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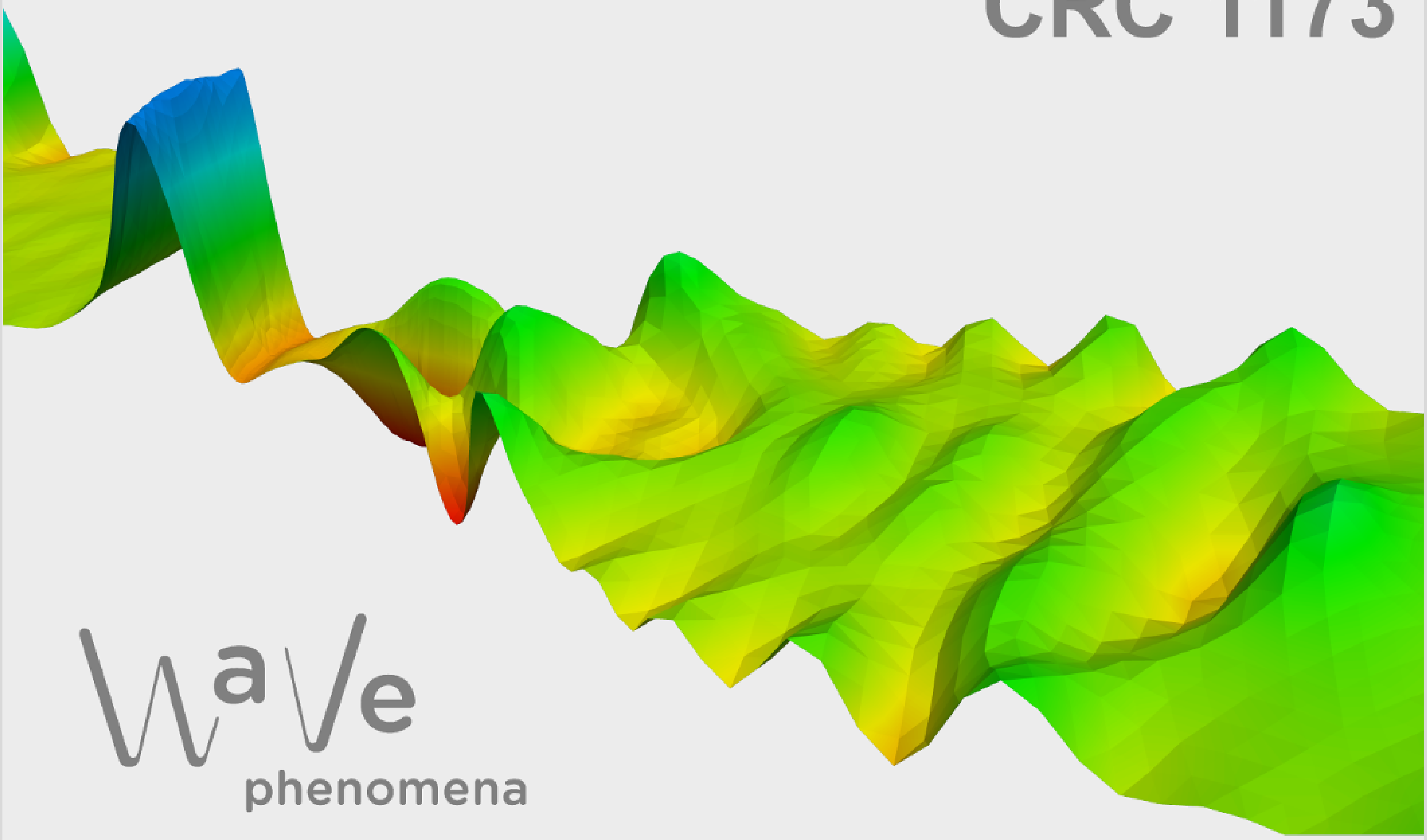

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Abstract In this paper we consider central fluxes discontinuous Galerkin space discretizations of a general class of wave-type equations of Friedrichs' type. This class includes important examples such as Maxwell's equations and wave equations. We prove an optimal error bound which holds under suitable regularity assumptions on the solution. Our analysis is performed in a framework of evolution equations on a Hilbert space and thus allows for the combination with various time integration schemes.

1 Introduction

The aim of this paper is to provide a rigorous error analysis of central fluxes discontinuous Galerkin (dG) space discretizations of a large class of linear wave-type equations of the following form. For a given initial value u^0 we seek a solution u such that

$$\begin{cases} M\partial_t u = \mathcal{L}u + g, & \mathbb{R}_+ \times \Omega, \quad (1a) \\ u(0) = u^0, & \Omega, \quad (1b) \end{cases}$$

supplied with suitable boundary conditions, which will be specified later. Here, Ω is an open, bounded and connected Lipschitz domain in $\mathbb{R}^d$, M is a symmetric positive definite material tensor, and g is a source term. Further, $\mathcal{L}$ is a *Friedrichs' operator* [8] given by

$$\mathcal{L}u = \sum_{i=1}^{d} L_i \partial_i u + L_0 u, \qquad L_i \in \mathbb{R}^{m\times m}, \quad i = 0, \dots, d, \tag{2}$$

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where $L_1, \dots, L_d$ are symmetric and the symmetric part of L_0 is negative semi-definite, i.e., $x^T(L_0 + L_0^T)x \le 0$ for all $x \in \mathbb{R}^m$. For the sake of presentation, we restrict ourselves to constant matrix coefficients L_i although our results also hold for space dependent coefficients under certain additional assumptions. We refer to [1, 14] for the more general case.

Important examples of this class of problems are the wave equation in first order formulation, Maxwell's equations, and the advection equation, see, e.g., [1, 2, 3, 14].

Partial differential equations governed by Friedrichs' operators have been studied intensively in the series of papers [5, 6, 7] and Chapter 7 of the book [3]. However, the results therein are only applicable to stationary problems or by treating the problem as a space-time problem, where the temporal variable is incorporated into the Friedrichs' operator.

In contrast to this work, our analysis is performed in a framework of evolution equations and can thus be combined with various time integration schemes. This then ultimately leads to full discretization error bounds, as has been shown in the thesis [14] for the particular choice of a Peaceman–Rachford ADI scheme. A proof of the wellposedness of (1) supplied with suitable initial and boundary conditions was recently provided in [1]. This analysis covers the special case of $M = I$, which means that the material parameters are incorporated into the coefficients of the differential operator $\mathcal{L}$. Unfortunately, this excludes materials with sharp interfaces as the coefficients of $\mathcal{L}$ need to fulfill certain regularity restrictions, see the second remark in [1, Sec. 3.2].

Hence, we follow a slightly different approach by incorporating the material tensor into the inner product of the state space. This allows us to weaken the restrictions on the regularity of the material parameters. Moreover, we treat boundary conditions as in [5] since this fits better to the dG discretization than the approach in [1].

Semi-discretizations of more general hyperbolic problems were considered in a unified error analysis in [11]. In this analysis, error bounds are given in terms of various discretization defects, interpolation errors, errors in the approximations of the spatial domain, the bilinear forms, and starting values. To apply this analysis to a particular application and discretization, one has to check that the continuous and the discretized problem both fit into the very general framework and to provide bounds for all these approximation errors. This constitutes the main work in our paper. Although we could then apply the general result of [11], we present proofs of the final error bound in Section 4, since they are relatively short for our application and this keeps the paper self-contained.

To the best of our knowledge, such bounds for hyperbolic evolution equations in first order formulation are only available for Maxwell's equations, cf. [15]. For the wave equation in second order formulation similar results were derived in [9], where the Laplace operator was discretized by a symmetric interior penalty dG method.

Our main result shows that the solution of the spatially discrete evolution equation has an error of order h^k in the L^2-norm induced by the material tensor M.

The paper is organized as follows. In Section 2 we provide the analytical framework for our paper. In particular, we collect properties of Friedrichs' operators and show the wellposedness of the linear wave-type problem (1). Section 3 is devoted to the dG discretization of linear wave-type equations written as a Friedrichs' system. We show various properties of the discretized Friedrichs' operator which are crucial for the following error analysis given in Section 4.

Notation

Throughout this paper we use the following notation:

We use $d \in \mathbb{N}$ as the spatial dimension and $m \in \mathbb{N}$ as a generic positive integer, usually being the number of components of vector-valued functions. The indicator function of a set $S \subset \mathbb{R}^d$ is denoted as $\mathbb{1}_S$.

