x) \qquad \text{where } \phi(x) = |x|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\omega|x|) \quad \text{for some } \omega \in \mathbb{R}. \tag{3}$$

Here, $J_{(n-2)/2}$ denotes the Bessel function of the first kind and ϕ solves the linear Helmholtz equation $\Delta\phi + \omega^2\phi = 0$ in $\mathbb{R}^n$. Aiming for a more general result in this direction, we first consider initial profiles of the form $\phi(x) = e^{i\omega|x|}\phi_\omega(|x|)$ where ϕ_ω belongs to the function spaces X respectively Y_m for some $m \in \{0,\dots,n\}$ that we introduce now. The space X is defined to be the completion of $C_c^\infty(\mathbb{R}_{\geq 0};\mathbb{C})$ with respect to the norm $\|\cdot\|_X := \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$ given by

$$\|f\|_{X_1} := \sup_{z>0} z^{\frac{1-n}{2}} \int_0^z \left(|f(r)|r^{n-2} + |f'(r)|r^{n-1}\right) dr,$$

$$\|f\|_{X_2} := \int_0^\infty \left|\frac{d}{dr}\left(f(r) r^{\frac{n-1}{2}}\right)\right| dr + \sup_{z>0}\left(z\int_z^\infty |f(r)| r^{\frac{n-5}{2}}\, dr\right).$$

Similarly, we define Y_m to be the completion of $C_c^\infty(\mathbb{R}_{\geq 0};\mathbb{C})$ with respect to the norm

$$\|f\|_{Y_m} := \sum_{k=0}^m \int_0^\infty |f^{(k)}(r)| r^{n-m+k-1}\, dr \qquad \text{if } m \in \{0,\dots,n-1\},$$

$$\|f\|_{Y_n} := \sum_{k=1}^n \int_0^\infty |f^{(k)}(r)| r^{k-1}\, dr + \sup_{z>0} z^{-2}\int_0^z f(r) r\, dr + |f(0)|.$$

One main feature of these spaces is that slow decay rates of its elements are admissible only provided that their derivatives decay fast enough. For instance, we have that $r \mapsto (1+r)^{-\alpha}$ lies in X if and only if $\alpha \geq \frac{n-1}{2}$ and in Y_m if and only if $\alpha > n-m$ whereas $r \mapsto e^{ir}(1+r)^{-\alpha}$ belongs to X if and only if $\alpha > \frac{n+1}{2}$ and to Y_m if and only if $\alpha > n$. Using these spaces we find a local well-posedness theory for at most linearly oscillating radial initial data.

Theorem 2. *Let $n \in \mathbb{N}, n \geq 2, m \in \{0,\dots,n\}$ and $\phi(x) = \phi_\omega(|x|)e^{i\omega|x|}$ for some $\omega \in \mathbb{R}$.*

(i) If $\phi_\omega \in Y_m$ *then* (1) *has a unique global mild solution* ψ *satisfying*

$$|\psi(x,t)| \le C(\sqrt{t})^{m-n}\|\phi_\omega\|_{Y_m}.$$

(ii) If $\phi_\omega \in X$ *then* (1) *has a unique global mild solution* ψ *satisfying*

$$|\psi(x,t)| \le C|x|^{\frac{1-n}{2}}\|\phi_\omega\|_X.$$

In (i) and (ii) the constant C *does not depend on* ω.

Combining the estimates (i) and (ii) we deduce the following.

Corollary 1. *Let* $n \in \mathbb{N}, n \ge 2$ *and assume* $\phi(x) = \int_{\mathbb{R}} \phi_\omega(|x|)e^{i\omega|x|}\,d\mu(\omega)$ *for some Borel measure* μ *on* $\mathbb{R}$. *Then* (1) *has a unique global mild solution satisfying*

$$|\psi(x,t)| \le C(1+|x|)^{\frac{1-n}{2}} \int_{\mathbb{R}} \left(\|\phi_\omega\|_X + \|\phi_\omega\|_{Y_n}\right) d\mu(\omega)$$

provided the right hand side is finite.

Corollary 2. *Let* $n \in \mathbb{N}, n \ge 2, m \in \{0,\dots,n\}$ *and assume* $\phi(x) = \int_{\mathbb{R}} \phi_\omega(|x|)e^{i\omega|x|}\,d\mu(\omega)$ *for some Borel measure* μ *on* $\mathbb{R}$. *Then* (1) *has a unique global mild solution satisfying*

$$|\psi(x,t)| \le C(1+t)^{-\frac{m}{2}} \int_{\mathbb{R}} \left(\|\phi_\omega\|_{Y_{n-m}} + \|\phi_\omega\|_{Y_n}\right) d\mu(\omega)$$

provided the right hand side is finite.

Remark 1.

(a) *Corollary 1 applies to superpositions of radially symmetric Herglotz waves given by some density* $a \in L^1(\mathbb{R})$. *In fact, using the asymptotic expansions of the Bessel functions at infinity (see Proposition 2 below) one can find functions* $\eta_\omega \in X \cap Y_n$ *such that*

$$\begin{aligned}\phi(x) &:= \int_{\mathbb{R}} a(\omega)|x|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\omega|x|)\,d\omega \\ &= \int_{\mathbb{R}} a(\omega)\left(\eta_\omega(|x|)e^{i\omega|x|} + \overline{\eta_\omega(|x|)}e^{-i\omega|x|}\right) d\omega \\ &= \int_{\mathbb{R}} \phi_\omega(|x|)e^{i\omega|x|}\,d\omega\end{aligned}$$

where $\phi_\omega := a(\omega)\eta_\omega + a(-\omega)\overline{\eta_\omega} \in X \cap Y_n$. *In particular, Corollary 1 generalizes the observation that the solutions of the Schrödinger equation with initial data given by superpositions of radially symmetric Herglotz waves as in* (3) *remain bounded in time and uniformly localized in space.*

(b) *Theorem 2 (i) is a generalized version of the fact that integrable initial data yield bounded solutions. Indeed, the latter statement corresponds to* $m = 0$ *in the theorem. So we get that less integrability of the initial datum is still sufficient for the absence of finite time blowup provided that the derivatives decay sufficiently fast. Notice that some kind of control on the derivatives seems necessary given that there are initial* *data in $L^s(\mathbb{R}^n)$ for any given $s>1$ the corresponding solutions of which blow up in finite time, see [1, Lemma 2.1 and Remark 2.2].*

