Inverse problems for abstract evolution equations II: higher order differentiability for viscoelasticity

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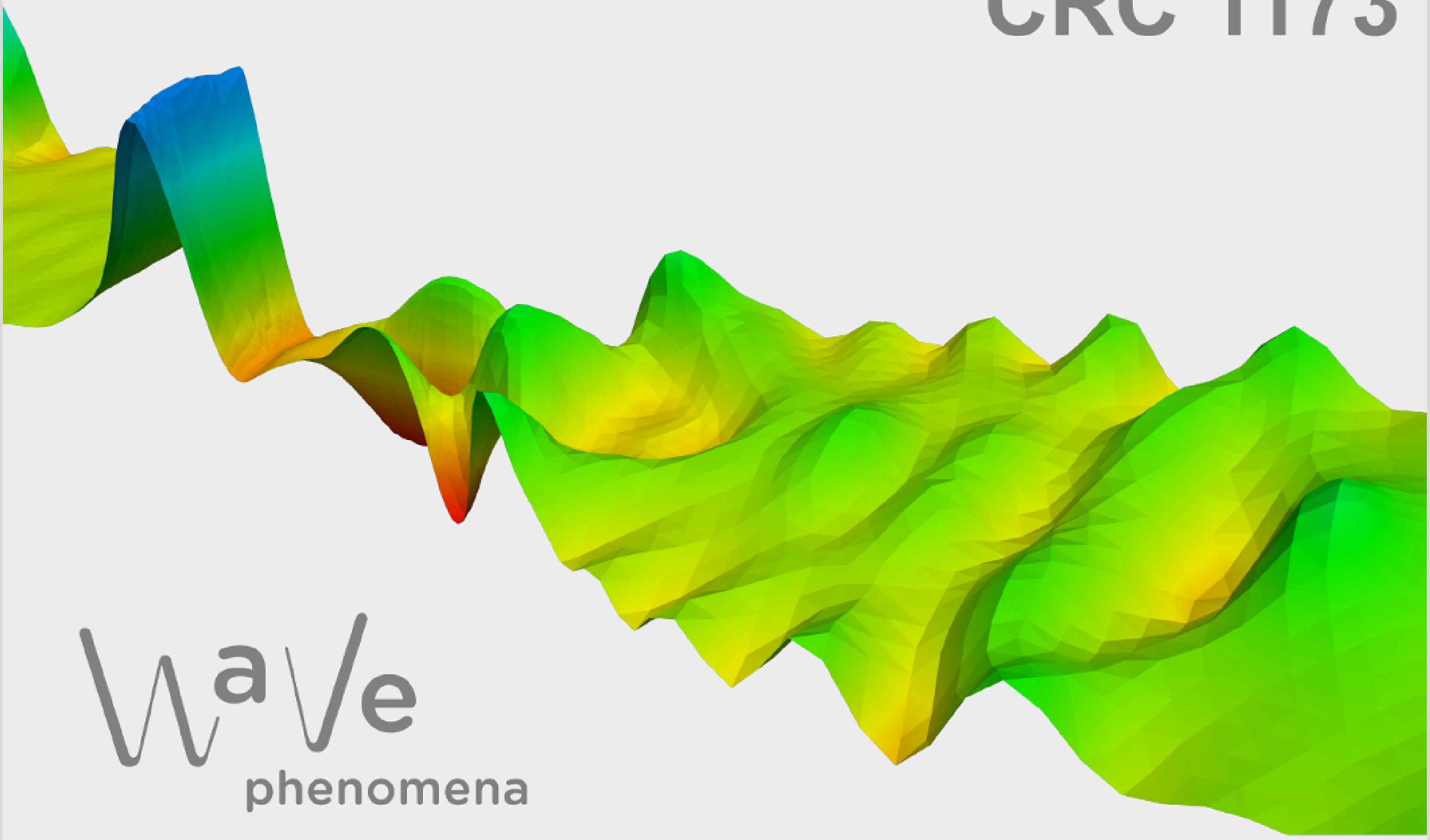
CRC Preprint 2019/10 (revised), October 2019

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



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Funded by

DFG

ISSN 2365-662X

INVERSE PROBLEMS FOR ABSTRACT EVOLUTION EQUATIONS II: HIGHER ORDER DIFFERENTIABILITY FOR VISCOELASTICITY

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Abstract. In this follow-up of [*Inverse Problems* 32 (2016) 085001] we generalize our previous abstract results so that they can be applied to the viscoelastic wave equation which serves as a forward model for full waveform inversion (FWI) in seismic imaging including dispersion and attenuation. FWI is the nonlinear inverse problem of identifying parameter functions of the viscoelastic wave equation from measurements of the reflected wave field. Here we rigorously derive rather explicit analytic expressions for the Fréchet derivative and its adjoint (adjoint state method) of the underlying parameter-to-solution map. These quantities enter crucially Newton-like gradient decent solvers for FWI. Moreover, we provide the second Fréchet derivative and a related adjoint as ingredients to second-degree solvers.

1. Introduction

Full waveform inversion (FWI) is the leading-edge technique in geophysical exploration using the full information content (amplitude and phase) of the seismic recordings to reconstruct the parameters in the underlying wave propagation model, see, e.g, [6, 13]. Waves propagating in realistic material encounter dispersion and attenuation which have to be taken into account by a viscoelastic model. There are several of these models described in the literature, see [6, Chap. 5] for an overview and references and see [16, Chap. 2] for how these models are related to each other. The model we consider here is the viscoelastic wave equation in the velocity stress formulation based on the generalized standard linear solid rheology, see (2) below.

In [10] we provided an abstract framework for the nonlinear inverse problem of FWI which applies to the elastic but not directly to the viscoelastic wave equation. The present paper is driven by the wish to slightly adjust our abstract framework such that it finally fits to the viscoelastic equation. So we are indeed able to give analytic expressions for the Fréchet derivative and its adjoint of the full waveform forward operator Φ which maps the parameters of the viscoelatic model (density, wave speeds, scaling factors) to the wave field.

Date: October 24, 2019.

2000 *Mathematics Subject Classification.* 35F10, 35R30, 86A22.

Key words and phrases. full waveform seismic inversion, viscoelastic wave equation, adjoint state method, nonlinear inverse and ill-posed problem, higher order Fréchet derivative.

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

We thank our colleagues Thomas Bohlen, Christian Wieners, and Uwe Zeltmann for sharing with us their insights into the viscoelastic model. Further, we are indebted to Lingli Gao for checking our tedious calculations. Last but not least: we thank the anonymous referees for their helpful feedback on a previous version of this work.

Moreover, we present the second Fréchet derivative of Φ which is needed for Newton-like solvers of second degree, see, e.g., [8]. Second-degree methods are of interest for FWI to mitigate an effect known as 'cross-talk' or 'parameter trade-off'. These terms refer to a coupling phenomenon: for some parameter combinations, the update of one parameter value affects the other parameter values, see, e.g., [5] for a numerical demonstration.

For the reader's convenience we now sketch our contribution in the context of second-degree methods. Assume for the time being that Φ incorporates the measurement process and let $\mathbf{y}$ be the measurements (seismograms). Then, FWI entails the solution of

$$\Phi(\mathbf{p}) = \mathbf{y}$$

for the parameter vector $\mathbf{p}$. The second-degree iteration of Hettlich and Rundell [8] starts with a guess $\mathbf{p}_0$ and updates the current iterate $\mathbf{p}_k$ by

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{s}_k \tag{1a}$$

where $\mathbf{s}_k$ is a regularized solution to

$$\Phi'(\mathbf{p}_k)\mathbf{s} + \frac{1}{2}\Phi''(\mathbf{p}_k)[\mathbf{h}_k, \mathbf{s}] = \mathbf{y}_k - \Phi(\mathbf{p}_k). \tag{1b}$$

The above needed value for $\mathbf{h}_k$ is obtained by solving the Newton equation

$$\Phi'(\mathbf{p}_k)\mathbf{h} = \mathbf{y}_k - \Phi(\mathbf{p}_k). \tag{1c}$$

The two linear systems which determine $\mathbf{s}_k$ are typically solved by iterative regularization schemes like the Landweber or the conjugate gradient iterations. Their implementation requires not only the evaluation of the first and second derivatives but also of the adjoint operators. For all these objects we give explicit representations in a functional analytic framework.

We need to emphasize that this equation-based approach to FWI differs slightly from the usual optimization-based methods in geophysics where a misfit functional J is minimized by Newton-like techniques. Here the second derivative ('Hessian') of J is needed which is related to Φ'' in the following (formal) way: Let $J(\mathbf{p}) = \frac{1}{2}\|\mathbf{y} - \Phi(\mathbf{p})\|^2$ ($\|\cdot\|$ is a Hilbert space norm for the ease presentation). Then,

$$J''(\mathbf{p})[\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2] = \langle \mathcal{H}(\mathbf{p})\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2\rangle \text{ with } \mathcal{H}(\mathbf{p})\widehat{\mathbf{p}} = \Phi'(\mathbf{p})^*\Phi(\mathbf{p})\widehat{\mathbf{p}} - \Phi''(\mathbf{p})^*[\widehat{\mathbf{p}}, \cdot](y - \Phi(\mathbf{p})).$$

Our paper is organized as follows. In the next section we introduce the viscoelastic model in its original formulation for three spatial dimensions. After a transformation of the state variables we arrive at the version which we investigate in an abstract framework. This is done in Section 3 where we will rely on [10]. Then, we return to the concrete viscoelastic model and validate all required properties to apply the abstract results to the full waveform forward operator Φ (Section 4). Our results cannot directly be applied to the viscoelastic model in two spatial dimensions. Since our first numerical test will doubtlessly be performed in the 2D setting we present the corresponding results in an appendix.

Zeltmann [16] also considered a viscoelastic model using techniques akin to ours. In principle, first order differentiability of Φ could have been obtained from his results as well. However, this is an involved task indeed as his setting includes further and different parameters. Moreover, our main objective was to validate second order differentiablity. We therefore generalized our clear framework from [10] and the first order result is thus merely a by-product.

Finally we would like to mention that there are rather generic and formal derivations of the second derivative in the geophysics literature, see [6, Chap. 9.3] and [7].

2. Viscoelasticity

The viscoelastic wave equation in the velocity stress formulation based on the generalized standard linear solid (GSLS) rheology reads: In a Lipschitz domain $D \subset \mathbb{R}^3$ we determine the velocity field $\mathbf{v}\colon [0,T] \times D \to \mathbb{R}^3$, the stress tensor $\boldsymbol{\sigma}\colon [0,T] \times D \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, and memory tensors $\boldsymbol{\eta}_l\colon [0,T] \times D \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, $l = 1, \dots, L$, from the first-order system

$$\rho\,\partial_t \mathbf{v} = \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} \qquad \text{in }]0,T[\times D, \tag{2a}$$

$$\partial_t \boldsymbol{\sigma} = C\big((1 + L\tau_{\mathrm{S}})\mu_0, (1 + L\tau_{\mathrm{P}})\pi_0\big)\, \varepsilon(\mathbf{v}) + \sum_{l=1}^{L} \boldsymbol{\eta}_l \qquad \text{in }]0,T[\times D, \tag{2b}$$

$$-\tau_{\boldsymbol{\sigma},l}\partial_t \boldsymbol{\eta}_l = C\big(L\tau_{\mathrm{S}}\mu_0, L\tau_{\mathrm{P}}\pi_0\big)\, \varepsilon(\mathbf{v}) + \boldsymbol{\eta}_l, \qquad l = 1, \dots, L, \qquad \text{in }]0,T[\times D. \tag{2c}$$

Here, $\mathbf{f}$ denotes the external volume force density and ρ is the mass density. The linear maps $C(m,p)$ for $m, p \in \mathbb{R}$ are defined as

$$C(m,p)\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}, \qquad C(m,p)\mathbf{M} = 2m\mathbf{M} + (p - 2m)\operatorname{tr}(\mathbf{M})\mathbf{I}, \tag{3}$$

with $\mathbf{I} \in \mathbb{R}^{3\times 3}$ being the identity matrix and $\operatorname{tr}(\mathbf{M})$ denotes the trace of $\mathbf{M} \in \mathbb{R}^{3\times 3}$. Further,

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}\big[(\nabla_x \mathbf{v})^\top + \nabla_x \mathbf{v}\big]$$

is the linearized strain rate. In formulation (2) two independent GSLS are used to describe the propagation of pressure and shear waves (P- and S-waves). The parameters μ_0 and π_0 denote the relaxed P- and S-wave modulus, respectively. Further, τ_{P} and τ_{S} are scaling factors for the relaxed moduli. They have been introduced for the first time by [1] and are now widely used to quantify attenuation and phase velocity dispersion in viscoelastic media, see e.g. [6, 14].

Wave propagation in viscoelastic media is frequency-dependent over a bounded frequency band with center frequency ω_0. Within this band the Q-factor, which is the rate of the full energy over the dissipated energy, remains nearly constant. This fact is used to determine the stress relaxation times $\tau_{\boldsymbol{\sigma},l} > 0$ by a least-squares approach [2, 3] where up to $L = 5$ relaxation mechanisms may be required. Now we obtain the following frequency-dependent phase velocities of P- and S-waves:

$$v_{\mathrm{P}}^2 = \frac{\pi_0}{\rho}(1 + \tau_{\mathrm{P}}\alpha) \quad \text{and} \quad v_{\mathrm{S}}^2 = \frac{\mu_0}{\rho}(1 + \tau_{\mathrm{S}}\alpha) \quad \text{with } \alpha = \alpha(\omega_0) = \sum_{l=1}^{L} \frac{\omega_0^2 \tau_{\boldsymbol{\sigma},l}^2}{1 + \omega_0^2 \tau_{\boldsymbol{\sigma},l}^2}. \tag{4}$$

Full waveform inversion (FWI) in seismic imaging entails the inverse problem of reconstructing the five spatially dependent parameters $(\rho, v_{\mathrm{S}}, \tau_{\mathrm{S}}, v_{\mathrm{P}}, \tau_{\mathrm{P}})$ from wave field measurements.

Using the transformation

$$\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma}_0 \\ \boldsymbol{\sigma}_1 \\ \vdots \\ \boldsymbol{\sigma}_L \end{pmatrix} := \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} + \sum_{l=1}^{L} \tau_{\boldsymbol{\sigma},l}\boldsymbol{\eta}_l \\ -\tau_{\boldsymbol{\sigma},1}\boldsymbol{\eta}_1 \\ \vdots \\ -\tau_{\boldsymbol{\sigma},L}\boldsymbol{\eta}_1 \end{pmatrix}$$

discovered and explored by Zeltmann [16] we reformulate (2) equivalently into

$$\partial_t \mathbf{v} = \frac{1}{\rho}\operatorname{div}\Big(\sum_{l=0}^{L}\boldsymbol{\sigma}_l\Big) + \frac{1}{\rho}\mathbf{f} \qquad \text{in }]0,T[\times D, \tag{5a}$$

$$\partial_t \boldsymbol{\sigma}_0 = C\big(\mu_0,\pi_0\big)\,\varepsilon(\mathbf{v}) \qquad \text{in }]0,T[\times D, \tag{5b}$$

$$\partial_t \boldsymbol{\sigma}_l = C\big(L\tau_{\mathrm{S}}\mu_0, L\tau_{\mathrm{P}}\pi_0\big)\,\varepsilon(\mathbf{v}) - \frac{1}{\tau_{\boldsymbol{\sigma},l}}\boldsymbol{\sigma}_l, \qquad l=1,\dots,L, \qquad \text{in }]0,T[\times D. \tag{5c}$$

Let $X = L^2(D,\mathbb{R}^3)\times L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})^{1+L}$. For suitable[1] $w = (\mathbf{w},\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_L)\in X$ we define the operators A, B, and Q mapping into X by

$$Aw = -\begin{pmatrix}\operatorname{div}\big(\sum_{l=0}^{L}\boldsymbol{\psi}_l\big)\\ \varepsilon(\mathbf{w})\\ \vdots\\ \varepsilon(\mathbf{w})\end{pmatrix},\quad B^{-1}w = \begin{pmatrix}\frac{1}{\rho}\mathbf{w}\\ C\big(\mu_0,\pi_0\big)\boldsymbol{\psi}_0\\ L\,C\big(\tau_{\mathrm{S}}\mu_0,\tau_{\mathrm{P}}\pi_0\big)\boldsymbol{\psi}_1\\ \vdots\\ L\,C\big(\tau_{\mathrm{S}}\mu_0,\tau_{\mathrm{P}}\pi_0\big)\boldsymbol{\psi}_L\end{pmatrix},\quad Qw = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \frac{1}{\tau_{\boldsymbol{\sigma},1}}\boldsymbol{\psi}_1\\ \vdots\\ \frac{1}{\tau_{\boldsymbol{\sigma},L}}\boldsymbol{\psi}_L\end{pmatrix}. \tag{6}$$

With these operators the system (5) can be rewritten as

$$Bu'(t) + Au(t) + BQu(t) = f(t)$$

where $u = (\mathbf{v},\boldsymbol{\sigma}_0,\dots,\boldsymbol{\sigma}_L)$ and $f = (\mathbf{f},\mathbf{0},\dots,\mathbf{0})$.