Let $(X, (\cdot \mid \cdot)_X)$ and $(Y, (\cdot \mid \cdot)_Y)$ be real Hilbert spaces. The identity operator on a Hilbert space X is denoted by $\mathcal{I}$. By $\mathcal{B}(X, Y)$ we denote the set of all bounded operators from X to Y and we abbreviate $\mathcal{B}(X) = \mathcal{B}(X, X)$. The dual space of a Hilbert space X is denoted as X' and we use the notation $\langle \cdot \mid \cdot \rangle \colon X' \times X \to \mathbb{R}$ for the canonical dual pairing between a Hilbert space and its dual space.

Let $K \subset \mathbb{R}^d$ open. Then we denote the space of infinitely differentiable functions, which have compact support on K as $C_c^\infty(K)$. For vector-valued functions $u, v \in L^2(K)^m$, the $L^2(K)$-inner product is denoted by

$$(u \mid v)_K = \int_K u \cdot v \, \mathrm{d}x,$$

and for $F \subset \partial K$ and $u|_F, v|_F \in L^2(F)^m$ we write

$$(u \mid v)_F = \int_F u|_F \cdot v|_F \, \mathrm{d}\sigma.$$

The norms induced by these inner products are denoted by $\| \cdot \|_K$ and $\| \cdot \|_F$. We denote the $H^q(K)$-norm and seminorm by $\| \cdot \|_{q,K}$ and $| \cdot |_{q,K}$, respectively.

Let $M \in L^\infty(K)^{m\times m}$ be a square matrix-valued field on K. We denote the essential supremum of the spectral norm of M by

$$\|M\|_{\infty,K} = \operatorname*{ess\,sup}_{x\in K} \|M(x)\|,$$

where $\| \cdot \|$ is the spectral norm.

2 Analytical properties of Friedrichs' systems

The *graph space of* a Friedrichs' operator $\mathcal{L}$ defined in (2) is given by

$$H(\mathcal{L}) = \{v \in L^2(\Omega)^m \mid \mathcal{L}v \in L^2(\Omega)^m\},$$

and, endowed with the *graph norm* $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_{\Omega} + \|\mathcal{L}\cdot\|_{\Omega}$, is a Hilbert space [3, Lem. 7.2]. Note that, by definition, we have $\mathcal{L} \in \mathcal{B}(H(\mathcal{L}), L^2(\Omega)^m)$. The notation $H(\mathcal{L})$ is chosen based on the spaces $H(\mathrm{div})$ and $H(\mathrm{curl})$, which are the corresponding concepts for the divergence and curl operator, respectively.

Definition 2.1 We call $\mathcal{L}^{\circledast} \in \mathcal{B}(H(\mathcal{L}), L^2(\Omega)^m)$ defined by

$$\mathcal{L}^{\circledast} u = -\sum_{i=1}^{d} L_i \partial_i u + L_0^T u \tag{3}$$

the *formal adjoint of* $\mathcal{L}$.

Functions in $H(\mathcal{L})$ are not necessarily smooth enough to admit L^2-traces on the boundary. To still obtain access to boundary values in this weak setting, we follow [5] and introduce the following abstract boundary operator.

Definition 2.2 We call $\mathcal{L}_{\partial}\colon H(\mathcal{L}) \to H(\mathcal{L})'$ defined by

$$\langle \mathcal{L}_{\partial} u \,|\, v \rangle = (\mathcal{L} u \,|\, v)_{\Omega} - (u \,|\, \mathcal{L}^{\circledast} v)_{\Omega} \qquad \text{for all } u, v \in H(\mathcal{L}) \tag{4}$$

the *boundary operator associated with* $\mathcal{L}$.

We point out that (4) can be seen as a generalization of the integration by parts formula. Further, by [5, Sec. 2.1] we have $\mathcal{L}_{\partial} \in \mathcal{B}(H(\mathcal{L}), H(\mathcal{L})')$ and that $\mathcal{L}_{\partial}$ is self-adjoint.

Next, we implement boundary conditions into the abstract setting. In particular, we consider a class of homogeneous conditions that can be treated by incorporating them into the space on which the wave-type problem (1) is considered. Again, we follow [5] and pose the following assumption.

Assumption 2.3 We assume there exists a bounded operator $\mathcal{L}_{\Gamma} \in \mathcal{B}(H(\mathcal{L}), H(\mathcal{L})')$ fulfilling

$$\langle \mathcal{L}_{\Gamma} v \,|\, v \rangle \le 0 \quad \text{for all } v \in H(\mathcal{L}), \tag{5a}$$

$$H(\mathcal{L}) = \ker(\mathcal{L}_{\partial} - \mathcal{L}_{\Gamma}) + \ker(\mathcal{L}_{\partial} + \mathcal{L}_{\Gamma}). \tag{5b}$$

Note that both $\ker(\mathcal{L}_{\partial} - \mathcal{L}_{\Gamma})$ and $\ker(\mathcal{L}_{\partial} + \mathcal{L}_{\Gamma})$ are Hilbert spaces if endowed with the graph norm of $\mathcal{L}$, as they are the kernels of bounded operators on $H(\mathcal{L})$.

Theorem 2.4 *The restriction of* $\mathcal{L}$ *to* $\ker(\mathcal{L}_{\partial} - \mathcal{L}_{\Gamma})$ *is maximal dissipative.*

Proof Let $v \in \ker(\mathcal{L}_{\partial} - \mathcal{L}_{\Gamma})$. By Definitions 2.2 and 2.1 of the boundary operator and the formal adjoint, respectively, we have

$$\begin{aligned}
2(\mathcal{L} v \,|\, v)_{\Omega} &= (\mathcal{L} v \,|\, v)_{\Omega} + (\mathcal{L}^{\circledast} v \,|\, v)_{\Omega} + (\mathcal{L} v \,|\, v)_{\Omega} - (\mathcal{L}^{\circledast} v \,|\, v)_{\Omega} \\
&= ((L_0 + L_0^T) v \,|\, v)_{\Omega} + \langle \mathcal{L}_{\partial} v \,|\, v \rangle \\
&\le \langle (\mathcal{L}_{\partial} - \mathcal{L}_{\Gamma}) v \,|\, v \rangle + \langle \mathcal{L}_{\Gamma} v \,|\, v \rangle \\
&\le 0,
\end{aligned}$$

where the first inequality follows since the symmetric part of L_0 is negative semi-definite and the second because of $v \in \ker(\mathcal{L}_\partial - \mathcal{L}_\Gamma)$ and (5a). Hence, $\mathcal{L}$ is dissipative on $\ker(\mathcal{L}_\partial - \mathcal{L}_\Gamma)$. The maximality is a direct consequence of [5, Theorem 2.5], which shows that $(\mathcal{I} - \lambda\mathcal{L})\colon \ker(\mathcal{L}_\partial - \mathcal{L}_\Gamma) \to L^2(\Omega)^m$ is an isomorphism for $\lambda > 0$. □

Next, we show wellposedness of (1) by using semigroup theory. Hence, we define the *domain of* $\mathcal{L}$ as $D(\mathcal{L}) \coloneqq \ker(\mathcal{L}_\partial - \mathcal{L}_\Gamma)$. We assume that $M \in L^\infty(\Omega)^{m\times m}$ is symmetric positive definite a.e. on Ω, and that the source term satisfies $g \in C(\mathbb{R}_+; L^2(\Omega)^m)$. Then, Theorem 2.4 already yields wellposedness of (1) on $D(\mathcal{L})$ for suitable initial conditions if $M = I$. This is due to the fact that $\mathcal{L}|_{D(\mathcal{L})}$ is the generator of a contraction semigroup w.r.t. $\|\cdot\|_\Omega$ by the Lumer–Phillips Theorem [4, Thm. II.3.15, Cor. II.3.20].

If we have $M \neq I$, we define the weighted inner product $(\,\cdot \mid \cdot\,)_M$ by

$$(u \mid v)_M = (Mu \mid v)_\Omega, \qquad u, v\colon \Omega \to \mathbb{R}^d$$

and denote the induced norm by $\|\cdot\|_M$. Note that since this inner product is equivalent to the standard L^2 inner product, $L^2(\Omega)^m$ is again a Hilbert space if endowed with $(\,\cdot \mid \cdot\,)_M$.