(c) *The decay rates from Corollary 1 improve once we add regularity assumptions on μ. In the simplest situation $d\mu(\omega)=a(\omega)\,d\omega$ for $a\in W_0^{k,1}(\mathbb{R})$ and $\omega\mapsto\phi_\omega\in W^{k,\infty}(\mathbb{R};X\cap Y_n)$, the decay rate improves to $(1+|x|)^{\frac{1-n}{2}-k}$. This is proved using integration by parts as in the method of stationary phase. In the Appendix we discuss the densities $a(\omega)=(\omega-1)^{-\delta}\mathbb{1}_{[1,2]}(\omega)$ with $\delta\in(0,1)$ and find the intermediate decay rates $(1+|x|)^{\frac{1-n}{2}-(1-\delta)}$.*

(d) *Theorem 2 (i) tells us that non-dispersive solutions of the Schrödinger equation* (1) *can only occur for radial initial data $\phi(x)=\phi_{\rm rad}(|x|)$ that satisfy $\|\phi_{\rm rad}\|_{Y_m}=\infty$ for all $m\in\{0,\dots,n-1\}$. For smooth initial data this essentially means that for some $k\in\{0,\dots,n-1\}$ the function $|\phi_{\rm rad}^{(k)}(r)|$ does not decay faster than $(1+r)^{-1-k}$ as $r\to\infty$. We conclude that the lack of dispersion is a phenomenon related to slowly decaying or heavily oscillating initial data.*

2. Proof of Theorem 2

In the following let ψ denote the unique mild solution of the Schrödinger equation (1) with initial datum ϕ given by $\phi(x)=\phi_{\rm rad}(|x|)=\phi_\omega(|x|)e^{i\omega|x|}$. By density, il suffices to prove the estimates for $\phi_\omega\in C_c^\infty(\mathbb{R}_{\geq 0};\mathbb{C})$. In the following we use the abbreviations

$$f(r):=\phi_{\rm rad}(r)r^{\frac{n}{2}}J_{\frac{n-2}{2}}\left(\frac{r|x|}{2t}\right)\qquad\text{and}\qquad g(\rho):=f(2\sqrt{t}\rho). \tag{4}$$

We first recall the representation formula of the solution for radial initial data.

Proposition 1. *We have for all $x\in\mathbb{R}^n, t>0$*

$$\psi(x,t)=|x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}e^{i(\frac{|x|^2}{4t}-\frac{n\pi}{4})}\int_0^\infty g(\rho)e^{i\rho^2}\,d\rho. \tag{5}$$

Proof. From (4.2) in [6] or (2.2.5) in [2] we get

$$\begin{aligned}
\psi(x,t)&=\frac{1}{(4i\pi t)^{n/2}}\int_{\mathbb{R}^n}e^{i\frac{|x-y|^2}{4t}}\phi(y)\,dy\\
&=\frac{1}{(4i\pi t)^{n/2}}\int_0^\infty\phi_{\rm rad}(r)r^{n-1}e^{i\frac{|x|^2+r^2}{4t}}\left(\int_{\partial B_1(0)}e^{i\frac{r\langle x,\omega\rangle}{2t}}\,d\sigma(\omega)\right)dr\\
&=\frac{1}{(4i\pi t)^{n/2}}\int_0^\infty\phi_{\rm rad}(r)r^{n-1}e^{i\frac{|x|^2+r^2}{4t}}\left(\int_{\partial B_1(0)}e^{i\frac{r|x|\omega_1}{2t}}\,d\sigma(\omega)\right)dr\\
&=\frac{1}{(4i\pi t)^{n/2}}\int_0^\infty\phi_{\rm rad}(r)r^{n-1}e^{i\frac{|x|^2+r^2}{4t}}\cdot|\partial B_1(0)|\Gamma(n/2)2^{\frac{n-2}{2}}\left(\frac{r|x|}{2t}\right)^{\frac{2-n}{2}}J_{\frac{n-2}{2}}\left(\frac{r|x|}{2t}\right)dr\\
&=|\partial B_1(0)|\Gamma(n/2)(4\pi i)^{-n/2}2^{n-2}|x|^{\frac{2-n}{2}}t^{-1}e^{i\frac{|x|^2}{4t}}\int_0^\infty f(r)e^{i\frac{r^2}{4t}}\,dr\\
&=|\partial B_1(0)|\Gamma(n/2)(4\pi i)^{-n/2}2^{n-1}|x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}e^{i\frac{|x|^2}{4t}}\int_0^\infty f(2\sqrt{t}\rho)e^{i\rho^2}\,d\rho.
\end{aligned}$$

So the claim follows from $|\partial B_1(0)| = 2\pi^{n/2}/\Gamma(n/2)$. □

It will be convenient to split the integrand in (5) into three parts $g = g_1 + g_2 + g_3$. The function g_1 will be identical to g for small arguments and the corresponding estimates rely on the behaviour of the Bessel function $J_{(n-2)/2}$ on the interval $[0,1]$. The sum $g_2 + g_3$ represents g for large arguments and their definitions are based on the asymptotic expansion of the Bessel function at infinity. To be more precise, we fix some cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $[0, \frac{1}{2}]$ and $\chi \equiv 0$ on $[1, \infty]$. Then, similar as in [9, p.202], we write

$$z^{n/2} J_{\frac{n-2}{2}}(z) = A_n(z) + e^{iz} B_n(z) + e^{-iz}\overline{B_n(z)}. \tag{6}$$

where the functions A_n, B_n are given by

$$A_n(z) := \chi(z) z^{n/2} J_{\frac{n-2}{2}}(z), \qquad B_n(z) := (1-\chi(z)) e^{-i\frac{(n-1)\pi}{4}} \sum_{k=0}^\infty \alpha_k z^{\frac{n-1}{2}-k} \tag{7}$$

and the coefficients $\alpha_k \in \mathbb{C}$ are $\alpha_0 := \frac{1}{\sqrt{2\pi}}$ and for $k \in \mathbb{N}$

$$\begin{aligned} \alpha_k &:= \frac{1}{\sqrt{2\pi}} \left(\frac{n-2}{2}, k\right) \left(\frac{i}{2}\right)^k \qquad \text{where} \\ (\nu, k) &:= \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \cdot \ldots \cdot (4\nu^2 - (2k-1)^2)}{4^k k!}, \end{aligned} \tag{8}$$

see [9, p.199]. The motivation for this decomposition is that A_n, B_n and its derivatives satisfy useful uniform estimates that we provide next.