Please note: The five parameters to be reconstructed by FWI enter only the operator B via, see (4),

$$\pi_0 = \frac{\rho\, v_{\mathrm{P}}^2}{1+\tau_{\mathrm{P}}\alpha} \quad\text{and}\quad \mu_0 = \frac{\rho\, v_{\mathrm{S}}^2}{1+\tau_{\mathrm{S}}\alpha}. \tag{7}$$

3. Abstract framework

We consider an abstract evolution equation in a Hilbert space X of the form

$$Bu'(t) + Au(t) + BQu(t) = f(t), \quad t\in]0,T[, \quad u(0)=u_0, \tag{8}$$

under the following general hypotheses: $T>0$, $u_0\in X$,

B belongs to the Banach space $\mathcal{L}^*(X) = \{J\in\mathcal{L}(X) : J^* = J\}$ and satisfies $\langle Bx,x\rangle_X = \langle x,Bx\rangle_X \ge \beta\|x\|_X^2$ for some $\beta>0$ and for all $x\in X$,

$A\colon \mathsf{D}(A)\subset X\to X$ is a maximal monotone operator: $\langle Ax,x\rangle_X\ge 0$ for all $x\in X$ and $I+A\colon \mathsf{D}(A)\to X$ is onto (I is the identity),

$Q\in\mathcal{L}(X)$, and $f\in L^1([0,T],X)$.

Later we will show that the three operators from (6) are well defined and satisfy our general hypotheses in a precise mathematical setting.

In [10] we explored (8) with $Q=0$. Existence and regularity results of this paper apply correspondingly. Let us be more precise: equation (8) can be transformed equivalently in

$$u'(t) + (B^{-1}A+Q)u(t) = B^{-1}f(t), \quad t\in]0,T[, \quad u(0)=u_0,$$

where $B^{-1}A$ with $\mathsf{D}(B^{-1}A)=\mathsf{D}(A)$ generates a contraction semigroup on $(X,\langle\cdot,\cdot\rangle_B)$ with weighted inner product $\langle\cdot,\cdot\rangle_B := \langle B\cdot,\cdot\rangle_X$ where the induced norm $\|\cdot\|_B$ is equivalent to the

[1] A rigorous mathematical formulation will be given in Section 4 below.

original norm on X. Further, $B^{-1}A+Q$ is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ with

$$\|S(t)\|_B \leq \exp(\|Q\|_B t),$$

see, e.g., Theorem 3.1.1 of [12]. Thus, (8) has a unique mild/weak solution in $\mathcal{C}([0,T],X)$ given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)B^{-1}f(s)\,\mathrm{d}s.$$

On the basis of the above comments, both Theorems 2.4 and 2.6 of [10] carry over to (8) when we replace f by $B^{-1}f$ and compensate the use of $\|\cdot\|_X$ by an additional constant depending on $\|B\|$, $\|B^{-1}\|$, $\|Q\|$, and T. Thus, we have the continuous dependence of u on the data:

$$\|u\|_{\mathcal{C}([0,T],X)} \lesssim \|u_0\|_X + \|f\|_{L^1([0,T],X)}{}^{2} \tag{9}$$

as well as the following regularity result which has been shown in [10, Theorem 2.6] for $Q=0$ under more general assumptions on f and u_0.

Theorem 3.1. *For some $k\in\mathbb{N}$, let $f\in W^{k,1}(]0,T[,X)$ with $f^{(\ell)}(0)=0$, $\ell=0,\dots,k-1$ (note that $f^{(\ell)}$ is continuous). Let $B\in\mathsf{D}(F)$ and let u be the unique mild solution of (8) with $u_0=0$. Then $u\in\mathcal{C}^k([0,T],X)\cap\mathcal{C}^{k-1}([0,T],\mathsf{D}(A))$ and*

$$\|u\|_{\mathcal{C}^k([0,T],X)} \lesssim \|f\|_{W^{k,1}(]0,T[,X)} \tag{10}$$

where the constant depends on T, Q, $\|B\|$, and $\|B^{-1}\|$.

3.1. **Abstract parameter-to-solution map.** We define the following parameter-to-solution map related to (8):

$$F\colon \mathsf{D}(F)\subset\mathcal{L}^*(X)\to\mathcal{C}([0,T],X),\quad B\mapsto u, \tag{11}$$

where

$$\mathsf{D}(F)=\{B\in\mathcal{L}^*(X):\beta_-\|x\|_X^2\leq\langle Bx,x\rangle_X\leq\beta_+\|x\|_X^2\}$$

for given $0<\beta_-<\beta_+<\infty$.

Transferring the techniques of proof of [10, Theorem 3.6] straightforwardly to F yields the following result.

Theorem 3.2. *Let $T>0$, $f\in W^{1,1}\big(]0,T[,X\big)$, and $u_0\in\mathsf{D}(A)$. Then, the mild solution of (8) is a classical solution, i.e., $u\in\mathcal{C}^1\big([0,T],X\big)\cap\mathcal{C}\big([0,T],\mathsf{D}(A)\big)$, and F is Fréchet differentiable at $B\in\operatorname{int}(\mathsf{D}(F))$ with $F'(B)H=\overline{u}$, $H\in\mathcal{L}^*(X)$, where $\overline{u}\in\mathcal{C}\big([0,T],X\big)$ is the mild solution of*

$$B\overline{u}'(t)+A\overline{u}(t)+BQ\overline{u}(t)=-H(u'(t)+Qu(t)),\ t\in]0,T[,\quad \overline{u}(0)=0. \tag{12}$$

The representation of the adjoint of the Fréchet derivative carries over as well, see [10, Theorem 3.8].

Theorem 3.3. *Under the notation and assumptions of Theorem 3.2 we have*

$$\big[F'(B)^*g\big]H=\int_0^T\big\langle H(u'(t)+Qu(t)),w(t)\big\rangle_X\mathrm{d}t,\quad g\in L^2([0,T],X),\ H\in\mathcal{L}^*(X),$$

where $w\in\mathcal{C}([0,T],X)$ is the mild solution of the adjoint evolution equation

$$Bw'(t)-A^*w(t)-Q^*Bw(t)=g(t),\ t\in]0,T[,\quad w(T)=0. \tag{13}$$

[2] $A\lesssim B$ indicates the existence of a generic constant $c>0$ such that $A\leq c\,B$.

Remark 3.4. *Setting $\widetilde{w}(t) = w(T-t)$ and $\widetilde{g}(t) = g(T-t)$ we rewrite (13) as initial value problem*

$$B\widetilde{w}'(t) + A^*\widetilde{w}(t) + Q^*B\widetilde{w}(t) = -\widetilde{g}(t),\ t \in]0,T[, \quad \widetilde{w}(0) = 0,$$

which is of the same structure as our original equation (8) since A^* is maximal monotone as well. Further, in our concrete setting of the viscoelastic wave equation we have $A^* = -A$ (see the next section) so that basically the same numerical solver can be used for the state and the adjoint state equation.

This remark applies also to the situation of Theorem 4.8 below.

Next we investigate second order differentiability of F.

Theorem 3.5. *Let $f \in W^{3,1}(]0,T[,X)$, $u_0 = 0$, and $f(0) = f'(0) = f''(0) = 0$. Then, F is twice Fréchet differentiable at $B \in \operatorname{int}(\mathsf{D}(F))$ with $F''(B)[H_1,H_2] = \overline{\overline{u}}$, $H_i \in \mathcal{L}^*(X)$, $i = 1,2$, where $\overline{\overline{u}} \in \mathcal{C}([0,T],X)$ is the mild (in fact the classical) solution of*

$$B\overline{\overline{u}}'(t) + A\overline{\overline{u}}(t) + BQ\overline{\overline{u}}(t) = -H_1(\overline{u}'(t) + Q\overline{u}(t)), \quad \overline{\overline{u}}(0) = 0. \tag{14}$$

Here, $\overline{u} \in \mathcal{C}^2([0,T],X) \cap \mathcal{C}^1([0,T],\mathsf{D}(A))$ is the classical solution of (12) with H replaced by H_2:

$$B\overline{u}'(t) + A\overline{u}(t) + BQ\overline{u}(t) = -H_2(u'(t) + Qu(t)), \quad \overline{u}(0) = 0. \tag{15}$$

Further, $u \in \mathcal{C}^3([0,T],X) \cap \mathcal{C}^2([0,T],\mathsf{D}(A))$ solves (8).

Proof. We need to show that

$$\sup_{H_2 \in \mathcal{L}^*(X)} \frac{\|F'(B+H_1)H_2 - F'(B)H_2 - F''(B)[H_1,H_2]\|_{\mathcal{C}([0,T],X)}}{\|H_1\|_{\mathcal{L}(X)}\|H_2\|_{\mathcal{L}(X)}} \xrightarrow{H_1 \to 0} 0.$$

Set $\widetilde{u} := F'(B+H_1)H_2$ which is well defined for H_1 sufficiently small. We have

$$\begin{aligned} B\overline{u}' + (A+BQ)\overline{u} &= -H_2(u' + Qu),\\ (B+H_1)\widetilde{u}' + \big(A + (B+H_1)Q\big)\widetilde{u} &= -H_2(u' + Qu),\\ B\overline{\overline{u}}' + (A+BQ)\overline{\overline{u}} &= -H_1(\overline{u}' + Q\overline{u}). \end{aligned}$$

Then, $\widetilde{u} - \overline{u}$ and $v := \widetilde{u} - \overline{u} - \overline{\overline{u}}$ satisfy

$$B(\widetilde{u} - \overline{u})' + (A+BQ)(\widetilde{u} - \overline{u}) = -H_1(\widetilde{u}' + Q\widetilde{u}) \tag{16}$$

and

$$Bv' + (A+BQ)v = -H_1\big[(\widetilde{u} - \overline{u})' + Q(\widetilde{u} - \overline{u})\big],$$

respectively, with homogeneous initial conditions. Using the continuous dependence of v on the right hand side, see (9), we get

$$\|v\|_{\mathcal{C}([0,T],X)} \lesssim \|H_1\|_{\mathcal{L}(X)}\, \|\widetilde{u} - \overline{u}\|_{\mathcal{C}^1([0,T],X)}. \tag{17}$$

Now we apply the regularity estimate (10) repeatedly for $k = 1$ to $\widetilde{u} - \overline{u}$ in (16), then for $k = 2$ to $\widetilde{u}$ and finally for $k = 3$ to u:

$$\begin{aligned} \|\widetilde{u} - \overline{u}\|_{\mathcal{C}^1([0,T],X)} \lesssim \|H_1\|_{\mathcal{L}(X)}\|\widetilde{u}\|_{\mathcal{C}^2([0,T],X)} &\lesssim \|H_1\|_{\mathcal{L}(X)}\|H_2\|_{\mathcal{L}(X)}\|u\|_{\mathcal{C}^3([0,T],X)}\\ &\lesssim \|H_1\|_{\mathcal{L}(X)}\|H_2\|_{\mathcal{L}(X)}\|f\|_{W^{3,\infty}(]0,T[,X)}. \end{aligned}$$

Substituting the latter bound into (17) yields

$$\frac{1}{\|H_1\|_{\mathcal{L}(X)}} \sup_{H_2 \in \mathcal{L}(X)} \frac{\|\widetilde{u} - \overline{u} - \overline{\overline{u}}\|_{\mathcal{C}([0,T],X)}}{\|H_2\|_{\mathcal{L}(X)}} \lesssim \|H_1\|_{\mathcal{L}(X)}\|f\|_{W^{3,\infty}(]0,T[,X)}$$

which finishes the proof. $\square$

Remark 3.6. *In seismic exploration, where (8) is the viscoacoustic or viscoelastic wave equation, we can assume the environment to be at rest before firing the source. In other words, the assumptions on u_0 and f from the above theorem are justified.*

The mindful reader might have noticed an unbalanced increase of the smoothness assumptions on f and u_0 from Theorem 3.2 ($f \in W^{1,1}$) to Theorem 3.5 ($f \in W^{3,1}$) compared to the increase of smoothness of F: two additional differentiation orders for f gain only one order for F. This is because in (17) we need convergence of $\|\widetilde{u} - \overline{u}\|_{\mathcal{C}^1([0,T],X)} \to 0$ as $H_1 \to 0$ *uniformly* in H_2. At least we get $F \in \mathcal{C}^{2,1}$, that is, F'' is uniformly Lipschitz continuous.

Theorem 3.7. *Under the assumptions of Theorem 3.5 we have that*[3]

$$\|F''(B) - F''(\widetilde{B})\|_{\mathcal{L}^2(\mathcal{L}^*(X),\mathcal{C}([0,T],X))} \lesssim \|B - \widetilde{B}\|_{\mathcal{L}(X)}$$

uniformly in $\operatorname{int}(\mathsf{D}(F))$. *The constant in the above estimate only depends on β_-, β_+, T, Q, and f.*

Proof. For $H_i \in \mathcal{L}^*(X)$, $i = 1, 2$, we estimate $\|\overline{\overline{u}} - \overline{\overline{v}}\|_{\mathcal{C}([0,T],X)}$ where $\overline{\overline{v}} = F''(B + \delta B)[H_1, H_2]$, $\overline{\overline{u}} = F''(B)[H_1, H_2]$. From (14) we get

$$B(\overline{\overline{v}}\,' - \overline{\overline{u}}\,') + (A + BQ)(\overline{\overline{v}} - \overline{\overline{u}}) = -H_1(\overline{v}\,' - \overline{u}\,' + Q(\overline{v} - \overline{u})) - \delta B(\overline{\overline{v}}\,' + Q\overline{\overline{v}})$$

where $\overline{u}$ is the solution of (15) and $\overline{v}$ solves (15) with B replaced by $B + \delta B$ and u by v, the latter being the solution of (8) with $B + \delta B$ instead of B and $v(0) = 0$. As before, by the continuous dependence on the right hand side,

$$\|\overline{\overline{v}} - \overline{\overline{u}}\|_{\mathcal{C}([0,T],X)} \lesssim \|H_1\|_{\mathcal{L}(X)}\|\overline{v} - \overline{u}\|_{\mathcal{C}^1([0,T],X)} + \|\delta B\|_{\mathcal{L}(X)}\|\overline{\overline{v}}\|_{\mathcal{C}^1([0,T],X)} \tag{18}$$

where the involved constant only depends on β_-, β_+, T, and Q. All constants in this proof, which are not explicitly given, only depend on these four quantities.