By multiplying (1a) with M^{-1}, we obtain the (equivalent) abstract evolution problem

$$\begin{cases} \partial_t u = \widetilde{\mathcal{L}} u + f, \qquad \mathbb{R}_+ \times \Omega, & \text{(6a)} \\ u(0) = u^0, & \text{(6b)} \end{cases}$$

with $\widetilde{\mathcal{L}} \coloneqq M^{-1}\mathcal{L}$ and $f \coloneqq M^{-1}g$. We can now use Theorem 2.4 to show that the restriction of $\widetilde{\mathcal{L}}$ to $D(\mathcal{L})$ is maximal dissipative w.r.t. the weighted inner product $(\,\cdot \mid \cdot\,)_M$.

Theorem 2.5 *The restriction of* $\widetilde{\mathcal{L}}$ *to* $D(\mathcal{L})$ *is maximal dissipative.*

Proof The dissipativity of $\widetilde{\mathcal{L}}$ directly follows from the dissipativity of $\mathcal{L}$ as we have

$$(\widetilde{\mathcal{L}} u \mid v)_M = (M\widetilde{\mathcal{L}} u \mid v)_\Omega = (\mathcal{L} u \mid v)_\Omega \leq 0. \tag{7}$$

Maximality again follows as a consequence of [5, Theorem 2.5], which yields that $(M - \lambda\mathcal{L})\colon D(\mathcal{L}) \to L^2(\Omega)^m$ is an isomorphism for all $\lambda > 0$. Since M is positive definite and bounded, this is equivalent to $(\mathcal{I} - \lambda\widetilde{\mathcal{L}})\colon D(\mathcal{L}) \to L^2(\Omega)^m$ being an isomorphism, yielding the desired range condition. □

Hence, by the Lumer–Phillips Theorem, the restriction of $\widetilde{\mathcal{L}}$ to $D(\mathcal{L})$ generates a contraction semigroup w.r.t. $\|\cdot\|_M$, which we denote by $(\,\mathrm{e}^{t\widetilde{\mathcal{L}}}\,)_{t\geq 0}$.

Corollary 2.6 *Let* $f \in C^1(\mathbb{R}_+; L^2(\Omega)^m) \cup C(\mathbb{R}_+; D(\mathcal{L}))$*. Then, for given initial value* $u^0 \in D(\mathcal{L})$*, there exists a unique solution* $u \in C^1(\mathbb{R}_+; L^2(\Omega)^m) \cap C(\mathbb{R}_+; D(\mathcal{L}))$ *of* (6) *given by the variation-of-constants formula*

$$u(t) = \mathrm{e}^{t\widetilde{\mathcal{L}}} u^0 + \int_0^t \mathrm{e}^{(t-s)\widetilde{\mathcal{L}}} f(s)\,\mathrm{d}s.$$

Remark 2.7 For the sake of presentation, we only consider Friedrichs' operators with constant coefficients. However, all of the above can be extended to more general coefficients, e.g., Lipschitz coefficients. We refer to [14, Chapter 2] for the more general case.

Further, the assumption of negative semi-definiteness of L_0 can be dropped. This leads to $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ being shift-dissipative rather than dissipative on $D(\mathcal{L})$, see also the remark on the positivity condition (F2) in [1, Section 2.1].

Let us also point out that the restriction of the formal adjoint operator $\mathcal{L}^{\circledast}$ to $\ker(\mathcal{L}_\partial + \mathcal{L}_\Gamma^*)$ is maximal dissipative. This can be shown with the exact same strategy of proof. In fact, this is the Hilbert space adjoint of $\mathcal{L}$ restricted to $D(\mathcal{L})$.

3 Spatial discretization

To obtain a spatially discretized version of (6) we discretize the differential operator $\widetilde{\mathcal{L}}$ using a central fluxes dG approximation [3, 10]. As $\widetilde{\mathcal{L}}$ is defined via the Friedrichs' operator $\mathcal{L}$ we start by discretizing the latter and then define the discrete version of $\widetilde{\mathcal{L}}$ analogously to the continuous case.

To avoid technicalities, we assume that the domain Ω is a polyhedron, meaning we can decompose Ω into a polyhedral mesh. However, we refer to [11] for a way to take errors made by approximating non-polyhedral domains into account.

For the sake of readability, we postpone some of the longer proofs in this section. They can be found in the appendix.

3.1 Discrete setting

Before we define discrete Friedrichs' operators, we introduce some notation and the discrete setting. Let $\mathcal{T}$ be a mesh of Ω. For each (open) mesh element $K \in \mathcal{T}$, we denote the diameter of K by h_K. To write down mesh-dependent norms more concisely, we further define the piecewise constant function $h \in L^\infty(\Omega)$ by $h|_K \equiv h_K$ for all $K \in \mathcal{T}$. The maximal diameter $\hbar = \max_{K\in\mathcal{T}} h_K$ of all elements in $\mathcal{T}$ is called the meshsize of $\mathcal{T}$ and we use the notation $\mathcal{T}_\hbar$ for a mesh with meshsize $\hbar$. In order to investigate the convergence of the method, we consider a mesh sequence $\mathcal{T}_\mathcal{H} = (\mathcal{T}_\hbar)_{\hbar\in\mathcal{H}}$, where $\mathcal{H}$ is a countable collection of positive numbers with 0 as only accumulation point. We assume that we have $\hbar < 1$ for all $\hbar \in \mathcal{H}$ and that $\mathcal{T}_\mathcal{H}$ is admissible in the sense of [3, Def. 1.57], meaning it is shape- and contact regular and has optimal polynomial approximation properties, cf., [3, Def. 1.38, 1.55]. We denote the mesh regularity parameter by ρ.

We gather the faces of a mesh $\mathcal{T}_\hbar$ in the set $\mathcal{F}_\hbar = \mathcal{F}_\hbar^{\mathrm{int}} \cup \mathcal{F}_\hbar^{\mathrm{bnd}}$, where $\mathcal{F}_\hbar^{\mathrm{int}}$ contains the interior faces and $\mathcal{F}_\hbar^{\mathrm{bnd}}$ contains the boundary faces. For each $K \in \mathcal{T}_\hbar$, we denote the faces composing the boundary of an element K by $\mathcal{F}_\hbar^K = \mathcal{F}_\hbar^{K,\mathrm{int}} \cup \mathcal{F}_\hbar^{K,\mathrm{bnd}}$, again decomposed into interior and boundary faces. The maximum number of faces per element in $\mathcal{T}_\hbar$ is denoted by $N_\partial = \max_{K \in \mathcal{T}_\hbar} |\mathcal{F}_\hbar^K|$. Note that by [3, Lemma 1.41], N_∂ is bounded independently of $\hbar \in \mathcal{H}$.

The outward unit normal vector to an element $K \in \mathcal{T}_\hbar$ is denoted by n^K. Further, for each interface $F \in \mathcal{F}_\hbar^{\mathrm{int}}$, we arbitrarily denote the two neighboring elements, whose boundaries contain F, as K_1^F and K_2^F. We fix this choice and define the face normal vector n^F as the outward unit normal vector to K_1^F. For all boundary faces $F \in \mathcal{F}_\hbar^{\mathrm{bnd}}$, we define n^F as the outward unit normal vector to Γ.

To approximate functions in space, we consider the discrete approximation space

$$V_\hbar = \{ \boldsymbol{v} \in L^2(\Omega) \mid \boldsymbol{v}|_K \in \mathbb{Q}_d^k(K) \text{ for all } K \in \mathcal{T}_\hbar \}^m,$$

where $\mathbb{Q}_d^k(K)$ denotes the set of polynomials on K of degree at most k in each variable.

Remark 3.1 For the sake of presentation, we use the same polynomial degree on all elements $K \in \mathcal{T}_\hbar$. However, we point out that the method is flexible enough to easily allow varying polynomial degrees on each element. Note also that other choices for the discrete approximation space are possible. We refer to [3, Sec. 1.2.4.3] for further details.

We will frequently need the L^2-orthogonal projection $\pi_h \colon L^2(\Omega)^m \to V_\hbar$ onto $V_\hbar$, defined such that for $v \in L^2(\Omega)^m$ we have

$$(v - \pi_h v \,|\, \boldsymbol{\varphi})_\Omega = 0 \qquad \text{for all } \boldsymbol{\varphi} \in V_\hbar. \tag{8}$$

Using the L^2-orthogonal projection, by

$$e_\pi^v = v - \pi_h v$$

we denote the projection error of a function $v \in L^2(\Omega)^m$.