Proposition 2. *We have* $\operatorname{supp}(A_n) \subset [0,1], \operatorname{supp}(B_n) \subset [\frac{1}{2}, \infty)$ *and for all* $j \in \mathbb{N}, z \in \mathbb{R}$

$$|A_n^{(j)}(z)| \lesssim \begin{cases} |z|^{n-1-j} & , \text{if } j \in \{0, \ldots, n-1\}, \\ |z| & , \text{if } j \in \{n, n+2, n+4, \ldots\}, \\ 1 & , \text{if } j \in \{n+1, n+3, \ldots\}, \end{cases}$$

$$\left| \frac{d^j}{dz^j} \left(B_n(z) z^{\frac{1-n}{2}} \right) \right| \lesssim \begin{cases} 1 & , \text{if } j = 0, \\ |z|^{-1-j} & , \text{if } j \geq 1. \end{cases}$$

Proof. The estimate for A_n follows from

$$\begin{aligned} A_n(z) &= \chi(z) z^{\frac{n}{2}} J_{\frac{n-2}{2}}(z) \\ &= \chi(z) z^{\frac{n}{2}} \sum_{m=0}^\infty \frac{(-1)^m}{m! \Gamma(m + \frac{n}{2})} \left(\frac{z}{2}\right)^{\frac{n-2}{2}+2m} \\ &= \chi(z) \sum_{m=0}^\infty \frac{(-1)^m}{2^{\frac{n-2}{2}+2m} m! \Gamma(m+\frac{n}{2})} z^{n-1+2m}. \end{aligned} \tag{9}$$

The estimate for B_n follows from its series representation (7), see also [9, p.206]. □

Given the definition of g in (4) and the splitting (6) of the Bessel function, we decompose the integrand g according to $g = g_1 + g_2 + g_3$ where, for $r := 2\sqrt{t}\rho$,

$$\begin{aligned}
g_1(\rho) &:= \phi_{\mathrm{rad}}(r) A_n\left(\frac{r|x|}{2t}\right)\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}} = e^{2i\sqrt{t}\omega\rho}\phi_\omega(r) A_n\left(\frac{r|x|}{2t}\right)\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}, \\
g_2(\rho) &:= e^{i\frac{r|x|}{2t}}\phi_{\mathrm{rad}}(r) B_n\left(\frac{r|x|}{2t}\right)\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}} = e^{2i\sqrt{t}(\omega+\frac{|x|}{2t})\rho}\phi_\omega(r) B_n\left(\frac{r|x|}{2t}\right)\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}, \\
g_3(\rho) &:= e^{-i\frac{r|x|}{2t}}\phi_{\mathrm{rad}}(r) \overline{B_n\left(\frac{r|x|}{2t}\right)}\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}} = e^{2i\sqrt{t}(\omega-\frac{|x|}{2t})\rho}\phi_\omega(r) \overline{B_n\left(\frac{r|x|}{2t}\right)}\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}.
\end{aligned}$$

We now remove the linear phase factors by putting $g_{j,a_j}(\rho) := g_j(\rho)e^{-ia_j\rho}$ for

$$a_1 := 2\sqrt{t}\omega, \qquad a_2 := 2\sqrt{t}\left(\omega + \frac{|x|}{2t}\right), \qquad a_3 := 2\sqrt{t}\left(\omega - \frac{|x|}{2t}\right). \tag{10}$$

This implies, again for $r := 2\sqrt{t}\rho$,

$$\begin{aligned}
g_{1,a_1}(\rho) &= \phi_\omega(r) A_n\left(\frac{r|x|}{2t}\right)\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}, \\
g_{2,a_2}(\rho) &= \phi_\omega(r) B_n\left(\frac{r|x|}{2t}\right)\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}, \\
g_{3,a_3}(\rho) &= \phi_\omega(r) \overline{B_n\left(\frac{r|x|}{2t}\right)}\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}.
\end{aligned} \tag{11}$$

So we infer from Proposition 1

$$\psi(x,t)|x|^{\frac{n-2}{2}}\sqrt{t}e^{-i(\frac{|x|^2}{4t}-\frac{n\pi}{4})} = \int_0^\infty g(\rho)e^{i\rho^2}\,d\rho = \sum_{j=1}^3 \int_0^\infty g_{j,a_j}(\rho)e^{i(\rho^2+a_j\rho)}\,d\rho. \tag{12}$$

In order to estimate these terms, we make use of the following auxiliary result.

Proposition 3. *Let $a \in \mathbb{R}$ and $\Xi_a^m \in C^\infty(\mathbb{R};\mathbb{C})$ for $m \in \mathbb{N}_0$ be inductively defined by*

$$\Xi_a^0(s) := \int_s^\infty e^{i(\rho^2+a\rho)}\,d\rho, \qquad \Xi_a^m(s) := \int_s^\infty \Xi_a^{m-1}(\rho)\,d\rho.$$

Then $|\Xi_a^m(s)| \leq C_m$ for all $a \in \mathbb{R}, s \geq 0$.

Proof. This follows from $\Xi_a^m(s) = e^{-ia^2/4}\Xi_0^m(s+\frac{a}{2})$ once we have proved the estimate

$$|\Xi^m(s)| \leq C_m(1+s_+)^{-m-1} \qquad \text{for all } s \in \mathbb{R}$$

where $\Xi^m := \Xi_0^m$. The existence of the improper Fresnel integral $\Xi^0(s)$ is a well-known consequence of the Residue Theorem. Moreover, l'Hôpital's rule gives

$$\Xi^0(s)(-2is)e^{-is^2} \to 1 \qquad \text{and} \qquad s^2(1+2is\Xi^0(s)e^{-is^2}) \to \frac{i}{2} \quad \text{as } s \to \infty.$$

(For the second limit one may proceed as we do below in the computation of z_k.) Since the improper integral $\int_s^\infty \frac{1}{\rho} e^{i\rho^2}\, d\rho$ exists for $s > 1$ (again by the Residue Theorem), we obtain from the previous statement that the integral $\Xi^1(s)$ exists and

$$\Xi^1(s)(-2is)^2 e^{-is^2} \to 1 \qquad \text{and} \qquad s^2(1 - (-2is)^2 \Xi^1(s) e^{-is^2}) \to \frac{3i}{2} \quad \text{as } s \to \infty.$$

