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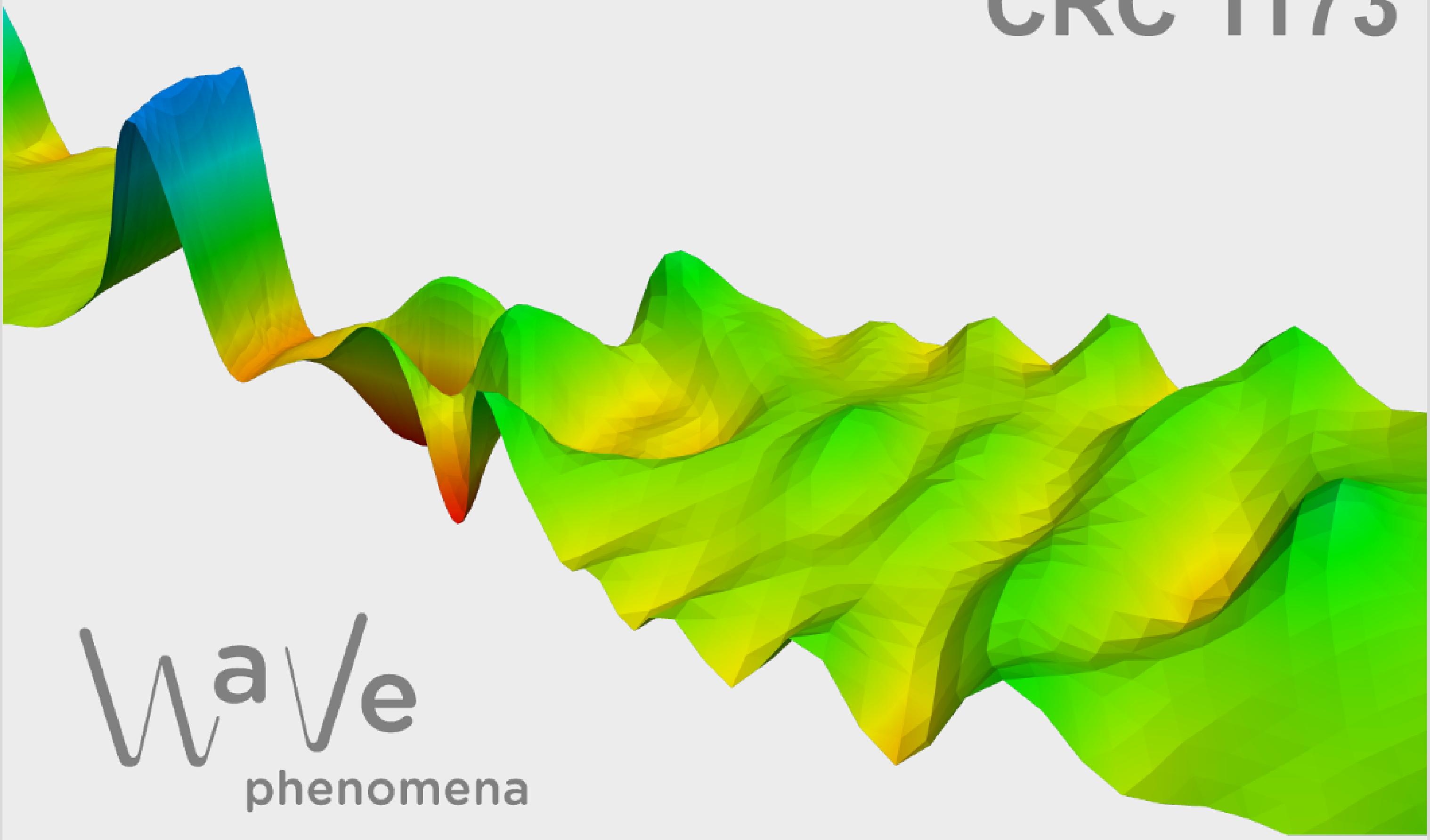

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HETEROGENEOUS MULTISCALE METHOD FOR MAXWELL'S EQUATIONS*

MARLIS HOCHBRUCK†, BERNHARD MAIER†, AND CHRISTIAN STOHRER

Abstract. We present a Finite Element Heterogeneous Multiscale Method (FE-HMM) for time-dependent Maxwell's equations in first-order formulation in highly oscillatory materials using Nédélec's edge elements. Based on a uniform approach for the error analysis of non-conforming space discretizations [17], we prove an error bound for the semidiscrete scheme. We further present error bounds for the fully discrete scheme, where we consider time discretization using algebraically stable Runge–Kutta methods, the Crank–Nicolson method and the leapfrog method. These error bounds are confirmed by numerical experiments.

Key words. Heterogeneous Multiscale Method, First order time-dependent Maxwell's equations, Fully discrete error analysis

AMS subject classifications. Primary: 65M12, 65M15, 74Q15 Secondary: 65M20, 65M60.

1. Introduction.

Our goal is the numerical solution of time-dependent linear Maxwell's equations in a highly oscillatory material at reasonable computational cost. Therefore, we approximate the effective behavior of the electromagnetic field without resolving the fine scale behavior. This is done using a Finite Element Heterogeneous Multiscale Method (FE-HMM) introduced in [15]. Such methods are known to be efficient and offer a good framework for the error analysis for the simulation of homogenization problems in highly oscillatory media [4]. Highly oscillatory means that the material parameters, i.e., the magnetic permeability μ^η and the electric permittivity ε^η, depend on a characteristic microscopic length η that is assumed to be very small in comparison to the diameter of the domain Ω. As the material parameters depend on η, the magnetic field $\boldsymbol{H}^\eta$ and the electric field $\boldsymbol{E}^\eta$ inherit this dependency on the microscopic structure of the material, indicated by the superscript η. Accordingly, we consider the following problem:

$$
\left\{
\begin{aligned}
&\text{Find } \boldsymbol{H}^\eta : [0,T] \to H(\operatorname{curl},\Omega) \text{ and } \boldsymbol{E}^\eta : [0,T] \to H_0(\operatorname{curl},\Omega), \text{ such that}\\
&\qquad\mu^\eta(x)\partial_t \boldsymbol{H}^\eta(t,x) = -\operatorname{curl} \boldsymbol{E}^\eta(t,x),\\
&\qquad\varepsilon^\eta(x)\partial_t \boldsymbol{E}^\eta(t,x) = \operatorname{curl} \boldsymbol{H}^\eta(t,x) - \boldsymbol{J}_{\mathrm{ext}}(t,x),\\
&\qquad\boldsymbol{H}^\eta(0) = \boldsymbol{H}_0, \qquad \boldsymbol{E}^\eta(0) = \boldsymbol{E}_0.
\end{aligned}
\right. \tag{1.1}
$$

We will call this set of equations Maxwell's equations in first order formulation, as there are only first derivatives in space. To our knowledge, all previously presented FE-HMMs for Maxwell's equations were based on the second order formulation, also known as the curl-curl problem. In [12] and [16], Heterogeneous Multiscale Methods for time-harmonic Maxwell's equations in second order formulation are introduced. An HMM for time-dependent Maxwell's equations is analyzed in [21]. These methods have the drawback of introducing new micro problems, whereas the approach presented in this work is only based on the micro problems already known from FE-HMMs for elliptic problems [2]. Besides HMM there are only few multiscale methods

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for the time-dependent first order formulation of Maxwell's equations. In [10], a multiscale scheme based on an asymptotic expansion has been presented. Recently, this idea has been adapted to the time-dependent Maxwell-Schrödinger system [11]. In [24], the multiscale hybrid-mixed method was extended to Maxwell's equation (1.1).

As our main results, we present error estimates for the fully discrete FE-HMM. This includes an analysis of the discretization in time with algebraically stable Runge–Kutta methods, the Crank–Nicolson scheme and the leapfrog scheme. Note that this was already done for parabolic problems, e.g., the application of the implicit Euler method is analyzed in [5] and a class of higher order Runge–Kutta methods is considered in [6].

1.1. Outline. The general setup (1.1) is discussed in Section 2. We further recap a homogenization result from [28], which basically states that the multiscale solutions $\boldsymbol{H}^\eta$ and $\boldsymbol{E}^\eta$ converge to the solutions $\boldsymbol{H}^{\mathrm{eff}}$ and $\boldsymbol{E}^{\mathrm{eff}}$ of the homogenized equations. These equations, however, use homogenized material parameters whose computation in general includes the solution of infinitely many elliptic differential equations. As we introduce space discretization and derive the HMM in Section 3, we reduce these micro problems to a finite number. We also derive the so-called HMM-material parameters, as explained in [3]. These parameters allow us to write the HMM in a form which is equivalent to the homogenized system, which is fundamental for the error analysis in Section 4. There, we prove a convergence result for the semidiscrete discretization using tools from [17]. Furthermore, we generalize these convergence results to the fully discrete case in Section 5, where we consider time discretization with algebraically stable Runge–Kutta methods of arbitrary high order, the Crank–Nicolson method and the leapfrog scheme. Finally, we present some numerical results verifying our theoretical results in Section 6.

1.2. Notation. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain and $Y^\kappa(x) = x + (-\frac{\kappa}{2}, \frac{\kappa}{2})^3$ for $\kappa > 0$ and $x \in \Omega$. To keep notation simple, we omit the argument if $x = 0$ and omit the subscript if $\kappa = 1$. For $s \in \mathbb{N}_0$ we denote by $\|\cdot\|_{s,\Omega}$ the standard norm of the Sobolev space $W^{s,2}(\Omega)$ with $L^2(\Omega) = W^{0,2}(\Omega)$. Analogously, we denote by $\|\cdot\|_{s,\infty,\Omega}$ the norms of the Sobolev space $W^{s,\infty}(\Omega)$. By $W_0^{1,2}(\Omega)$, we denote the space consisting of all elements of $W^{1,2}(\Omega)$ with vanishing trace on the boundary of Ω. To simplify the notation, we use the same expressions for the norm of vector- or matrix-valued Sobolev spaces. We further use the spaces

$$\begin{aligned}
H(\mathrm{curl}, \Omega) &= \{f \in L^2(\Omega)^3 \mid \mathrm{curl}\, f \in L^2(\Omega)^3\}, \\
H(\mathrm{div}, \Omega) &= \{f \in L^2(\Omega)^3 \mid \mathrm{div}\, f \in L^2(\Omega)^3\}, \\
H(\mathrm{curl}^2, \Omega) &= \{f \in H(\mathrm{curl}, \Omega) \mid \mathrm{curl}\, f \in H(\mathrm{curl}, \Omega)\},
\end{aligned}$$

and $H_0(\mathrm{curl}, \Omega)$, which consists of all functions in $H(\mathrm{curl}, \Omega)$ with vanishing tangential trace. Note that this corresponds to the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H(\mathrm{curl}, \Omega)$. To indicate that a function space $F(Y^\eta)$ contains only periodic functions, we write $F_\#(Y^\eta)$. Finally, we use a generic constant $C > 0$, which may have different values on any occurrence.

2. Multiscale model problem and homogenization. We consider Maxwell's equations (1.1) for locally periodic material parameters.

DEFINITION 2.1. *Let $\eta > 0$. A tensor $\alpha^\eta : \Omega \to \mathbb{R}^{3\times 3}$ is called locally periodic if there is a tensor $\alpha : \Omega \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$, which is Y-periodic in its second argument and $\alpha^\eta(x) = \alpha(x, \frac{x}{\eta})$ holds for almost every $x \in \Omega$.*

Furthermore, the magnetic permeability μ^η and the electric permittivity ε^η are assumed to be symmetric, uniformly positive definite and uniformly bounded. More precisely, we assume that $\mu^\eta, \varepsilon^\eta \in L^\infty(\Omega)^{3\times 3}$ and that there are constants $\lambda,\ \Lambda > 0$, such that

$$\lambda|\xi|^2 \leq \mu^\eta(x)\xi\cdot\xi \leq \Lambda|\xi|^2 \qquad \text{and} \qquad \lambda|\xi|^2 \leq \varepsilon^\eta(x)\xi\cdot\xi \leq \Lambda|\xi|^2, \tag{2.1}$$

for all $\xi \in \mathbb{R}^3$ and almost every $x \in \Omega$, which implies that all materials considered have a positive refractive index for all $\eta > 0$.

From (1.1), we get the following variational multiscale Maxwell's equation:

$$\begin{cases} \text{Find } u^\eta : [0,T] \to V = H(\mathrm{curl},\Omega)\times H_0(\mathrm{curl},\Omega), \text{ such that for all } \xi \in V \\ \qquad m^\eta(\partial_t u^\eta(t),\xi) = s(u^\eta(t),\xi) + m^\eta(f^\eta(t),\xi), \\ \qquad u^\eta(0) = u_0. \end{cases} \tag{2.2}$$

The solution $u^\eta = (\boldsymbol{H}^\eta, \boldsymbol{E}^\eta)$ consists of the magnetic field $\boldsymbol{H}^\eta \in H(\mathrm{curl},\Omega)$ and the electric field $\boldsymbol{E}^\eta \in H_0(\mathrm{curl},\Omega)$. The bilinear forms are given by

$$m^\eta\left(\begin{pmatrix}\psi\\ \varphi\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}\\ \widetilde{\varphi}\end{pmatrix}\right) = (\mu^\eta\widetilde{\psi},\psi)_{0,\Omega} + (\varepsilon^\eta\widetilde{\varphi},\varphi)_{0,\Omega},$$

$$s\left(\begin{pmatrix}\psi\\ \varphi\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}\\ \widetilde{\varphi}\end{pmatrix}\right) = (\mathrm{curl}\,\widetilde{\psi},\varphi)_{0,\Omega} - (\mathrm{curl}\,\psi,\widetilde{\varphi})_{0,\Omega},$$

for all $\psi, \widetilde{\psi} \in H(\mathrm{curl},\Omega)$ and $\varphi, \widetilde{\varphi} \in H_0(\mathrm{curl},\Omega)$. Depending on the given electric current $\boldsymbol{J}_{\mathrm{ext}} \in C^1(0,T;H_0(\mathrm{curl}^2,\Omega)) \cap C(0,T;H(\mathrm{div},\Omega))$, we further define the function $f^\eta\colon [0,T]\times\Omega \to \mathbb{R}^6$ by

$$m^\eta\left(f^\eta(t), \begin{pmatrix}\psi\\ \varphi\end{pmatrix}\right) = \left(\begin{pmatrix}0\\ -\boldsymbol{J}_{\mathrm{ext}}(t)\end{pmatrix}, \begin{pmatrix}\psi\\ \varphi\end{pmatrix}\right)_{0,\Omega},$$

for all $\psi \in H(\mathrm{curl},\Omega)$ and $\varphi \in H_0(\mathrm{curl},\Omega)$.