Assumption 3.2 We assume that the material tensor M is piecewise constant and that for all $\hbar \in \mathcal{H}$, the mesh $\mathcal{T}_\hbar$ is matched to the material, i.e., for all $K \in \mathcal{T}_\hbar$ we have $M|_K \equiv M_K$ with constant $M_K \in \mathbb{R}^{m \times m}$.

It is easy to see that for $v \in L^2(\Omega)^m$ we have

$$(e_\pi^v \,|\, \boldsymbol{\varphi})_M = 0 \qquad \text{for all } \boldsymbol{\varphi} \in V_\hbar$$

because of Assumption 3.2.

Since we assumed the mesh sequence to be admissible, we can infer some important properties of the discrete spaces. Namely, the inverse inequality [3, Lem. 1.44]

$$\|\nabla \boldsymbol{v}\|_K \le C'_{\mathrm{inv}} \|h^{-1} \boldsymbol{v}\|_K, \tag{9}$$

and the discrete trace inequality [3, Lem. 1.46]

$$\|\boldsymbol{v}\|_F \le C_{\mathrm{tr}} \|h^{-1/2} \boldsymbol{v}\|_K \tag{10}$$

hold as a consequence of the shape- and contact regularity. From the inverse inequality (9), we can easily deduct a similar inequality for the Friedrichs' operator $\mathcal{L}$ instead of the gradient, namely

$$\|\mathcal{L}\boldsymbol{v}\|_K \leq C_{\mathcal{L}} C_{\mathrm{inv}} \|h^{-1}\boldsymbol{v}\|_K, \tag{11}$$

where $C_{\mathcal{L}} = \max_{i=0,\dots,d} \|L_i\|$ and $C_{\mathrm{inv}} = \sqrt{d}C'_{\mathrm{inv}} + 1$.

Further, the mesh sequence $\mathcal{T}_{\mathcal{H}}$ has optimal polynomial approximation properties in the sense of [3, Def. 1.55]. This means that for all $\hbar \in \mathcal{H}$, $K \in \mathcal{T}_\hbar$, $F \in \mathcal{F}_\hbar^K$ and $v \in H^{q+1}(K)$ the projection error of v satisfies

$$\|e_\pi^v\|_K \leq C_\pi |h^{q+1} v|_{q+1,K}, \qquad \|e_\pi^v\|_F \leq C_{\pi,\partial} |h^{q+1/2} v|_{q+1,K}, \tag{12}$$

where C_π and $C_{\pi,\partial}$ are independent of both K and $\hbar$.

The space $V_\hbar$ consists of functions that are polynomials on the elements of $\mathcal{T}_\hbar$. Hence, they can be used to approximate functions that are sufficiently smooth on these elements. Such functions are gathered in the broken Sobolev spaces

$$H^q(\mathcal{T}_\hbar) = \{\, v \in L^2(\Omega) \mid v|_K \in H^q(K) \text{ for all } K \in \mathcal{T}_\hbar \,\}, \qquad q \in \mathbb{N},$$

which are Hilbert spaces if endowed with the norm

$$\|v\|_{q,\mathcal{T}_\hbar}^2 = \sum_{j=0}^{q} |v|_{j,\mathcal{T}_\hbar}^2, \qquad |v|_{q,\mathcal{T}_\hbar}^2 = \sum_{K \in \mathcal{T}_\hbar} |v|_{q,K}^2.$$

Functions in both $V_\hbar$ and $H^q(\mathcal{T}_\hbar)$ are only piecewise smooth, i.e., smooth on every mesh element $K \in \mathcal{T}_\hbar$, but not necessarily on the whole domain Ω. Hence, they may have discontinuities across the faces of the mesh, which is why we define the average and the jump of a function v across an interior face $F \in \mathcal{F}_\hbar^{\mathrm{int}}$ as

$$\{\!\{v\}\!\}_F = \frac{v|_{K_1^F} + v|_{K_2^F}}{2} \qquad \text{and} \qquad [\![v]\!]_F = v|_{K_1^F} - v|_{K_2^F},$$

respectively. Here and in the following, the restriction of v to an element $K \in \mathcal{T}_\hbar$ evaluated on a face $F \in \mathcal{F}_\hbar^K$ is understood as the limit of v approaching F from K. For vector and matrix fields these operations act componentwise.

3.2 Friedrichs' operators in the discrete setting

Up until now, we only got hold of the boundary operators in an abstract way, since functions in the graph space of a Friedrichs' operator $\mathcal{L}$ are not necessarily smooth enough to admit square-integrable traces. However, to define the discrete operators and to implement them in the full discretization scheme, it is convenient to access boundary values in a more explicit way. This can be achieved by assuming a bit more regularity, which enables us to use the integration by parts formulas.

Lemma 3.3 *Let $S \subset \Omega$ with outward unit normal vector n^S and let $\mathcal{L}^{\circledast}$ be the formal adjoint of $\mathcal{L}$ defined in* (3)*. Then for $v, w \in D(\mathcal{L})$ with $v|_S, w|_S \in L^2(\partial S)^m$ we have*

$$(\mathcal{L}v \,|\, w)_S - (v \,|\, \mathcal{L}^{\circledast} w)_S = (\sum_{i=1}^{d} \mathrm{n}_i^S L_i v \,|\, w)_{\partial S},$$

and, in particular,

$$\langle \mathcal{L}_\partial v \,|\, w \rangle = (\sum_{i=1}^{d} \mathrm{n}_i^\Omega L_i v \,|\, w)_\Gamma.$$

Proof By the definition of $\mathcal{L}$, we have

$$(\mathcal{L}v \,|\, w)_S = \sum_{i=1}^{d} (\partial_i v \,|\, L_i w)_S + (v \,|\, L_0^T w)_S.$$

Using integration by parts and the fact that the coefficients of $\mathcal{L}$ are constant yields

$$\begin{aligned}(\mathcal{L}v \,|\, w)_S &= \sum_{i=1}^{d} \Big((v \,|\, -\partial_i (L_i w))_S + (\mathrm{n}_i^K v \,|\, L_i w)_{\partial S} \Big) + (v \,|\, L_0^T w)_S \\ &= (v \,|\, \mathcal{L}^{\circledast} w)_S + (\sum_{i=1}^{d} \mathrm{n}_i^K L_i v \,|\, w)_{\partial S}\end{aligned}$$

by definition of $\mathcal{L}^{\circledast}$. □

Note that, in particular, the assumptions of Lemma 3.3 are fulfilled for elements $S = K \in \mathcal{T}_h$ if $v, w \in D(\mathcal{L}) \cap H^1(\mathcal{T}_h)^m$.

Definition 3.4 For $K \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$ we define the *boundary operators* $\mathcal{L}_\partial^K \in \mathcal{B}(L^2(\partial K)^m)$ *associated with an element of the mesh* and $\mathcal{L}_\partial^F \in \mathcal{B}(L^2(F)^m)$ *associated with a face of the mesh* by

$$\mathcal{L}_\partial^K = \sum_{i=1}^{d} \mathrm{n}_i^K L_i \qquad \text{and} \qquad \mathcal{L}_\partial^F = \sum_{i=1}^{d} \mathrm{n}_i^F L_i,$$

respectively.

The definition of $\mathcal{L}_\partial^K$ is motivated by Lemma 3.3, which relates it to the boundary term of the integration by parts formula on each element of the mesh. Further, the operator $\mathcal{L}_\partial^F$ will allow for a more concise notation in the definition and handling of the discrete operator. The latter is well-defined as we consider constant coefficients L_i, $i = 1, \ldots, d$, and hence, their traces are single-valued on each face.

To get hold of the abstract boundary operator $\mathcal{L}_\Gamma$ defined in Assumption 2.3 in a similar way, we make the following assumption.

Assumption 3.5 We assume that we have $\mathcal{L}_\Gamma \in \mathcal{B}(L^2(\Gamma)^m)$, i.e., for $v, w \in L^2(\Gamma)^m$ we have

$$\langle \mathcal{L}_\Gamma v \,|\, w \rangle = (\mathcal{L}_\Gamma v \,|\, w)_\Gamma .$$

We point out that this assumption is not very restricting as it is fulfilled in many applications, see [5, Section 5].

Before we define the discrete Friedrichs' operator, we prove two auxiliary results, which are needed to show crucial properties of the discrete operators. The first one relates the boundary operators $\mathcal{L}_\partial^K$ and $\mathcal{L}_\partial^F$.