By induction, we find that for all $k \in \mathbb{N}, k \geq 2$ the (proper) integral $\Xi^k(s)$ exists and

$$\begin{aligned}
z_k &:= \lim_{s\to\infty} s^2\big(1 - (-2is)^{k+1}\Xi^k(s)e^{-is^2}\big) \\
&= \lim_{s\to\infty} \frac{e^{is^2}(-2is)^{-k-1} - \Xi^k(s)}{e^{is^2}s^{-2}(-2is)^{-k-1}} \\
&= \lim_{s\to\infty} \frac{-e^{is^2}(-2is)^{-k} + 2(k+1)ie^{is^2}(-2is)^{-k-2} - (\Xi^k)'(s)}{-e^{is^2}s^{-2}(-2is)^{-k} - 2e^{is^2}s^{-3}(-2is)^{-k-1} + 2(k+1)ie^{is^2}s^{-2}(-2is)^{-k-2}} \\
&= \lim_{s\to\infty} \frac{-(-2is)^{-k} + 2(k+1)i(-2is)^{-k-2} + \Xi^{k-1}(s)e^{-is^2}}{-s^{-2}(-2is)^{-k} - 2s^{-3}(-2is)^{-k-1} + 2(k+1)is^{-2}(-2is)^{-k-2}} \\
&= \lim_{s\to\infty} \frac{-(k+1)is^{-2} - 2 + 2(-2is)^k\Xi^{k-1}(s)e^{-is^2}}{-2s^{-2} - 2is^{-4} - (k+1)is^{-4}} \\
&= \lim_{s\to\infty} \frac{(k+1)i + 2s^2(1 - (-2is)^k\Xi^{k-1}(s)e^{-is^2})}{2 + 2is^{-2} + (k+1)is^{-2}} \\
&= \frac{(k+1)i}{2} + z_{k-1}
\end{aligned}$$

implying

$$z_k = \lim_{s\to\infty} s^2\big(1 - (-2is)^{k+1}\Xi^k(s)e^{-is^2}\big) = \frac{(k+2)(k+1)i}{4}.$$

This yields the bounds for $\Xi^m(s)$ and we are done. □

Let us remark that the proof actually yields the stronger estimate $|\Xi_a^m(s)| \leq C_m$ for $0 \leq s \leq a_-$ and $|\Xi_a^m(s)| \leq 2^{m+1}C_m(1+s)^{-m-1}$ for $s \geq a_-$. However, given that these estimates depend on a, it seems difficult to make use of them. Moreover, the independence of a guarantees that our estimates below do not depend on ω since the latter is completely absorbed in the definition of a_1, a_2, a_3 from (10). From (12) and Proposition 3 we deduce the following estimate for the solution ψ.

Proposition 4. *For all $x \in \mathbb{R}^n, t \in \mathbb{R}$, $m \in \{0, \dots, n\}$ we have*

$$(13)\quad |\psi(x,t)| \leq C_{m-1}|x|^{\frac{2-n}{2}} t^{-\frac{1}{2}} \left(\delta_{m,n}|g_{1,a_1}^{(n-1)}(0)| + \int_0^\infty |g_{1,a_1}^{(m)}(\rho)| + |g_{2,a_2}^{(m)}(\rho)| + |g_{3,a_3}^{(m)}(\rho)|\, d\rho \right)$$

where $g_{1,a_1}, g_{2,a_2}, g_{3,a_3}$ are given by (11).

Proof. In the following integration-by-parts scheme we use $(\Xi_{a_j}^m)' = -\Xi_{a_j}^{m-1}$ as well as

$$(14)\qquad g_{j,a_j}(0) = g'_{j,a_j}(0) = \ldots = g_{j,a_j}^{(n-2)}(0) = 0, \qquad |g_{2,a_2}^{(n-1)}(0)| = |g_{3,a_3}^{(n-1)}(0)| = 0,$$

which follows from (11) and Proposition 2. Recall that the support of B_n is contained in $[\frac{1}{2},\infty)$ by choice of the cut-off function χ so that the above estimate is actually trivial for $j\in\{2,3\}$. So we have for $m\in\{0,\ldots,n-1\}$

$$\begin{aligned}\int_0^\infty g_{j,a_j}(\rho)e^{i(\rho^2+a_j\rho)}\,d\rho &\overset{(14)}{=} \lim_{M\to\infty}\int_0^\infty\left(\int_0^\rho g_{j,a_j}'(t)\,dt\right)e^{i(\rho^2+a_j\rho)}\,d\rho\\ &=\lim_{M\to\infty}\int_0^M g_{j,a_j}'(t)\left(\int_t^M e^{i(\rho^2+a_j\rho)}\,d\rho\right)dt\\ &=\int_0^\infty g_{j,a_j}'(t)\Xi_{a_j}^0(t)\,dt\\ &\overset{(14)}{=}\int_0^\infty\left(\int_0^t g_{j,a_j}''(s)\,ds\right)\Xi_{a_j}^0(t)\,dt\\ &=\int_0^\infty g_{j,a_j}''(s)\Xi_{a_j}^1(s)\,ds\\ &=\ldots\\ &=\int_0^\infty g_{j,a_j}^{(m)}(\rho)\Xi_{a_j}^{m-1}(\rho)\,d\rho.\end{aligned}\tag{15}$$

Notice that the limit $M\to\infty$ passes under the integral because g_{j,a_j} has compact support and the $\Xi_{a_j}^k$ are bounded by Proposition 3. So we obtain for $m\in\{0,\ldots,n-1\}$

$$\begin{aligned}|\psi(x,t)| &\overset{(12)}{\leq} |x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}\sum_{j=1}^3\left|\int_0^\infty g_{j,a_j}^{(m)}(\rho)\Xi_{a_j}^{m-1}(\rho)\,d\rho\right|\\ &\leq C_{m-1}|x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}\sum_{j=1}^3\int_0^\infty |g_{j,a_j}^{(m)}(\rho)|\,d\rho.\end{aligned}$$