Further, by applying (10) again repeatedly for $k = 1$, $k = 2$, and $k = 3$, we obtain

$$\begin{aligned}\|\overline{\overline{v}}\|_{\mathcal{C}^1([0,T],X)} \lesssim \|H_1\|_{\mathcal{L}(X)}\|\overline{v}\|_{\mathcal{C}^2([0,T],X)} &\lesssim \|H_1\|_{\mathcal{L}(X)}\|H_2\|_{\mathcal{L}(X)}\|v\|_{\mathcal{C}^3([0,T],X)} \\ &\lesssim \|H_1\|_{\mathcal{L}(X)}\|H_2\|_{\mathcal{L}(X)}\|f\|_{W^{3,\infty}(]0,T[,X)}.\end{aligned} \tag{19}$$

In view of (18) it remains to investigate $\|\overline{v} - \overline{u}\|_{\mathcal{C}^1([0,T],X)}$. We can use the same approach as above: Set $\overline{d} = \overline{v} - \overline{u}$ and $d = v - u$. Then, $\overline{d}(0) = 0$ and

$$B\overline{d}\,' + (A + BQ)\overline{d} = -H_2(d' + Qd) - \delta B(\overline{v}\,' + Q\overline{v}).$$

By (10) as well as the second and third estimate from (19),

$$\|\overline{d}\|_{\mathcal{C}^1([0,T],X)} \lesssim \|H_2\|_{\mathcal{L}(X)}\big(\|d\|_{\mathcal{C}^2([0,T],X)} + \|\delta B\|_{\mathcal{L}(X)}\|f\|_{W^{3,\infty}(]0,T[,X)}\big).$$

We are left with estimating $\|d\|_{\mathcal{C}^2([0,T],X)}$. Note that

$$Bd' + (A + BQ)d = -\delta B(v' + Qv)$$

and (10) delivers

$$\|d\|_{\mathcal{C}^2([0,T],X)} \lesssim \|\delta B\|_{\mathcal{L}(X)}\|v\|_{\mathcal{C}^3([0,T],X)} \lesssim \|\delta B\|_{\mathcal{L}(X)}\|f\|_{W^{3,\infty}(]0,T[,X)}.$$

So we found that

$$\|\overline{v} - \overline{u}\|_{\mathcal{C}^1([0,T],X)} \lesssim \|H_2\|_{\mathcal{L}(X)}\|\delta B\|_{\mathcal{L}(X)}\|f\|_{W^{3,1}(]0,T[,X)}.$$

[3] $\mathcal{L}^2(V, W)$ denotes the space of bounded bilinear mappings from V to W.

Plugging this bound together with (19) into (18) results in

$$\sup_{H_1,H_2\in\mathcal{L}^*(X)} \frac{\|\overline{\overline{v}}-\overline{\overline{u}}\|_{\mathcal{C}([0,T],X)}}{\|H_1\|_{\mathcal{L}(X)}\|H_2\|_{\mathcal{L}(X)}} \lesssim \|f\|_{W^{3,\infty}(]0,T[,X)}\|\delta B\|_{\mathcal{L}(X)}$$

and we are done. □

3.2. **Local ill-posedness.** We recall briefly the concept of local ill-posedness from [9]: Let $\Psi\colon \mathsf{D}(\Psi)\subset\mathcal{X}\to\mathcal{Y}$ be a mapping between infinite dimensional normed spaces. Then, the equation $\Psi(\cdot)=y$ is *locally ill-posed* at $x^+\in\mathsf{D}(\Psi)$ if in any neighborhood U of x^+ there exists a sequence $\{\xi_k\}\subset U\cap\mathsf{D}(\Psi)$ with $\lim_{k\to\infty}\|\Psi(\xi_k)-y\|_{\mathcal{Y}}=0$ but $\{\xi_k\}$ does not converge to x^+ in $\mathcal{X}$.

Here, we consider (11) as a mapping with the larger image space $L^2([0,T],X)$. Theorem 4.1 of [10] applies directly to (8) and (11). The proof only needs a slight and obvious modification.

Theorem 3.8. *Let u be the classical solution of (8) for $u_0\in\mathsf{D}(A)$ and $f\in W^{1,1}(]0,T[,X)$. Then the equation $F(B)=u$ is locally ill-posed at any $\widehat{B}\in\mathsf{D}(F)$ satisfying $F(\widehat{B})=u$ if for any $r\in(0,1]$ there exists $\widehat{r}\in(0,r)$ and a sequence of bounded, symmetric and monotone operators $E_k\colon X\to X$ such that $\widehat{B}+E_k\in\mathsf{D}(F)$, $\widehat{r}\le\|E_k\|_{\mathcal{L}(X)}\le r$ for all $k\in\mathbb{N}$, and $\lim_{k\to\infty}E_kv=0$ for all $v\in X$.*

4. Application to the viscoelastic wave equation

We apply the abstract results to the viscoelastic wave equation in the formulation (5). The underlying Hilbert space is

$$X=L^2(D,\mathbb{R}^3)\times L^2(D,\mathbb{R}^{3\times3}_{\text{sym}})^{1+L}$$

with inner product

$$\big\langle(\mathbf{v},\boldsymbol{\sigma}_0,\dots,\boldsymbol{\sigma}_L),(\mathbf{w},\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_L)\big\rangle_X=\int_D\Big(\mathbf{v}\cdot\mathbf{w}+\sum_{l=0}^L\boldsymbol{\sigma}_l:\boldsymbol{\psi}_l\Big)\,\mathrm{d}x$$

where the colon indicates the Frobenius inner product on $\mathbb{R}^{3\times3}$.

To define the domain $\mathsf{D}(A)$ of A (6) we split the boundary ∂D of the bounded Lipschitz domain D into disjoint parts $\partial D=\partial D_D\,\dot\cup\,\partial D_N$. Let $\mathbf{n}$ be the outer normal vector on ∂D_N. Then,

$$\mathsf{D}(A)=\Big\{(\mathbf{w},\boldsymbol{\psi}_0,\dots\boldsymbol{\psi}_L)\in H^1_D\times H(\operatorname{div}):\sum_{l=0}^L\boldsymbol{\psi}_l\mathbf{n}=0\text{ on }\partial D_N\Big\}$$

with $H^1_D=\{\mathbf{v}\in H^1(D,\mathbb{R}^3):\mathbf{v}=0\text{ on }\partial D_D\}$ and $H(\operatorname{div})=\Big\{\boldsymbol{\sigma}\in L^2\big(D,\mathbb{R}^{3\times3}_{\text{sym}}\big)^{1+L}:\operatorname{div}\big(\sum_{l=0}^L\boldsymbol{\sigma}_l\big)\in L^2(D,\mathbb{R}^3),\Big\}$.[4]

Lemma 4.1. *The operator A as defined in (6) with $\mathsf{D}(A)\subset X$ from above is maximal monotone.*

Proof. Since

$$\big\langle A(\mathbf{v},\boldsymbol{\sigma}_0,\dots,\boldsymbol{\sigma}_L),(\mathbf{w},\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_L)\big\rangle_X=-\int_D\Big[\operatorname{div}\Big(\sum_{l=0}^L\boldsymbol{\sigma}_l\Big)\cdot\mathbf{w}+\boldsymbol{\varepsilon}(\mathbf{v}):\Big(\sum_{l=0}^L\boldsymbol{\psi}_l\Big)\Big]\,\mathrm{d}x$$

[4] The traces $\sum_{l=0}^L\boldsymbol{\sigma}_l\mathbf{n}$ exist in a suitable space, see, e.g., [11].

we can proceed exactly as in the proof of Lemma 6.1 from [10] to show skew-symmetry of A. Hence, $\langle Aw, w\rangle_X = 0$ for all $w \in \mathsf{D}(A)$.

Next we show that $I + A$ is onto adapting arguments of [10]. We will be brief therefore. For $(\mathbf{f}, \mathbf{g}_0, \ldots, \mathbf{g}_L) \in X$ we need to find $(\mathbf{v}, \boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_L) \in \mathsf{D}(A)$ satisfying

$$\mathbf{v} - \operatorname{div}\Big(\sum_{l=0}^{L} \boldsymbol{\sigma}_l\Big) = \mathbf{f}, \qquad \boldsymbol{\sigma}_l - \boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{g}_l, \quad l = 0, \ldots, L.$$

We multiply the equation on the left by a $\mathbf{w} \in H^1_D$, integrate over D and use the divergence theorem to get

$$\int_D \Big(\mathbf{v}\cdot\mathbf{w} + \Big(\sum_{l=0}^{L} \boldsymbol{\sigma}_l\Big) : \nabla\mathbf{w}\Big)\mathrm{d}x = \int_D \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x.$$

Now we sum up the $L+1$ equations on the right, use the relation $\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\sigma} = \nabla\mathbf{v} : \boldsymbol{\sigma}$ for $\boldsymbol{\sigma} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$, and arrive at

$$\int_D \big(\mathbf{v}\cdot\mathbf{w} + (L+1)\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w})\big)\mathrm{d}x = \int_D \Big(\mathbf{f}\cdot\mathbf{w} - \sum_{l=0}^{L} \mathbf{g}_l : \nabla\mathbf{w}\Big)\mathrm{d}x \quad \text{for all } \mathbf{w} \in H^1_D.$$

This is a standard variational problem (cf. displacement ansatz in elasticity) admitting a unique solution $\mathbf{v} \in H^1_D$.

Set $\boldsymbol{\sigma}_l = \mathbf{g}_l + \boldsymbol{\varepsilon}(\mathbf{v})$ and follow [10] to verify $(\mathbf{v}, \boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_L) \in \mathsf{D}(A)$. $\square$

Next we show that $B \in \mathcal{L}(X)$ from (6) is well defined with the required properties. As in [10] we consider C of (3) as a mapping from $\mathsf{D}(C) = \big\{(m, p) \in \mathbb{R}^2 : \underline{\mathrm{m}} \le m \le \overline{\mathrm{m}},\ \underline{\mathrm{p}} \le p \le \overline{\mathrm{p}}\big\}$ into $\mathrm{Aut}(\mathbb{R}^{3\times3}_{\mathrm{sym}})$[5] with constants $0 < \underline{\mathrm{m}} < \overline{\mathrm{m}}$ and $0 < \underline{\mathrm{p}} < \overline{\mathrm{p}}$ such that $3\underline{\mathrm{p}} > 4\overline{\mathrm{m}}$.[6] For $(m, p) \in \mathsf{D}(C)$,

$$\widetilde{C}(m, p) := C(m, p)^{-1} = C\left(\frac{1}{4m}, \frac{p - m}{m(3p - 4m)}\right). \tag{20}$$

Moreover, $C(m, p)\mathbf{M} : \mathbf{N} = \mathbf{M} : C(m, p)\mathbf{N}$ and

$$\min\{2\underline{\mathrm{m}}, 3\underline{\mathrm{p}} - 4\overline{\mathrm{m}}\}\, \mathbf{M} : \mathbf{M} \le C(m, p)\mathbf{M} : \mathbf{M} \le \max\{2\overline{\mathrm{m}}, 3\overline{\mathrm{p}} - 4\underline{\mathrm{m}}\}\, \mathbf{M} : \mathbf{M},$$

see, e.g., [16, Lemma 50]. Provided $\rho(x) > 0$, $\big(\mu_0(x), \pi_0(x)\big), \big(\tau_{\mathrm{S}}(x)\mu_0(x), \tau_{\mathrm{P}}(x)\pi_0(x)\big) \in \mathsf{D}(C)$ for almost all $x \in D$ we conclude that

$$B\begin{pmatrix} \mathbf{w} \\ \boldsymbol{\psi}_0 \\ \boldsymbol{\psi}_1 \\ \vdots \\ \boldsymbol{\psi}_L \end{pmatrix} = \begin{pmatrix} \rho\,\mathbf{w} \\ \widetilde{C}\big(\mu_0, \pi_0\big)\boldsymbol{\psi}_0 \\ \frac{1}{L}\,\widetilde{C}\big(\tau_{\mathrm{S}}\mu_0, \tau_{\mathrm{P}}\pi_0\big)\boldsymbol{\psi}_1 \\ \vdots \\ \frac{1}{L}\,\widetilde{C}\big(\tau_{\mathrm{S}}\mu_0, \tau_{\mathrm{P}}\pi_0\big)\boldsymbol{\psi}_L \end{pmatrix} \tag{21}$$

yielding a uniformly positive $B \in \mathcal{L}^*(X)$ in the sense of our general hypotheses from the beginning of Section 3. Hence, the general hypotheses are satisfied for the viscoelastic wave equation.

[5] This is the space of linear maps from $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ into itself (space of automorphisms).

[6] Note that in [10] and [16] different C's are used.

4.1. **Full waveform forward operator.** In FWI the five parameters $(\rho, v_{\mathrm{S}}, \tau_{\mathrm{S}}, v_{\mathrm{P}}, \tau_{\mathrm{P}})$ are of interest. Therefore we will define a parameter-to-solution map Φ which takes these parameters as arguments. A physically meaningful domain of definition for Φ is

$$\mathsf{D}(\Phi) = \big\{(\rho, v_{\mathrm{S}}, \tau_{\mathrm{S}}, v_{\mathrm{P}}, \tau_{\mathrm{P}}) \in L^\infty(D)^5 : \rho_{\min} \le \rho(\cdot) \le \rho_{\max},\ v_{\mathrm{P},\min} \le v_{\mathrm{P}}(\cdot) \le v_{\mathrm{P},\max},$$
$$v_{\mathrm{S},\min} \le v_{\mathrm{S}}(\cdot) \le v_{\mathrm{S},\max},\ \tau_{\mathrm{P},\min} \le \tau_{\mathrm{P}}(\cdot) \le \tau_{\mathrm{P},\max},\ \tau_{\mathrm{S},\min} \le \tau_{\mathrm{S}}(\cdot) \le \tau_{\mathrm{S},\max} \text{ a.e. in } D\big\}$$

with suitable positive bounds $0 < \rho_{\min} < \rho_{\max} < \infty$, etc.

In view of (4) we set

$$\mu_{\min} := \frac{\rho_{\min}\, v_{\mathrm{S},\min}^2}{1 + \tau_{\mathrm{S},\max}\alpha} \quad \text{and} \quad \mu_{\max} := \frac{\rho_{\max}\, v_{\mathrm{S},\max}^2}{1 + \tau_{\mathrm{S},\min}\alpha}$$

which are induced lower and upper bounds for μ_0. We set the bounds $\pi_{\min}$ and $\pi_{\max}$ for π_0 accordingly by replacing s by p. Next we define $\underline{\mathrm{p}}$, $\overline{\mathrm{p}}$, $\underline{\mathrm{m}}$, and $\overline{\mathrm{m}}$ such that $(\mu_0, \pi_0), (\tau_{\mathrm{S}}\mu_0, \tau_{\mathrm{P}}\pi_0)$ as functions of $(\rho, v_{\mathrm{P}}, v_{\mathrm{S}}, \tau_{\mathrm{P}}, \tau_{\mathrm{S}}) \in \mathsf{D}(\Phi)$ are in $\mathsf{D}(C)$. Indeed,

$$\underline{\mathrm{p}} := \pi_{\min} \min\{1, \tau_{\mathrm{P},\min}\} \quad \text{and} \quad \overline{\mathrm{p}} := \pi_{\max} \max\{1, \tau_{\mathrm{P},\max}\}$$

with $\underline{\mathrm{m}}$ and $\overline{\mathrm{m}}$ set correspondingly will do the job. The restriction $3\underline{\mathrm{p}} > 4\overline{\mathrm{m}}$ translates into

$$\frac{4}{3}\,\frac{\rho_{\max}}{\rho_{\min}}\,\frac{1 + \tau_{\mathrm{P},\max}\alpha}{1 + \tau_{\mathrm{S},\min}\alpha}\,\frac{\max\{1, \tau_{\mathrm{S},\max}\}}{\min\{1, \tau_{\mathrm{P},\min}\}} < \frac{v_{\mathrm{P},\min}^2}{v_{\mathrm{S},\max}^2}$$

which reflects in a way the physical fact that pressure waves propagate considerably faster than shear waves.