For an initial value $u_0 \in V$ and a fixed η, the wellposedness of this model problem was proven for example in [18, Prop. 3.5]. In order to consider the corresponding homogenized problem, the uniform boundedness of the solutions with respect to η has to be shown additionally.

Proposition 2.2 (cf. [28, Prop. 5.3]). *For $\eta > 0$ let $u^\eta = (\boldsymbol{H}^\eta, \boldsymbol{E}^\eta) \in V$ be the solution of (2.2). The functions $\boldsymbol{H}^\eta$, $\boldsymbol{E}^\eta$, $\partial_t\boldsymbol{H}^\eta$, $\partial_t\boldsymbol{E}^\eta$, $\mathrm{curl}\,\boldsymbol{H}^\eta$ and $\mathrm{curl}\,\boldsymbol{E}^\eta$ are bounded in $L^\infty(0,T;L^2(\Omega)^3)$ independently of η.*

Based on this result, the effective Maxwell's equation can be derived, which models the behavior of u^η in the limit $\eta \to 0$.

Theorem 2.3 (cf. [28, Thm. 3.2]). *For $\eta > 0$ let $u^\eta \in V$ be the solution of (2.2). Then*

$$u^\eta \rightharpoonup u^{\mathrm{eff}} \qquad \text{weakly in } L^2(0,T;V),$$

where $u^{\mathrm{eff}} \in C^1(0,T;L^2(\Omega)^6) \cap C(0,T;V)$ is the solution of the following variational effective Maxwell's equation:

$$\begin{cases} \text{Find } u^{\mathrm{eff}} : [0,T] \to V, \text{ such that for all } \xi \in V \\ m^{\mathrm{eff}}(\partial_t u^{\mathrm{eff}}(t),\xi) = s(u^{\mathrm{eff}}(t),\xi) + m^{\mathrm{eff}}(f^{\mathrm{eff}}(t),\xi), \\ \qquad u^{\mathrm{eff}}(0) = u_0. \end{cases} \tag{2.3}$$

For $\psi, \widetilde{\psi} \in H(\mathrm{curl}, \Omega)$ and $\varphi, \widetilde{\varphi} \in H_0(\mathrm{curl}, \Omega)$ the effective bilinear form m^{eff} is given by

$$m^{\mathrm{eff}}\left(\begin{pmatrix}\psi\\ \varphi\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}\\ \widetilde{\varphi}\end{pmatrix}\right) = \int_\Omega \mu^{\mathrm{eff}}(x)\widetilde{\psi}(x)\cdot\psi(x) + \varepsilon^{\mathrm{eff}}(x)\widetilde{\varphi}(x)\cdot\varphi(x)\,dx$$

with

$$\mu^{\mathrm{eff}}(x) = \frac{1}{|Y^\eta(x)|}\int_{Y^\eta(x)} \big(I - D_y\chi_\mu(x,y)\big)^T \mu\big(x, \tfrac{y}{\eta}\big)\big(I - D_y\chi_\mu(x,y)\big)\,dy, \tag{2.4a}$$

$$\varepsilon^{\mathrm{eff}}(x) = \frac{1}{|Y^\eta(x)|}\int_{Y^\eta(x)} \big(I - D_y\chi_\varepsilon(x,y)\big)^T \varepsilon\big(x, \tfrac{y}{\eta}\big)\big(I - D_y\chi_\varepsilon(x,y)\big)\,dy, \tag{2.4b}$$

for $x \in \Omega$, where I is the identity matrix. The unknown functions $\chi_\mu(x,\cdot), \chi_\varepsilon(x,\cdot) \in \big(W^{1,2}_\#(Y^\eta(x))/\mathbb{R}\big)^3$ are the uniquely defined solutions of the local problems

$$\int_{Y^\eta(x)} \big(I - D_y\chi_\mu(x,y)\big)^T \mu\big(x, \tfrac{y}{\eta}\big)\nabla_y v(y)\,dy = 0 \qquad \forall v \in W^{1,2}_\#(Y^\eta(x)), \tag{2.5a}$$

$$\int_{Y^\eta(x)} \big(I - D_y\chi_\varepsilon(x,y)\big)^T \varepsilon\big(x, \tfrac{y}{\eta}\big)\nabla_y v(y)\,dy = 0 \qquad \forall v \in W^{1,2}_\#(Y^\eta(x)). \tag{2.5b}$$

The function $f^{\mathrm{eff}} \colon [0,T]\times\Omega \to \mathbb{R}^6$ is defined by

$$m^{\mathrm{eff}}\left(f^{\mathrm{eff}}(t), \begin{pmatrix}\psi\\ \varphi\end{pmatrix}\right) = \left(\begin{pmatrix}0\\ -\boldsymbol{J}_{\mathrm{ext}}(t)\end{pmatrix}, \begin{pmatrix}\psi\\ \varphi\end{pmatrix}\right)_{0,\Omega}, \tag{2.6}$$

for all $\psi \in H(\mathrm{curl}, \Omega)$ and $\varphi \in H_0(\mathrm{curl}, \Omega)$.

Following ideas of [23, Chapter 1.4] and [9, Chapter 2.3], one can show that the effective material parameters μ^{eff} and $\varepsilon^{\mathrm{eff}}$ are still symmetric, positive definite, and bounded. Even more, the corresponding bounds remain valid with the same constants λ and Λ. Therefore, the effective system of Maxwell's equations is wellposed following the same arguments as for the multiscale problem (2.2).

Remark 2.4. For further insight into the physical meaning of the effective fields, we refer to [28]. There, the authors use the concept of two-scale convergence, which was originally introduced in [27], to characterize the effective fields as local means of the corresponding two-scale limits. A brief outline of the results can also be found in [12, Prop. 1, Cor. 2] for second order Maxwell's equations.

3. Spatial discretization. As for most heterogeneous multiscale methods, two spatial discretizations are required: a macro discretization of the domain Ω and a micro discretization of $Y^\eta(x)$ for the micro problems (2.5). In view of the numerical experiments implemented in `deal.II` [8], we focus on hexahedral elements. There is however no principle obstruction to use other finite elements. Let the domain Ω be partitioned by a shape regular mesh $\mathcal{T}_H$ consisting of parallelepipeds. The subscript H denotes the maximum over the edge lengths of all cells $K \in \mathcal{T}_H$.

We need quadrature formulas with nodes x_j^K and weights ω_j^K ($j = 1, \dots, J_K$) for every element $K \in \mathcal{T}_H$. Summing over all $K \in \mathcal{T}_H$ yields a quadrature formula for the whole domain Ω. Let $\mathcal{Q}^{i,j,\ell}$ be the space of polynomials of degree at most $i, j, \ell \in \mathbb{N}$ in the respective variables. We assume all quadrature formulas to be exact for polynomials in $\mathcal{Q}^{2\ell,2\ell,2\ell}$, where $\ell \in \mathbb{N}$ is the order of the finite elements of the macro discretization introduced in the next section. Hence we have for all $p \in \mathcal{Q}^{2\ell,2\ell,2\ell}$

$$\int_\Omega p(x)\,dx = \sum_{K\in\mathcal{T}_H} \sum_{j=1}^{J_K} \omega_j^K p(x_j^K), \tag{3.1}$$

which means that products of polynomials of degree at most ℓ in each variable are integrated exactly.

3.1. Edge elements. For the macro discretization, we use Nédélec's $H(\mathrm{curl},\Omega)$-conforming elements of first type for hexahedral elements, which were introduced in [26]. We first recall some important results for these finite elements. With

$$\mathcal{Q}_\mathcal{N}^\ell = \mathcal{Q}^{\ell-1,\ell,\ell} \times \mathcal{Q}^{\ell,\ell-1,\ell} \times \mathcal{Q}^{\ell,\ell,\ell-1},$$

we denote by

$$V_H(\mathrm{curl},\mathcal{T}_H) = \{v_H \in H(\mathrm{curl},\Omega) \mid v_H|_K \in \mathcal{Q}_\mathcal{N}^\ell \;\forall K \in \mathcal{T}_H\} \tag{3.2a}$$

the space of Nédélec's $H(\mathrm{curl},\Omega)$-conforming elements of first type of order $\ell \in \mathbb{N}$. We further write

$$V_{H,0}(\mathrm{curl},\mathcal{T}_H) = H_0(\mathrm{curl},\Omega) \cap V_H(\mathrm{curl},\mathcal{T}_H). \tag{3.2b}$$

Nédélec showed that the corresponding interpolation operator satisfies the following error bound.

Theorem 3.1 (cf. [26, Thm. 6]). *Let $u \in W^{\ell+1,2}(\Omega)$. The interpolation operator $\mathcal{I}_H : W^{\ell+1,2}(\Omega) \to V_H(\mathrm{curl},\mathcal{T}_H)$ for Nédélec elements of first type satisfies*

$$\|u - \mathcal{I}_H u\|_{H(\mathrm{curl},\Omega)} \le C H^\ell |u|_{\ell+1,\Omega},$$

where $|\cdot|_{\ell+1,\Omega}$ denotes the $W^{\ell+1,2}(\Omega)^3$-seminorm.

3.2. Heterogeneous multiscale method. Our goal is to approximate the solution u^{eff} of the effective Maxwell's equation (2.3). In order to evaluate the bilinear form m^{eff} exactly, the effective parameters μ^{eff}, $\varepsilon^{\mathrm{eff}}$ have to be known. These matrix-valued functions are given pointwise in terms of the solution of the micro problems (2.5). As analytic expressions thereof are not available in general, we use an approximated bilinear form instead. This procedure is detailed in the following.

To construct an HMM for Maxwell's equations, we replace in (2.3) the function spaces $H(\mathrm{curl},\Omega)$ and $H_0(\mathrm{curl},\Omega)$ with the corresponding discrete counterparts defined in (3.2a) and (3.2b), respectively. Furthermore we use the quadrature formula (3.1) to approximate the bilinear forms m^{eff} and s. This yields the following discrete effective Maxwell's equation:

$$\begin{cases} \text{Find } u_H^{\mathrm{eff}} : [0,T] \to V_H = V_H(\mathrm{curl},\mathcal{T}_H) \times V_{H,0}(\mathrm{curl},\mathcal{T}_H), \text{ such that for all } \xi_H \in V_H \\ \qquad m_H^{\mathrm{eff}}(\partial_t u_H^{\mathrm{eff}}(t), \xi_H) = s_H(u_H^{\mathrm{eff}}(t), \xi_H) + m_H^{\mathrm{eff}}(f_H^{\mathrm{eff}}(t), \xi_H), \\ \qquad u_H^{\mathrm{eff}}(0) = u_{0,H} \end{cases}$$

with an approximation $u_{0,H} \in V_H$ to the initial values $u_0 \in V$. We further use the discretized versions of the bilinear forms

$$
\begin{aligned}
m_H^{\mathrm{eff}}\left(\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix},\begin{pmatrix}\widetilde\psi_H\\ \widetilde\varphi_H\end{pmatrix}\right) &= \sum_{K,j}\omega_j^K\big(\mu^{\mathrm{eff}}(x_j^K)\widetilde\psi_H(x_j^K)\cdot\psi_H(x_j^K)\\
&\qquad + \varepsilon^{\mathrm{eff}}(x_j^K)\widetilde\varphi_H(x_j^K)\cdot\varphi_H(x_j^K)\big),\\
s_H\left(\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix},\begin{pmatrix}\widetilde\psi_H\\ \widetilde\varphi_H\end{pmatrix}\right) &= \sum_{K,j}\omega_j^K\big(\varphi_H(x_j^K)\cdot\operatorname{curl}\widetilde\psi_H(x_j^K)\\
&\qquad - \widetilde\varphi_H(x_j^K)\cdot\operatorname{curl}\psi_H(x_j^K)\big)
\end{aligned}
\tag{3.3}
$$

and the function $f_H^{\mathrm{eff}} \in C(0,T;V_H)$ defined by

$$
m_H^{\mathrm{eff}}\left(f_H^{\mathrm{eff}}(t),\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix}\right) = \left(\begin{pmatrix}0\\ -\boldsymbol{J}_{\mathrm{ext},H}(t)\end{pmatrix},\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix}\right)_{0,\Omega},
$$

for all $\psi_H,\widetilde\psi_H \in V_H(\mathrm{curl},\mathcal{T}_H)$ and $\varphi_H,\widetilde\varphi_H \in V_{H,0}(\mathrm{curl},\mathcal{T}_H)$, where $\boldsymbol{J}_{\mathrm{ext},H} : [0,T] \to V_H(\mathrm{curl},\mathcal{T}_H)$ is an approximation to the electric displacement $\boldsymbol{J}_{\mathrm{ext}}$. The next step is to insert the equations for the effective material parameters (2.4) into m_H^{eff}. Therefore, we define the abbreviations

$$
\begin{aligned}
\psi_{H,\chi}(x,y) &= \big(I - D_y\chi_\mu(x,y)\big)\psi_H(x), \quad \varphi_{H,\chi}(x,y) = \big(I - D_y\chi_\varepsilon(x,y)\big)\varphi_H(x),\\
\widetilde\psi_{H,\chi}(x,y) &= \big(I - D_y\chi_\mu(x,y)\big)\widetilde\psi_H(x), \quad \widetilde\varphi_{H,\chi}(x,y) = \big(I - D_y\chi_\varepsilon(x,y)\big)\widetilde\varphi_H(x).
\end{aligned}
$$