Moreover, using (15) for $m=n-1$ we get

$$\begin{aligned}|\psi(x,t)| &= |x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}\left|\sum_{j=1}^3\int_0^\infty g_{j,a_j}^{(n-1)}(\rho)\Xi_{a_j}^{n-2}(\rho)\,d\rho\right|\\ &=|x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}\left|\sum_{j=1}^3\left(g_{j,a_j}^{(n-1)}(0)\int_0^\infty\Xi_{a_j}^{n-2}(\rho)\,d\rho+\int_0^\infty\left(\int_0^t g_{j,a_j}^{(n)}(\rho)\,d\rho\right)\Xi_{a_j}^{n-2}(t)\,dt\right)\right|\\ &=|x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}\left|\sum_{j=1}^3\left(g_{j,a_j}^{(n-1)}(0)\Xi_{a_j}^{n-1}(0)+\int_0^\infty g_{j,a_j}^{(n)}(\rho)\Xi_{a_j}^{n-1}(\rho)\,d\rho\right)\right|\\ &\overset{(14)}{\leq} C_{n-1}|x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}\left(|g_{1,a_1}^{(n-1)}(0)|+\sum_{j=1}^3\int_0^\infty|g_{j,a_j}^{(n)}(\rho)|\,d\rho\right).\end{aligned}$$

□

For notational convenience we write $x \lesssim y$ respectively $x \gtrsim y$ instead of $x \le c\,y$ respectively $x \ge c\,y$ for positive numbers c that are independent of $\omega, |x|, t, r$ but may depend on $m \in \{0,\ldots,n\}$ or the space dimension $n \in \mathbb{N}$.

Proposition 5. *Let $m \in \{0,\ldots,n\}$. Then the functions $g_{1,a_1}, g_{2,a_2}, g_{3,a_3}$ from (11) satisfy the following estimates for all $\rho \ge 0$ and $r = 2\sqrt{t}\rho$:*

$$
\begin{aligned}
|g_{1,a_1}^{(m)}(\rho)| &\lesssim |x|^{\frac{n-2}{2}}(\sqrt{t})^{m-n+2}\sum_{k=0}^{m}|\phi_\omega^{(k)}(r)|r^{n-m+k-1}\cdot \mathbb{1}_{[0,1]}\left(\frac{r|x|}{2t}\right) \qquad \text{if } m<n\\
|g_{1,a_1}^{(n)}(\rho)| &\lesssim \left(|x|^{\frac{n-2}{2}}(\sqrt{t})^{2}\sum_{k=1}^{n}|\phi_\omega^{(k)}(r)|r^{k-1} + |x|^{\frac{n+2}{2}}(\sqrt{t})^{-2}|\phi_\omega(r)|r\right)\cdot \mathbb{1}_{[0,1]}\left(\frac{r|x|}{2t}\right)\\
|g_{2,a_2}^{(m)}(\rho)| + |g_{3,a_3}^{(m)}(\rho)| &\lesssim |x|^{-\frac{1}{2}}(\sqrt{t})^{1+m}\sum_{k=0}^{m}|\phi_\omega^{(k)}(r)|r^{\frac{n-1}{2}-m+k}\cdot \mathbb{1}_{[1/2,\infty)}\left(\frac{r|x|}{2t}\right).
\end{aligned}
$$

Proof. We get for $r = 2\sqrt{t}\rho$ and $m \in \{0,\ldots,n-1\}$

$$
\begin{aligned}
|g_{1,a_1}^{(m)}(\rho)| &\overset{(11)}{=} (2\sqrt{t})^m\left|\frac{d^m}{dr^m}\left(\phi_\omega(r)A_n\left(\frac{r|x|}{2t}\right)\right)\right|\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}\\
&\lesssim (\sqrt{t})^m\sum_{k=0}^{m}\left|\phi_\omega^{(k)}(r)A_n^{(m-k)}\left(\frac{r|x|}{2t}\right)\right|\left(\frac{|x|}{2t}\right)^{m-k-\frac{n}{2}}\\
&\overset{\text{Prop. }2}{\lesssim} (\sqrt{t})^m\sum_{k=0}^{m}|\phi_\omega^{(k)}(r)|\left(\frac{r|x|}{2t}\right)^{n-1-m+k}\left(\frac{|x|}{2t}\right)^{m-k-\frac{n}{2}}\cdot \mathbb{1}_{[0,1]}\left(\frac{r|x|}{2t}\right)\\
&\lesssim |x|^{\frac{n-2}{2}}(\sqrt{t})^{m-n+2}\sum_{k=0}^{m}|\phi_\omega^{(k)}(r)|r^{n-1-m+k}\cdot \mathbb{1}_{[0,1]}\left(\frac{r|x|}{2t}\right).
\end{aligned}
$$

This implies the first estimate. For $m = n$ we use the estimate for $A_n^{(j)}$ from Proposition 2 for $j \in \{0,\ldots,n\}$ and obtain

$$
\begin{aligned}
|g_{1,a_1}^{(n)}(\rho)| &\overset{(11)}{=} (2\sqrt{t})^n\left|\frac{d^n}{dr^n}\left(\phi_\omega(r)A_n\left(\frac{r|x|}{2t}\right)\right)\right|\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}\\
&\lesssim (\sqrt{t})^n\sum_{k=0}^{n}\left|\phi_\omega^{(k)}(r)A_n^{(n-k)}\left(\frac{r|x|}{2t}\right)\right|\left(\frac{|x|}{2t}\right)^{\frac{n}{2}-k}\\
&\lesssim (\sqrt{t})^n\left(\sum_{k=1}^{n}|\phi_\omega^{(k)}(r)|\left(\frac{r|x|}{2t}\right)^{-1+k}\left(\frac{|x|}{2t}\right)^{\frac{n}{2}-k} + |\phi_\omega(r)|\frac{r|x|}{2t}\left(\frac{|x|}{2t}\right)^{\frac{n}{2}}\right)\cdot \mathbb{1}_{[0,1]}\left(\frac{r|x|}{2t}\right)\\
&\lesssim \left(|x|^{\frac{n-2}{2}}(\sqrt{t})^{2}\sum_{k=1}^{n}|\phi_\omega^{(k)}(r)|r^{k-1} + |x|^{\frac{n+2}{2}}(\sqrt{t})^{-2}|\phi_\omega(r)|r\right)\cdot \mathbb{1}_{[0,1]}\left(\frac{r|x|}{2t}\right).
\end{aligned}
$$

This yields the second estimate. The third estimate results from

$$|B_n^{(j)}(z)| \lesssim |z|^{\frac{n-1}{2}-j}, \tag{16}$$

which is a consequence of Proposition 2.