Lemma 3.6 *Let $v, w \in H^1(\mathcal{T}_h)^m$. Then we have*

$$\sum_{K \in \mathcal{T}_h} (\mathcal{L}_\partial^K v \,|\, w)_{\partial K} = \sum_{F \in \mathcal{F}_h^{\mathrm{int}}} \Big((\mathcal{L}_\partial^F \{\!\{v\}\!\}_F \,|\, [\![w]\!]_F)_F + (\mathcal{L}_\partial^F [\![v]\!]_F \,|\, \{\!\{w\}\!\}_F)_F \Big) + \sum_{F \in \mathcal{F}_h^{\mathrm{bnd}}} (\mathcal{L}_\partial^F v \,|\, w)_F .$$

Proof Using the definition of the boundary operators $\mathcal{L}_\partial^K$ and $\mathcal{L}_\partial^F$ and the directions of the element and face normals n^K and n^F, respectively, we calculate

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} (\mathcal{L}_\partial^K v \,|\, w)_{\partial K} &= \sum_{F \in \mathcal{F}_h^{\mathrm{int}}} \Big((\mathcal{L}_\partial^F v|_{K_1^F} \,|\, w|_{K_1^F})_F - (\mathcal{L}_\partial^F v|_{K_2^F} \,|\, w|_{K_2^F})_F \Big) \\ &\quad + \sum_{F \in \mathcal{F}_h^{\mathrm{bnd}}} (\mathcal{L}_\partial^F v \,|\, w)_F \\ &= \sum_{F \in \mathcal{F}_h^{\mathrm{int}}} ([\![(\mathcal{L}_\partial^F v) \cdot w]\!]_F \,|\, 1)_F + \sum_{F \in \mathcal{F}_h^{\mathrm{bnd}}} (\mathcal{L}_\partial^F v \,|\, w)_F . \end{aligned}$$

Using the identity $[\![f \cdot g]\!]_F = \{\!\{f\}\!\}_F \cdot [\![g]\!]_F + [\![f]\!]_F \cdot \{\!\{g\}\!\}_F$ for all $f, g \colon \Omega \to \mathbb{R}^m$ concludes the proof. □

The next result characterizes functions in the graph space of $\mathcal{L}$, or rather its intersection with the broken Sobolev space $H^1(\mathcal{T}_h)^m$. It states that the traces of such functions corresponding to $\mathcal{L}$ vanish across interfaces of the mesh. Further, if these functions are additionally contained in the domain $D(\mathcal{L})$, they fulfill the corresponding boundary condition. The proof can be found in the appendix.

Lemma 3.7 *Let $v \in H^1(\mathcal{T}_h)^m$. Then we have $v \in H(\mathcal{L})$ if and only if*

$$\mathcal{L}_\partial^F [\![v]\!]_F = 0 \qquad \text{a.e. on } F \text{ for all } F \in \mathcal{F}_h^{\mathrm{int}}. \tag{13}$$

Additionally, for $v \in D(\mathcal{L}) \cap H^1(\mathcal{T}_h)^m$, we have

$$(\mathcal{L}_\partial^F - \mathcal{L}_\Gamma) v = 0 \qquad \text{a.e. on } F \text{ for all } F \in \mathcal{F}_h^{\mathrm{bnd}}. \tag{14}$$

3.3 Discrete Friedrichs' operators

In this section we define and investigate the central flux discretization of a Friedrichs' operator $\mathcal{L}$. Naturally, we would define this discrete operator on the discrete approximation space $V_{\hbar}$. However, in view of the error analysis, it is convenient to extend the definition to the space $D(\mathcal{L}) \cap H^1(\mathcal{T}_{\hbar})^m$. We combine both spaces in the *discrete operator domain* associated with $\mathcal{L}$ given by

$$V_{\hbar}^{\mathcal{L}} = V_{\hbar} + (D(\mathcal{L}) \cap H^1(\mathcal{T}_{\hbar})^m)$$

and define the discrete operator as follows.

Definition 3.8 The *central fluxes dG discretization of* $\mathcal{L}$ is the operator $\boldsymbol{\mathcal{L}} \colon V_{\hbar}^{\mathcal{L}} \to V_{\hbar}$ defined as

$$\begin{aligned}(\boldsymbol{\mathcal{L}} v \mid \boldsymbol{\varphi})_{\Omega} = \sum_{K \in \mathcal{T}_{\hbar}} (\mathcal{L} v \mid \boldsymbol{\varphi})_K - \sum_{F \in \mathcal{F}_{\hbar}^{\mathrm{int}}} (\mathcal{L}_{\partial}^F [\![v]\!]_F \mid \{\!\{\boldsymbol{\varphi}\}\!\}_F)_F \\ - \tfrac{1}{2} \sum_{F \in \mathcal{F}_{\hbar}^{\mathrm{bnd}}} ((\mathcal{L}_{\partial}^F - \mathcal{L}_{\Gamma}) v \mid \boldsymbol{\varphi})_F \qquad \text{for all } \boldsymbol{\varphi} \in V_{\hbar}.\end{aligned} \tag{15}$$

Remark 3.9 The average used in (15) can be replaced by a weighted average $\{\!\{v\}\!\}_F^{\Lambda} = \{\!\{\Lambda\}\!\}_F^{-1} \{\!\{\Lambda v\}\!\}_F$ with $\Lambda \in L^{\infty}(\Omega)^{m \times m}$ being symmetric and uniformly positive a.e. on Ω. The following results then still hold, albeit with different constants involving the weights. If the weight is chosen in a suitable way this can improve the constants, see e.g., [15] for isotropic Maxwell's equations.

We next gather some important properties of the discrete Friedrichs' operator. The first one is a consistency property that shows that the discrete operator in some sense indeed approximates its continuous counterpart.

Proposition 3.10 *The discrete Friedrichs' operator* $\boldsymbol{\mathcal{L}}$ *fulfills the consistency property*

$$\boldsymbol{\mathcal{L}} v = \pi_h \mathcal{L} v \qquad \textit{for all } v \in D(\mathcal{L}) \cap H^1(\mathcal{T}_{\hbar})^m.$$

Proof Let $v \in D(\mathcal{L}) \cap H^1(\mathcal{T}_{\hbar})^m$. By Lemma 3.7 the interface and boundary terms in (15) vanish. Hence, we have

$$(\boldsymbol{\mathcal{L}} v \mid \boldsymbol{\varphi})_{\Omega} = (\mathcal{L} v \mid \boldsymbol{\varphi})_{\Omega} = (\pi_h \mathcal{L} v \mid \boldsymbol{\varphi})_{\Omega} \qquad \text{for all } \boldsymbol{\varphi} \in V_{\hbar}$$

by the definition of π_h in (8). □

In addition, the discrete Friedrichs' operator inherits the dissipativity of $\mathcal{L}$ if restricted to the discrete approximation space. To show this, we proceed as in the continuous case.

Lemma 3.11 *The adjoint operator $\mathcal{L}^{\circledast}\colon V_{\hbar} \to V_{\hbar}$ of $\mathcal{L}|_{V_{\hbar}}$ is given by*

$$(\mathcal{L}^{\circledast} v \,|\, \boldsymbol{\varphi})_{\Omega} = \sum_{K \in \mathcal{T}_{\hbar}} (\mathscr{L}^{\circledast} v \,|\, \boldsymbol{\varphi})_{K} + \sum_{F \in \mathcal{F}_{\hbar}^{\text{int}}} (\mathscr{L}_{\partial}^{F} [\![v]\!]_{F} \,|\, \{\!\{\boldsymbol{\varphi}\}\!\}_{F})_{F} + \frac{1}{2} \sum_{F \in \mathcal{F}_{\hbar}^{\text{bnd}}} ((\mathscr{L}_{\partial}^{F} + \mathscr{L}_{\Gamma}^{T}) v \,|\, \boldsymbol{\varphi})_{F} \qquad \text{for all } \boldsymbol{\varphi} \in V_{\hbar},$$

and satisfies

$$(\mathcal{L}\boldsymbol{v} \,|\, \boldsymbol{\varphi})_{\Omega} + (\mathcal{L}^{\circledast}\boldsymbol{v} \,|\, \boldsymbol{\varphi})_{\Omega} = \big((L_0 + L_0^T)\boldsymbol{v} \,|\, \boldsymbol{\varphi}\big)_{\Omega} + \tfrac{1}{2}\big((\mathscr{L}_{\Gamma} + \mathscr{L}_{\Gamma}^{T})\boldsymbol{v} \,|\, \boldsymbol{\varphi}\big)_{\Gamma} \qquad (16)$$

for all $\boldsymbol{\varphi} \in V_{\hbar}$.