This yields an equivalent representation of the bilinear form (3.3), namely

$$
\begin{aligned}
m_H^{\mathrm{eff}}\left(\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix},\begin{pmatrix}\widetilde\psi_H\\ \widetilde\varphi_H\end{pmatrix}\right) = \sum_{K,j}\frac{\omega_j^K}{|Y^\eta(x_j^K)|}\Big(&\int_{Y^\eta(x_j^K)}\mu\big(x_j^K,\tfrac{y}{\eta}\big)\widetilde\psi_{H,\chi}(x_j^K,y)\cdot\psi_{H,\chi}(x_j^K,y)\,dy\\
&+\int_{Y^\eta(x_j^K)}\varepsilon\big(x_j^K,\tfrac{y}{\eta}\big)\widetilde\varphi_{H,\chi}(x_j^K,y)\cdot\varphi_{H,\chi}(x_j^K,y)\,dy\Big).
\end{aligned}
$$

Based on the local problems (2.5) we introduce elliptic PDEs which uniquely describe the unknown functions $\psi_{H,\chi}$, $\varphi_{H,\chi}$, $\widetilde\psi_{H,\chi}$, $\widetilde\varphi_{H,\chi}$. Since all functions can be treated analogously, the procedure is only presented for $\psi_{H,\chi}$. We introduce

$$
\Psi_{H,\mathrm{lin}}(x,y) = (y-x)\cdot\psi_H(x) \qquad \text{and} \qquad \Psi_{H,\#}(x,y) = -\eta\chi_\mu(x,y)\cdot\psi_H(x).
$$

It is easy to see that

$$
\Psi_H(x,y) = \Psi_{H,\mathrm{lin}}(x,y) + \Psi_{H,\#}(x,y)
$$

is a potential of $\psi_{H,\chi}$, i.e., $\nabla_y\Psi_H = \psi_{H,\chi}$. While $\Psi_{H,\mathrm{lin}}$ is known explicitly, $\Psi_{H,\#}$ depends on the (unknown) solutions χ_μ of the local micro problem (2.5a). It is easy to check that $\Psi_{H,\#}$ is the solution of the following problem:

$$
\left\{
\begin{aligned}
&\text{Find } \Psi_{H,\#}(x_j^K,\cdot) \in W_\#^{1,2}(Y^\eta(x_j^K))/\mathbb{R}, \text{ such that for all } v \in W_\#^{1,2}(Y^\eta(x_j^K))\\
&\int_{Y^\eta(x_j^K)}\mu\big(x_j^K,\tfrac{y}{\eta}\big)\nabla_y\Psi_{H,\#}(x_j^K,y)\cdot\nabla_y v(y)\,dy =\\
&\qquad\qquad -\int_{Y^\eta(x_j^K)}\mu\big(x_j^K,\tfrac{y}{\eta}\big)\nabla_y\Psi_{H,\mathrm{lin}}(x_j^K,y)\cdot\nabla_y v(y)\,dy.
\end{aligned}
\right.
$$

This weak form is already known from HMMs for elliptic problems, see [1].

The final step in the construction of the HMM is the discretization of the micro problems stated above. To do so, we first introduce for $\delta \geq \eta$ the sampling domains $Y^\delta(x_j^K)$, which are used to approximate the effective material parameters. We use hexahedral Lagrange elements and denote the mesh, which partitions $Y^\delta(x_j^K)$ by $\mathcal{T}_h$. The corresponding discrete function space of piecewise polynomials of degree at most $k \in \mathbb{N}$ in each variable is denoted by $\mathcal{S}^k_\#(\mathcal{T}_h)$ for periodic boundary conditions or $\mathcal{S}^k_0(\mathcal{T}_h)$ in the case of homogeneous Dirichlet boundary conditions. Note that the use of periodic boundary conditions is favorable if η is known explicitly, as these are the natural boundary conditions from (2.5). Nevertheless, we want to emphasize that with the use of Dirichlet boundary conditions, our scheme is also suitable for applications, where η is mostly unknown. As our error analysis covers both cases, we use the notation $\mathcal{S}^k_{0/\#}(\mathcal{T}_h)$, whenever the results are independen of the specific choice of the boundary conditions. Finally, we get the following elliptic HMM micro problems:

$$
\left\{
\begin{aligned}
&\text{Find } \Psi_{H,h,\#}(x_j^K,\cdot) \in \mathcal{S}^k_{0/\#}(\mathcal{T}_h), \text{ such that for all } v_h \in \mathcal{S}^k_{0/\#}(\mathcal{T}_h)\\
&\int_{Y^\delta(x_j^K)} \mu\big(x_j^K, \tfrac{y}{\eta}\big) \nabla_y \Psi_{H,h,\#}(x_j^K, y) \cdot \nabla_y v_h(y)\, dy = \\
&\qquad\qquad - \int_{Y^\delta(x_j^K)} \mu\big(x_j^K, \tfrac{y}{\eta}\big) \nabla_y \Psi_{H,\mathrm{lin}}(x_j^K, y) \cdot \nabla_y v_h(y)\, dy.
\end{aligned}
\right.
\tag{3.4}
$$

Remark 3.2. Note that the solution of (3.4) with $\mathcal{S}^k_{0/\#}(\mathcal{T}_h) = \mathcal{S}^k_\#(\mathcal{T}_h)$ is only unique up to an additional constant. Nevertheless, for the sake of presentation, we will omit the use of quotient spaces here and in the following to treat both boundary conditions without further differentiation. Also note that our scheme does not rely on the solution itself, but only its gradient.

By solving these micro problems on every element for all quadrature nodes and all basis functions, we find an approximation m^{HMM} to m_H^{eff}:

$$
\begin{aligned}
m^{\mathrm{HMM}}\left(\begin{pmatrix}\psi\\ \varphi\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}\\ \widetilde{\varphi}\end{pmatrix}\right) = \sum_{K,j} \frac{\omega_j^K}{|Y^\delta(x_j^K)|} \Big(&\int_{Y^\delta(x_j^K)} \mu\big(x_j^K, \tfrac{y}{\eta}\big) \widetilde{\psi}_{H,h}(x_j^K, y) \cdot \psi_{H,h}(x_j^K, y)\, dy \\
&+ \int_{Y^\delta(x_j^K)} \varepsilon\big(x_j^K, \tfrac{y}{\eta}\big) \widetilde{\varphi}_{H,h}(x_j^K, y) \cdot \varphi_{H,h}(x_j^K, y)\, dy \Big),
\end{aligned}
\tag{3.5}
$$

where we use the solutions of the HMM micro problems (3.4) to compute

$$
\begin{aligned}
\psi_{H,h}(x,y) &= \nabla_y \Psi_{H,\mathrm{lin}} + \nabla_y \Psi_{H,h,\#}, &\qquad \varphi_{H,h}(x,y) &= \nabla_y \Phi_{H,\mathrm{lin}} + \nabla_y \Phi_{H,h,\#},\\
\widetilde{\psi}_{H,h}(x,y) &= \nabla_y \widetilde{\Psi}_{H,\mathrm{lin}} + \nabla_y \widetilde{\Psi}_{H,h,\#}, &\qquad \widetilde{\varphi}_{H,h}(x,y) &= \nabla_y \widetilde{\Phi}_{H,\mathrm{lin}} + \nabla_y \widetilde{\Phi}_{H,h,\#},
\end{aligned}
$$

for all $\psi_H, \widetilde{\psi}_H \in V_H(\mathrm{curl}, \mathcal{T}_H)$ and $\varphi_H, \widetilde{\varphi}_H \in V_{H,0}(\mathrm{curl}, \mathcal{T}_H)$. Together with the discrete bilinear form s_H, we finally derive the HMM for Maxwell's equation:

$$
\left\{
\begin{aligned}
&\text{Find } u_H^{\mathrm{HMM}} : [0,T] \to V_H, \text{ such that for all } \xi_H \in V_H\\
&m^{\mathrm{HMM}}(\partial_t u_H^{\mathrm{HMM}}(t), \xi_H) = s_H(u_H^{\mathrm{HMM}}(t), \xi_H) + (f_H^{\mathrm{HMM}}(t), \xi_H)_{0,\Omega},\\
&\qquad\qquad u_H^{\mathrm{HMM}}(0) = u_{0,H}.
\end{aligned}
\right.
\tag{3.6}
$$

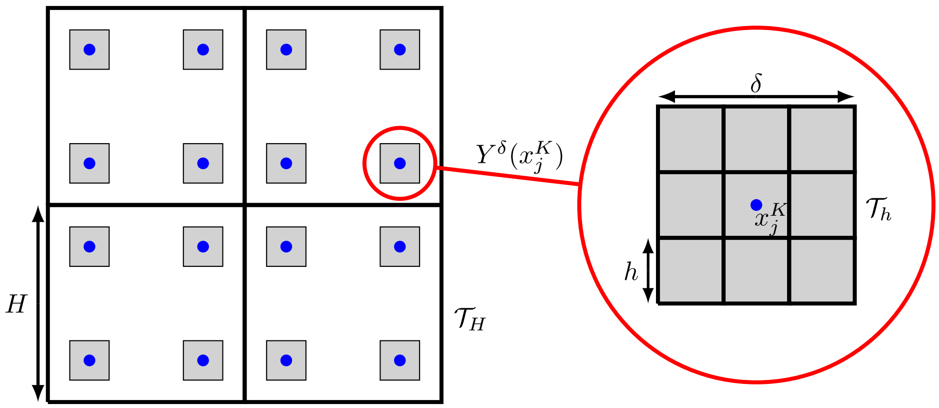

FIG. 3.1. *HMM scheme in 2D*

The function $f_H^{\mathrm{HMM}} \in C(0,T;V_H)$ is defined by

$$m^{\mathrm{HMM}}\left(f_H^{\mathrm{HMM}}(t), \begin{pmatrix}\psi_H \\ \varphi_H\end{pmatrix}\right) = \left(\begin{pmatrix}0 \\ -\boldsymbol{J}_{\mathrm{ext},H}(t)\end{pmatrix}, \begin{pmatrix}\psi_H \\ \varphi_H\end{pmatrix}\right)_{0,\Omega}, \tag{3.7}$$

for all $\psi_H \in V_H(\mathrm{curl}, \mathcal{T}_H)$ and $\varphi_H \in V_{H,0}(\mathrm{curl}, \mathcal{T}_H)$.

Figure 3.1 shows an overview of the general procedure of heterogeneous multiscale methods. For every quadrature node (bullets) of the macro mesh $\mathcal{T}_H$, there is a micro cell Y^δ (colored in gray), on which the HMM micro problems (3.4) are solved using a micro mesh $\mathcal{T}_h$.

Remark 3.3. As our implementation of the HMM scheme is based on the FE-library `deal.II` [8], which supports only hexahedral meshes, we restrict the analysis to such triangulations. Nevertheless we want to emphasize that the results presented in this work can be easily extended to tetrahedral meshes using different polynomial spaces and quadrature formulas. In addition it is also possible to mix the elements, e.g., one could use tetrahedral elements on the macro scale and hexahedral ones to solve the micro problems.

3.3. HMM material parameters. As an intermediate step between the introduction of the HMM and the semidiscrete error analysis in the next section, we follow an idea of [3, Chapter 5.1] to define the so-called HMM material parameters. Those parameters allow the reformulation of (3.6) into an equivalent differential equation with a similar structure as (2.3). Although the HMM material parameters are not used in numerical computations, they are a fundamental tool for the later error analysis.

A close look at the HMM scheme shows that up to the introduction of the micro problems in (3.4), we only transformed the bilinear form in an equivalent way. The idea for the formulation of the HMM parameters is similar, as the continuous local problems (2.5) are replaced by their discrete counterparts (3.8) in the definition of the effective parameters (2.4). This results in the following pointwise definitions of the HMM material parameters.

$$
\begin{aligned}
\mu^{\mathrm{HMM}}(x) &= \frac{1}{|Y^\delta(x)|}\int_{Y^\delta(x)} \big(I - D_y\chi_{\mu,h}(x,y)\big)^T \mu\big(x,\tfrac{y}{\eta}\big)\big(I - D_y\chi_{\mu,h}(x,y)\big)\,dy,\\
\varepsilon^{\mathrm{HMM}}(x) &= \frac{1}{|Y^\delta(x)|}\int_{Y^\delta(x)} \big(I - D_y\chi_{\varepsilon,h}(x,y)\big)^T \varepsilon\big(x,\tfrac{y}{\eta}\big)\big(I - D_y\chi_{\varepsilon,h}(x,y)\big)\,dy,
\end{aligned}
$$

where $\chi_{\mu,h}(x,\cdot),\ \chi_{\varepsilon,h}(x,\cdot) \in \mathcal{S}^k_{0/\#}(\mathcal{T}_h)^3$ are the solutions of the discrete local problems

$$
\begin{aligned}
&\int_{Y^\delta(x)} \big(I - D_y\chi_{\mu,h}(x,y)\big)^T \mu\big(x,\tfrac{y}{\eta}\big)\nabla_y v_h(y)\,dy = 0 \quad \forall v_h \in \mathcal{S}^k_{0/\#}(\mathcal{T}_h),\\
&\int_{Y^\delta(x)} \big(I - D_y\chi_{\varepsilon,h}(x,y)\big)^T \varepsilon\big(x,\tfrac{y}{\eta}\big)\nabla_y v_h(y)\,dy = 0 \quad \forall v_h \in \mathcal{S}^k_{0/\#}(\mathcal{T}_h),
\end{aligned}
\tag{3.8}
$$

for almost every $x \in \Omega$. These definitions allow an equivalent reformulation of (3.5) as

$$
\begin{aligned}
m^{\mathrm{HMM}}\left(\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}_H\\ \widetilde{\varphi}_H\end{pmatrix}\right) = \sum_{K,j} \omega_j^K \big(&\mu^{\mathrm{HMM}}(x_j^K)\widetilde{\psi}_H(x_j^K)\cdot\psi_H(x_j^K)\\
&+ \varepsilon^{\mathrm{HMM}}(x_j^K)\widetilde{\varphi}_H(x_j^K)\cdot\varphi_H(x_j^K)\big).
\end{aligned}
$$

Following an analogous approach as for the effective material parameters, one can show that the HMM parameters are again symmetric, uniformly positive definite, uniformly bounded and satisfy the same bounds as in (2.1). This yields the wellposedness of the HMM scheme (3.6). For details, we refer to [25, Lem. 6.4 and Thm. 6.6].