$$\begin{aligned}
|g_{2,a_2}^{(m)}(\rho)| + |g_{2,a_3}^{(m)}(\rho)| &\overset{(11)}{=} 2(2\sqrt{t})^m \left|\frac{d^m}{dr^m}\left(\phi_\omega(r) B_n\left(\frac{r|x|}{2t}\right)\right)\right| \left(\frac{|x|}{2t}\right)^{-\frac{n}{2}} \\
&\lesssim (\sqrt{t})^m \sum_{k=0}^m |\phi_\omega^{(k)}(r)| \left|B_n^{(m-k)}\left(\frac{r|x|}{2t}\right)\right| \left(\frac{|x|}{2t}\right)^{m-k-\frac{n}{2}} \\
&\overset{(16)}{\lesssim} (\sqrt{t})^m \sum_{k=0}^m |\phi_\omega^{(k)}(r)| \left(\frac{r|x|}{2t}\right)^{\frac{n-1}{2}-m+k} \left(\frac{|x|}{2t}\right)^{m-k-\frac{n}{2}} \cdot \mathbb{1}_{[1/2,\infty)}\left(\frac{r|x|}{2t}\right) \\
&\lesssim |x|^{-\frac{1}{2}} (\sqrt{t})^{m+1} \sum_{k=0}^m |\phi_\omega^{(k)}(r)| r^{\frac{n-1}{2}-m+k} \cdot \mathbb{1}_{[1/2,\infty)}\left(\frac{r|x|}{2t}\right).
\end{aligned}$$

□

Proof of Theorem 2 (i): We combine the estimates from Proposition 4 and Proposition 5. Under the assumption $\phi_\omega \in C_c^\infty(\mathbb{R}_{\geq 0};\mathbb{C})$ we get for all $m \in \{0,\ldots,n-1\}$

$$\begin{aligned}
|\psi(x,t)| &\overset{\text{Prop. 4}}{\lesssim} |x|^{\frac{2-n}{2}} (\sqrt{t})^{-1} \int_0^\infty |g_{1,a_1}^{(m)}(\rho)| + |g_{2,a_2}^{(m)}(\rho)| + |g_{3,a_3}^{(m)}(\rho)|\, d\rho \\
&\overset{\text{Prop. 5}}{\lesssim} (\sqrt{t})^{m-n+1} \sum_{k=0}^m \int_0^{\frac{\sqrt{t}}{|x|}} |\phi_\omega^{(k)}(2\sqrt{t}\rho)| (2\sqrt{t}\rho)^{n-m+k-1}\, d\rho \\
&\qquad + |x|^{\frac{1-n}{2}} (\sqrt{t})^m \sum_{k=0}^m \int_{\frac{\sqrt{t}}{2|x|}}^\infty |\phi_\omega^{(k)}(2\sqrt{t}\rho)| (2\sqrt{t}\rho)^{\frac{n-1}{2}-m+k}\, d\rho \\
&\lesssim (\sqrt{t})^{m-n} \sum_{k=0}^m \int_0^{\frac{2t}{|x|}} |\phi_\omega^{(k)}(r)| r^{n-m+k-1}\, dr \\
&\qquad + (\sqrt{t})^{m-n} \left(\frac{t}{|x|}\right)^{\frac{n-1}{2}} \sum_{k=0}^m \int_{\frac{t}{|x|}}^\infty |\phi_\omega^{(k)}(r)| r^{\frac{n-1}{2}-m+k}\, dr \\
&\lesssim (\sqrt{t})^{m-n} \sum_{k=0}^m \left(\int_0^{\frac{2t}{|x|}} |\phi_\omega^{(k)}(r)| r^{n-m+k-1}\, dr + \int_{\frac{t}{|x|}}^\infty |\phi_\omega^{(k)}(r)| r^{n-m+k-1}\, dr \right) \\
&\lesssim (\sqrt{t})^{m-n} \|\phi_\omega\|_{Y_m}.
\end{aligned}$$

In the case $m = n$ we use the second estimate in Proposition 5 instead of the first one. By density of $C_c^\infty(\mathbb{R}_{\geq 0};\mathbb{C})$ in Y_m the result follows.

Proof of Theorem 2 (ii): For $r = 2\sqrt{t}\rho$ we use

$$|g'_{1,a_1}(\rho)| \lesssim |x|^{\frac{n-2}{2}}(\sqrt{t})^{3-n}\left(|\phi_\omega(r)|r^{n-2} + |\phi'_\omega(r)|r^{n-1}\right)\cdot \mathbb{1}_{[0,1]}\left(\frac{r|x|}{2t}\right)$$

as well as

$$\begin{aligned}
&|g'_{2,a_2}(\rho)| + |g'_{2,a_3}(\rho)| \\
&\overset{(11)}{=} 4\sqrt{t}\left|\frac{d}{dr}\left(\phi_\omega(r)B_n\left(\frac{r|x|}{2t}\right)\right)\right|\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}} \\
&\lesssim \sqrt{t}\left|\frac{d}{dr}\left(\phi_\omega(r)r^{\frac{n-1}{2}}\right)\right|\left|r^{\frac{1-n}{2}}B_n\left(\frac{r|x|}{2t}\right)\right|\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}} \\
&\quad + \sqrt{t}|\phi_\omega(r)|r^{\frac{n-1}{2}}\left|\frac{d}{dr}\left(r^{\frac{1-n}{2}}B_n\left(\frac{r|x|}{2t}\right)\right)\right|\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}} \\
&\overset{\text{Prop. } 2}{\lesssim} \sqrt{t}\left|\frac{d}{dr}\left(\phi_\omega(r)r^{\frac{n-1}{2}}\right)\right| r^{\frac{1-n}{2}}\left(\frac{r|x|}{2t}\right)^{\frac{n-1}{2}}\left(\frac{|x|}{2t}\right)^{-\frac{n}{2}}\cdot \mathbb{1}_{[1/2,\infty)}\left(\frac{r|x|}{2t}\right) \\
&\quad + \sqrt{t}|\phi_\omega(r)|r^{\frac{n-1}{2}}\left(\frac{r|x|}{2t}\right)^{-2}\left(\frac{|x|}{2t}\right)^{\frac{1}{2}}\cdot \mathbb{1}_{[1/2,\infty)}\left(\frac{r|x|}{2t}\right) \\
&\lesssim \left(|x|^{-\frac{1}{2}}(\sqrt{t})^2\left|\frac{d}{dr}\left(\phi_\omega(r)r^{\frac{n-1}{2}}\right)\right| + |x|^{-\frac{3}{2}}(\sqrt{t})^4|\phi_\omega(r)|r^{\frac{n-5}{2}}\right)\cdot \mathbb{1}_{[1/2,\infty)}\left(\frac{r|x|}{2t}\right).
\end{aligned}$$