For $\mathbf{f} \in W^{1,1}(]0,T[, L^2(D,\mathbb{R}^3))$ and $u_0 = (\mathbf{v}(0), \boldsymbol{\sigma}_0(0), \ldots, \boldsymbol{\sigma}_L(0)) \in \mathsf{D}(A)$ the *full waveform forward operator*

$$\Phi\colon \mathsf{D}(\Phi) \subset L^\infty(D)^5 \to L^2([0,T], X), \quad (\rho, v_{\mathrm{S}}, \tau_{\mathrm{S}}, v_{\mathrm{P}}, \tau_{\mathrm{P}}) \mapsto (\mathbf{v}, \boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_L),$$

is well defined where $(\mathbf{v}, \boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_L)$ is the unique classical solution of (5) with initial value u_0.

To benefit from our abstract results we factorize $\Phi = F \circ V$ where F is as in (11) and

$$V\colon \mathsf{D}(\Phi) \subset L^\infty(D)^5 \to \mathcal{L}^*(X), \quad (\rho, v_{\mathrm{S}}, \tau_{\mathrm{S}}, v_{\mathrm{P}}, \tau_{\mathrm{P}}) \mapsto B,$$

where B is defined in (21) via (7).

Remark 4.2. *Note that the image of V is in $\mathsf{D}(F)$ by an appropriate choice of β_- and β_+ in terms of $\rho_{\min}$, $\rho_{\max}$, $\underline{\mathrm{p}}$, $\overline{\mathrm{p}}$, $\underline{\mathrm{m}}$, and $\overline{\mathrm{m}}$.*

The inverse problem of FWI in the viscoelastic regime is locally ill-posed. This can be proved using Theorem 3.8, compare the proof of Theorem 6.7 of [10]. We give a direct proof though.

Theorem 4.3. *The inverse problem $\Phi(\cdot) = (\mathbf{v}, \boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_L)$ is locally ill-posed at any interior point of $\mathbf{p} = (\rho, v_S, \tau_S, v_P, \tau_P) \in \mathsf{D}(\Phi)$.*

Proof. Fix a point $\xi \in D$ and define balls $K_n = \{y \in \mathbb{R}^3 : |y - \xi| \le \delta/n\}$ with a $\delta > 0$ so small that $K_n \subset D$ for all $n \in \mathbb{N}$. Let χ_n be the indicator function of K_n. Further, for any $r > 0$ such that $\mathbf{p}_n := \mathbf{p} + r(\chi_n, \chi_n, \chi_n, \chi_n, \chi_n) \in \mathsf{D}(\Phi)$ we have that $\|\mathbf{p}_n - \mathbf{p}\|_{L^\infty(D)^5} = r$, that is, $\mathbf{p}_n$ does not converge to $\mathbf{p}$. However, $\lim_{n\to\infty} \|\Phi(\mathbf{p}_n) - \Phi(\mathbf{p})\|_{L^2([0,T],X)} = 0$ as we demonstrate now.

Let $u_n = \Phi(\mathbf{p}_n)$ and $u = \Phi(\mathbf{p})$. Then, $d_n = u_n - u$ satisfies

$$V(\mathbf{p}_n)d_n' + Ad_n + V(\mathbf{p}_n)Qd_n = \big(V(\mathbf{p}) - V(\mathbf{p}_n)\big)(u' + Qu), \quad d_n(0) = 0.$$

By the continuous dependence of d_n on the data, see (9), we obtain

$$\|d_n\|_{L^2([0,T],X)} \lesssim \big\|\big(V(\mathbf{p}) - V(\mathbf{p}_n)\big)(u' + Qu)\big\|_{L^1([0,T],X)}$$

where the constant is independent of n, see Remark 4.2. Next one shows $\lim_{n\to\infty} \|\big(V(\mathbf{p}) - V(\mathbf{p}_n)\big)v\|_X = 0$ for any $v \in X$ using $\mathbf{p}_n \to \mathbf{p}$ pointwise a.e. in D as $n \to \infty$ and the dominated convergence theorem. Since $\|V(\mathbf{p}_n)\|_X \lesssim 1$ for all $n \in \mathbb{N}$ a further application of the dominated convergence theorem with respect to the time domain yields

$$\int_0^T \big\|\big(V(\mathbf{p}) - V(\mathbf{p}_n)\big)\big(u'(t) + Qu(t)\big)\big\|_X \mathrm{d}t \xrightarrow{n\to\infty} 0$$

and finishes the proof. □

4.2. **First order differentiability.** To derive the first order Fréchet derivative of Φ we provide the Fréchet derivative of V. Its formulation needs the derivative of $\widetilde{C}$ which we take from [10, Lemma 6.3]:

$$\widetilde{C}'(m,p)\begin{bmatrix}\widehat{m}\\ \widehat{p}\end{bmatrix} = -\widetilde{C}(m,p) \circ C(\widehat{m},\widehat{p}) \circ \widetilde{C}(m,p) \tag{22}$$

for $(m,p) \in \operatorname{int}(\mathsf{D}(C))$ and $(\widehat{m},\widehat{p}) \in \mathbb{R}^2$.

Let $\mathbf{p} = (\rho, v_{\mathrm{S}}, \tau_{\mathrm{S}}, v_{\mathrm{P}}, \tau_{\mathrm{P}}) \in \operatorname{int}(\mathsf{D}(\Phi))$ and $\widehat{\mathbf{p}} = (\widehat{\rho}, \widehat{v}_{\mathrm{S}}, \widehat{\tau}_{\mathrm{S}}, \widehat{v}_{\mathrm{P}}, \widehat{\tau}_{\mathrm{P}}) \in L^\infty(D)^5$. Then, $V'(\mathbf{p})\widehat{\mathbf{p}} \in \mathcal{L}^*(X)$ is given by

$$V'(\mathbf{p})\widehat{\mathbf{p}}\begin{pmatrix}\mathbf{w}\\ \boldsymbol{\psi}_0\\ \vdots\\ \boldsymbol{\psi}_L\end{pmatrix} = \begin{pmatrix} \widehat{\rho}\,\mathbf{w}\\ -\frac{\widehat{\rho}}{\rho^2}\widetilde{C}(\mu,\pi)\boldsymbol{\psi}_0 + \frac{1}{\rho}\widetilde{C}'(\mu,\pi)\begin{bmatrix}\widetilde{\mu}\\ \widetilde{\pi}\end{bmatrix}\boldsymbol{\psi}_0\\ -\frac{\widehat{\rho}}{L\rho^2}\widetilde{C}(\tau_{\mathrm{S}}\mu, \tau_{\mathrm{P}}\pi)\boldsymbol{\psi}_1 + \frac{1}{L\rho}\widetilde{C}'(\tau_{\mathrm{S}}\mu, \tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}\\ \widehat{\pi}\end{bmatrix}\boldsymbol{\psi}_1\\ \vdots\\ -\frac{\widehat{\rho}}{L\rho^2}\widetilde{C}(\tau_{\mathrm{S}}\mu, \tau_{\mathrm{P}}\pi)\boldsymbol{\psi}_L + \frac{1}{L\rho}\widetilde{C}'(\tau_{\mathrm{S}}\mu, \tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}\\ \widehat{\pi}\end{bmatrix}\boldsymbol{\psi}_L \end{pmatrix} \tag{23}$$

where $\mu = \mu_0/\rho$, $\pi = \pi_0/\rho$, see (7), and

$$\widetilde{\mu} = \frac{2v_{\mathrm{S}}}{1+\tau_{\mathrm{S}}\alpha}\widehat{v}_{\mathrm{S}} - \frac{\alpha\, v_{\mathrm{S}}^2}{(1+\tau_{\mathrm{S}}\alpha)^2}\widehat{\tau}_{\mathrm{S}}, \qquad \widetilde{\pi} = \frac{2v_{\mathrm{P}}}{1+\tau_{\mathrm{P}}\alpha}\widehat{v}_{\mathrm{P}} - \frac{\alpha\, v_{\mathrm{P}}^2}{(1+\tau_{\mathrm{P}}\alpha)^2}\widehat{\tau}_{\mathrm{P}}, \tag{24}$$

$$\widehat{\mu} = \frac{2\tau_{\mathrm{S}}\, v_{\mathrm{S}}}{1+\tau_{\mathrm{S}}\alpha}\widehat{v}_{\mathrm{S}} + \frac{v_{\mathrm{S}}^2}{(1+\tau_{\mathrm{S}}\alpha)^2}\widehat{\tau}_{\mathrm{S}}, \qquad \widehat{\pi} = \frac{2\tau_{\mathrm{P}}\, v_{\mathrm{P}}}{1+\tau_{\mathrm{P}}\alpha}\widehat{v}_{\mathrm{P}} + \frac{v_{\mathrm{P}}^2}{(1+\tau_{\mathrm{P}}\alpha)^2}\widehat{\tau}_{\mathrm{P}}. \tag{25}$$

Theorem 4.4. *Under the assumptions made in this section the full waveform forward operator Φ is Fréchet differentiable at any interior point $\mathbf{p} = (\rho, v_S, \tau_S, v_P, \tau_P)$ of $\mathsf{D}(\Phi)$: For $\widehat{\mathbf{p}} = (\widehat{\rho}, \widehat{v}_S, \widehat{\tau}_S, \widehat{v}_P, \widehat{\tau}_P) \in L^\infty(D)^5$ we have $\Phi'(\mathbf{p})\widehat{\mathbf{p}} = \overline{u}$ where $\overline{u} = (\overline{\mathbf{v}}, \overline{\boldsymbol{\sigma}}_0, \ldots, \overline{\boldsymbol{\sigma}}_L) \in \mathcal{C}([0,T],X)$ with $\overline{u}(0) = 0$ is the mild solution of*

$$\rho\,\partial_t\overline{\mathbf{v}} = \operatorname{div}\Big(\sum_{l=0}^{L}\overline{\boldsymbol{\sigma}}_l\Big) - \widehat{\rho}\,\partial_t\mathbf{v}, \tag{26a}$$

$$\partial_t\overline{\boldsymbol{\sigma}}_0 = C(\mu_0,\pi_0)\boldsymbol{\varepsilon}(\overline{\mathbf{v}}) + \big(\widehat{\rho}\,C(\mu,\pi) + \rho\,C(\widetilde{\mu},\widetilde{\pi})\big)\varepsilon(\mathbf{v}), \tag{26b}$$

$$\partial_t\overline{\boldsymbol{\sigma}}_l = L\,C(\tau_S\mu_0, \tau_P\pi_0)\boldsymbol{\varepsilon}(\overline{\mathbf{v}}) \tag{26c}$$

$$-\frac{1}{\tau_{\boldsymbol{\sigma},L}}\overline{\boldsymbol{\sigma}}_l + \big(\widehat{\rho}\,L\,C(\tau_S\mu,\tau_P\pi) + \rho\,C(\widehat{\mu},\widehat{\pi})\big)\varepsilon(\mathbf{v}), \quad l = 1,\dots,L,$$

where $(\mathbf{v},\boldsymbol{\sigma}_0,\dots,\boldsymbol{\sigma}_L)$ *is the classical solution of* (5).

Proof. We apply Theorem 3.2 to $\Phi'(\mathbf{p})\widehat{\mathbf{p}} = F'(V(\mathbf{p}))V'(\mathbf{p})\widehat{\mathbf{p}}$ and get the system

$$\begin{pmatrix} \rho\,\partial_t\overline{\mathbf{v}} \\ \frac{1}{\rho}\,\widetilde{C}(\mu,\pi)\partial_t\overline{\boldsymbol{\sigma}}_0 \\ \frac{1}{L\rho}\,\widetilde{C}(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\partial_t\overline{\boldsymbol{\sigma}}_1 \\ \vdots \\ \frac{1}{L\rho}\,\widetilde{C}(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\partial_t\overline{\boldsymbol{\sigma}}_L \end{pmatrix} = \begin{pmatrix} \operatorname{div}\big(\sum_{l=0}^L\overline{\boldsymbol{\sigma}}_l\big) \\ \varepsilon(\overline{\mathbf{v}}) \\ \vdots \\ \varepsilon(\overline{\mathbf{v}}) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \frac{1}{L\,\rho\,\tau_{\boldsymbol{\sigma},1}}\,\widetilde{C}\big(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi\big)\overline{\boldsymbol{\sigma}}_1 \\ \vdots \\ \frac{1}{L\,\rho\,\tau_{\boldsymbol{\sigma},L}}\,\widetilde{C}\big(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi\big)\overline{\boldsymbol{\sigma}}_L \end{pmatrix}$$

$$- V'(\mathbf{p})\widehat{\mathbf{p}}\left[\begin{pmatrix} \partial_t\mathbf{v} \\ \partial_t\boldsymbol{\sigma}_0 \\ \partial_t\boldsymbol{\sigma}_1 \\ \vdots \\ \partial_t\boldsymbol{\sigma}_L \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \frac{1}{\tau_{\boldsymbol{\sigma},1}}\,\boldsymbol{\sigma}_1 \\ \vdots \\ \frac{1}{\tau_{\boldsymbol{\sigma},L}}\,\boldsymbol{\sigma}_L \end{pmatrix}\right]$$

which is equivalent to (26) in view of (5b), (5c), (22), and (23). □

Theorem 4.5. *The assumptions are as in Theorem 4.4. Then, the adjoint* $\Phi'(\mathbf{p})^* \in \mathcal{L}\big(L^2([0,T],X),(L^\infty(D)^5)'\big)$ *at* $\mathbf{p} = (\rho, v_S, \tau_S, v_P, \tau_P) \in \mathsf{D}(\Phi)$ *is given by*

$$\Phi'(\mathbf{p})^*\mathbf{g} = \begin{pmatrix} \int_0^T\big(\partial_t\mathbf{v}\cdot\mathbf{w} - \frac{1}{\rho}\,\boldsymbol{\varepsilon}(\mathbf{v}) : (\boldsymbol{\varphi}_0 + \boldsymbol{\Sigma})\big)\mathrm{d}t \\ \frac{2}{v_S}\int_0^T\big(-\boldsymbol{\varepsilon}(\mathbf{v}) : (\boldsymbol{\varphi}_0 + \boldsymbol{\Sigma}) + \pi\,\mathrm{tr}(\boldsymbol{\Sigma}^v)\operatorname{div}\mathbf{v}\big)\mathrm{d}t \\ \frac{1}{1+\alpha\tau_S}\int_0^T\big(\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\Sigma}^\tau_{S,2} + \pi\,\mathrm{tr}(\boldsymbol{\Sigma}^\tau_{S,1})\operatorname{div}\mathbf{v}\big)\mathrm{d}t \\ -\frac{2\pi}{v_P}\int_0^T\mathrm{tr}(\boldsymbol{\Sigma}^v)\operatorname{div}\mathbf{v}\,\mathrm{d}t \\ \frac{\pi}{1+\alpha\tau_P}\int_0^T\mathrm{tr}(\boldsymbol{\Sigma}^\tau_P)\operatorname{div}\mathbf{v}\,\mathrm{d}t \end{pmatrix} \in L^1(D)^5$$

for $\mathbf{g} = (\mathbf{g}_{-1},\mathbf{g}_0,\dots,\mathbf{g}_L) \in L^2\big([0,T],L^2(D,\mathbb{R}^3)\times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})^{1+L}\big)$ *where* $\mathbf{v}$ *is the first component of the solution of* (5), $\boldsymbol{\Sigma} = \sum_{l=1}^L\boldsymbol{\varphi}_l$, *and*

$$\boldsymbol{\Sigma}^v = \frac{1}{3\pi-4\mu}\,\boldsymbol{\varphi}_0 + \frac{\tau_P}{3\tau_P\pi - 4\tau_S\mu}\,\boldsymbol{\Sigma},$$

$$\boldsymbol{\Sigma}^\tau_{S,1} = -\frac{\alpha}{3\pi-4\mu}\,\boldsymbol{\varphi}_0 + \frac{\tau_P}{\tau_S(3\tau_P\pi - 4\tau_S\mu)}\,\boldsymbol{\Sigma}, \quad \boldsymbol{\Sigma}^\tau_{S,2} = \alpha\,\boldsymbol{\varphi}_0 - \frac{1}{\tau_S}\,\boldsymbol{\Sigma},$$

$$\boldsymbol{\Sigma}^\tau_P = \frac{\alpha}{3\pi-4\mu}\,\boldsymbol{\varphi}_0 - \frac{1}{3\tau_P\pi - 4\tau_S\mu}\,\boldsymbol{\Sigma},$$

and $w = (\mathbf{w},\boldsymbol{\varphi}_0,\dots,\boldsymbol{\varphi}_L) \in \mathcal{C}([0,T],X)$ *uniquely solves*

$$\partial_t\mathbf{w} = \frac{1}{\rho}\operatorname{div}\Big(\sum_{l=0}^L\boldsymbol{\varphi}_l\Big) + \frac{1}{\rho}\,\mathbf{g}_{-1}, \tag{27a}$$

$$\partial_t\boldsymbol{\varphi}_0 = C\big(\mu_0,\pi_0\big)\big(\varepsilon(\mathbf{w}) + \mathbf{g}_0\big), \tag{27b}$$

$$\partial_t \boldsymbol{\varphi}_l = LC\big(\tau_S\mu_0, \tau_P\pi_0\big)\big(\varepsilon(\mathbf{w}) + \mathbf{g}_l\big) + \frac{1}{\tau_{\boldsymbol{\sigma},l}}\,\boldsymbol{\varphi}_l, \qquad l = 1, \dots, L, \tag{27c}$$

with $w(T) = 0$.