As final preparation for the error analysis in the following section, we now present bounds for the differences between the HMM parameters and the effective parameters. These bounds are based on the introduction of auxiliary parameters $\mu^{\mathrm{eff},\delta}$, $\varepsilon^{\mathrm{eff},\delta}$, which are defined equivalently to (2.4) (2.5) with $Y^\eta(x)$ and $W^{1,2}_{\#}(Y^\eta(x))$ replaced by $Y^\delta(x)$ and $W^{1,2}_{0/\#}(Y^\eta(x))$, respectively. For the definition of the space $W^{1,2}_{0/\#}(Y^\eta(x))$, we use the same boundary conditions as for $\mathcal{S}^k_{0/\#}(\mathcal{T}_h)$.

Based on this definition, we split the differences between the HMM parameters and the effective parameters at a quadrature node x_j^K

$$
\begin{aligned}
\|\mu^{\mathrm{HMM}}(x_j^K) - \mu^{\mathrm{eff}}(x_j^K)\|_F &\le \|\mu^{\mathrm{HMM}}(x_j^K) - \mu^{\mathrm{eff},\delta}(x_j^K)\|_F + \|\mu^{\mathrm{eff},\delta}(x_j^K) - \mu^{\mathrm{eff}}(x_j^K)\|_F,\\
\|\varepsilon^{\mathrm{HMM}}(x_j^K) - \varepsilon^{\mathrm{eff}}(x_j^K)\|_F &\le \|\varepsilon^{\mathrm{HMM}}(x_j^K) - \varepsilon^{\mathrm{eff},\delta}(x_j^K)\|_F + \|\varepsilon^{\mathrm{eff},\delta}(x_j^K) - \varepsilon^{\mathrm{eff}}(x_j^K)\|_F
\end{aligned}
\tag{3.9}
$$

where the first term arises due to the discretization of the local problems, whereas the second term covers the introduction of the sampling domains and if present the homogeneous Dirichlet boundary conditions.

For the first term, we present the following bound on the difference of the effective and the HMM parameters shown in [3, Cor. 5.3].

Lemma 3.4. *Let $k \in \mathbb{N}$, $\mathcal{S}^k_{0/\#}(\mathcal{T}_h)$ be the finite element space used to solve the micro problems (3.4) and assume that*

$$
|\chi_\mu|_{k+1,Y^\delta(x_j^K)}, |\chi_\varepsilon|_{k+1,Y^\delta(x_j^K)} \le C\eta^{-k}\sqrt{|Y^\delta(x_j^K)|}
\tag{3.10}
$$

holds for all quadrature nodes x_j^K of the mesh $\mathcal{T}_H$. Then we have the following estimates on all quadrature nodes x_j^K

$$\begin{aligned}\sup_{K,j}\|\mu^{\mathrm{HMM}}(x_j^K) - \mu^{\mathrm{eff},\delta}(x_j^K)\|_F &\le C\big(\tfrac{h}{\eta}\big)^{2k},\\ \sup_{K,j}\|\varepsilon^{\mathrm{HMM}}(x_j^K) - \varepsilon^{\mathrm{eff},\delta}(x_j^K)\|_F &\le C\big(\tfrac{h}{\eta}\big)^{2k}\end{aligned}\tag{3.11}$$

with the Frobenius norm $\|\cdot\|_F$ and a constant $C > 0$ independent of h and η.

As pointed out in [3, Remark 5.1], a sufficient condition for the case $k = 1$ to show (3.10) is that the multiscale material parameters are sufficiently smooth on every mesh element of the macro discretization, i.e.,

$$\mu^\eta|_K, \varepsilon^\eta|_K \in W^{1,\infty}(K)^{3\times 3}, \qquad |\mu^\eta|_{W^{1,\infty}(K)^{3\times3}}, |\varepsilon^\eta|_{W^{1,\infty}(K)^{3\times3}} \le C\eta^{-1},$$

for all $K \in \mathcal{T}_H$ with $C > 0$. For the case $k > 1$ a similar estimate based on higher order Sobolev spaces depending on the regularity of the multiscale material parameters is also indicated there.

To bound the second term of (3.9), which is the main contribution to the so-called modeling error, we cite results from [4, Lem. 4.8, Thm. 4.9].

LEMMA 3.5. *(a) If the local problems (3.4) are solved with periodic boundary values, we have $\frac{\delta}{\eta} \in \mathbb{N}$*

$$\sup_{K,j}\|\varepsilon^{\mathrm{eff},\delta}(x_j^K) - \varepsilon^{\mathrm{eff}}(x_j^K)\|_F = 0$$

for all $K \in \mathcal{T}_H$.
(b) If the local problems (3.4) are solved with homoeegeneous Dirichlet boundary conditions, we have for $\delta > \eta$

$$\sup_{K,j}\|\varepsilon^{\mathrm{eff},\delta}(x_j^K) - \varepsilon^{\mathrm{eff}}(x_j^K)\|_F \le C\Big(\frac{\eta}{\delta} + \delta\Big)$$

for all $K \in \mathcal{T}_H$ with a constant $C > 0$ independent of h and η.

Remark 3.6. Note that our framework is not limited to the discrete micro problems (3.8) presented above, e.g., one may also use different boundary conditions or different averaging methods to discretize (2.5). For further discussion of the resulting discretization errors, we refer to [29]. Another approach is presented in [7], where the authors suggest the usage of hyperbolic micro problems together with averaging kernels to improve the bounds for the modeling error.

4. Semidiscrete a priori error analysis. The following error analysis for the semidiscrete approximation (3.6) to the continuous problem (2.3) is based on the unified error analysis [17, Thm. 3.3]. Our analysis makes use of the following function spaces equipped with the corresponding inner products.

$$\begin{aligned} X &= L^2(\Omega)^6, & (\xi, \widetilde{\xi})_X &= m^{\mathrm{eff}}(\xi, \widetilde{\xi}),\\ V &= H(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega), & (\zeta, \widetilde{\zeta})_V &= (\zeta, \widetilde{\zeta})_{H(\mathrm{curl},\Omega)^2},\\ V_H &= V_H(\mathrm{curl}, \mathcal{T}_H) \times V_{H,0}(\mathrm{curl}, \mathcal{T}_H), & (\xi_H, \widetilde{\xi}_H)_{V_H} &= m^{\mathrm{HMM}}(\xi_H, \widetilde{\xi}_H).\end{aligned}$$

We further introduce the space $Z = W^{\ell+1,2}(\Omega)^6$ equipped with the standard norm. All these spaces are Hilbert spaces and the norms that are induced by the bilinear forms m^{eff} and m^{HMM} are equivalent to the standard $L^2(\Omega)^6$-norm. Hence for all $\xi \in X$ and all $\xi_H \in V_H$, we have

$$\sqrt{\lambda}\|\xi\|_{0,\Omega} \le \|\xi\|_X \quad \le \sqrt{\Lambda}\|\xi\|_{0,\Omega}, \tag{4.1a}$$

$$\sqrt{\lambda}\|\xi_H\|_{0,\Omega} \le \|\xi_H\|_{V_H} \le \sqrt{\Lambda}\|\xi_H\|_{0,\Omega}, \tag{4.1b}$$

as μ^{eff}, ε^{eff}, μ^{HMM} and ε^{HMM} all satisfy the bounds from (2.1).

In order to prove an error estimate for the HMM solution, we have to bound the errors in the bilinear forms

$$\Delta m : V_H \times V_H \to \mathbb{R}, \qquad \Delta m(\xi_H, \widetilde{\xi}_H) = |m^{\text{eff}}(\xi_H, \widetilde{\xi}_H) - m^{\text{HMM}}(\xi_H, \widetilde{\xi}_h)|,$$
$$\Delta s : V_H \times V_H \to \mathbb{R}, \qquad \Delta s(\xi_H, \widetilde{\xi}_H) = |s(\xi_H, \widetilde{\xi}_H) - s_H(\xi_H, \widetilde{\xi}_h)|,$$

which is done in the following lemma.

Lemma 4.1. *(a) Let assumption (3.10) be fulfilled and assume*

$$\mu^{\text{eff}}|_K, \varepsilon^{\text{eff}}|_K \in W^{\ell+1,\infty}(K)^{3\times 3}, \qquad \|\mu^{\text{eff}}\|_{\ell+1,\infty,K}, \|\varepsilon^{\text{eff}}\|_{\ell+1,\infty,K} \le \widetilde{C}, \tag{4.2}$$

for all $K \in \mathcal{T}_H$ with a constant $\widetilde{C} > 0$ independent of η and H. Then, we get for all $\xi \in Z$, $\xi_H \in V_H$

$$\Delta m(\xi, \xi_H) \le C\Big(H^\ell + \big(\tfrac{h}{\eta}\big)^{2k} + e_{\text{mod}}\Big)\|\xi\|_{\ell+1,\Omega}\|\xi_H\|_{0,\Omega},$$

with $e_{\text{mod}} = 0$ under the assumptions of Lemma 3.5(a) or $e_{\text{mod}} < \frac{\eta}{\delta} + \delta$ under the assumptions of Lemma 3.5(b).
(b) For all $\xi_H, \widetilde{\xi}_H \in V_H$, it holds

$$\Delta s(\xi_H, \widetilde{\xi}_H) = 0.$$

Remark 4.2. As the space Z is continuously embedded into $C(\Omega)$ for $\ell \ge 1$, we can extend m_H^{eff} and m^{HMM} to $Z \times Z$.

Proof of Lemma 4.1. As part (a) is more involved, we will first prove part (b). The estimate for Δs is trivial, since inserting the definitions of the bilinear forms yields for all $\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}_H\\ \widetilde{\varphi}_H\end{pmatrix} \in V_H$

$$\Delta s\left(\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}_H\\ \widetilde{\varphi}_H\end{pmatrix}\right) \le \sum_K \left|\int_K \widetilde{\varphi}_H(x)\cdot \operatorname{curl}\psi_H(x)\,dx - \sum_j \omega_j^K \widetilde{\varphi}_H(x_j^K)\cdot \operatorname{curl}\psi_H(x_j^K)\right|$$
$$+ \sum_K \left|\int_K \varphi_H(x)\cdot \operatorname{curl}\widetilde{\psi}_H(x)\,dx - \sum_j \omega_j^K \varphi_H(x_j^K)\cdot \operatorname{curl}\widetilde{\psi}_H(x_j^K)\right|,$$

which is just the sum of two quadrature errors. We assumed in (3.1) the exactness of the quadrature formula for polynomials in $\mathcal{Q}^{2\ell,2\ell,2\ell}$. Therefore these quadrature errors both vanish, which yields the result.

The proof of part (a) is not as simple, since the effective parameters and the HMM parameters appear in these differences. We start by splitting Δm into an HMM error a quadrature error and a modeling error.

$$\Delta m(\xi, \xi_H) \le \Delta m_{\text{HMM}}(\xi, \xi_H) + \Delta m_{\text{Quad}}(\xi, \xi_H) + e_{\text{mod}},$$

where

$$\Delta m_{\mathrm{HMM}}(\xi,\xi_H) = |m^{\mathrm{HMM}}(\xi,\xi_H) - m_H^{\mathrm{eff}}(\xi,\xi_H)|,$$
$$\Delta m_{\mathrm{Quad}}(\xi,\xi_H) = |m^{\mathrm{eff}}(\xi,\xi_H) - m_H^{\mathrm{eff}}(\xi,\xi_H)|,$$

for all $\xi = \begin{pmatrix}\psi\\ \varphi\end{pmatrix} \in Z$, $\xi_H = \begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix} \in V_H$.

The next step is to prove a bound for Δm_{HMM} using the estimates (3.11) for the HMM material parameters. Inserting the definitions of m^{HMM} and m_H^{eff} and using the triangle inequality yields

$$\begin{aligned}\Delta m_{\mathrm{HMM}}\left(\begin{pmatrix}\psi\\ \varphi\end{pmatrix}, \begin{pmatrix}\psi_H\\ \phi_H\end{pmatrix}\right) &\le C \sup_{\alpha\in\{\mu,\varepsilon\}} \sup_{K,j} \|\alpha^{\mathrm{eff}}(x_j^K) - \alpha^{\mathrm{HMM}}(x_j^K)\|_F \\ &\sum_K \left(\left| \sum_j \omega_j^K \zeta^\alpha(x_j^K)\cdot\zeta_H^\alpha(x_j^K) - \int_K \zeta^\alpha(x)\zeta_H^\alpha(x)\,dx \right| + \left| \int_K \zeta^\alpha(x)\zeta_H^\alpha(x)\,dx \right| \right),\end{aligned} \tag{4.3}$$

where we used the notation $\zeta^\mu = \psi$, $\zeta^\varepsilon = \varphi$. From [13, Thm. 4.1.5] we have the bound

$$\left| \sum_j \omega_j^K \zeta^\alpha(x_j^K)\cdot\zeta_H^\alpha(x_j^K) - \int_K \zeta^\alpha(x)\cdot\zeta_H^\alpha(x)\,dx \right| \le CH^{\ell+1}\|\zeta^\alpha\|_{\ell+1,K}\|\zeta_H^\alpha\|_{1,K} \tag{4.4}$$

and with the inverse inequality (cf. [13, Thm. 3.2.6]), this yields

$$\left| \sum_j \omega_j^K \zeta^\alpha(x_j^K)\cdot\zeta_H^\alpha(x_j^K) - \int_K \zeta^\alpha(x)\cdot\zeta_H^\alpha(x)\,dx \right| \le CH^{\ell}\|\zeta^\alpha\|_{\ell+1,K}\|\zeta_H^\alpha\|_{0,K}. \tag{4.5}$$

For the last term in (4.3), we use the Cauchy-Schwarz inequality to get

$$\left| \int_K \zeta^\alpha(x)\cdot\zeta_H^\alpha(x)\,dx \right| \le \|\zeta^\alpha\|_{0,K}\|\zeta_H^\alpha\|_{0,K}.$$

Inserting these results into (4.3) and using again the Cauchy-Schwarz inequality together with Lemma 3.4 and Lemma 3.5 yields the desired bound for Δm_{HMM}.