This implies

$$\begin{aligned}
|\psi(x,t)| &\overset{\text{Prop. } 4}{\lesssim} |x|^{\frac{2-n}{2}}(\sqrt{t})^{-1}\int_0^\infty |g^{(m)}_{1,a_1}(\rho)| + |g^{(m)}_{2,a_2}(\rho)| + |g^{(m)}_{3,a_3}(\rho)|\, d\rho \\
&\overset{\text{Prop. } 5}{\lesssim} (\sqrt{t})^{2-n}\int_0^{\frac{\sqrt{t}}{|x|}}\left(|\phi_\omega(2\sqrt{t}\rho)|(2\sqrt{t}\rho)^{n-2} + |\phi'_\omega(2\sqrt{t}\rho)|(2\sqrt{t}\rho)^{n-1}\right)\, d\rho \\
&\quad + |x|^{\frac{1-n}{2}}\sqrt{t}\int_{\frac{\sqrt{t}}{2|x|}}^\infty \left|\frac{d}{dr}\left(\phi_\omega(r)r^{\frac{n-1}{2}}\right)\right|_{r=2\sqrt{t}\rho}\, d\rho \\
&\quad + |x|^{\frac{-1-n}{2}}(\sqrt{t})^3\int_{\frac{\sqrt{t}}{2|x|}}^\infty |\phi_\omega(2\sqrt{t}\rho)|(2\sqrt{t}\rho)^{\frac{n-5}{2}}\, d\rho \\
&\lesssim |x|^{\frac{1-n}{2}}\left(\frac{t}{|x|}\right)^{\frac{1-n}{2}}\int_0^{\frac{2t}{|x|}}\left(|\phi_\omega(r)|r^{n-2} + |\phi'_\omega(r)|r^{n-1}\right)\, dr \\
&\quad + |x|^{\frac{1-n}{2}}\left(\int_{\frac{t}{|x|}}^\infty\left|\frac{d}{dr}\left(\phi_\omega(r)r^{\frac{n-1}{2}}\right)\right|\, dr + \frac{t}{|x|}\int_{\frac{t}{|x|}}^\infty |\phi_\omega(r)|r^{\frac{n-5}{2}}\, dr\right) \\
&\lesssim |x|^{\frac{1-n}{2}}\|\phi_\omega\|_X.
\end{aligned}$$

So we get the result by density of $C_c^\infty(\mathbb{R}_{\geq 0};\mathbb{C})$ in X. ☐

3. Proof of Theorem 1

We estimate the solution of the Schrödinger equation for the initial datum

$$\phi(x) = \phi_{\mathrm{rad}}(|x|) \quad \text{where } \phi_{\mathrm{rad}}(\rho) := e^{-i\frac{\rho^2}{4}}\mathbb{1}_{\rho\geq 1}\rho^{-\sigma}$$

and σ is chosen according to $\frac{n-3}{2} < \sigma < n$. In this case, the formula (5) is well-defined and provides a solution of the initial value problem (1). In [1, Section 2.1] it was shown that the solution ψ blows up in $L^\infty(\mathbb{R}^n)$ as $t \to 1$ provided that $\frac{n}{2} < \sigma < n$ holds. In fact, in this case the function $a(x) := \mathbb{1}_{|x|\geq 1}|x|^{-\sigma}$ lies in $L^2(\mathbb{R}^n)$ but not in $L^1(\mathbb{R}^n)$ so that [1, Remark 2.2] applies. We now generalize this analysis to the range $\frac{n-3}{2} < \sigma < n$ and detect a selfsimilar blow-up in $L^q(\mathbb{R}^n)$ for all $q > \frac{n}{n-\sigma}$ and a lower estimate for the corresponding blow-up rate then implies $\|\psi\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))} = \infty$ for $p \geq \frac{2q}{((n-\sigma)q-n)_+}$. From this we will finally deduce the nonvalidity of Strichartz estimates for initial data $\phi \in L^r(\mathbb{R}^n)$ where $r > 2$.

We set $k_t := \sqrt{\frac{1}{4t} - \frac{1}{4}}$ for $0 \leq t < 1$ and write $\psi(x) = \psi_{\mathrm{rad}}(|x|)$. We get for $|x| = 2tk_t z$

$$\begin{aligned}
2|\psi(x,t)|k_t^{n-\sigma} &\overset{(5)}{=} |x|^{\frac{2-n}{2}} t^{-1} k_t^{n-\sigma} \left| \int_0^\infty J_{\frac{n-2}{2}}\left(\frac{\rho|x|}{2t}\right) \phi_{\mathrm{rad}}(\rho)\rho^{\frac{n}{2}} e^{i\frac{\rho^2}{4t}}\, d\rho \right| \\
&= |x|^{\frac{2-n}{2}} t^{-1} k_t^{n-\sigma} \left| \int_1^\infty J_{\frac{n-2}{2}}\left(\frac{\rho|x|}{2t}\right) \rho^{\frac{n}{2}-\sigma} e^{i\rho^2 k_t^2}\, d\rho \right| \\
&= (2tk_t z)^{\frac{2-n}{2}} t^{-1} k_t^{n-\sigma} \left| \int_1^\infty J_{\frac{n-2}{2}}(\rho k_t z) \rho^{\frac{n}{2}-\sigma} e^{i\rho^2 k_t^2}\, d\rho \right| \\
&= (2tk_t z)^{\frac{2-n}{2}} t^{-1} k_t^{\frac{n-2}{2}} \left| \int_{k_t}^\infty J_{\frac{n-2}{2}}(sz) s^{\frac{n}{2}-\sigma} e^{is^2}\, ds \right| \\
&\to (2z)^{\frac{2-n}{2}} \left| \int_0^\infty J_{\frac{n-2}{2}}(sz) s^{\frac{n}{2}-\sigma} e^{is^2}\, ds \right| \qquad \text{as } t \to 1
\end{aligned}$$

and the convergence is locally uniform in $z \in (0,\infty)$ due to $\frac{n-3}{2} < \sigma < n$. Since the right hand side is not identically zero we may find $\delta > 0$ and radii $0 < R_1 < R_2$ such that

$$|\psi(x,t)| \gtrsim k_t^{\sigma-n} \qquad \text{for } R_1 k_t \leq |x| \leq R_2 k_t \text{ and } 1-\delta < t < 1.$$