Proof Using the integration by parts formula from Lemma 3.3 on each element K and Lemma 3.6 on the arising interface terms readily yields that $\mathcal{L}^{\circledast}$ is in fact the adjoint of $\mathcal{L}$. Identity (16) follows by a straightforward calculation. □

Proposition 3.12 *The restriction of the discrete Friedrichs' operator $\mathcal{L}$ to $V_{\hbar}$ is dissipative, i.e., we have*

$$(\mathcal{L} v \,|\, v)_{\Omega} \leq 0 \qquad \text{for all } v \in V_{\hbar}.$$

Proof By the adjointness of $\mathcal{L}$ and $\mathcal{L}^{\circledast}$ we have

$$(\mathcal{L}\boldsymbol{v} \,|\, \boldsymbol{v})_{\Omega} = \tfrac{1}{2}\Big((\mathcal{L}\boldsymbol{v} \,|\, \boldsymbol{v})_{\Omega} + (\mathcal{L}^{\circledast}\boldsymbol{v} \,|\, \boldsymbol{v})_{\Omega}\Big) \leq 0,$$

where we have used (16) together with the dissipativity of $\mathscr{L}_{\Gamma}$ and the negative semidefiniteness of L_0. □

Similar to the continuous operator fulfilling (11), the discrete Friedrichs' operator satisfies an inverse inequality.

Proposition 3.13 *Let $\boldsymbol{v} \in V_{\hbar}$. Then, the discrete Friedrichs' operator $\mathcal{L}$ fulfills the inverse inequality*

$$\|\mathcal{L}\boldsymbol{v}\|_{\Omega} \leq C_{\text{inv},\mathscr{L}} \|h^{-1}\boldsymbol{v}\|_{\Omega}.$$

The constant is given by $C_{\text{inv},\mathscr{L}} = C_{\mathscr{L}} C_{\text{inv}} + \frac{1}{2} C_{\text{tr}}^2 (C_{\Gamma,\mathscr{L}} + N_{\partial} C_{\mathscr{L}} (1 + \rho^{1/2}))$ with $C_{\Gamma,\mathscr{L}} = \max_{F \in \mathcal{F}_{\hbar}^{\text{bnd}}} \|\mathscr{L}_{\partial}^{F} - \mathscr{L}_{\Gamma}\|_{\infty,F}$.

Lastly, we have a result on the approximation properties of the discrete Friedrichs' operator. It gives a bound on the application of $\mathcal{L}$ to the projection error of a function in $D(\mathscr{L}) \cap H^{q+1}(\mathcal{T}_{\hbar})^m$.

Proposition 3.14 *Let $v \in D(\mathscr{L}) \cap H^{q+1}(\mathcal{T}_{\hbar})^m$ for $0 \leq q \leq k$. Then we have*

$$\|\mathcal{L} e_{\pi}^{v}\|_{\Omega} \leq C_{\pi,\mathscr{L}} |h^q v|_{q+1,\mathcal{T}_{\hbar}}. \qquad (17)$$

The constant is given by $C_{\pi,\mathscr{L}} = \frac{1}{2} N_{\partial} C_{\text{tr}} C_{\pi,\partial} (C_{\Gamma,\mathscr{L}} + C_{\mathscr{L}} (1 + \rho^{1/2}))$.

The proofs of both Proposition 3.13 and 3.14 are given in the appendix.

3.4 Spatial discretization of the wave-type problem

We are now able to formulate the spatially semi-discrete version of the wave-type problem (6). To this end, we define the operator $\widetilde{\mathcal{L}}\colon V_{\hbar}^{\mathcal{L}} \to V_{\hbar}$ analogously to the continuous case by $\widetilde{\mathcal{L}} = M^{-1}\mathcal{L}$. Note that, owing to Assumption 3.2, $\widetilde{\mathcal{L}}$ exhibits the same consistency property as $\mathcal{L}$, namely

$$\widetilde{\mathcal{L}} v = \pi_h \widetilde{\mathcal{L}} v \qquad \text{for all } v \in D(\mathcal{L}) \cap H^1(\mathcal{T}_{\hbar})^m. \tag{18}$$

Using $\widetilde{\mathcal{L}}$ we can state the spatially discrete wave-type problem

$$\begin{cases} \partial_t \boldsymbol{u} = \widetilde{\mathcal{L}}\boldsymbol{u} + \boldsymbol{f}_\pi, \qquad \mathbb{R}_+ \times \Omega, & \text{(19a)} \\ \boldsymbol{u}(0) = \boldsymbol{u}_\pi^0, & \text{(19b)} \end{cases}$$

with $\boldsymbol{f}_\pi := \pi_h f$ and initial value $\boldsymbol{u}_\pi^0 := \pi_h u^0$.

Since $\mathcal{L}$ inherits the dissipativity of the continuous operator on the discrete approximation space $V_{\hbar}$ by Proposition 3.12, $\widetilde{\mathcal{L}}$ is dissipative w.r.t. the weighted inner product $(\,\cdot \mid \cdot\,)_M$. This can easily be seen since (7) also holds for the discrete operators. Further, both $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are maximal as $V_{\hbar}$ is finite-dimensional. Hence, by the Lumer–Phillips Theorem, the restriction of $\widetilde{\mathcal{L}}$ to $V_{\hbar}$ generates a contraction semigroup w.r.t. $\|\cdot\|_M$, which we denote by $\big(\mathrm{e}^{t\widetilde{\mathcal{L}}}\big)_{t\geq 0}$.

Corollary 3.15 *There exists a unique solution* $\boldsymbol{u} \in C^1(\mathbb{R}_+; V_{\hbar})$ *of* (19) *given by the variation-of-constants formula*

$$\boldsymbol{u}(t) = \mathrm{e}^{t\widetilde{\mathcal{L}}}\, \boldsymbol{u}_\pi^0 + \int_0^t \mathrm{e}^{(t-s)\widetilde{\mathcal{L}}}\, \boldsymbol{f}_\pi(s)\,\mathrm{d}s. \tag{20}$$

4 Error analysis of the spatially semi-discrete problem

We are now able to analyze the *spatially semi-discrete error*

$$e = u - \boldsymbol{u},$$

where $\boldsymbol{u}$ denotes the semi-discrete approximation given by (19) and u is the exact solution of (6). We split this error into

$$e = e_\pi + \boldsymbol{e}_h = u - \pi_h u + \pi_h u - \boldsymbol{u}, \tag{21}$$

where e_π is the *projection error* and $\boldsymbol{e}_h$ is the *space discretization error*.

By (12) and the boundedness of M we have the following bound on the projection error

$$\|e_\pi(t)\|_M \leq C_{\pi,M} |h^{k+1} u(t)|_{k+1,\mathcal{T}_h} \tag{22}$$

with $C_{\pi,M} = \|M\|_{\infty,\Omega}^{1/2} C_\pi$.

Hence, it remains to bound the space discretization error $\boldsymbol{e}_h$. We do this by showing that $\boldsymbol{e}_h$ satisfies the semi-discrete problem (19) with zero initial value and the right hand side given by a defect stemming from the spatial discretization. Using the variation-of-constants formula (20) and the stability of the semi-discrete scheme (owing to the contractivity of the semigroup $(\mathrm{e}^{t\widetilde{\mathcal{L}}})_{t\geq 0}$) we can then bound the discretization error by this defect. Lastly, the approximation property (17) of the discrete operator $\mathcal{L}$ provides a bound on the defect and thus on the discretization error.

Lemma 4.1 *Assume that the exact solution of* (6) *fulfills* $u \in C^1(\mathbb{R}_+; L^2(\Omega)^m) \cap C(\mathbb{R}_+; D(\mathcal{L}) \cap H^1(\mathcal{T}_h)^m)$. *Then the space discretization error* $\boldsymbol{e}_h = \pi_h u - \boldsymbol{u}$ *satisfies*

$$\begin{cases} \partial_t \boldsymbol{e}_h(t) = \widetilde{\mathcal{L}} \boldsymbol{e}_h(t) + \boldsymbol{d}_\pi(t), & t \in \mathbb{R}_+, \\ \boldsymbol{e}_h(0) = 0, \end{cases} \tag{23}$$

where the defect $\boldsymbol{d}_\pi : \mathbb{R}_+ \to \mathbb{R}^m$ *is given by*

$$\boldsymbol{d}_\pi = \widetilde{\mathcal{L}} e_\pi. \tag{24}$$

Proof We begin by inserting the projected exact solution $\pi_h u$ into the semi-discrete equation (19a) and defining the error made by this as $\boldsymbol{d}_\pi$, yielding

$$\partial_t \pi_h u = \widetilde{\mathcal{L}} \pi_h u + \boldsymbol{f}_\pi + \boldsymbol{d}_\pi. \tag{25}$$

Subtracting the semi-discrete scheme (19a) from (25) readily implies (23).