Finally, we treat the quadrature error Δm_{Quad} following [13, Thm. 4.1.4]. Using the triangle inequality, we get

$$\begin{aligned}\Delta m_{\mathrm{Quad}}(\xi,\xi_H) \le{}& \left| \int_\Omega \mu^{\mathrm{eff}}(x)\psi(x)\cdot\psi_H(x)\,dx - \sum_{K,j}\omega_j^K \mu^{\mathrm{eff}}(x_j^K)\psi(x_j^K)\cdot\psi_H(x_j^K) \right| \\ &+ \left| \int_\Omega \varepsilon^{\mathrm{eff}}(x)\varphi(x)\cdot\varphi_H(x)\,dx - \sum_{K,j}\omega_j^K \varepsilon^{\mathrm{eff}}(x_j^K)\varphi(x_j^K)\cdot\varphi_H(x_j^K) \right|.\end{aligned}$$

Application of [13, Thm. 4.1.5] and using the same techniques as in (4.4) and (4.5) then yields the result. □

Remark 4.3. Similar to the comment following Lemma 3.4, it is also possible to reduce the assumptions (4.2) to regularity assumptions for the multiscale parameters. This follows from the definition of the effective parameters (2.4), since these parameters only depend on the multiscale parameters and the solutions of the local problems.

As final preparation for the semidiscrete error bound, we have to prove the interpolation property from Theorem 3.1 in the norm of the discrete space V_H.

LEMMA 4.4. *For $\xi \in Z$ and the identity operator $\mathcal{I}$, the following estimate holds.*

$$\|(\mathcal{I}-\mathcal{I}_H)\xi\|_{V_H} \le CH^\ell |\xi|_{\ell+1,\Omega}.$$

Proof. The idea of this proof is to apply the norm equivalence (4.1b) and the interpolation property from Theorem 3.1. However, as $(\mathcal{I}-\mathcal{I}_H)\xi$ is in general not in V_H, we cannot apply (4.1b) directly.

As a workaround, we define an interpolation operator $\mathcal{I}_{V_H} : Z \to V_H(\mathrm{curl}, \mathcal{T}_H)^2$, such that for all $\xi \in Z,\ \xi_H \in V_H$

$$(\mathcal{I}_{V_H}\xi, \xi_H)_{V_H} = (\xi, \xi_H)_{V_H}$$

holds. Since the inner product on V_H contains only nodal evaluations at the quadrature points, $\mathcal{I}_{V_H}$ is the nodal interpolation at the quadrature points. As a direct consequence, we get

$$\|(\mathcal{I}-\mathcal{I}_{V_H})\xi\|_{V_H} = 0.$$

Together with the triangle inequality, this yields

$$\|(\mathcal{I}-\mathcal{I}_H)\xi\|_{V_H} \le \|(\mathcal{I}-\mathcal{I}_{V_H})\xi\|_{V_H} + \|(\mathcal{I}_{V_H}-\mathcal{I}_H)\xi\|_{V_H} = \|(\mathcal{I}_{V_H}-\mathcal{I}_H)\xi\|_{V_H}.$$

Using the norm equivalence (4.1b) and again the triangle inequality, we get

$$\|(\mathcal{I}-\mathcal{I}_H)\xi\|_{V_H} \le C\|(\mathcal{I}-\mathcal{I}_{V_H})\xi\|_{0,\Omega} + C\|(\mathcal{I}-\mathcal{I}_H)\xi\|_{0,\Omega}.$$

From [14, Sec. 3.6.2, Thm. 7] we get a bound for the first term.

$$\|(\mathcal{I}-\mathcal{I}_{V_H})\xi\|_{0,\Omega} \le CH^{\ell+1}|\xi|_{\ell+1,\Omega}.$$

Bounding the second term with Theorem 3.1 yields the result. ∎

With all preliminary results at hand, we now state the error estimate for the semidiscrete HMM method.

THEOREM 4.5. *Let $u^{\mathrm{eff}} \in C^1(0,T;Z)$ be the solution of (2.3) and let $u_H^{\mathrm{HMM}} \in L^\infty(0,T;V_H)$ be the solution of (3.6) at time $t \in (0,T)$. If (3.10) and (4.2) hold, we get the semidiscrete error estimate*

$$\begin{aligned}\|u_H^{\mathrm{HMM}}(t)-u^{\mathrm{eff}}(t)\|_{0,\Omega} \le C(1+t)\Big(&\|u_{0,H}-\mathcal{I}_H u_0\|_{0,\Omega} + \|\boldsymbol{J}_{\mathrm{ext},H}-\boldsymbol{J}_{\mathrm{ext}}\|_{L^\infty(0,t;L^2(\Omega)^3)} \\ &+ H^\ell\big(\|u^{\mathrm{eff}}\|_{L^\infty(0,t;Z)} + |\partial_t u^{\mathrm{eff}}|_{L^\infty(0,t;Z)}\big) + \Big(\big(\tfrac{h}{\eta}\big)^{2k} + e_{\mathrm{mod}}\Big)\|\partial_t u^{\mathrm{eff}}\|_{L^\infty(0,t;Z)}\Big),\end{aligned}$$

with $e_{\mathrm{mod}} = 0$ under the assumptions of Lemma 3.5(a) or $e_{\mathrm{mod}} < \frac{\eta}{\delta} + \delta$ under the assumptions of Lemma 3.5(b).

Proof. We use [17, Thm. 3.3] to prove this result. Interpretation of the HMM method as a non-conforming method then yields the following estimate.

(4.6)

$$\begin{aligned}\|u_H^{\mathrm{HMM}}(t)-u^{\mathrm{eff}}(t)\|_X \le C(1+t)\Big(&\|u_{0,H}-\mathcal{I}_H u_0^{\mathrm{eff}}\|_{V_H} + \|f_H^{\mathrm{HMM}}-\mathcal{P}_H f^{\mathrm{eff}}\|_{L^\infty(0,t;V_H)} \\ &+ \|(\mathcal{I}-\mathcal{I}_H)u^{\mathrm{eff}}\|_{L^\infty(0,t;V)} + \sup_{\tau\in[0,t]}\sup_{\|v_H\|_{V_H}=1} \Delta m(\mathcal{I}_H\partial_t u^{\mathrm{eff}}(\tau), v_H) \\ &+ \|(\mathcal{I}-\mathcal{I}_H)\partial_t u^{\mathrm{eff}}\|_{L^\infty(0,T;X)} + \sup_{\tau\in[0,t]}\sup_{\|v_H\|_{V_H}=1} \Delta s(\mathcal{I}_H u^{\mathrm{eff}}(\tau), v_H)\Big),\end{aligned}$$

with $\mathcal{P}_H : X \to V_H$, such that for all $\xi \in X,\ \xi_H \in V_H$

$$(\mathcal{P}_H \xi, \xi_H)_{V_H} = (\xi, \xi_H)_X \tag{4.7}$$

holds.

Now we bound the terms in (4.6) separately. From Theorem 3.1 we get

$$\sup_{\tau\in[0,t]} \|(\mathcal{I}-\mathcal{I}_H)u^{\text{eff}}(\tau)\|_V \le CH^\ell |u^{\text{eff}}|_{L^\infty(0,t;Z)},$$
$$\sup_{\tau\in[0,t]} \|(\mathcal{I}-\mathcal{I}_H)\partial_t u^{\text{eff}}(\tau)\|_X \le CH^\ell |\partial_t u^{\text{eff}}|_{L^\infty(0,t;Z)},$$

and Lemma 4.1(b) yields

$$\sup_{\tau\in[0,t]} \sup_{\|v_H\|_{V_H}=1} \Delta s(\mathcal{I}_H u^{\text{eff}}(\tau), v_H) = 0.$$

Finally, we have to bound Δm. For $v_H \in V_H$ the triangle inequality yields

$$\begin{aligned}
\Delta m(\mathcal{I}_H\partial_t u^{\text{eff}}(\tau), v_H) &= |m^{\text{eff}}(\partial_t u^{\text{eff}}(\tau), v_H) - m^{\text{HMM}}(\partial_t u^{\text{eff}}(\tau), v_H) \\
&\quad + m^{\text{eff}}(\mathcal{I}_H\partial_t u^{\text{eff}}(\tau), v_H) - m^{\text{eff}}(\partial_t u^{\text{eff}}(\tau), v_H) \\
&\quad + m^{\text{HMM}}(\partial_t u^{\text{eff}}(\tau), v_H) - m^{\text{HMM}}(\mathcal{I}_H\partial_t u^{\text{eff}}(\tau), v_H)| \\
&\le \Delta m(\partial_t u^{\text{eff}}(\tau), v_H) + |((\mathcal{I}-\mathcal{I}_H)\partial_t u^{\text{eff}}(\tau), v_H)_X| \\
&\quad + |((\mathcal{I}-\mathcal{I}_H)\partial_t u^{\text{eff}}(\tau), v_H)_{V_H}|.
\end{aligned}$$

Using Theorem 3.1 and Lemma 4.4 we further get

$$\begin{aligned}
&\sup_{\|v_H\|_{V_H}=1} \Delta m(\mathcal{I}_H\partial_t u^{\text{eff}}(\tau), v_H) \\
&\quad \le \sup_{\|v_H\|_{V_H}=1} \Delta m(\partial_t u^{\text{eff}}(\tau), v_H) + C\|(\mathcal{I}-\mathcal{I}_H)\partial_t u^{\text{eff}}(\tau)\|_X + \|(\mathcal{I}-\mathcal{I}_H)\partial_t u^{\text{eff}}(\tau)\|_{V_H} \\
&\quad \le \sup_{\|v_H\|_{V_H}=1} \Delta m(\partial_t u^{\text{eff}}(\tau), v_H) + C(H^\ell + H^{\ell+1})|\partial_t u^{\text{eff}}(\tau)|_{\ell+1,\Omega}.
\end{aligned}$$

We use Lemma 4.1(a) to get

$$\sup_{\tau\in[0,t]} \sup_{\|v_H\|_{V_H}=1} \Delta m(\partial_t u^{\text{eff}}(\tau), v_H) \le C\Big(\big(H^\ell + \tfrac{h}{\eta}\big)^{2k} + e_{\text{mod}}\Big)\|\partial_t u^{\text{eff}}\|_{L^\infty(0,t;Z)}.$$

Finally, we bound the error in the right-hand sides using (3.7), (4.7), (2.6) and (4.1b).

$$\begin{aligned}
\|f_H^{\text{HMM}} - \mathcal{P}_H f^{\text{eff}}\|_{V_H} &= \sup_{\|v_H\|_{V_H}=1} \big(m^{\text{HMM}}(f_H^{\text{HMM}}, v_H) - m^{\text{HMM}}(\mathcal{P}_H f^{\text{eff}}, v_H)\big) \\
&= \sup_{\|v_H\|_{V_H}=1} \big((\boldsymbol{J}_{\text{ext},H}, v_H)_{0,\Omega} - m^{\text{eff}}(f^{\text{eff}}, v_H)\big) \\
&\le \tfrac{1}{\sqrt{\lambda}}\|\boldsymbol{J}_{\text{ext},H} - \boldsymbol{J}_{\text{ext}}\|_{0,\Omega}
\end{aligned}$$

Inserting these results into (4.6) finishes this proof. ∎

Remark 4.6. Instead of continuous finite elements, it is also possible to use the discontinuous Galerkin (dG) approach presented in [22] to discretize the spatial domain Ω. We want to emphasize that for this setting an error estimate analogous to

Theorem 4.5 can be derived, as the difficulties arising from the dG approach affecting s_H and those from the HMM scheme affecting m^{HMM} are well separated. Indeed, the bilinear form s_H does not depend on the material parameters and can therefore be bounded as explained in [17, Chapter 3.2.2]. The bound for Δm follows as before, using the discrete spaces and the interpolation operator from the dG approach.

5. Full discretization. In this section we provide the analysis of the full discretization composed of the HMM scheme and different time integration methods. In the first part, we focus on algebraically stable Runge–Kutta methods, whereas we consider the Crank–Nicolson and the leapfrog method in the second part. Although the leapfrog method is an explicit scheme and therefore more commonly used in applications, we want to emphasize that implicit schemes like the algebraically stable Runge–Kutta methods or the Crank–Nicolson method can outperform explicit schemes not only in artificial examples, but also in applications. This was shown in [20], where the authors discuss the efficiency of several explicit and implicit Runge–Kutta schemes for Maxwell's equations.