Remark 4.6. *Please note that* $\Phi'(\mathbf{p})^*$ *actually maps into* $L^1(D)^5$ *which is a subspace of* $(L^\infty(D)^5)'$. *This remark applies also to the adjoints considered in Theorems 4.9 and 4.10 below.*

Proof of Theorem 4.5. Using $A^* = -A$ (skew-symmetry), $Q^* = Q$, and $QB = BQ$ we convince ourselves that (27) is the concrete version of the abstract equation (13). Further, by Theorem 3.3,

$$\begin{aligned}\big\langle \Phi'(\mathbf{p})^*\mathbf{g}, \widehat{\mathbf{p}}\big\rangle_{(L^\infty(D)^5)'\times L^\infty(D)^5} &= \big\langle F'(V(\mathbf{p}))^*\mathbf{g}, V'(\mathbf{p})\widehat{\mathbf{p}}\big\rangle_{\mathcal{L}(X)'\times\mathcal{L}(X)} \\ &= \int_0^T \big\langle V'(\mathbf{p})\widehat{\mathbf{p}}\big(u'(t) + Qu(t)\big), w(t)\big\rangle_X \,\mathrm{d}t\end{aligned} \tag{28}$$

where $u = (\mathbf{v}, \boldsymbol{\sigma}_0, \dots, \boldsymbol{\sigma}_L)$ is the classical solution of (5). We are now going to evaluate the above integrand suppressing its t-dependence. Using (23) and (22) we find for $\widehat{\mathbf{p}} = (\widehat{\rho}, \widehat{v}_\mathrm{S}, \widehat{\tau}_\mathrm{S}, \widehat{v}_\mathrm{P}, \widehat{\tau}_\mathrm{P})$ that

$$\big\langle V'(\mathbf{p})\widehat{\mathbf{p}}\big(u' + Qu\big), w\big\rangle_X = \int_D \big(\widehat{\rho}\,\partial_t\mathbf{v}\cdot\mathbf{w} + S_0 + S_1 + \dots + S_L\big)\mathrm{d}x \tag{29}$$

with

$$S_0 = \Big[-\frac{\widehat{\rho}}{\rho^2}\widetilde{C}(\mu,\pi)\partial_t\boldsymbol{\sigma}_0 - \frac{1}{\rho}\widetilde{C}(\mu,\pi)C(\widetilde{\mu},\widetilde{\pi})\widetilde{C}(\mu,\pi)\partial_t\boldsymbol{\sigma}_0\Big] : \boldsymbol{\varphi}_0$$

and, for $l = 1, \dots, L$,

$$\begin{aligned}S_l = \Big[&-\frac{\widehat{\rho}}{L\rho^2}\widetilde{C}(\tau_\mathrm{S}\mu, \tau_\mathrm{P}\pi)\Big(\partial_t\boldsymbol{\sigma}_l + \frac{\boldsymbol{\sigma}_l}{\tau_{\boldsymbol{\sigma},l}}\Big) \\ &- \frac{1}{L\rho}\widetilde{C}(\tau_\mathrm{S}\mu,\tau_\mathrm{P}\pi)C(\widehat{\mu},\widehat{\pi})\widetilde{C}(\tau_\mathrm{S}\mu,\tau_\mathrm{P}\pi)\Big(\partial_t\boldsymbol{\sigma}_l + \frac{\boldsymbol{\sigma}_l}{\tau_{\boldsymbol{\sigma},l}}\Big)\Big] : \boldsymbol{\varphi}_l.\end{aligned}$$

In view of (5b) we may write

$$S_0 = \Big[-\frac{\widehat{\rho}}{\rho}\boldsymbol{\varepsilon}(\mathbf{v}) - \widetilde{C}(\mu,\pi)C(\widetilde{\mu},\widetilde{\pi})\boldsymbol{\varepsilon}(\mathbf{v})\Big] : \boldsymbol{\varphi}_0 = -\frac{\widehat{\rho}}{\rho}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_0 - C(\widetilde{\mu},\widetilde{\pi})\boldsymbol{\varepsilon}(\mathbf{v}) : \widetilde{C}(\mu,\pi)\boldsymbol{\varphi}_0$$

and, similarly by (5c) ,

$$S_l = -\frac{\widehat{\rho}}{\rho}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_l - C(\widehat{\mu},\widehat{\pi})\boldsymbol{\varepsilon}(\mathbf{v}) : \widetilde{C}(\tau_\mathrm{S}\mu,\tau_\mathrm{P}\pi)\boldsymbol{\varphi}_l, \quad l = 1, \dots, L.$$

Next, using (20), we compute

$$\begin{aligned}&C(\widetilde{\mu},\widetilde{\pi})\boldsymbol{\varepsilon}(\mathbf{v}) : \widetilde{C}(\mu,\pi)\boldsymbol{\varphi}_0 \\ &\quad= \big(2\widetilde{\mu}\,\boldsymbol{\varepsilon}(\mathbf{v}) + (\widetilde{\pi} - 2\widetilde{\mu})\operatorname{div}\mathbf{v}\,\mathbf{I}\big) : \Big(\frac{1}{2\mu}\,\boldsymbol{\varphi}_0 + \frac{2\mu - \pi}{2\mu(3\pi - 4\mu)}\,\mathrm{tr}(\boldsymbol{\varphi}_0)\mathbf{I}\Big) \\ &\quad= \widetilde{\mu}\Big(\frac{1}{\mu}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_0 - \frac{\pi}{\mu(3\pi - 4\mu)}\operatorname{div}\mathbf{v}\;\mathrm{tr}(\boldsymbol{\varphi}_0)\Big) + \frac{\widetilde{\pi}}{3\pi - 4\mu}\operatorname{div}\mathbf{v}\;\mathrm{tr}(\boldsymbol{\varphi}_0)\end{aligned} \tag{30}$$

yielding

$$S_0 = -\frac{\widehat{\rho}}{\rho}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_0$$

$$+\,\widetilde{\mu}\Big(-\frac{1}{\mu}\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_0+\frac{\pi}{\mu(3\pi-4\mu)}\,\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0)\Big)-\frac{\widetilde{\pi}}{3\pi-4\mu}\,\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0).$$

Analogously,

$$\begin{aligned}
S_l &= -\frac{\widehat{\rho}}{\rho}\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_l\\
&+\widehat{\mu}\Big(-\frac{1}{\tau_{\mathrm{S}}\mu}\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_l+\frac{\tau_{\mathrm{P}}\pi}{\tau_{\mathrm{S}}\mu(3\tau_{\mathrm{P}}\pi-4\tau_{\mathrm{S}}\mu)}\,\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_l)\Big)-\frac{\widehat{\pi}}{3\tau_{\mathrm{P}}\pi-4\tau_{\mathrm{S}}\mu}\,\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_l).
\end{aligned}$$

Next we group the terms in the sum (29) belonging to the five components of $\widehat{\mathbf{p}}$. To this end we replace $\widetilde{\mu}$, $\widetilde{\pi}$, $\widehat{\mu}$, and $\widehat{\pi}$ by their respective expressions from (24) and (25) which we slightly rewrite introducing μ and π:

$$\widetilde{\mu}=\frac{2\mu}{v_{\mathrm{S}}}\widehat{v}_{\mathrm{S}}-\frac{\alpha\,\mu}{1+\tau_{\mathrm{S}}\alpha}\widehat{\tau}_{\mathrm{S}},\qquad \widetilde{\pi}=\frac{2\pi}{v_{\mathrm{P}}}\widehat{v}_{\mathrm{P}}-\frac{\alpha\,\pi}{1+\tau_{\mathrm{P}}\alpha}\widehat{\tau}_{\mathrm{P}},\tag{31}$$

$$\widehat{\mu}=\frac{2\tau_{\mathrm{S}}\,\mu}{v_{\mathrm{S}}}\widehat{v}_{\mathrm{S}}+\frac{\mu}{1+\tau_{\mathrm{S}}\alpha}\widehat{\tau}_{\mathrm{S}},\qquad \widehat{\pi}=\frac{2\tau_{\mathrm{P}}\,\pi}{v_{\mathrm{P}}}\widehat{v}_{\mathrm{P}}+\frac{\pi}{1+\tau_{\mathrm{P}}\alpha}\widehat{\tau}_{\mathrm{P}}.\tag{32}$$

After some algebra we get

$$\begin{aligned}
\big\langle V'(\mathbf{p})\widehat{\mathbf{p}}\big(u'+Qu\big),\overline{u}\big\rangle_X &= \int_D\Big[\widehat{\rho}\Big(\partial_t\mathbf{v}\cdot\mathbf{w}-\frac{1}{\rho}\boldsymbol{\varepsilon}(\mathbf{v}):(\boldsymbol{\varphi}_0+\boldsymbol{\Sigma})\Big)\\
&\quad+\widehat{v}_{\mathrm{S}}\frac{2}{v_{\mathrm{S}}}\Big(-\boldsymbol{\varepsilon}(\mathbf{v}):(\boldsymbol{\varphi}_0+\boldsymbol{\Sigma})+\pi\operatorname{tr}(\boldsymbol{\Sigma}^v)\operatorname{div}\mathbf{v}\Big)\\
&\quad+\frac{\widehat{\tau}_{\mathrm{S}}}{1+\alpha\tau_{\mathrm{S}}}\Big(\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\Sigma}^\tau_{\mathrm{S},2}+\pi\operatorname{tr}(\boldsymbol{\Sigma}^\tau_{\mathrm{S},1})\operatorname{div}\mathbf{v}\Big)\\
&\quad-\widehat{v}_{\mathrm{P}}\frac{2\pi}{v_{\mathrm{P}}}\operatorname{tr}(\boldsymbol{\Sigma}^v)\operatorname{div}\mathbf{v}+\widehat{\tau}_{\mathrm{P}}\frac{\pi}{1+\alpha\tau_{\mathrm{P}}}\operatorname{tr}(\boldsymbol{\Sigma}^\tau_{\mathrm{P}})\operatorname{div}\mathbf{v}\Big]\,\mathrm{d}x
\end{aligned}$$

which ends the proof. □

4.3. **Second order differentiability.** The second derivative of Φ is given by

$$\Phi''(\mathbf{p})[\widehat{\mathbf{p}}_1,\widehat{\mathbf{p}}_2]=F''(V(\mathbf{p}))[V'(\mathbf{p})\widehat{\mathbf{p}}_1,V'(\mathbf{p})\widehat{\mathbf{p}}_2]+F'(V(\mathbf{p}))V''(\mathbf{p})[\widehat{\mathbf{p}}_1,\widehat{\mathbf{p}}_2]\tag{33}$$

using the chain and product rules, see, e.g., [15, Section 4.3]. In a first step we need to find V''. Differentiating (23) at $\mathbf{p}=(\rho,v_{\mathrm{S}},\tau_{\mathrm{S}},v_{\mathrm{P}},\tau_{\mathrm{P}})\in\operatorname{int}(\mathsf{D}(\Phi))$ we obtain

$$V''(\mathbf{p})[\widehat{\mathbf{p}}_1,\widehat{\mathbf{p}}_2]\begin{pmatrix}\mathbf{w}\\ \boldsymbol{\psi}_0\\ \vdots\\ \boldsymbol{\psi}_L\end{pmatrix}=\tag{34}$$

$$\begin{pmatrix} \mathbf{0} \\ \Big(\frac{\widehat{\rho}_1\widehat{\rho}_2}{\rho^3}\widetilde{C}(\mu,\pi) - \frac{\widehat{\rho}_1}{\rho^2}\widetilde{C}'(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_2\\ \widetilde{\pi}_2\end{bmatrix} - \frac{\widehat{\rho}_2}{\rho^2}\widetilde{C}'(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_1\\ \widetilde{\pi}_1\end{bmatrix} + \frac{1}{\rho}\widetilde{C}''(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_1\\ \widetilde{\pi}_1\end{bmatrix}\begin{bmatrix}\widetilde{\mu}_2\\ \widetilde{\pi}_2\end{bmatrix}\Big)\boldsymbol{\psi}_0 \\ \Big(\frac{\widehat{\rho}_1\widehat{\rho}_2}{L\rho^3}\widetilde{C}(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi) - \frac{\widehat{\rho}_1}{L\rho^2}\widetilde{C}'(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_2\\ \widehat{\pi}_2\end{bmatrix} - \frac{\widehat{\rho}_2}{L\rho^2}\widetilde{C}'(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_1\\ \widehat{\pi}_1\end{bmatrix} \\ \qquad + \frac{1}{L\rho}\widetilde{C}''(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_1\\ \widehat{\pi}_1\end{bmatrix}\begin{bmatrix}\widehat{\mu}_2\\ \widehat{\pi}_2\end{bmatrix}\Big)\boldsymbol{\psi}_1 \\ \vdots \\ \Big(\frac{\widehat{\rho}_1\widehat{\rho}_2}{L\rho^3}\widetilde{C}(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi) - \frac{\widehat{\rho}_1}{L\rho^2}\widetilde{C}'(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_2\\ \widehat{\pi}_2\end{bmatrix} - \frac{\widehat{\rho}_2}{L\rho^2}\widetilde{C}'(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_1\\ \widehat{\pi}_1\end{bmatrix} \\ \qquad + \frac{1}{L\rho}\widetilde{C}''(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_1\\ \widehat{\pi}_1\end{bmatrix}\begin{bmatrix}\widehat{\mu}_2\\ \widehat{\pi}_2\end{bmatrix}\Big)\boldsymbol{\psi}_L \end{pmatrix}$$

for $\widehat{\mathbf{p}}_i = (\widehat{\rho}_i, \widehat{v}_{\mathrm{S},i}, \widehat{\tau}_{\mathrm{S},i}, \widehat{v}_{\mathrm{P},i}, \widehat{\tau}_{\mathrm{P},i}) \in L^\infty(D)^5$, $i = 1, 2$. Further, $\widetilde{\mu}_i$, $\widetilde{\pi}_i$, and $\widehat{\mu}_i$, $\widehat{\pi}_i$ are defined as in (24) and (25), respectively, plugging in the respective components of $\widehat{\mathbf{p}}_i$. We close the expression for V'' by

$$\widetilde{C}''(m,p)\begin{bmatrix}\widehat{m}_1\\ \widehat{p}_1\end{bmatrix}\begin{bmatrix}\widehat{m}_2\\ \widehat{p}_2\end{bmatrix} = \widetilde{C}(m,p)\circ C(\widehat{m}_1,\widehat{p}_1)\circ\widetilde{C}(m,p)\circ C(\widehat{m}_2,\widehat{p}_2)\circ\widetilde{C}(m,p) \\ + \widetilde{C}(m,p)\circ C(\widehat{m}_2,\widehat{p}_2)\circ\widetilde{C}(m,p)\circ C(\widehat{m}_1,\widehat{p}_1)\circ\widetilde{C}(m,p). \tag{35}$$

The proof of (35) requires straightforward but lengthy calculations.