To show (24), we use that ∂_t and the L^2-projection commute and that u solves the continuous problem (6) to obtain

$$\partial_t \pi_h u = \pi_h \partial_t u = \pi_h(\widetilde{\mathcal{L}} u + f) = \widetilde{\mathcal{L}} u + \boldsymbol{f}_\pi.$$

Here, we have used the consistency property (18) in the last step. Equating this with (25) and solving for $\boldsymbol{d}_\pi$ yields

$$\boldsymbol{d}_\pi = \widetilde{\mathcal{L}} u - \widetilde{\mathcal{L}} \pi_h u = \widetilde{\mathcal{L}} e_\pi,$$

concluding the proof. □

Having derived an evolution equation for the error, we can now solve it to obtain a bound on the space discretization error. Together with the already mentioned bound on the projection error (22) we can thus bound the spatially semi-discrete error.

Theorem 4.2 *Assume that the exact solution u of the wave-type problem* (6) *satisfies* $u \in C^1(\mathbb{R}_+; L^2(\Omega)^m) \cap C(\mathbb{R}_+; D(\mathcal{L}) \cap H^{k+1}(\mathcal{T}_h)^m)$. *Then, for* $t \in \mathbb{R}_+$, *the spatially semi-discrete error satisfies*

$$\begin{aligned}\|u(t) - \boldsymbol{u}(t)\|_M &\le C_{\pi,M} |h^{k+1} u(t)|_{k+1,\mathcal{T}_h} + C_{\pi,\mathcal{L},M} \int_0^t |h^k u(s)|_{k+1,\mathcal{T}_h} \,\mathrm{d}s \\ &\le C h^k,\end{aligned}$$

where $C_{\pi,\mathcal{L},M} = \|M^{-1}\|_{\infty,\Omega}^{1/2} C_{\pi,\mathcal{L}}$ *and* C *only depends on* $C_{\pi,M}$, $C_{\pi,\mathcal{L},M}$ *and* $|u(s)|_{k+1,\mathcal{T}_h}$, $s \in [0,t]$.

Proof We use Corollary 3.15 to solve the error equation (23), which yields

$$\boldsymbol{e}_h(t) = \int_0^t \mathrm{e}^{(t-s)\widetilde{\mathcal{L}}} \, \boldsymbol{d}_\pi(s) \,\mathrm{d}s.$$

By the contractivity of the semigroup $(\mathrm{e}^{t\widetilde{\mathcal{L}}})_{t\ge 0}$ in the $\|\cdot\|_M$-norm we obtain

$$\|\boldsymbol{e}_h(t)\|_M \le \int_0^t \|\boldsymbol{d}_\pi(s)\|_M \,\mathrm{d}s = \int_0^t \|\widetilde{\mathcal{L}} e_\pi(s)\|_M \,\mathrm{d}s.$$

It remains to bound $\|\widetilde{\mathcal{L}} e_\pi(s)\|_M$. To do so, we use the boundedness of M and the approximation property from Proposition 3.14 applied to $\mathcal{L}$, yielding

$$\begin{aligned}\|\widetilde{\mathcal{L}} e_\pi(s)\|_M &= \|M^{-1/2} \mathcal{L} e_\pi(s)\|_\Omega \\ &\le \|M^{-1}\|_{\infty,\Omega}^{1/2} \|\mathcal{L} e_\pi(s)\|_\Omega \\ &\le \|M^{-1}\|_{\infty,\Omega}^{1/2} C_{\pi,\mathcal{L}} |h^k u(s)|_{k+1,\mathcal{T}_h}.\end{aligned}$$

Taking norms and using the triangle inequality in the error splitting (21) together with the already established bound (22) on e_π proves the claim. □

Concluding remarks

In this paper we presented a rigorous error analysis of the spatial discretization of a large class of wave-type problems by discontinuous Galerkin methods. This class includes Maxwell's equations and the acoustic wave equation, for instance. It has been shown in [12] that such a space discretization on cuboids and tensorial grids can be combined with a Peaceman–Rachford (ADI) time integration scheme in such a way that it has optimal (linear) complexity for suitable problems. The full discretization error of the resulting schemeis studied in [14].

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Appendix: proofs from Section 3

In this appendix we collect the proofs we postponed in Section 3.

Proof (Lemma 3.7) Let $v \in H^1(\mathcal{T}_h)^m$.

(i) We first prove that $v \in H(\mathcal{L})$ follows from (13) by showing the boundedness of the mapping

$$C_c^\infty(\Omega)^m \to \mathbb{R}, \quad \varphi \mapsto (v \,|\, \mathcal{L}^\circledast \varphi)_\Omega. \tag{26}$$

Let $\varphi \in C_c^\infty(\Omega)^m$ so that $[\![\varphi]\!]_F = 0$ and $\{\!\{\varphi\}\!\}_F = \varphi|_F$ for all $F \in \mathcal{F}_h^{\mathrm{int}}$ and $\varphi|_F = 0$ for all $F \in \mathcal{F}_h^{\mathrm{bnd}}$. By applying the integration by parts formula from Lemma 3.3 on each element we have

$$\begin{aligned}(v \,|\, \mathcal{L}^\circledast \varphi)_\Omega &= \sum_{K \in \mathcal{T}_h} (v \,|\, \mathcal{L}^\circledast \varphi)_K \\ &= \sum_{K \in \mathcal{T}_h} (\mathcal{L} v \,|\, \varphi)_K + \sum_{K \in \mathcal{T}_h} (\mathcal{L}_\partial^K v \,|\, \varphi)_{\partial K} \\ &= \sum_{K \in \mathcal{T}_h} (\mathcal{L} v \,|\, \varphi)_K + \sum_{F \in \mathcal{F}_h^{\mathrm{int}}} (\mathcal{L}_\partial^F [\![v]\!]_F \,|\, \varphi)_F \\ &= \sum_{K \in \mathcal{T}_h} (\mathcal{L} v \,|\, \varphi)_K,\end{aligned} \tag{27}$$

where we have used Lemma 3.6 in the third and (13) in the last step. Applying the Cauchy–Schwarz inequality we obtain the boundedness of (26) and hence $v \in H(\mathcal{L})$.

(ii) Next, let $v \in H(\mathcal{L}) \cap H^1(\mathcal{T}_h)^m$. By [13, Theorem 1.2] we have that $H(\mathcal{L}) \cap C^\infty(\Omega)^m$ is dense in $H(\mathcal{L})$. Hence, we can choose a sequence $(v_n)_{n\in\mathbb{N}}$ in $H(\mathcal{L}) \cap C^\infty(\Omega)^m$ with

$$v_n \to v, \qquad \mathcal{L} v_n \to \mathcal{L} v \qquad \text{in } L^2(\Omega)^m.$$

For arbitrary $\varphi \in C_c^\infty(\Omega)^m$ and with n^Ω denoting the outward unit normal vector to Γ, Lemma 3.3 yields

$$
\begin{aligned}
(\mathcal{L}v \,|\, \varphi)_\Omega &= \lim_{n\to\infty} (\mathcal{L}v_n \,|\, \varphi)_\Omega \\
&= \lim_{n\to\infty} \Big((v_n \,|\, \mathcal{L}^{\circledast}\varphi)_\Omega + \big(\sum_{i=1}^{d} \mathrm{n}_i^\Omega L_i v_n \,|\, \varphi)_\Gamma \Big) \\
&= (v \,|\, \mathcal{L}^{\circledast}\varphi)_\Omega .
\end{aligned}
$$

Comparing this with the third line in (27) yields

$$
\sum_{F \in \mathcal{F}_{\hbar}^{\mathrm{int}}} (\mathcal{L}_\partial^F [\![v]\!]_F \,|\, \varphi)_F = 0.
$$

Since φ was arbitrary, this in particular holds for $\operatorname{supp}\varphi$ only intersecting a single interface, which implies (13).