5.1. Error analysis for algebraically stable Runge–Kutta methods. We consider implicit s-stage Runge–Kutta methods of order p with Runge–Kutta matrix $\mathcal{A} = (a_{ij})_{i,j=1}^s$, weights $b = (b_i)_{i=1}^s$, and nodes $c = (c_i)_{i=1}^s$. Moreover, we denote the time step by $\tau > 0$. We assume that the c_i's are pairwise distinct and satisfy $0 \leq c_i \leq 1$ for all i. Furthermore, we introduce the operators $\mathcal{C}_{\mathrm{HMM}} : V_H \to V_H$, $\mathcal{C}^{\boldsymbol{H}}_{\mathrm{HMM}} : V_H(\mathrm{curl}, \mathcal{T}_H) \to V_H(\mathrm{curl}, \mathcal{T}_H)$ and $\mathcal{C}^{\boldsymbol{E}}_{\mathrm{HMM}} : V_{H,0}(\mathrm{curl}, \mathcal{T}_H) \to V_{H,0}(\mathrm{curl}, \mathcal{T}_H)$ (cf. (3.2a), (3.2b)) via

$$\left(\mathcal{C}_{\mathrm{HMM}}\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}_H\\ \widetilde{\varphi}_H\end{pmatrix}\right)_{V_H} = \left(\begin{pmatrix}-\mathcal{C}^{\boldsymbol{E}}_{\mathrm{HMM}}\varphi_H\\ \mathcal{C}^{\boldsymbol{H}}_{\mathrm{HMM}}\psi_H\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}_H\\ \widetilde{\varphi}_H\end{pmatrix}\right)_{V_H} = s_H\left(\begin{pmatrix}\psi_H\\ \varphi_H\end{pmatrix}, \begin{pmatrix}\widetilde{\psi}_H\\ \widetilde{\varphi}_H\end{pmatrix}\right), \tag{5.1}$$

for all $\psi_H, \widetilde{\psi}_H \in V_H(\mathrm{curl}, \mathcal{T}_H)$ and $\varphi_H, \widetilde{\varphi}_H \in V_{H,0}(\mathrm{curl}, \mathcal{T}_H)$. Note that $\mathcal{C}_{\mathrm{HMM}}$ is skew-symmetric, since we have for all $v_H \in V_H$

$$(\mathcal{C}_{\mathrm{HMM}} v_H, v_H)_{V_H} = s_H(v_H, v_H) = 0.$$

With this definition, we can rewrite the evolution equation for the HMM for Maxwell's equation (3.6) as

$$\partial_t u_H^{\mathrm{HMM}}(t) = \mathcal{C}_{\mathrm{HMM}} u_H^{\mathrm{HMM}}(t) + f_H^{\mathrm{HMM}}(t). \tag{5.2}$$

Using the abbreviation $f_H^{ni} = f_H^{\mathrm{HMM}}(t_n + c_i\tau)$, the approximations $u_H^n = (\boldsymbol{H}_H^n, E_H^n)$ to the solution $u_H^{\mathrm{HMM}}(t_n) = (\boldsymbol{H}_H^{\mathrm{HMM}}(t_n), \boldsymbol{E}_H^{\mathrm{HMM}}(t_n))$ of the semidiscrete HMM problem (3.6) at time $t_n = n\tau$ are given by the following Runge–Kutta method

$$\dot{U}_H^{ni} = \mathcal{C}_{\mathrm{HMM}} U_H^{ni} + f_H^{ni}, \qquad i = 1, \dots, s, \tag{5.3a}$$

$$U_H^{ni} = u_H^n + \tau \sum_{j=1}^s a_{ij} \dot{U}_H^{nj}, \qquad i = 1, \dots, s, \tag{5.3b}$$

$$u_H^{n+1} = u_H^n + \tau \sum_{i=1}^s b_i \dot{U}_H^{ni} \quad \text{and} \quad u_H^0 = u_{0,H}. \tag{5.3c}$$

We consider only Runge–Kutta methods with the following properties

(5.4a) $p \geq s+1$,

(5.4b) $b_i > 0, i = 1, \dots, s$ and $(b_i a_{ij} + b_j a_{ji} - b_i b_j)_{i,j=1}^s$ is positive semidefinite.

(5.4c) There exist $\beta > 0$ and a diagonal, positive definite matrix $\mathcal{D}$, such that

$$(\mathcal{D}\mathcal{Q}^{-1}v) \cdot v \geq \beta(\mathcal{D}v) \cdot v, \qquad \text{for all } v \in \mathbb{R}^s.$$

Assumption (5.4b) means that the Runge–Kutta method is algebraically stable and assumption (5.4c) that it is coercive.

Let u^{eff} be the solution of the effective Maxwell's system (2.3). In order to prevent overloading the notation, we drop the superscript and simply write u. Using the notation

$$u^n = u(t_n), \qquad U^{ni} = u(t_n + c_i\tau), \qquad \dot{U}^{ni} = \partial_t u(t_n + c_i\tau),$$

with $n = 0, \dots, N$, $i = 1, \dots, s$, it is easy to see that these quantities solve the perturbed Runge–Kutta equations

$$\dot{U}^{ni} = \mathcal{C}_{\mathrm{eff}} U^{ni} + f^{ni}, \qquad i = 1, \dots, s, \tag{5.5a}$$

$$U^{ni} = u^n + \tau \sum_{j=1}^s a_{ij} \dot{U}^{nj} + \Delta^{ni}, \qquad i = 1, \dots, s, \tag{5.5b}$$

$$u^{n+1} = u^n + \tau \sum_{i=1}^s b_i \dot{U}^{ni} + \delta^{n+1} \quad \text{and} \quad u^0 = u_0. \tag{5.5c}$$

The operator $\mathcal{C}_{\mathrm{eff}} : V \to X$ is defined analogously to (5.1) by

$$(\mathcal{C}_{\mathrm{eff}} v_H, \widetilde{v}_H)_X = s(v_H, \widetilde{v}_H),$$

for all $v_H, \widetilde{v}_H \in V$. We further define $f^{ni} = f^{\mathrm{eff}}(t_n + c_i\tau)$ and the defects Δ^{ni}, δ^{n+1} are implicitly given by (5.5b) and (5.5c).

In order to study the error of the fully discrete scheme, we apply the interpolation operator $\mathcal{I}_H$ to (5.5) and subtract these equations from (5.3). With

$$e_H^n = u_H^n - \mathcal{I}_H u^n, \qquad E_H^{ni} = U_H^{ni} - \mathcal{I}_H U^{ni}, \qquad \dot{E}_H^{ni} = \dot{U}_H^{ni} - \mathcal{I}_H \dot{U}^{ni},$$

we find for the first equation

$$\dot{E}_H^{ni} = \mathcal{C}_{\mathrm{HMM}} U_H^{ni} - \mathcal{I}_H \mathcal{C}_{\mathrm{eff}} U^{ni} + f_H^{ni} - \mathcal{I}_H f^{ni}, \qquad i = 1, \dots, s.$$

Introduction of the operator $\mathcal{R}_H = \mathcal{C}_{\mathrm{HMM}} \mathcal{I}_H - \mathcal{P}_H \mathcal{C}_{\mathrm{eff}}$ (with $\mathcal{P}_H$ as defined in (4.7)) yields

$$\dot{E}_H^{ni} = \mathcal{C}_{\mathrm{HMM}} E_H^{ni} + \mathcal{R}_H U^{ni} + (\mathcal{P}_H - \mathcal{I}_H)\mathcal{C}_{\mathrm{eff}} U^{ni} + f_H^{ni} - \mathcal{I}_H f^{ni}.$$

Defining

$$g_H^{ni} = \mathcal{R}_H U^{ni} + (\mathcal{P}_H - \mathcal{I}_H)\mathcal{C}_{\mathrm{eff}} U^{ni} + f_H^{ni} - \mathcal{I}_H f^{ni}$$

yields the following system of equations for the error terms.

$$\dot{E}_H^{ni} = \mathcal{C}_{\mathrm{HMM}} E_H^{ni} + g_H^{ni}, \qquad i = 1 \dots s, \tag{5.6a}$$

$$E_H^{ni} = e_H^n + \tau \sum_{j=1}^s a_{ij} \dot{E}_H^{nj} - \mathcal{I}_H \Delta^{ni}, \qquad i = 1 \dots s, \tag{5.6b}$$

$$e_H^{n+1} = e_H^n + \tau \sum_{i=1}^s b_i \dot{E}_H^{ni} - \mathcal{I}_H \delta^{n+1} \quad \text{and} \quad e_H^0 = u_{0,H} - \mathcal{I}_H u_0. \tag{5.6c}$$

We present a stability estimate for (5.6) based on [19, Lem. 3.3 and Thm. 3.5].

THEOREM 5.1. *Let $u \in C^1(0,T;Z)$ be the solution of (2.3) and assume further $\partial_t^{s+1}u, \partial_t^{s+2}u \in L^2(0,T;W^{2,2}(\Omega))$. The approximations u_H^n to $u(t_n)$ $(n=1,\dots,N)$ obtained by application of (5.3) with step size $\tau>0$ sufficiently small satisfy*

$$\|e_H^n\|_{0,\Omega}^2 \le C\|e_H^0\|_{0,\Omega}^2 + C(1+T)\left(\tau^{2(s+1)}B(u,s,t_n)^2 + \tau\sum_{r=0}^{n-1}\sum_{i=1}^{s}\|g_H^{ri}\|_{V_H}^2\right), \tag{5.7}$$

with

$$B(u,s,t_*)^2 = \int_0^{t_*} \|\partial_t^{s+1}u(t)\|_{2,\Omega}^2 + \|\partial_t^{s+2}u(t)\|_{2,\Omega}^2\,dt.$$

Proof. Taking the V_H-inner product of (5.6c) with $e_H^{n+1}+e_H^n$ and using again (5.6c) on the right side of the equation yields

$$\begin{aligned}
\|e_H^{n+1}\|_{V_H}^2 - \|e_H^n\|_{V_H}^2 &= (\tau\sum_{i=1}^{s} b_i\dot E_H^{ni} - \mathcal{I}_H\delta^{n+1}, 2e_H^n + \tau\sum_{i=1}^{s} b_i\dot E_H^{ni} - \mathcal{I}_H\delta^{n+1})_{V_H}\\
&= \|\mathcal{I}_H\delta^{n+1}\|_{V_H}^2 - 2(\mathcal{I}_H\delta^{n+1}, e_H^n + \tau\sum_{i=1}^{s} b_i\dot E_H^{ni})_{V_H}\\
&\quad + 2\tau\sum_{i=1}^{s} b_i(\dot E_H^{ni}, e_H^n)_{V_H} + \tau^2\sum_{i,j=1}^{s} b_ib_j(\dot E_H^{ni}, \dot E_H^{nj})_{V_H}\\
&= \|\mathcal{I}_H\delta^{n+1}\|_{V_H}^2 - 2(\mathcal{I}_H\delta^{n+1}, e_H^n + \tau\sum_{i=1}^{s} b_i\dot E_H^{ni})_{V_H}\\
&\quad + 2\tau\sum_{i=1}^{s} b_i(\dot E_H^{ni}, E_H^{ni} + \mathcal{I}_H\Delta^{ni})_{V_H}\\
&\quad - \tau^2\sum_{i,j=1}^{s}(b_ia_{ij} + b_ja_{ji} - b_ib_j)(\dot E_H^{ni}, \dot E_H^{nj})_{V_H},
\end{aligned}$$

where we used (5.6b) to replace e_H^n and (5.4b). Since the method is algebraically stable, the sum on the last line is non-negative and we get the following estimate.

$$\begin{aligned}
\|e_H^{n+1}\|_{V_H}^2 - \|e_H^n\|_{V_H}^2 \le \|\mathcal{I}_H\delta^{n+1}\|_{V_H}^2 &- 2(\mathcal{I}_H\delta^{n+1}, e_H^n + \tau\sum_{i=1}^{s} b_i\dot E_H^{ni})_{V_H}\\
&+ 2\tau\sum_{i=1}^{s} b_i(\dot E_H^{ni}, E_H^{ni} + \mathcal{I}_H\Delta^{ni})_{V_H}.
\end{aligned} \tag{5.8}$$

We bound the remaining terms separately starting with the latest one. Using (5.6a) and Young's inequality with a constant $\gamma>0$ yields

$$\begin{aligned}
(\dot E_H^{ni}, E_H^{ni} + \mathcal{I}_H\Delta^{ni})_{V_H} &= (\mathcal{C}_{\mathrm{HMM}}E_H^{ni} + g_H^{ni}, E_H^{ni} + \mathcal{I}_H\Delta^{ni})_{V_H}\\
&= -(E_H^{ni}, \mathcal{C}_{\mathrm{HMM}}\mathcal{I}_H\Delta^{ni})_{V_H} + (g_H^{ni}, E_H^{ni})_{V_H} + (g_H^{ni}, \mathcal{I}_H\Delta^{ni})_{V_H}\\
&\le \gamma\|g_H^{ni}\|_{V_H}^2 + \tfrac{1}{\gamma}\|E_H^{ni}\|_{V_H}^2 + \tfrac{1}{2\gamma}\|\mathcal{I}_H\Delta^{ni}\|_{V_H}^2 + \tfrac{\gamma}{2}\|\mathcal{C}_{\mathrm{HMM}}\mathcal{I}_H\Delta^{ni}\|_{V_H}^2.
\end{aligned}$$

From [19, (3.18)] we further get

$$(\mathcal{I}_H \delta^{n+1}, e_H^n + \tau \sum_{i=1}^s b_i \dot{E}_H^{ni})_{V_H} \leq \tfrac{C}{\gamma} \tau \Big(\|e_H^n\|_{V_H}^2 + \sum_{i=1}^s \|E_H^{ni}\|_{V_H}^2 + \sum_{i=1}^s \|\mathcal{I}_H \Delta^{ni}\|_{V_H}^2 \Big) + \gamma\tau \|\tfrac{1}{\tau} \mathcal{I}_H \delta^{n+1}\|_{V_H}^2 .$$

Following the proof of [19, Lem. 3.4], we bound the inner stages E_H^{ni} with a constant $C = C(\mathcal{Q}, s, \mathcal{D}, \beta)$ as

$$\sum_{i=1}^s \|E_H^{ni}\|_{V_H}^2 \leq C \Big(\|e_H^n\|_{V_H}^2 + \sum_{i=1}^s \|\mathcal{I}_H \Delta^{ni}\|_{V_H}^2 + \tau^2 \sum_{i=1}^s \|g_H^{ni}\|_{V_H}^2 \Big).$$

We insert these bounds into (5.8) with $\gamma = 1 + T$.

$$\begin{aligned} \|e_H^{n+1}\|_{V_H}^2 - \|e_H^n\|_{V_H}^2 \leq \frac{C\tau}{1+T} \|e_H^n\|_{V_H}^2 + C(1+T)\tau \Big(\sum_{i=1}^s \|g_H^{ni}\|_{V_H}^2 \\ + \|\tfrac{1}{\tau} \mathcal{I}_H \delta^{n+1}\|_{V_H}^2 + \sum_{i=1}^s \|\mathcal{I}_H \Delta^{ni}\|_V^2 \Big). \end{aligned} \tag{5.9}$$

Using the interpolation property from Theorem 3.1, an inverse estimate and [19, (3.6) and (3.7)], we have

$$\tau \sum_{r=0}^n \Big(\|\tfrac{1}{\tau} \mathcal{I}_H \delta^{r+1}\|_{V_H}^2 + \sum_{i=1}^s \|\mathcal{I}_H \Delta^{ri}\|_V^2 \Big) \leq C \tau^{2(s+1)} B(u, s, t_{n+1})^2 .$$

Taking the sum over n in (5.9) and using this result yields

$$\begin{aligned} \|e_H^n\|_{V_H}^2 \leq \|e_H^0\|_{V_H}^2 + \frac{C\tau}{1+T} \sum_{r=0}^{n-1} \|e_H^r\|_{V_H}^2 + C(1+T)\tau^{2(s+1)} B(u, s, t_n)^2 \\ + C(1+T)\tau \sum_{r=0}^{n-1} \sum_{i=1}^s \|g_H^{ri}\|_{V_H}^2 . \end{aligned}$$

Finally, using the discrete Gronwall Lemma and the norm equivalence (4.1b) yields the result. ∎

In the next lemma, we bound the right-hand sides g_H^{ni}.