Theorem 4.7. *Let* $\mathbf{f}$ *be in* $W^{3,1}(]0,T[, L^2(D,\mathbb{R}^3))$ *with* $\mathbf{f}(0) = \mathbf{f}'(0) = \mathbf{f}''(0) = 0$*. Further, let* $u_0 = 0$ *and adopt the assumptions and notation made in this section.*

Then, the full waveform forward operator Φ *is twice Fréchet differentiable at any interior point* $\mathbf{p} = (\rho, v_S, \tau_S, v_P, \tau_P)$ *of* $\mathsf{D}(\Phi)$*: For* $\widehat{\mathbf{p}}_i = (\widehat{\rho}_i, \widehat{v}_{S,i}, \widehat{\tau}_{S,i}, \widehat{v}_{P,i}, \widehat{\tau}_{P,i}) \in L^\infty(D)^5$*,* $i = 1, 2$*, we have* $\Phi''(\mathbf{p})[\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2] = v + \overline{\overline{u}}$ *where* $v = (\mathbf{w}, \boldsymbol{\psi}_0, \dots, \boldsymbol{\psi}_L)$ *and* $\overline{\overline{u}} = (\overline{\overline{\mathbf{v}}}, \overline{\overline{\boldsymbol{\sigma}}}_0, \dots, \overline{\overline{\boldsymbol{\sigma}}}_L)$ *are both in* $\mathcal{C}([0,T], X)$*. They are uniquely determined as mild solutions of the following viscoelastic equations.*

The equations for $\overline{\overline{u}}$ *are* $\overline{\overline{u}}(0) = 0$ *and*

$$\rho\,\partial_t\overline{\overline{\mathbf{v}}} = \operatorname{div}\Big(\sum_{l=0}^{L}\overline{\overline{\boldsymbol{\sigma}}}_l\Big) - \widehat{\rho}_1\,\partial_t\overline{\mathbf{v}},$$

$$\partial_t\overline{\overline{\boldsymbol{\sigma}}}_0 = C(\mu_0,\pi_0)\boldsymbol{\varepsilon}(\overline{\overline{\mathbf{v}}}) + \big(\widehat{\rho}_1\,C(\mu,\pi) + \rho\,C(\widetilde{\mu}_1,\widetilde{\pi}_1)\big)\varepsilon(\overline{\mathbf{v}}),$$

$$\partial_t\overline{\overline{\boldsymbol{\sigma}}}_l = L\,C(\tau_S\mu_0,\tau_P\pi_0)\boldsymbol{\varepsilon}(\overline{\overline{\mathbf{v}}}) \\ - \frac{1}{\tau_{\boldsymbol{\sigma},L}}\overline{\overline{\boldsymbol{\sigma}}}_l + \big(\widehat{\rho}_1\,L\,C(\tau_S\mu,\tau_P\pi) + \rho\,C(\widehat{\mu}_1,\widehat{\pi}_1)\big)\varepsilon(\overline{\mathbf{v}}), \quad l = 1,\dots,L,$$

with $\overline{\mathbf{v}}$ *being the first component of the solution of* (26) *where the parameters* $\widehat{\mathbf{p}}$ *have to be replaced by* $\widehat{\mathbf{p}}_2$*.*

The equations for v *are* $v(0) = 0$ *and*

$$\rho\,\partial_t\mathbf{w} = \operatorname{div}\Big(\sum_{l=0}^{L}\boldsymbol{\psi}_l\Big),$$

$$\partial_t \boldsymbol{\psi}_0 = C(\mu_0,\pi_0)\boldsymbol{\varepsilon}(\mathbf{w}) - \Big(\frac{\widehat{\rho}_1\widehat{\rho}_2}{\rho^2} C(\mu,\pi) + \widehat{\rho}_1 C(\widetilde{\mu}_1,\widetilde{\pi}_1) + \widehat{\rho}_2 C(\widetilde{\mu}_2,\widetilde{\pi}_2)$$
$$+ \rho C(\widetilde{\mu}_1,\widetilde{\pi}_1)\widetilde{C}(\mu,\pi)C(\widetilde{\mu}_2,\widetilde{\pi}_2) + \rho C(\widetilde{\mu}_2,\widetilde{\pi}_2)\widetilde{C}(\mu,\pi)C(\widetilde{\mu}_1,\widetilde{\pi}_1)\Big)\boldsymbol{\varepsilon}(\mathbf{v}),$$

$$\partial_t \boldsymbol{\psi}_l = L\, C(\tau_S\mu_0,\tau_P\pi_0)\boldsymbol{\varepsilon}(\mathbf{w}) - \frac{1}{\tau_{\boldsymbol{\sigma},l}}\boldsymbol{\psi}_l - L\Big(\frac{\widehat{\rho}_1\widehat{\rho}_2}{\rho^2} C(\tau_S\mu,\tau_P\pi) + \widehat{\rho}_1 C(\widehat{\mu}_1,\widehat{\pi}_1) + \widehat{\rho}_2 C(\widehat{\mu}_2,\widehat{\pi}_2)$$
$$+ \rho C(\widehat{\mu}_1,\widehat{\pi}_1)\widetilde{C}(\tau_S\mu,\tau_P\pi)C(\widehat{\mu}_2,\widehat{\pi}_2) + \rho C(\widehat{\mu}_2,\widehat{\pi}_2)\widetilde{C}(\tau_S\mu,\tau_P\pi)C(\widehat{\mu}_1,\widehat{\pi}_1)\Big)\boldsymbol{\varepsilon}(\mathbf{v}),$$

$l = 1,\dots,L$, *where* $\mathbf{v}$ *is the first component of the solution of* (5).

Proof. By (33), $\Phi''(\mathbf{p})[\widehat{\mathbf{p}}_1,\widehat{\mathbf{p}}_2] = v + \overline{\overline{u}}$ where

$$v := F'(V(\mathbf{p}))V''(\mathbf{p})[\widehat{\mathbf{p}}_1,\widehat{\mathbf{p}}_2] \quad \text{and} \quad \overline{\overline{u}} := F''(V(\mathbf{p}))[V'(\mathbf{p})\widehat{\mathbf{p}}_1, V'(\mathbf{p})\widehat{\mathbf{p}}_2].$$

We apply Theorems 3.2 and 3.5 to specify the equations for v and $\overline{\overline{u}}$, respectively.

We start with $\overline{\overline{u}}$ which is determined by two coupled equations of type (26). These equations only differ in the plugged in parameters and right hand sides.

Theorem 3.2 yields the following system for v:

$$\begin{pmatrix} \rho\,\partial_t \mathbf{w} \\ \frac{1}{\rho}\widetilde{C}(\mu,\pi)\partial_t\boldsymbol{\psi}_0 \\ \frac{1}{L\rho}\widetilde{C}(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\partial_t\boldsymbol{\psi}_1 \\ \vdots \\ \frac{1}{L\rho}\widetilde{C}(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\partial_t\boldsymbol{\psi}_L \end{pmatrix} = \begin{pmatrix} \operatorname{div}\big(\sum_{l=0}^L \boldsymbol{\psi}_l\big) \\ \varepsilon(\mathbf{w}) \\ \vdots \\ \varepsilon(\mathbf{w}) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \frac{1}{L\,\rho\,\tau_{\boldsymbol{\sigma},1}}\widetilde{C}\big(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi\big)\boldsymbol{\psi}_1 \\ \vdots \\ \frac{1}{L\,\rho\,\tau_{\boldsymbol{\sigma},L}}\widetilde{C}\big(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi\big)\boldsymbol{\psi}_L \end{pmatrix}$$
$$- V''(\mathbf{p})[\widehat{\mathbf{p}}_1,\widehat{\mathbf{p}}_2]\left[\begin{pmatrix} \partial_t \mathbf{v} \\ \partial_t \boldsymbol{\sigma}_0 \\ \partial_t\boldsymbol{\sigma}_1 \\ \vdots \\ \partial_t\boldsymbol{\sigma}_L \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \frac{1}{\tau_{\boldsymbol{\sigma},1}}\boldsymbol{\sigma}_1 \\ \vdots \\ \frac{1}{\tau_{\boldsymbol{\sigma},L}}\boldsymbol{\sigma}_L \end{pmatrix}\right].$$

Applying (5b), (5c), (22), (34), and (35) leads to the equations for v. □

4.4. **An additional adjoint.** As explained in the introduction, second-degree Newton solvers might resolve the cross-talk effect. In our group we plan to implement a variant of the second-degree Newton method (1) for viscolesatic FWI. There one needs to solve a linear system containing the operator $\Phi''(\mathbf{p})[\widehat{\mathbf{p}},\cdot]$, see (1b). Our regularization method of choice is the conjugate gradient iteration which needs the adjoint operator. In this subsection we derive an explicit expression for it.

Recall from (33) that

$$\Phi''(\mathbf{p})[\widehat{\mathbf{p}},\cdot] = F''(V(\mathbf{p}))[V'(\mathbf{p})\widehat{\mathbf{p}}, V'(\mathbf{p})\,\cdot] + F'(V(\mathbf{p}))V''(\mathbf{p})[\widehat{\mathbf{p}},\cdot]. \tag{36}$$

In a first step we therefore consider $F''(B)[H,\cdot]\colon \mathcal{L}^*(X) \to L^2([0,T],X)$ for $B \in \mathsf{D}(F)$ and $H \in \mathcal{L}^*(X)$.

Theorem 4.8. *Under the assumptions of Theorem* 3.5 *we have*

$$\big[F''(B)[H_1,\cdot]^* g\big] H_2 = \int_0^T \big\langle H_1(\overline{u}'(t) + Q\overline{u}(t)), w(t)\big\rangle_X \mathrm{d}t$$

for $g \in L^2([0,T],X)$, $H_i \in \mathcal{L}^*(X)$, $i = 1,2$, *where* $\overline{u} = F'(B)H_2$ *is the solution of* (15). *Further,* $w \in \mathcal{C}([0,T],X)$ *is the mild solution of the adjoint evolution equation*

$$Bw'(t) - A^*w(t) - Q^*Bw(t) = g(t),\ t \in]0,T[, \quad w(T) = 0.$$

Proof. Since $\big[F''(B)[H_1,\cdot]^*g\big]H_2 = \langle \overline{\overline{u}}, g\rangle_{L^2([0,T],X)}$ where $\overline{\overline{u}} \in \mathcal{C}^1([0,T],X)$ solves (14) we argue similar to the proof of Theorem 3.8 in [10]: Assume $g \in W^{1,1}(]0,T[,X)$. Then, $w \in \mathcal{C}^1([0,T],X)$. Using the selfadjointness of B and integration by parts we compute

$$\begin{aligned}\langle \overline{\overline{u}}, g\rangle_{L^2([0,T],X)} &= \int_0^T \langle \overline{\overline{u}}(t), g(t)\rangle_X \mathrm{d}t = \int_0^T \langle \overline{\overline{u}}(t), Bw'(t) - A^*w(t) - Q^*Bw(t)\rangle_X \mathrm{d}t \\ &= -\int_0^T \langle B\overline{\overline{u}}'(t) + A\overline{\overline{u}}(t) + BQ\overline{\overline{u}}(t), w(t)\rangle_X \mathrm{d}t \\ &= \int_0^T \langle H_1(\overline{u}'(t) + Q\overline{u}(t)), w(t)\rangle_X \mathrm{d}t.\end{aligned}$$

The assertion follows since $W^{1,1}(]0,T[,X)$ is dense in $L^2(]0,T[,X)$. □

Theorem 4.9. *Under the assumptions of Theorem* 4.7 *we have that the adjoint*

$$F''(V(\mathbf{p}))[V'(\mathbf{p})\widehat{\mathbf{p}}, V'(\mathbf{p})\,\cdot]^* \in \mathcal{L}\big(L^2([0,T],X), (L^\infty(D)^5)'\big)$$

at $\mathbf{p} = (\rho, v_S, \tau_S, v_P, \tau_P) \in \mathsf{D}(\Phi)$ *and* $\widehat{\mathbf{p}} = (\widehat{\rho}, \widehat{v}_S, \widehat{\tau}_S, \widehat{v}_P, \widehat{\tau}_P) \in L^\infty(D)^5$ *is given by*

$$F''(V(\mathbf{p}))[V'(\mathbf{p})\widehat{\mathbf{p}}, V'(\mathbf{p})\cdot]^*\mathbf{g} = \begin{pmatrix} \int_0^T \big(\partial_t\overline{\mathbf{v}}\cdot\mathbf{w} - \frac{1}{\rho}\,\boldsymbol{\varepsilon}(\overline{\mathbf{v}}) : (\boldsymbol{\varphi}_0 + \boldsymbol{\Sigma})\big)\mathrm{d}t \\ \frac{2}{v_S}\int_0^T \big(-\boldsymbol{\varepsilon}(\overline{\mathbf{v}}) : (\boldsymbol{\varphi}_0 + \boldsymbol{\Sigma}) + \pi\,\mathrm{tr}(\boldsymbol{\Sigma}^v)\,\mathrm{div}\,\overline{\mathbf{v}}\big)\mathrm{d}t \\ \frac{1}{1+\alpha\tau_S}\int_0^T \big(\boldsymbol{\varepsilon}(\overline{\mathbf{v}}) : \boldsymbol{\Sigma}^\tau_{S,2} + \pi\,\mathrm{tr}(\boldsymbol{\Sigma}^\tau_{S,1})\,\mathrm{div}\,\overline{\mathbf{v}}\big)\mathrm{d}t \\ -\frac{2\pi}{v_P}\int_0^T \mathrm{tr}(\boldsymbol{\Sigma}^v)\,\mathrm{div}\,\overline{\mathbf{v}}\,\mathrm{d}t \\ \frac{\pi}{1+\alpha\tau_P}\int_0^T \mathrm{tr}(\boldsymbol{\Sigma}^\tau_P)\,\mathrm{div}\,\overline{\mathbf{v}}\,\mathrm{d}t \end{pmatrix} \in L^1(D)^5$$

for $\mathbf{g} = (\mathbf{g}_{-1}, \mathbf{g}_0, \dots, \mathbf{g}_L) \in L^2\big([0,T], L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^{3\times3}_{\mathrm{sym}})^{1+L}\big)$ *where* $\overline{\mathbf{v}}$ *is the first component of the solution of* (26), $w = (\mathbf{w}, \boldsymbol{\varphi}_0, \dots, \boldsymbol{\varphi}_L)$ *solves* (27) *with* $w(T) = 0$, *and* $\boldsymbol{\Sigma} = \sum_{l=1}^L \boldsymbol{\varphi}_l$. *The quantities* $\boldsymbol{\Sigma}^v$, $\boldsymbol{\Sigma}^\tau_{S,1}$, $\boldsymbol{\Sigma}^\tau_{S,2}$, *and* $\boldsymbol{\Sigma}^\tau_P$ *are exactly those from Theorem* 4.5.