(iii) To show the last assertion, let $v \in D(\mathcal{L}) \cap H^1(\mathcal{T}_{\hbar})^m$ and $F \in \mathcal{F}_{\hbar}^{\mathrm{bnd}}$. Since $v|_F \in L^2(F)^m$, by Lemma 3.3 and Assumption 3.5 we have

$$
((\mathcal{L}_\partial^F - \mathcal{L}_\Gamma) v \,|\, \varphi)_F = 0 \qquad \text{for all } \varphi \in C^\infty(\overline{\Omega})^m,
$$

which shows (14). □

Proof (Proposition 3.13) We begin by deriving an elementwise representation of $\boldsymbol{\mathcal{L}}$. Namely, since we have $(\boldsymbol{\mathcal{L}}\boldsymbol{v} \,|\, \boldsymbol{\varphi})_K = (\boldsymbol{\mathcal{L}}\boldsymbol{v} \,|\, \mathbb{1}_K \boldsymbol{\varphi})_\Omega$ for all $\boldsymbol{\varphi} \in V_{\hbar}$, a straightforward calculation yields

$$
\begin{aligned}
(\boldsymbol{\mathcal{L}}\boldsymbol{v} \,|\, \boldsymbol{\varphi})_K = (\mathcal{L}\boldsymbol{v} \,|\, \boldsymbol{\varphi})_K &- \tfrac{1}{2} \sum_{F \in \mathcal{F}_{\hbar}^{K,\mathrm{int}}} (\mathcal{L}_\partial^F [\![\boldsymbol{v}]\!]_F \,|\, \boldsymbol{\varphi}|_K)_F \\
&- \tfrac{1}{2} \sum_{F \in \mathcal{F}_{\hbar}^{K,\mathrm{bnd}}} ((\mathcal{L}_\partial^F - \mathcal{L}_\Gamma)\boldsymbol{v} \,|\, \boldsymbol{\varphi})_F .
\end{aligned}
\tag{28}
$$

We now bound the element, interface and boundary face terms individually, beginning with the former. Using the Cauchy–Schwarz inequality and the inverse inequality (11) yields

$$
(\mathcal{L}\boldsymbol{v} \,|\, \boldsymbol{\varphi})_K \le \|\mathcal{L}\boldsymbol{v}\|_K \|\boldsymbol{\varphi}\|_K \le C_{\mathcal{L}} C_{\mathrm{inv}} \|h^{-1}\boldsymbol{v}\|_K \|\boldsymbol{\varphi}\|_K .
$$

The boundary terms are treated similarly by again using the Cauchy–Schwarz inequality and this time the boundedness of $\mathcal{L}_\partial^F$ and $\mathcal{L}_\Gamma$ and the trace inequality (10) to obtain

$$
\begin{aligned}
((\mathcal{L}_\partial^F - \mathcal{L}_\Gamma)\boldsymbol{v} \,|\, \boldsymbol{\varphi})_F &\le C_{\Gamma,\mathcal{L}} \|\boldsymbol{v}\|_F \|\boldsymbol{\varphi}\|_F \\
&\le C_{\Gamma,\mathcal{L}} C_{\mathrm{tr}} \|h^{-1/2}\boldsymbol{v}\|_K \, C_{\mathrm{tr}} \|h^{-1/2}\boldsymbol{\varphi}\|_K \\
&= C_{\Gamma,\mathcal{L}} C_{\mathrm{tr}}^2 \|h^{-1}\boldsymbol{v}\|_K \|\boldsymbol{\varphi}\|_K ,
\end{aligned}
$$

where we used that h is piecewise constant. To bound the interface terms, we first rewrite the jump $[\![\boldsymbol{v}]\!]_F$ as

$$(\mathcal{L}_\partial^F [\![\boldsymbol{v}]\!]_F \mid \boldsymbol{\varphi}|_K)_F = \varepsilon_{K,F} \Big((\mathcal{L}_\partial^F \boldsymbol{v}|_K \mid \boldsymbol{\varphi}|_K)_F - (\mathcal{L}_\partial^F \boldsymbol{v}|_{K_F} \mid \boldsymbol{\varphi}|_K)_F \Big),$$

where $\varepsilon_{K,F} = \mathrm{n}^K \cdot \mathrm{n}^F = \pm 1$. The first term can thus be bounded completely analogously to the boundary term by

$$(\mathcal{L}_\partial^F \boldsymbol{v}|_K \mid \boldsymbol{\varphi}|_K)_F \le C_{\mathcal{L}} C_{\mathrm{tr}}^2 \|h^{-1} \boldsymbol{v}\|_K \|\boldsymbol{\varphi}\|_K.$$

To bound the second term, we additionally use $h_K^{-1} \le \rho h_{K_F}^{-1}$ (see [3, Lem. 1.43]) and thus

$$(\mathcal{L}_\partial^F \boldsymbol{v}|_{K_F} \mid \boldsymbol{\varphi}|_K)_F \le \rho^{1/2} C_{\mathcal{L}} C_{\mathrm{tr}}^2 \|h^{-1} \boldsymbol{v}\|_{K_F} \|\boldsymbol{\varphi}\|_K.$$

Assembling all these bounds and taking into account that each element has at most N_∂ neighboring elements and at most one boundary face yields

$$(\mathcal{L}\boldsymbol{v} \mid \boldsymbol{\varphi})_K \le \Big(C_{\mathcal{L},\mathrm{el},1} \|h^{-1} \boldsymbol{v}\|_K + C_{\mathcal{L},\mathrm{el},2} \sum_{F \in \mathcal{F}_h^{K,\mathrm{int}}} \|h^{-1} \boldsymbol{v}\|_{K_F} \Big) \|\boldsymbol{\varphi}\|_K \tag{29}$$

for all $\boldsymbol{\varphi} \in V_h$. The constants are given as $C_{\mathcal{L},\mathrm{el},1} = C_{\mathcal{L}} C_{\mathrm{inv}} + \frac{1}{2} C_{\mathrm{tr}}^2 (C_{\Gamma,\mathcal{L}} + N_\partial C_{\mathcal{L}})$ and $C_{\mathcal{L},\mathrm{el},2} = \frac{1}{2} \rho^{1/2} C_{\mathrm{tr}}^2 C_{\mathcal{L}}$.

It remains to put these elementwise bounds together to obtain a bound w.r.t. the whole domain Ω. Summing (29) over all elements $K \in \mathcal{T}_h$ yields

$$(\mathcal{L}\boldsymbol{v} \mid \boldsymbol{\varphi})_\Omega \le C_{\mathcal{L},\mathrm{el},1} \sum_{K \in \mathcal{T}_h} \|h^{-1} \boldsymbol{v}\|_K \|\boldsymbol{\varphi}\|_K + C_{\mathcal{L},\mathrm{el},2} \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_h^{K,\mathrm{int}}} \|h^{-1} \boldsymbol{v}\|_{K_F} \|\boldsymbol{\varphi}\|_K.$$

From here, the assertion follows by straightforward applications of the Cauchy–Schwarz and Young's inequality, respectively. □

Proof (Proposition 3.14) We proceed similarly to the proof of Proposition 3.13, meaning that we first work on the element-based formulation (28). Using integration by parts yields

$$\begin{aligned}
(\mathcal{L} e_\pi^v \mid \boldsymbol{\varphi})_K &= (\mathcal{L} e_\pi^v \mid \boldsymbol{\varphi})_K - \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^{K,\mathrm{int}}} (\mathcal{L}_\partial^F [\![e_\pi^v]\!]_F \mid \boldsymbol{\varphi}|_K)_F \\
&\quad - \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^{K,\mathrm{bnd}}} ((\mathcal{L}_\partial^F - \mathcal{L}_\Gamma) e_\pi^v \mid \boldsymbol{\varphi})_F \\
&= \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^{K,\mathrm{int}}} (\mathcal{L}_\partial^F \{\!\{ e_\pi^v \}\!\}_F \mid \boldsymbol{\varphi}|_K)_F + \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^{K,\mathrm{bnd}}} ((\mathcal{L}_\partial^F + \mathcal{L}_\Gamma) e_\pi^v \mid \boldsymbol{\varphi})_F
\end{aligned}$$

for all $\boldsymbol{\varphi} \in V_h$, where the element term vanishes because of the defining property of the L^2-projection (8) since $\mathcal{L}^{\circledast} \boldsymbol{\varphi}|_K \in \mathbb{Q}_d^k(K)$.

The rest of the proof is completely analogous to the corresponding part of the proof of Proposition 3.13. The only difference is that we use the second bound in (12) instead of the discrete trace inequality. □

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