Lemma 5.2. *Let $u \in C^1(0, T; Z)$. For $n = 0, \dots, N-1$ and $i = 1, \dots, s$, we have*

$$\begin{aligned} \|g_H^{ni}\|_{V_H} \leq \|f_H^{ni} - \mathcal{P}_H f^{ni}\|_{V_H} + C \sup_{t \in [t_n, t_{n+1}]} \sup_{\|v_H\|_{V_H} = 1} \Delta m(\mathcal{I}_H \partial_t u(t), v_H) \\ + C \|(\mathcal{I} - \mathcal{I}_H) \partial_t u\|_{L^\infty(t_n, t_{n+1}; X)} + C \|(\mathcal{I} - \mathcal{I}_H) u\|_{L^\infty(t_n, t_{n+1}; V)} . \end{aligned} \tag{5.10}$$

Proof. By definition of g_H^{ni} and the triangle inequality, we have

$$\|g_H^{ni}\|_{V_H} \leq \|\mathcal{R}_H U^{ni}\|_{V_H} + \|(\mathcal{I}_H - \mathcal{P}_H) \dot{U}^{ni}\|_{V_H} + \|f_H^{ni} - \mathcal{P}_H f^{ni}\|_{V_H} . \tag{5.11}$$

We bound these terms separately. Following the proof of [17, Thm. 3.3], using Theorem 3.1 and Lemma 4.1(b), we get

$$\|\mathcal{R}_H u\|_{L^\infty(t_n, t_{n+1}; V_H)} \leq C \|(\mathcal{I} - \mathcal{I}_H) u\|_{L^\infty(t_n, t_{n+1}; V)} .$$

We can further bound the second term using [17, Lem. 2.11] for the special case $c_x = 1, Q_h^* = \mathcal{P}_H, J_H = \mathcal{I}_H$.

$$\begin{aligned}\|(\mathcal{I}_H - \mathcal{P}_H)\dot{U}^{ni}\|_{V_H} &\leq \|(\mathcal{I}_H - \mathcal{P}_H)\partial_t u\|_{L^\infty(t_n,t_{n+1};V_H)} \\ &\leq \sup_{t\in[t_n,t_{n+1}]} \sup_{\|v_H\|_{V_H}=1} \Delta m(\mathcal{I}_H \partial_t u(t), v_H) \\ &\quad + \|(\mathcal{I} - \mathcal{I}_H)\partial_t u\|_{L^\infty(t_n,t_{n+1};X)}.\end{aligned}$$

Using these bounds in (5.11) yields the result. ∎

Now we state one of our main results.

THEOREM 5.3. *Let $u \in C^1(0,T;Z)$ be the solution of the effective Maxwell's equations (2.3) and assume $\partial_t^{s+1}u, \partial_t^{s+2}u \in L^2(0,T;W^{2,2}(\Omega))$. The approximation u_H^n to $u(t_n)$ ($n = 1,\ldots,N$) obtained from (5.3) with step size $\tau > 0$ sufficiently small (depending on the Runge–Kutta method and T) satisfies the following bound.*

$$\begin{aligned}\|u_H^n - u(t_n)\|_{0,\Omega} &\leq C\|u_{0,H} - u(0)\|_{0,\Omega} + C \max_{\substack{r=0,\ldots,n-1\\ i=1,\ldots,s}} \|f_H^{ri} - \mathcal{P}_H f^{ri}\|_{V_H} + C\tau^{s+1}B(u,s,t_n) \\ &\quad + C\Big(\big(\tfrac{h}{\eta}\big)^{2k} + e_{\mathrm{mod}}\Big)\|\partial_t u\|_{L^\infty(0,t_n;Z)} + CH^\ell\big(\|u\|_{L^\infty(0,t_n;Z)} + \|\partial_t u\|_{L^\infty(0,t_n;Z)}\big),\end{aligned}$$

with $e_{\mathrm{mod}} = 0$ under the assumptions of Lemma 3.5(a) or $e_{\mathrm{mod}} < \frac{\eta}{\delta} + \delta$ under the assumptions of Lemma 3.5(b).

Proof. First, we split the error into

$$\|u_H^n - u(t_n)\|_{0,\Omega} \leq \|e_H^n\|_{0,\Omega} + \|(\mathcal{I}_H - \mathcal{I})u(t_n)\|_{0,\Omega}.$$

From Theorem 3.1 we get

$$\|(\mathcal{I}_H - \mathcal{I})u(t_n)\|_{0,\Omega} \leq CH^\ell |u(t_n)|_{\ell+1,\Omega} \leq CH^\ell |u|_{L^\infty(0,t_n;Z)}.$$

In order to bound e_H^n, we insert (5.10) into the stability estimate (5.7).

$$\begin{aligned}\|e_H^n\|_{0,\Omega}^2 \leq C\|e_H^0\|_{0,\Omega}^2 + C(1+T)\bigg(&\tau^{2(s+1)}B(u,s,t_n)^2 \\ &+ \tau\sum_{r=0}^{n-1}\sum_{i=1}^{s}\|f_H^{ni} - \mathcal{P}_H f^{ni}\|_{V_H}^2 + \|(\mathcal{I} - \mathcal{I}_H)\partial_t u\|_{L^\infty(0,t_n;X)}^2 \\ &+ \|(\mathcal{I} - \mathcal{I}_H)u\|_{L^\infty(0,t_n;V)}^2 + \sup_{t\in[0,t_n]}\sup_{\|v_H\|_{V_H}=1} \Delta m(\mathcal{I}_H\partial_t u(t), v_H)^2\bigg),\end{aligned}$$

where we used $\tau n \leq T$. Following the proof of Theorem 4.5, we use Theorem 3.1, Lemma 4.1(a) to show

$$\begin{aligned}\|e_H^n\|_{0,\Omega}^2 &\leq C\|e_H^0\|_{V_H}^2 + C(1+T)\bigg(\tau^{2(s+1)}B(u,s,t_n)^2 + C\max_{\substack{r=0,\ldots,n-1\\ i=1,\ldots,s}}\|f_H^{ri} - \mathcal{P}_H f^{ri}\|_{V_H}^2 \\ &\quad + CH^\ell\big(\|u\|_{L^\infty(0,t_n;Z)}^2 + \|\partial_t u\|_{L^\infty(0,t_n;Z)}^2\big) + C\Big(\big(\tfrac{h}{\eta}\big)^{2k} + e_{\mathrm{mod}}\Big)\|\partial_t u\|_{L^\infty(0,t_n;Z)}^2\bigg).\end{aligned}$$

Taking the square root finally yields the result. ∎

5.2. Error analysis for the Crank–Nicolson and the leapfrog method. After studying a class of Runge–Kutta methods, we now consider the Crank–Nicolson and the leapfrog method for time integration. The Crank–Nicolson method applied to (5.2) reads

$$\widetilde{u}_H^{n+1} = \widetilde{u}_H^n + \tfrac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}(\widetilde{u}_H^{n+1} + \widetilde{u}_H^n) + \tfrac{\tau}{2}(f_H^{n+1} + f_H^n), \tag{5.12}$$

with $\widetilde{u}_H^0 = u_{0,H}$, $f_H^n = f_H^{\mathrm{HMM}}(t_n)$. The leapfrog method is given by

$$\begin{aligned} \widehat{\boldsymbol{H}}_H^{n+1/2} &= \widehat{\boldsymbol{H}}_H^n - \tfrac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}}\widehat{\boldsymbol{E}}_H^n, \\ \widehat{\boldsymbol{E}}_H^{n+1} &= \widehat{\boldsymbol{E}}_H^n + \tau\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}}\widehat{\boldsymbol{H}}_H^{n+1/2} + \tfrac{\tau}{2}(\widehat{f}_H^{n+1} + \widehat{f}_H^n), \\ \widehat{\boldsymbol{H}}_H^{n+1} &= \widehat{\boldsymbol{H}}_H^{n+1/2} - \tfrac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}}\widehat{\boldsymbol{E}}_H^{n+1}, \end{aligned} \tag{5.13}$$

with $(\widehat{\boldsymbol{H}}_H^0, \widehat{\boldsymbol{E}}_H^0) = u_{0,H}$, $(0, \widehat{f}_H^n) = f_H^{\mathrm{HMM}}(t_n)$ using the operators from (5.1).

Following an approach from [22, Sec. 3.2], we rewrite the Crank–Nicolson method (5.12) in the following form.

$$\mathcal{R}_L\widetilde{u}_H^{n+1} = \mathcal{R}_R\widetilde{u}_H^n + \tfrac{\tau}{2}(f_H^{n+1} + f_H^n), \quad \text{and} \quad \widetilde{u}_H^0 = u_{0,H}, \tag{5.14}$$

with

$$\mathcal{R}_L = \begin{pmatrix} \mathcal{I} & \frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} \\ -\frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}} & \mathcal{I} \end{pmatrix}, \quad \mathcal{R}_R = \begin{pmatrix} \mathcal{I} & -\frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} \\ \frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}} & \mathcal{I} \end{pmatrix}.$$

Similarly, the leapfrog method (5.13) can be rewritten as

$$\widehat{\mathcal{R}}_L\widehat{u}_H^{n+1} = \widehat{\mathcal{R}}_R\widehat{u}_H^n + \tfrac{\tau}{2}(f_H^{n+1} + f_H^n), \quad \text{and} \quad \widehat{u}_H^0 = u_{0,H}, \tag{5.15}$$

with

$$\widehat{\mathcal{R}}_L = \begin{pmatrix} \mathcal{I} & \frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} \\ -\frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}} & \mathcal{I} - \frac{\tau^2}{4}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} \end{pmatrix}, \quad \widehat{\mathcal{R}}_R = \begin{pmatrix} \mathcal{I} & -\frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} \\ \frac{\tau}{2}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}} & \mathcal{I} - \frac{\tau^2}{4}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}}\mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} \end{pmatrix}.$$

In this way the leapfrog method can be interpreted as a perturbed Crank–Nicolson scheme. Since the leapfrog method is an explicit scheme, the time step $\tau > 0$ needs to satisfy the CFL-condition

$$\tau \le \frac{2\theta}{C_{\mathrm{CFL}}c_\infty} \min_{K\in\mathcal{T}_H} H_K \tag{5.16}$$

for the scheme to be stable. Here $0 < \theta < 1$ is arbitrary, but fixed. The constant $C_{\mathrm{CFL}} > 0$ depends on the mesh $\mathcal{T}_H$ and the polynomial degree ℓ and $c_\infty = \|\mu^{\mathrm{HMM}}\varepsilon^{\mathrm{HMM}}\|_{0,\infty,\Omega}$ is an upper bound for the speed of light within the material.

We now state our main result on the error of the full discretization for both schemes.

THEOREM 5.4. *Let $u \in C(0,T;Z) \cap C^3(0,T;X)$ be the solution of the effective Maxwell's equations (2.3).*

(a) For the Crank–Nicolson approximation $\widetilde{u}_H^n$ to $u(t_n)$ $(n = 1,\dots,N)$ obtained from (5.14) with step size $\tau > 0$, we have the following bound.

$$\begin{aligned} \|\widetilde{u}_H^n - u(t_n)\|_{0,\Omega} &\le C\|u_{0,H} - u(0)\|_{0,\Omega} + C \max_{r=0,\dots,n-1}\|f_H^r - \mathcal{P}_H f^r\|_{V_H} \\ &\quad + C\tau^2\|\partial_t^3 u\|_{L^\infty(t_n,t_{n+1};L^2(\Omega)^6)} + C\Big(\big(\tfrac{h}{\eta}\big)^{2k} + e_{\mathrm{mod}}\Big)\|\partial_t u\|_{L^\infty(0,t_n;Z)} \\ &\quad + CH^\ell\big(\|u\|_{L^\infty(0,t_n;Z)} + \|\partial_t u\|_{L^\infty(0,t_n;Z)}\big), \end{aligned}$$

with $e_{\mathrm{mod}} = 0$ under the assumptions of Lemma 3.5(a) or $e_{\mathrm{mod}} < \frac{\eta}{\delta} + \delta$ under the assumptions of Lemma 3.5(b).

(b) *Let further $0 < \theta < 1$ and the CFL condition (5.16) be satisfied. The leapfrog approximation $\widehat{u}_H^n$ to $u(t_n)$ $(n = 1, \dots, N)$ obtained from (5.15) with step size $\tau > 0$ satisfies the following bound.*

$$\begin{aligned}\|\widehat{u}_H^n - u(t_n)\|_{0,\Omega} \le C\|u_{0,H} - u(0)\|_{0,\Omega} &+ C \max_{r=0,\dots,n-1} \|f_H^r - \mathcal{P}_H f^r\|_{V_H} \\ &+ C\tau^2\big(\|\partial_t^2 u\|_{L^\infty(t_n,t_{n+1};L^2(\Omega)^6)} + \|\partial_t^3 u\|_{L^\infty(t_n,t_{n+1};L^2(\Omega)^6)}\big) \\ &+ C\Big(\big(\tfrac{h}{\eta}\big)^{2k} + e_{\mathrm{mod}}\Big)\|\partial_t u\|_{L^\infty(0,t_n;Z)} \\ &+ CH^\ell\big(\|u\|_{L^\infty(0,t_n;Z)} + \|\partial_t u\|_{L^\infty(0,t_n;Z)}\big),\end{aligned}$$

with $e_{\mathrm{mod}} = 0$ under the assumptions of Lemma 3.5(a) or $e_{\mathrm{mod}} < \frac{\eta}{\delta} + \delta$ under the assumptions of Lemma 3.5(b).