Proof. The second order Fréchet derivative is symmetric, see, e.g, [4, (8.12.2)], that is,

$$\begin{aligned}\big(F''(V(\mathbf{p}))[V'(\mathbf{p})\widehat{\mathbf{p}}_1, V'(\mathbf{p})\,\cdot]^*\mathbf{g}\big)\widehat{\mathbf{p}}_2 &= \big(F''(V(\mathbf{p}))[V'(\mathbf{p})\widehat{\mathbf{p}}_2, \cdot]^*\mathbf{g}\big)V'(\mathbf{p})\widehat{\mathbf{p}}_1 \\ &= \int_0^T \big\langle V'(\mathbf{p})\widehat{\mathbf{p}}_2(\overline{u}'(t) + Q\overline{u}(t)), w(t)\big\rangle_X \mathrm{d}t\end{aligned}$$

where we applied the previous theorem to obtain the second equality. Note that here $\overline{u} = F'(V(\mathbf{p}))V'(\mathbf{p})\widehat{\mathbf{p}}_1$ solves (26) with $\widehat{\mathbf{p}} = \widehat{\mathbf{p}}_1$ and w solves (27). We are now exactly in the situation of the proof of Theorem 4.5, see (28), and proceed accordingly. □

Theorem 4.10. *Under the assumptions of Theorem* 4.7 *we have that the adjoint*

$$F'(V(\mathbf{p}))V''(\mathbf{p})[\widehat{\mathbf{p}}, \cdot]^* \in \mathcal{L}\big(L^2([0,T],X), (L^\infty(D)^5)'\big)$$

at $\mathbf{p} = (\rho, v_S, \tau_S, v_P, \tau_P) \in \mathsf{D}(\Phi)$ *and* $\widehat{\mathbf{p}} = (\widehat{\rho}, \widehat{v}_S, \widehat{\tau}_S, \widehat{v}_P, \widehat{\tau}_P) \in L^\infty(D)^5$ *is given by*

$$F'(V(\mathbf{p}))V''(\mathbf{p})[\widehat{\mathbf{p}}, \cdot]^*\mathbf{g} = \begin{pmatrix} \frac{1}{\rho}\int_0^T \big(\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\Upsilon}_1^\rho + \operatorname{tr}(\boldsymbol{\Upsilon}_2^\rho)\operatorname{div}\mathbf{v}\big)\mathrm{d}t \\ \frac{2}{v_S}\int_0^T \big(\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\Upsilon}_{S,1}^v + \operatorname{tr}(\boldsymbol{\Upsilon}_{S,2}^v)\operatorname{div}\mathbf{v}\big)\mathrm{d}t \\ \frac{1}{1+\alpha\tau_S}\int_0^T \big(\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\Upsilon}_{S,1}^\tau + \operatorname{tr}(\boldsymbol{\Upsilon}_{S,2}^\tau)\operatorname{div}\mathbf{v}\big)\mathrm{d}t \\ \frac{2\pi}{v_P}\int_0^T \operatorname{tr}(\boldsymbol{\Upsilon}_P^v)\operatorname{div}\mathbf{v}\,\mathrm{d}t \\ \frac{\pi}{1+\alpha\tau_P}\int_0^T \operatorname{tr}(\boldsymbol{\Upsilon}_P^\tau)\operatorname{div}\mathbf{v}\,\mathrm{d}t \end{pmatrix} \in L^1(D)^5$$

for $\mathbf{g} = (\mathbf{g}_{-1}, \mathbf{g}_0, \ldots, \mathbf{g}_L) \in L^2\big([0,T], L^2(D,\mathbb{R}^3) \times L^2(D,\mathbb{R}^{3\times 3}_{\mathrm{sym}})^{1+L}\big)$ *where* $\mathbf{v}$ *is the first component of the solution of* (5)*. Let* $w = (\mathbf{w}, \boldsymbol{\varphi}_0, \ldots, \boldsymbol{\varphi}_L)$ *solve* (27) *with* $w(T) = 0$ *and set* $\boldsymbol{\Sigma} = \sum_{l=1}^L \boldsymbol{\varphi}_l$*. Then,*

$$\boldsymbol{\Upsilon}_1^\rho = \Big(\frac{\widehat{\rho}}{\rho} + \frac{\widetilde{\mu}}{\mu}\Big)\boldsymbol{\varphi}_0 + \Big(\frac{\widehat{\rho}}{\rho} + \frac{\widehat{\mu}}{\tau_S\mu}\Big)\boldsymbol{\Sigma}, \quad \boldsymbol{\Upsilon}_2^\rho = \frac{\mu\widetilde{\pi} - \pi}{\mu(3\pi - 4\mu)}\boldsymbol{\varphi}_0 + \frac{\tau_S\mu\widehat{\pi} - \tau_P\pi}{\tau_S\mu(3\tau_P\pi - 4\tau_S\mu)}\boldsymbol{\Sigma},$$

$$\boldsymbol{\Upsilon}_{S,1}^v = \Big(\frac{\widehat{\rho}}{\rho} + \frac{2\widetilde{\mu}}{\mu}\Big)\boldsymbol{\varphi}_0 + \Big(\frac{\widehat{\rho}}{\rho} + \frac{2\widehat{\mu}}{\tau_S\mu}\Big)\boldsymbol{\Sigma},$$

$$\begin{aligned}\boldsymbol{\Upsilon}_{S,2}^v = \Big(2\,\frac{3\widetilde{\mu}\pi^2 - 4\widetilde{\pi}\mu^2}{\mu(3\pi - 4\mu)^2} - \frac{\widehat{\rho}}{\rho}\,\frac{\pi}{3\pi - 4\mu}\Big)\boldsymbol{\varphi}_0 \\ + \Big(2\,\frac{3\widehat{\mu}\tau_P^2\pi^2 - 4\widehat{\pi}\tau_S^2\mu^2}{\tau_S\mu(3\tau_P\pi - 4\tau_S\mu)^2} - \frac{\widehat{\rho}}{\rho}\,\frac{\tau_P\pi}{3\tau_P\pi - 4\tau_S\mu}\Big)\boldsymbol{\Sigma},\end{aligned}$$

$$\boldsymbol{\Upsilon}_{S,1}^\tau = -\alpha\Big(\frac{\widehat{\rho}}{\rho} + \frac{2\widetilde{\mu}}{\mu}\Big)\boldsymbol{\varphi}_0 + \Big(\frac{\widehat{\rho}}{\tau_S\rho} + \frac{2\widehat{\mu}}{\tau_S^2\mu}\Big)\boldsymbol{\Sigma},$$

$$\begin{aligned}\boldsymbol{\Upsilon}_{S,2}^\tau = -\alpha\Big(2\,\frac{3\widetilde{\mu}\pi^2 - 4\widetilde{\pi}\mu^2}{\mu(3\pi - 4\mu)^2} - \frac{\widehat{\rho}}{\rho}\,\frac{\pi}{3\pi - 4\mu}\Big)\boldsymbol{\varphi}_0 \\ + \Big(2\,\frac{3\widehat{\mu}\tau_P^2\pi^2 - 4\widehat{\pi}\tau_S^2\mu^2}{\tau_S^2\mu(3\tau_P\pi - 4\tau_S\mu)^2} - \frac{\widehat{\rho}}{\rho}\,\frac{\tau_P\pi}{\tau_S(3\tau_P\pi - 4\tau_S\mu)}\Big)\boldsymbol{\Sigma},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Upsilon}_P^v = \Big(\frac{\widehat{\rho}}{\rho}\,\frac{1}{3\pi - 4\mu} + 2\,\frac{3\widetilde{\pi}\pi^2 - 4\widetilde{\mu}\mu^2}{\mu^2(3\pi - 4\mu)^2}\Big)\boldsymbol{\varphi}_0 \\ + \tau_P\Big(\frac{\widehat{\rho}}{\rho}\,\frac{1}{3\tau_P\pi - 4\tau_S\mu} + 2\,\frac{3\widehat{\pi}\tau_P^2\pi^2 - 4\widehat{\mu}\tau_S^2\mu^2}{\tau_S^2\mu^2(3\tau_P\pi - 4\tau_S\mu)^2}\Big)\boldsymbol{\Sigma},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Upsilon}_P^\tau = -\alpha\Big(\frac{\widehat{\rho}}{\rho}\,\frac{1}{3\pi - 4\mu} + 2\,\frac{3\widetilde{\pi}\pi^2 - 4\widetilde{\mu}\mu^2}{\mu^2(3\pi - 4\mu)^2}\Big)\boldsymbol{\varphi}_0 \\ + \Big(\frac{\widehat{\rho}}{\rho}\,\frac{1}{3\tau_P\pi - 4\tau_S\mu} + 2\,\frac{3\widehat{\pi}\tau_P^2\pi^2 - 4\widehat{\mu}\tau_S^2\mu^2}{\tau_S^2\mu^2(3\tau_P\pi - 4\tau_S\mu)^2}\Big)\boldsymbol{\Sigma},\end{aligned}$$

with the abbreviations $\widetilde{\mu}$, $\widetilde{\pi}$, *and* $\widehat{\mu}$, $\widehat{\pi}$ *from* (31) *and* (32) *which depend on* $\widehat{\mathbf{p}}$.

Proof. Since

$$\big(F'(V(\mathbf{p}))V''(\mathbf{p})[\widehat{\mathbf{p}}_1, \cdot]^*\mathbf{g}\big)\widehat{\mathbf{p}}_2 \overset{(28)}{=} \int_0^T \langle V''(\mathbf{p})[\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2]\big(u'(t) + Qu(t)\big), w(t)\rangle_X\,\mathrm{d}t.$$

we are basically again in the situation of the proof of Theorem 4.5. Using (34) we find that

$$\langle V''(\mathbf{p})[\widehat{\mathbf{p}}_1,\widehat{\mathbf{p}}_2](u'+Qu),w\rangle_X=\int_D(S_0+S_1+\cdots+S_L)\,\mathrm{d}x$$

with

$$S_0=\Bigg(\frac{\widehat{\rho}_1\widehat{\rho}_2}{\rho^3}\widetilde{C}(\mu,\pi)-\frac{\widehat{\rho}_1}{\rho^2}\widetilde{C}'(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_2\\ \widetilde{\pi}_2\end{bmatrix}-\frac{\widehat{\rho}_2}{\rho^2}\widetilde{C}'(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_1\\ \widetilde{\pi}_1\end{bmatrix}\\ +\frac{1}{\rho}\widetilde{C}''(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_1\\ \widetilde{\pi}_1\end{bmatrix}\begin{bmatrix}\widetilde{\mu}_2\\ \widetilde{\pi}_2\end{bmatrix}\Bigg)\partial_t\boldsymbol{\sigma}_0:\boldsymbol{\varphi}_0$$

and

$$S_l=\Bigg(\frac{\widehat{\rho}_1\widehat{\rho}_2}{L\rho^3}\widetilde{C}(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)-\frac{\widehat{\rho}_1}{L\rho^2}\widetilde{C}'(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_2\\ \widehat{\pi}_2\end{bmatrix}-\frac{\widehat{\rho}_2}{L\rho^2}\widetilde{C}'(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_1\\ \widehat{\pi}_1\end{bmatrix}\\ +\frac{1}{L\rho}\widetilde{C}''(\tau_{\mathrm{S}}\mu,\tau_{\mathrm{P}}\pi)\begin{bmatrix}\widehat{\mu}_1\\ \widehat{\pi}_1\end{bmatrix}\begin{bmatrix}\widehat{\mu}_2\\ \widehat{\pi}_2\end{bmatrix}\Bigg)\Big(\partial_t\boldsymbol{\sigma}_l+\frac{\boldsymbol{\sigma}_l}{\tau_{\boldsymbol{\sigma},l}}\Big):\boldsymbol{\psi}_l,\quad l=1,\ldots,L.$$

First we simplify S_0. By (5b),

$$\frac{1}{\rho}\widetilde{C}(\mu,\pi)\partial_t\boldsymbol{\sigma}_0:\boldsymbol{\varphi}_0=\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_0.$$

Further, in view of (30),

$$-\frac{1}{\rho}\widetilde{C}'(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_i\\ \widetilde{\pi}_i\end{bmatrix}\partial_t\boldsymbol{\sigma}_0:\boldsymbol{\varphi}_0\\ =\widetilde{\mu}_i\Big(\frac{1}{\mu}\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_0-\frac{\pi}{\mu(3\pi-4\mu)}\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0)\Big)+\frac{\widetilde{\pi}_i}{3\pi-4\mu}\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0),\quad i=1,2.$$

Next, using (5b) and (35) we get

$$\frac{1}{\rho}\widetilde{C}''(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_1\\ \widetilde{\pi}_1\end{bmatrix}\begin{bmatrix}\widetilde{\mu}_2\\ \widetilde{\pi}_2\end{bmatrix}\partial_t\boldsymbol{\sigma}_0:\boldsymbol{\varphi}_0=\widetilde{C}(\mu,\pi)C(\widetilde{\mu}_1,\widetilde{\pi}_1)\boldsymbol{\varepsilon}(\mathbf{v}):C(\widetilde{\mu}_2,\widetilde{\pi}_2)\widetilde{C}(\mu,\pi)\boldsymbol{\varphi}_0\\ +\widetilde{C}(\mu,\pi)C(\widetilde{\mu}_2,\widetilde{\pi}_2)\boldsymbol{\varepsilon}(\mathbf{v}):C(\widetilde{\mu}_1,\widetilde{\pi}_1)\widetilde{C}(\mu,\pi)\boldsymbol{\varphi}_0.$$

We have

$$\widetilde{C}(\mu,\pi)C(\widetilde{\mu}_2,\widetilde{\pi}_2)\boldsymbol{\varepsilon}(\mathbf{v})=\frac{\widetilde{\mu}_2}{\mu}\boldsymbol{\varepsilon}(\mathbf{v})+\frac{\mu\widetilde{\pi}_2-\widetilde{\mu}_2\pi}{\mu(3\pi-4\mu)}\operatorname{div}\mathbf{v}\,\mathbf{I}$$

and

$$C(\widetilde{\mu}_1,\widetilde{\pi}_1)\widetilde{C}(\mu,\pi)\boldsymbol{\varphi}_0=\frac{\widetilde{\mu}_1}{\mu}\boldsymbol{\varphi}_0+\frac{\mu\widetilde{\pi}_1-\widetilde{\mu}_1\pi}{\mu(3\pi-4\mu)}\operatorname{tr}(\boldsymbol{\varphi}_0)\,\mathbf{I}$$

so that

$$\frac{1}{\rho}\widetilde{C}''(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_1\\ \widetilde{\pi}_1\end{bmatrix}\begin{bmatrix}\widetilde{\mu}_2\\ \widetilde{\pi}_2\end{bmatrix}\partial_t\boldsymbol{\sigma}_0:\boldsymbol{\varphi}_0=2\,\frac{\widetilde{\mu}_1\widetilde{\mu}_2}{\mu^2}\,\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_0\\ +2\,\frac{\widetilde{\mu}_2(3\widetilde{\mu}_1\pi^2-4\widetilde{\pi}_1\mu^2)+\widetilde{\pi}_2(3\widetilde{\pi}_1\pi^2-4\widetilde{\mu}_1\mu^2)}{\mu^2(3\pi-4\mu)^2}\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0).$$

Substituting above auxiliary results into the expression for S_0 yields

$$S_0 = \widehat{\rho}_2\bigg(\Big(\frac{\widehat{\rho}_1}{\rho^2} + \frac{\widetilde{\mu}_1}{\rho\mu}\Big)\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_0 + \frac{1}{\rho}\Big(\frac{\widetilde{\pi}_1}{3\pi - 4\mu} - \frac{\pi}{\mu(3\pi-4\mu)}\Big)\operatorname{div}\mathbf{v}\,\operatorname{tr}(\boldsymbol{\varphi}_0)\bigg)$$
$$+ \widetilde{\mu}_2\bigg(\Big(\frac{\widehat{\rho}_1}{\rho\mu} + \frac{2\widetilde{\mu}_1}{\mu^2}\Big)\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_0 + \Big(2\,\frac{3\widetilde{\mu}_1\pi^2 - 4\widetilde{\pi}_1\mu^2}{\mu^2(3\pi-4\mu)^2} - \frac{\widehat{\rho}_1}{\rho}\,\frac{\pi}{\mu(3\pi-4\mu)}\Big)\operatorname{div}\mathbf{v}\,\operatorname{tr}(\boldsymbol{\varphi}_0)\bigg)$$
$$+ \widetilde{\pi}_2\Big(\frac{\widehat{\rho}_1}{\rho}\,\frac{1}{\mu(3\pi-4\mu)} + 2\,\frac{3\widetilde{\pi}_1\pi^2 - 4\widetilde{\mu}_1\mu^2}{\mu^2(3\pi-4\mu)^2}\Big)\operatorname{div}\mathbf{v}\,\operatorname{tr}(\boldsymbol{\varphi}_0).$$