Hence we get

$$\begin{aligned}
\int_{1-\delta}^1 \left(\int_{B_{R_2k_t}(0)\setminus B_{R_1k_t}(0)} |\psi(x,t)|^q\, dx \right)^{p/q} dt &\gtrsim \int_{1-\delta}^1 \left(\int_{R_1k_t}^{R_2k_t} r^{n-1} k_t^{(\sigma-n)q}\, dr \right)^{p/q} dt \\
&\gtrsim \int_{1-\delta}^1 \left(k_t^{n+(\sigma-n)q} \right)^{p/q} dt \\
&\gtrsim \int_{1-\delta}^1 (1-t)^{\frac{p}{2q}(n+(\sigma-n)q)}\, dt.
\end{aligned}$$

This integral is finite if and only if $\frac{p}{2q}(n + (\sigma-n)q) > -1$. So $\psi \in L^p(\mathbb{R}; L^q(\mathbb{R}^n))$ can only hold for $p < \frac{2q}{((n-\sigma)q-n)_+}$. Moreover, the initial datum lies in $L^r(\mathbb{R}^n)$ if and only if $\sigma > \frac{n}{r}$. So, for any $r > 1$ we can consider the limit $\sigma \searrow \max\{\frac{n}{r}, \frac{n-3}{2}\}$ we find that the validity of the Strichartz estimate with initial datum in $L^r(\mathbb{R}^n), r > 1$ implies

$$p \leq \frac{2q}{((n - \max\{\frac{n}{r}, \frac{n-3}{2}\})q - n)_+} = \max\left\{\frac{2qr}{n((r-1)q - r)_+}, \frac{4q}{((n+3)q - 2n)_+}\right\}. \tag{17}$$

On the other hand, the scaling invariance of the Schrödinger equation implies $\frac{2}{p} + \frac{n}{q} = \frac{n}{r}$ and thus $p = \frac{2qr}{n(q-r)}$. Plugging this into (17) we obtain $r \leq 2$. Hence, the Strichartz estimate (3) cannot hold for any $r > 2$, which finishes the proof. □

4. Appendix

In this Appendix we briefly discuss the restriction $p \geq 2$ in the context of Strichartz estimates of the form

$$\|e^{it\Delta}\phi\|_{L^p_t(\mathbb{R};L^q(\mathbb{R}^n))} \lesssim \|\phi\|_{L^2(\mathbb{R}^n)},$$

which results from an abstract reasoning involving translation-invariant operators due to Hörmander [4], see [5, p.970-971]. Here we provide a family of explicit counterexamples that not only implies $p \geq 2$ for square integrable initial data, but even shows $p \geq \frac{2r}{(2n - r(n-1))_+}$ for initial data in $L^r(\mathbb{R}^n)$ with $r > \frac{2n}{n+1}$. In order to avoid lengthy computations involving oscillatory integrals, we only sketch the proofs. The starting point is a reasonable choice of an initial condition. We choose

$$\phi(x) = |x|^{\frac{2-n}{2}} \int_1^2 (\omega - 1)^{-\delta} J_{\frac{n-2}{2}}(\omega|x|)\, d\omega$$

for $\delta \in (0, 1)$. This function corresponds to a singular superposition of Herglotz waves, cf. Remark 1 (c). It is smooth and lenghty computations involving the van der Corput Lemma [7, p.334] reveal

$$|\phi(x)| \sim 2|x|^{\frac{1-n}{2} - (1-\delta)} \int_0^\infty \mathrm{Re}\left(e^{i\rho}\alpha_0 e^{i(|x| - \frac{(n-1)\pi}{4})}\right) \rho^{-\delta}\, d\rho \qquad \text{as } |x| \to \infty$$

where $\alpha_0 > 0$ is the domininant term in the series expansion of the Bessel funtion near infinity, see (8). In particular we get $\phi \in L^r(\mathbb{R}^n)$ if and only if $\delta < \frac{n+1}{2} - \frac{n}{r}$.

The above choice for the initial datum allows to write down the corresponding solution of the Schrödinger equation semi-explicitly via

$$\widehat{\psi(\cdot, t)}(\xi) = e^{-it|\xi|^2}\hat{\phi}(\xi) = e^{-it|\xi|^2}|\xi|^{-\frac{n}{2}}(|\xi| - 1)^{-\delta}\mathbb{1}_{[1,2]}(|\xi|).$$

Hence, one gets

$$\psi(x, t) = |x|^{\frac{2-n}{2}} \int_1^2 e^{-it\omega^2}(\omega - 1)^{-\delta} J_{\frac{n-2}{2}}(\omega|x|)\, d\omega$$

and the van der Corput Lemma implies $|\psi(x, t)| \gtrsim ct^{\delta-1}$ for small $|x|$ and large t. In particular, $\psi \in L^p(\mathbb{R}; L^q(\mathbb{R}^n))$ implies $p(1 - \delta) > 1$. So we conclude that for any $r \in (\frac{2n}{n+1}, \infty]$ we can consider the limit $\delta \nearrow \min\{1, \frac{n+1}{2} - \frac{n}{r}\}$ in the above computations and obtain that $\psi \in L^p(\mathbb{R}; L^q(\mathbb{R}^n))$ implies

$$p \geq \frac{2r}{(2n - r(n-1))_+} \qquad \text{provided } r > \frac{2n}{n+1}. \tag{18}$$

For $r = 2$, i.e., for square integrable initial data, this implies $p \geq 2$, which is all we wanted to demonstrate.

Let us mention that a detailed analysis of ψ reveals $\psi \in L^p(\mathbb{R}; L^q(\mathbb{R}^n))$ if and only if

$$\begin{aligned} q &> \max\left\{\frac{2n-1}{n-\delta}, \frac{2n}{n+1-2\delta}\right\} \qquad \text{and} \\ p &> \max\left\{\frac{1}{1-\delta}, \frac{2q}{q(n-\delta)+1-2n}, \frac{2q}{q(n+1-2\delta)-2n}\right\}. \end{aligned}$$

Keeping the scaling condition $\frac{2}{p} + \frac{n}{q} = \frac{n}{r}$ in mind, these a priori more restrictive conditions do however not result in stronger necessary conditions than (18), so that further necessary conditions cannot be deduced from this example.

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