Proof. (a) As in the proof of Theorem 5.3, we first split the error.

$$\|\widetilde{u}_H^n - u(t_n)\|_{0,\Omega} \le \|\widetilde{e}_H^n\|_{0,\Omega} + CH^\ell |u|_{L^\infty(0,t_n;Z)},$$

with $\widetilde{e}_H^n = \widetilde{u}_H^n - \mathcal{I}_H u(t_n)$. Following the proof of [22, Lem. 5.1], one can show that $\widetilde{e}_H^n$ satisfies a perturbed version of (5.14), namely

$$\mathcal{R}_L \widetilde{e}_H^{n+1} = \mathcal{R}_R \widetilde{e}_H^n + d^n, \tag{5.17}$$

where the defect d^n is given by

$$\begin{aligned} d^n = \tfrac{\tau}{2}(\mathcal{P}_H - \mathcal{I}_H)(u^{n+1} - u^n) &- \tfrac{\tau}{2}\mathcal{R}_H(u^{n+1} + u^n) \\ &+ \tfrac{\tau}{2}(f_H^{n+1} - \mathcal{P}_H f^{n+1} + f_H^n - \mathcal{P}_H f^n) + \tau^2 \mathcal{P}_H \delta^n \end{aligned}$$

and $\delta^n = (\delta_H^n, \delta_E^n)^T$ is defined via

$$\delta_{\boldsymbol{U}}^n = \int_{t_n}^{t_{n+1}} \frac{(t - t_n)(t_{n+1} - t)}{2\tau^2} \partial_t^3 \boldsymbol{U}(t)\,dt, \qquad \boldsymbol{U} = \boldsymbol{H}, \boldsymbol{E}.$$

Using (5.17) and taking the sum over n yields

$$\widetilde{e}_H^{n+1} = \mathcal{R}^{n+1}\widetilde{e}_H^0 + \sum_{m=0}^{n} \mathcal{R}^{n-m}\mathcal{R}_L^{-1} d^m,$$

with $\mathcal{R} = \mathcal{R}_L^{-1}\mathcal{R}_R$. From the triangle inequality and [22, Lem. 4.1] and [22, Lem. 4.2], we get

$$\|\widetilde{e}_H^{n+1}\|_{0,\Omega} \le C\|\widetilde{e}_H^0\|_{0,\Omega} + \sum_{m=0}^{n} \|d^m\|_{0,\Omega}. \tag{5.18}$$

As in the proof of Lemma 5.2, we then bound the defect d^n.

$$\begin{aligned}\tfrac{1}{\tau}\|d^n\|_{0,\Omega} \le \sup_{t\in[t_n,t_{n+1}]} \sup_{\|v_H\|_{V_H}=1} &\Delta m(\mathcal{I}_H \partial_t u(t), v_H) + C\|(\mathcal{I} - \mathcal{I}_H)\partial_t u\|_{L^\infty(t_n,t_{n+1};X)} \\ &+ C\|(\mathcal{I} - \mathcal{I}_H)u\|_{L^\infty(t_n,t_{n+1};V)} + C \max_{r=0,\dots,n-1} \|f_H^r - \mathcal{P}_H f^r\|_{V_H} \\ &+ C\tau^2 \|\partial_t^3 u\|_{L^\infty(t_n,t_{n+1};X)}.\end{aligned}$$

Inserting this result into (5.18) and using Lemma 4.1(a) to bound Δm yields the result.

(b) We also split the error of the leapfrog scheme.

$$\|\widehat{u}_H^n - u(t_n)\|_{0,\Omega} \leq \|\widehat{e}_H^n\|_{0,\Omega} + CH^\ell |u|_{L^\infty(0,t_n;Z)},$$

with $\widehat{e}_H^n = \widehat{u}_H^n - \mathcal{I}_H u(t_n)$. Again, one can show that $\widehat{e}_H^n$ satisfies a perturbed version of (5.15), namely

$$\widehat{\mathcal{R}}_L \widehat{e}_H^{n+1} = \widehat{\mathcal{R}}_R \widehat{e}_H^n + \widehat{d}^n,$$

with a perturbed defect $\widehat{d}^n$ given by

$$\widehat{d}^n = d^n - \frac{\tau^2}{4} \begin{pmatrix} 0 \\ \mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}} \mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} (\widehat{e}_{\pi,E}^{n+1} - \widehat{e}_{\pi,E}^n) + \mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{H}} \Big(\frac{\mu^{\mathrm{eff}}}{\mu^{\mathrm{HMM}}} (\partial_t H^{n+1} - \partial_t H^n) \Big) \end{pmatrix}.$$

However, the same approach applied to the leapfrog method does not yield the expected orders of convergence. Instead, the defect has to be investigated more carefully, as proposed in [22, Sec. 5.1]. The main idea is to split the defect $\widehat{d}^n$ into

$$\widehat{d}^n = d^n + (\widehat{\mathcal{R}}_L - \widehat{\mathcal{R}}_R)\xi^n,$$

with

$$\xi^n = \frac{\tau}{4} \begin{pmatrix} \mathcal{C}_{\mathrm{HMM}}^{\boldsymbol{E}} (\widehat{e}_{\pi,E}^{n+1} - \widehat{e}_{\pi,E}^n) + \frac{\mu^{\mathrm{eff}}}{\mu^{\mathrm{HMM}}} (\partial_t \boldsymbol{H}^{n+1} - \partial_t \boldsymbol{H}^n) \\ 0 \end{pmatrix}.$$

In this way, we get from (5.14)

$$\widehat{e}_H^{n+1} = \xi^n - \widehat{\mathcal{R}}^{n+1}\xi^0 + \sum_{m=0}^{n} \widehat{\mathcal{R}}^{n-m} \widehat{\mathcal{R}}_L^{-1} d^m - \sum_{m=0}^{n-1} \widehat{\mathcal{R}}^{n-m} \widehat{\mathcal{R}}_L^{-1} (\xi^{m+1} - \xi^m),$$

with $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_L^{-1} \widehat{\mathcal{R}}_R$. Using again the triangle inequality, [22, Lem. 4.1] and [22, Lem. 4.2], this yields

$$\|\widehat{e}_H^{n+1}\|_{0,\Omega} \leq \|\xi^n\|_{0,\Omega} + \|\xi^0\|_{0,\Omega} + \sum_{m=0}^{n} \|d^m\|_{0,\Omega} + \sum_{m=0}^{n-1} \|\xi^{m+1} - \xi^m\|_{0,\Omega}.$$

For ξ^n, we have the following bound (analogously for ξ^0).

$$\|\xi^n\|_{0,\Omega} \leq C\tau H^\ell \|\boldsymbol{E}\|_{L^\infty(t_n,t_{n+1};W^{\ell+1,2}(\Omega)^3)} + C\tau^2 \|\partial_t^2 \boldsymbol{H}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega)^3)}.$$

The differences can be bounded via

$$\|\xi^{m+1} - \xi^m\|_{0,\Omega} \leq C\tau H^\ell \|\boldsymbol{E}\|_{L^\infty(t_m,t_{m+2};W^{\ell+1,2}(\Omega)^3)} + C\tau^3 \|\partial_t^3 \boldsymbol{H}\|_{L^\infty(t_m,t_{m+2};L^2(\Omega)^3)}.$$

Using $n\tau \leq T$ and the fact that we have already shown a bound for the sum over d^m yields the fully discrete error estimate for the leapfrog method. $\square$

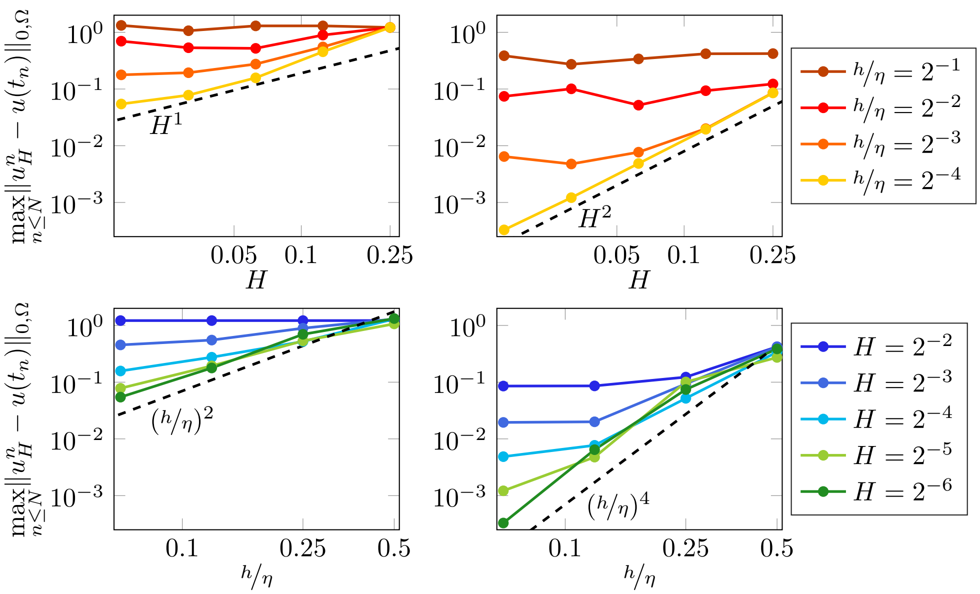

FIG. 6.1. *Maximal difference between the solution of the effective system (2.3) and the HMM-system (3.6) over all time steps. The computations were done with first order elements (left) and second order elements (right).*

6. Numerical experiments. We use the finite element library `deal.II` [8] for the implementation of the numerical schemes discussed in the previous sections.[1] Let the computational domain $\Omega = [0,1]^3$ be triangulated into uniform hexahedral meshes $\mathcal{T}_H$ of different mesh widths H. The electric permittivity and the magnetic permeability are given by

$$\mu^\eta(x) = \varepsilon^\eta(x) = \Big(\sqrt{2} + \sin\big(2\pi\tfrac{x_1}{\eta}\big)\Big)\Big(\sqrt{2} + \sin\big(2\pi\tfrac{x_2}{\eta}\big)\Big)\Big(\sqrt{2} + \sin\big(2\pi\tfrac{x_3}{\eta}\big)\Big),$$

with $\eta = 2^{-6}$. The corresponding effective parameters are $\mu^{\text{eff}} = \varepsilon^{\text{eff}} = 1$ (cf. [23]). Using a vanishing source term $\boldsymbol{J}_{\text{ext}} = 0$, the exact solutions of the effective Maxwell's equations (2.3) are given by

$$\boldsymbol{H}^{\text{eff}}(x,t) = \frac{1}{2\sqrt{3}}\begin{pmatrix} -4\sin(2\pi x_1)\cos(2\pi x_2)\cos(2\pi x_3)\sin(2\sqrt{3}\pi t) \\ -2\cos(2\pi x_1)\sin(2\pi x_2)\cos(2\pi x_3)\sin(2\sqrt{3}\pi t) \\ 5\cos(2\pi x_1)\cos(2\pi x_2)\sin(2\pi x_3)\sin(2\sqrt{3}\pi t) \end{pmatrix},$$

$$\boldsymbol{E}^{\text{eff}}(x,t) = \frac{1}{2}\begin{pmatrix} 2\cos(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3)\cos(2\sqrt{3}\pi t) \\ -3\sin(2\pi x_1)\cos(2\pi x_2)\sin(2\pi x_3)\cos(2\sqrt{3}\pi t) \\ 1\sin(2\pi x_1)\sin(2\pi x_2)\cos(2\pi x_3)\cos(2\sqrt{3}\pi t) \end{pmatrix}.$$

For the time integration, we use the leapfrog method with step size $\tau = 0.0025$ and final time $T = 1$.

Figure 6.1 shows the maximal $L^2(\Omega)$-error between u^{eff} and u_H^{HMM} for $\delta = \eta$ with periodic boundary conditions for the local problems. The left column was computed with linear elements, whereas the right column is based on quadratic elements. The upper row shows the errors over the macro mesh width H for different micro mesh

[1] Our source code is accessible under https://www.waves.kit.edu/hmm-maxwell.php.

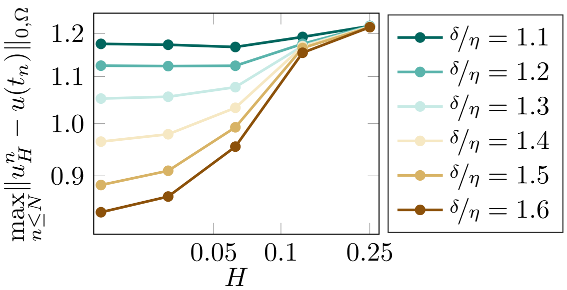

FIG. 6.2. *Maximal difference between the solution of the effective system* (2.3) *and the HMM-system* (3.6) *over all time steps for varying edge length* δ *of the sampling domain.*

widths h. The bottom row shows the same errors, but plotted over the micro mesh width h for different macro mesh widths H. As expected, we see first order convergence in H and second order convergence in h for linear elements. For quadratic elements, we get second order convergence in H and fourth order convergence in h.

For the use of homogeneous Dirichlet boundary conditions for the local problems with varying edge length δ of the sampling domain, we show in Figure 6.2 again the maximal $L^2(\Omega)$-error between u^{eff} and u_H^{HMM} over the macro mesh width H. We compute these results for the same material parameters as before but with $\eta = 2^{-12}$, using first order elements both in the macro and micro discretization with fixed micro mesh width $h/\eta = 0.05$ and time step size $\tau = 0.005$. As predicted, we observe that the error declines for δ increasing.

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