Similar computations for $l = 1, \ldots, L$ based on (5c) result in

$$S_l = \widehat{\rho}_2\bigg(\Big(\frac{\widehat{\rho}_1}{\rho^2} + \frac{\widehat{\mu}_1}{\rho\tau_{\mathrm{S}}\mu}\Big)\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_l + \frac{1}{\rho}\Big(\frac{\widehat{\pi}_1}{3\tau_{\mathrm{P}}\pi - 4\tau_{\mathrm{S}}\mu} - \frac{\tau_{\mathrm{P}}\pi}{\tau_{\mathrm{S}}\mu(3\tau_{\mathrm{P}}\pi - 4\tau_{\mathrm{S}}\mu)}\Big)\operatorname{div}\mathbf{v}\,\operatorname{tr}(\boldsymbol{\varphi}_l)\bigg)$$
$$+ \widehat{\mu}_2\bigg(\Big(\frac{\widehat{\rho}_1}{\rho\tau_{\mathrm{S}}\mu} + \frac{2\widehat{\mu}_1}{\tau_{\mathrm{S}}^2\mu^2}\Big)\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_l$$
$$+ \Big(2\,\frac{3\widehat{\mu}_1\tau_{\mathrm{P}}^2\pi^2 - 4\widehat{\pi}_1\tau_{\mathrm{S}}^2\mu^2}{\tau_{\mathrm{S}}^2\mu^2(3\tau_{\mathrm{P}}\pi - 4\tau_{\mathrm{S}}\mu)^2} - \frac{\widehat{\rho}_1}{\rho}\,\frac{\tau_{\mathrm{P}}\pi}{\tau_{\mathrm{S}}\mu(3\tau_{\mathrm{P}}\pi - 4\tau_{\mathrm{S}}\mu)}\Big)\operatorname{div}\mathbf{v}\,\operatorname{tr}(\boldsymbol{\varphi}_l)\bigg)$$
$$+ \widehat{\pi}_2\Big(\frac{\widehat{\rho}_1}{\rho}\,\frac{1}{\tau_{\mathrm{S}}\mu(3\tau_{\mathrm{P}}\pi - 4\tau_{\mathrm{S}}\mu)} + 2\,\frac{3\widehat{\pi}_1\tau_{\mathrm{P}}^2\pi^2 - 4\widehat{\mu}_1\tau_{\mathrm{S}}^2\mu^2}{\tau_{\mathrm{S}}^2\mu^2(3\tau_{\mathrm{P}}\pi - 4\tau_{\mathrm{S}}\mu)^2}\Big)\operatorname{div}\mathbf{v}\,\operatorname{tr}(\boldsymbol{\varphi}_l).$$

Next we replace $\widetilde{\mu}_2$, $\widetilde{\pi}_2$, and $\widehat{\mu}_2$, $\widehat{\pi}_2$ by their values from (31) and (32), respectively. Finally, we calculate $S_0 + \cdots + S_L$ and group the terms belonging to the components of $\widehat{\mathbf{p}}_2$. □

In view of (36) we have now derived an analytic expression for $\Phi''(\mathbf{p})[\widehat{\mathbf{p}}, \cdot]^*$ in rather basic terms.

Appendix A. Two spatial dimensions

The expressions for the Fréchet derivatives and their adjoints provided in the main part of this paper cannot directly be applied to the viscoelastic wave equation in two spatial dimensions. The differences to the 3D case which have to be taken into account are

$$\operatorname{tr}(\mathbf{I}) = 2 \quad \text{and} \quad \widetilde{C}(m,p)\mathbf{M} = C^{-1}(m,p)\mathbf{M} = \frac{1}{2m}\mathbf{M} + \frac{2m-p}{4m(p-m)}\operatorname{tr}(\mathbf{M})\mathbf{I}.$$

With these ingredients the derivatives and adjoints can be calculated exactly along the lines presented on the previous pages.

In this appendix we provide 2D versions of Theorems 4.5, 4.9, and 4.10.

Theorem A.1 (2D version of Theorem 4.5)**.**
The only quantities which have to be changed are $\boldsymbol{\Sigma}^v$, $\boldsymbol{\Sigma}^\tau_{S,1}$, and $\boldsymbol{\Sigma}^\tau_P$. With

$$\boldsymbol{\Sigma}^v = \frac{1}{2(\pi-\mu)}\,\boldsymbol{\varphi}_0 + \frac{\tau_P}{2(\tau_P\pi - \tau_S\mu)}\,\boldsymbol{\Sigma},$$
$$\boldsymbol{\Sigma}^\tau_{S,1} = -\frac{\alpha}{2(\pi-\mu)}\,\boldsymbol{\varphi}_0 + \frac{\tau_P}{2\,\tau_S(\tau_P\pi - \tau_S\mu)}\,\boldsymbol{\Sigma},$$
$$\boldsymbol{\Sigma}^\tau_P = \frac{\alpha}{2(\pi-\mu)}\,\boldsymbol{\varphi}_0 - \frac{1}{2(\tau_P\pi - \tau_S\mu)}\,\boldsymbol{\Sigma},$$

the statement of Theorem 4.5 can be copied without any further changes.

Proof. The only difference to the 3D proof concerns the computation of, compare (30),

$$\begin{aligned} &C(\widetilde{\mu},\widetilde{\pi})\boldsymbol{\varepsilon}(\mathbf{v}) : \widetilde{C}(\mu,\pi)\boldsymbol{\varphi}_0 \\ &\quad = \big(2\widetilde{\mu}\,\boldsymbol{\varepsilon}(\mathbf{v}) + (\widetilde{\pi}-2\widetilde{\mu})\operatorname{div}\mathbf{v}\,\mathbf{I}\big) : \Big(\frac{1}{2\mu}\boldsymbol{\varphi}_0 + \frac{2\mu-\pi}{4\mu(\pi-\mu)}\operatorname{tr}(\boldsymbol{\varphi}_0)\mathbf{I}\Big) \\ &\quad = \widetilde{\mu}\Big(\frac{1}{\mu}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_0 - \frac{\pi}{2\mu(\pi-\mu)}\operatorname{div}\mathbf{v}\operatorname{tr}(\boldsymbol{\varphi}_0)\Big) + \frac{\widetilde{\pi}}{2(\pi-\mu)}\operatorname{div}\mathbf{v}\operatorname{tr}(\boldsymbol{\varphi}_0). \end{aligned}$$

□

Theorem A.2 (2D version of Theorem 4.9).
Theorem 4.9 remains correct for the 2D case when the 2D versions of $\boldsymbol{\Sigma}^v$, $\boldsymbol{\Sigma}^\tau_{S,1}$, and $\boldsymbol{\Sigma}^\tau_P$ from the above theorem are taken.

Theorem A.3 (2D version of Theorem 4.10).
Theorem 4.10 remains correct for the 2D case when the definitions of the Υ's are replaced by

$$\begin{aligned} &\boldsymbol{\Upsilon}^\rho_1 = \Big(\frac{\widehat{\rho}}{\rho} + \frac{\widetilde{\mu}}{\mu}\Big)\boldsymbol{\varphi}_0 + \Big(\frac{\widehat{\rho}}{\rho} + \frac{\widehat{\mu}}{\tau_S\mu}\Big)\boldsymbol{\Sigma}, \quad \boldsymbol{\Upsilon}^\rho_2 = \frac{\widetilde{\pi}\mu-\pi}{2\mu(\pi-\mu)}\boldsymbol{\varphi}_0 + \frac{\widehat{\pi}\tau_S\mu-\tau_P\pi}{2\tau_S\mu(\tau_P\pi-\tau_S\mu)}\boldsymbol{\Sigma}, \\ &\boldsymbol{\Upsilon}^v_{S,1} = \Big(\frac{\widehat{\rho}}{\rho} + \frac{2\widetilde{\mu}}{\mu}\Big)\boldsymbol{\varphi}_0 + \Big(\frac{\widehat{\rho}}{\rho} + \frac{2\widehat{\mu}}{\tau_S\mu}\Big)\boldsymbol{\Sigma}, \quad \boldsymbol{\Upsilon}^v_{S,2} = \mathrm{K}_{S,\boldsymbol{\varphi}}\,\boldsymbol{\varphi}_0 + \mathrm{K}_{S,\boldsymbol{\Sigma}}\,\boldsymbol{\Sigma}, \\ &\boldsymbol{\Upsilon}^\tau_{S,1} = -\alpha\Big(\frac{\widehat{\rho}}{\rho} + \frac{2\widetilde{\mu}}{\mu}\Big)\boldsymbol{\varphi}_0 + \Big(\frac{\widehat{\rho}}{\tau_S\rho} + \frac{2\widehat{\mu}}{\tau_S^2\mu}\Big)\boldsymbol{\Sigma}, \quad \boldsymbol{\Upsilon}^\tau_{S,2} = -\alpha\mathrm{K}_{S,\boldsymbol{\varphi}}\,\boldsymbol{\varphi}_0 + \mathrm{K}_{S,\boldsymbol{\Sigma}}\,\boldsymbol{\Sigma}/\tau_S, \\ &\boldsymbol{\Upsilon}^v_P = \mathrm{K}_{P,\boldsymbol{\varphi}}\,\boldsymbol{\varphi}_0 + \tau_P\,\mathrm{K}_{P,\boldsymbol{\Sigma}}\,\boldsymbol{\Sigma}, \quad \boldsymbol{\Upsilon}^\tau_P = -\alpha\,\mathrm{K}_{P,\boldsymbol{\varphi}}\,\boldsymbol{\varphi}_0 + \mathrm{K}_{P,\boldsymbol{\Sigma}}\,\boldsymbol{\Sigma}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{K}_{S,\boldsymbol{\varphi}} &= \frac{2\pi\mu\widetilde{\mu} - \widetilde{\mu}\pi^2 - \widetilde{\pi}\mu^2}{\mu(\pi-\mu)^2} - \frac{\widehat{\rho}}{\rho}\,\frac{\pi}{2(\pi-\mu)}, \\ \mathrm{K}_{S,\boldsymbol{\Sigma}} &= \frac{2\tau_P\pi\tau_S\mu\widehat{\mu} - \widehat{\mu}\tau_P^2\pi^2 - \widehat{\pi}\tau_S^2\mu^2}{\tau_S\mu(\tau_P\pi-\tau_S\mu)^2} - \frac{\widehat{\rho}}{\rho}\,\frac{\tau_P\pi}{2(\tau_P\pi-\tau_S\mu)}, \\ \mathrm{K}_{P,\boldsymbol{\varphi}} &= \frac{\widehat{\rho}}{\rho}\,\frac{1}{2(\pi-\mu)} + \frac{\widetilde{\pi}-\widetilde{\mu}}{(\pi-\mu)^2}, \\ \mathrm{K}_{P,\boldsymbol{\Sigma}} &= \frac{\widehat{\rho}}{\rho}\,\frac{1}{2(\tau_P\pi-\tau_S\mu)} + \frac{\widehat{\pi}-\widehat{\mu}}{(\tau_P\pi-\tau_S\mu)^2}. \end{aligned}$$

Proof. We have

$$\widetilde{C}(\mu,\pi)C(\widetilde{\mu}_2,\widetilde{\pi}_2)\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{\widetilde{\mu}_2}{\mu}\boldsymbol{\varepsilon}(\mathbf{v}) + \frac{\mu\widetilde{\pi}_2 - \widetilde{\mu}_2\pi}{2\mu(\pi-\mu)}\operatorname{div}\mathbf{v}\,\mathbf{I}$$

and

$$C(\widetilde{\mu}_1,\widetilde{\pi}_1)\widetilde{C}(\mu,\pi)\boldsymbol{\varphi}_0 = \frac{\widetilde{\mu}_1}{\mu}\boldsymbol{\varphi}_0 + \frac{\mu\widetilde{\pi}_1 - \widetilde{\mu}_1\pi}{2\mu(\pi-\mu)}\operatorname{tr}(\boldsymbol{\varphi}_0)\,\mathbf{I}$$

so that

$$\frac{1}{\rho}\widetilde{C}''(\mu,\pi)\begin{bmatrix}\widetilde{\mu}_1\\ \widetilde{\pi}_1\end{bmatrix}\begin{bmatrix}\widetilde{\mu}_2\\ \widetilde{\pi}_2\end{bmatrix}\partial_t\boldsymbol{\sigma}_0 : \boldsymbol{\varphi}_0 = 2\,\frac{\widetilde{\mu}_1\widetilde{\mu}_2}{\mu^2}\,\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varphi}_0$$

$$+\frac{\widetilde{\mu}_2(2\pi\mu\widetilde{\mu}_1-\widetilde{\mu}_1\pi^2-\widetilde{\pi}_1\mu^2)+\widetilde{\pi}_2\mu^2(\widetilde{\pi}_1-\widetilde{\mu}_1)}{\mu^2(\pi-\mu)^2}\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0).$$

Let S_0 and S_l, $l=1,\dots,L$, be defined as in the proof of Theorem 4.10. Then,

$$\begin{aligned}S_0=\widehat{\rho}_2&\Big(\Big(\frac{\widehat{\rho}_1}{\rho^2}+\frac{\widetilde{\mu}_1}{\rho\mu}\Big)\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_0+\frac{1}{\rho}\,\frac{\widetilde{\pi}_1\mu-\pi}{2\mu(\pi-\mu)}\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0)\Big)\\&+\widetilde{\mu}_2\Big(\Big(\frac{\widehat{\rho}_1}{\rho\mu}+\frac{2\widetilde{\mu}_1}{\mu^2}\Big)\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_0+\Big(\frac{2\pi\mu\widetilde{\mu}_1-\widetilde{\mu}_1\pi^2-\widetilde{\pi}_1\mu^2}{\mu^2(\pi-\mu)^2}-\frac{\widehat{\rho}_1}{\rho}\,\frac{\pi}{2\mu(\pi-\mu)}\Big)\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0)\Big)\\&+\widetilde{\pi}_2\Big(\frac{\widehat{\rho}_1}{\rho}\,\frac{1}{2(\pi-\mu)}+\frac{\widetilde{\pi}_1-\widetilde{\mu}_1}{(\pi-\mu)^2}\Big)\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_0)\end{aligned}$$

and

$$\begin{aligned}S_l=\widehat{\rho}_2&\Big(\Big(\frac{\widehat{\rho}_1}{\rho^2}+\frac{\widehat{\mu}_1}{\rho\tau_{\mathrm{S}}\mu}\Big)\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_l+\frac{1}{\rho}\,\frac{\widehat{\pi}_1\tau_{\mathrm{S}}\mu-\tau_{\mathrm{P}}\pi}{2\tau_{\mathrm{S}}\mu(\tau_{\mathrm{P}}\pi-\tau_{\mathrm{S}}\mu)}\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_l)\Big)\\&+\widehat{\mu}_2\Big(\Big(\frac{\widehat{\rho}_1}{\rho\tau_{\mathrm{S}}\mu}+\frac{2\widehat{\mu}_1}{\tau_{\mathrm{S}}^2\mu^2}\Big)\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varphi}_l\\&\qquad+\Big(\frac{2\tau_{\mathrm{P}}\pi\tau_{\mathrm{S}}\mu\widehat{\mu}_1-\widehat{\mu}_1\tau_{\mathrm{P}}^2\pi^2-\widehat{\pi}_1\tau_{\mathrm{S}}^2\mu^2}{\tau_{\mathrm{S}}^2\mu^2(\tau_{\mathrm{P}}\pi-\tau_{\mathrm{S}}\mu)^2}-\frac{\widehat{\rho}_1}{\rho}\,\frac{\tau_{\mathrm{P}}\pi}{2\tau_{\mathrm{S}}\mu(\tau_{\mathrm{P}}\pi-\tau_{\mathrm{S}}\mu)}\Big)\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_l)\Big)\\&+\widehat{\pi}_2\Big(\frac{\widehat{\rho}_1}{\rho}\,\frac{1}{2(\tau_{\mathrm{P}}\pi-\tau_{\mathrm{S}}\mu)}+\frac{\widehat{\pi}_1-\widehat{\mu}_1}{(\tau_{\mathrm{P}}\pi-\tau_{\mathrm{S}}\mu)^2}\Big)\operatorname{div}\mathbf{v}\ \operatorname{tr}(\boldsymbol{\varphi}_l).\end{aligned}$$

The next steps are as in the proof of Theorem 4.10. □

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