Finite element error analysis of wave equations with dynamic boundary conditions: $L^2$ estimates

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Finite element error analysis of wave equations with dynamic boundary conditions: $L^2$ estimates

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$L^2$ norm error estimates of semi- and full discretisations, using bulk–surface finite elements and Runge–Kutta methods, of wave equations with dynamic boundary conditions are studied. The analysis resides on an abstract formulation and error estimates, via energy techniques, within this abstract setting. Four prototypical linear wave equations with dynamic boundary conditions are analysed which fit into the abstract framework. For problems with velocity terms, or with acoustic boundary conditions we prove surprising results: for such problems the spatial convergence order is shown to be less than two. These can also be observed in the presented numerical experiments.

Keywords: wave equations, dynamic boundary conditions, abstract error analysis, Ritz map, $L^2$ error estimates, Runge–Kutta methods.

1. Introduction

In this paper we study the $L^2$ error of semi- and full discretisations of wave equations with dynamic boundary conditions using bulk–surface finite elements and Gauss–Runge–Kutta methods.

Dynamic boundary conditions can account for the momentum of the wave on the boundary and, in particular, for tangential wave propagation along the boundary. As tangential wave propagation is inherently modelled on (piecewise) smooth boundaries, triangulations of the domains are possibly not exact. Therefore, finite element discretisations can become non-conforming which makes the error analysis more involved. This paper considers four prototypical examples for the class of linear wave-type problems with dynamic boundary conditions: a simple model problem with only second-order terms, problems with advective terms, problems with strong damping, and problems with acoustic boundary conditions. Albeit stating our main results for these four examples, the main part of our error analysis is done in an abstract setting, and can thus be applied to all linear second order wave equations fitting into this setting.

The modelling and analysis of wave equations with dynamic boundary conditions is an intensively researched field. Initially, dynamic boundary conditions for wave equations appeared in models of
vibrating elastic rods or beams with tip masses attached at their free ends, cf. Andrews et al. (1996) and references therein. However, the first derivation of dynamic boundary conditions, as considered in this paper, was given in Goldstein (2006) which also comments on their relation to Wentzell boundary conditions. This seminal work was recently complemented by Figotin & Reyes (2015) which presents a systematic approach to derive (dynamic) boundary conditions for conservative systems via a Lagrangian framework. Moreover, the analysis of such problems is quite developed. Let us mention here Vitillaro (2013) and Vitillaro (2017) where well-posedness of wave equations with (non-linear) dynamic boundary conditions is shown, Graber & Lasiecka (2014) which studies the regularity of problems with strong boundary damping, and Gal & Tebou (2017) which proves the Carleman inequality. Another important category are acoustic boundary conditions which arise in models for wave–structure interactions. First proposed in Beale & Rosencrans (1974), they continue to be a topic of intensive mathematical and physical research, see for example Gal et al. (2003), Mugnolo (2006), or Frota et al. (2011) for a non-linear version, as well as Vedurmudi et al. (2016).

Despite the long history of wave equations and, more general, partial differential equations (PDEs) with dynamic boundary conditions, the error analysis of their numerical approximations has mainly been developed during the last few years. Elliott & Ranner (2013) was the first paper to address the non-conformity of finite element approximations for bulk–surface PDEs in curved domains. It proposes and analyses an isoparametric bulk–surface finite element method for an elliptic coupled bulk–surface problem. Finite elements (and non-uniform rational B-splines for the approximation of curved domains) for elliptic problems with dynamic boundary conditions have been analysed in Kashiwabara et al. (2015). Although Fairweather (1979) already gave error estimates for (conforming) Galerkin methods for linear parabolic problems, it went unnoticed in the dynamic boundary conditions community, possibly due to the fact that the term dynamic has not appeared at all in his paper. We refer to Kovács & Lubich (2017) for a more complete numerical analysis of parabolic problems with dynamic boundary conditions, including surface differential operators, semi-linear problems and time integration. As for hyperbolic equations, Lescarret & Zuazua (2015) studies the numerical approximation of the special case of wave equations in two asymmetric half-spaces divided by a “wavy” surface. The first convergence estimates for general wave equations with dynamic boundary conditions and isoparametric finite element discretisations thereof were shown in Hipp (2017). These energy norm (\(H^1\)) estimates are derived using the unified theory for (possibly) non-conforming semi-discretisations of wave-type equations presented in Hipp et al. (2018). Apart from these two works, we are not aware of papers studying the numerical errors for general wave equations with dynamic boundary conditions.

In this paper we present \(L^2\) convergence rates for finite element approximations and for full discretisations with Gauss–Runge–Kutta methods of wave equations with dynamic boundary conditions by combining the ideas of Kovács & Lubich (2017) and Hipp et al. (2018), with those of Mansour (2015); Hochbruck & Pažur (2015). Our approach is based on energy techniques and an abstract formulation of second-order wave equations and their spatial semi-discretisations. It can be outlined as follows: Via energy estimates, we first reproduce a stability estimate in a weak norm for the continuous problem from Hipp (2017). These weak norm estimates entail a \(L^2\) norm stability result. For the \(L^2\) error analysis, we therefore derive an analogous stability estimate for the abstract semi-discrete problem in discrete weak norms. Then, using the abstract Ritz map from Kovács & Lubich (2017), we show an error estimate in terms of errors in the initial value and the semi-discrete defect, and further prove that the latter is bounded by geometric (in the abstract setting conformity, cf. Hipp (2017)) and approximation (i.e. interpolation and Ritz map) error estimates. Up to this point, the analysis does not use any specific information on the particular terms of the bilinear forms and, in particular, the boundary conditions. Finally, we obtain \(L^2\) convergence rates for each example separately, by studying the different error terms,
using properties of the wave equation and (geometric, interpolation and Ritz map) approximation results for the bulk–surface finite element method.

Surprisingly, the geometric approximation errors for the terms involving the velocity, do not allow optimal-order convergence rates for all cases. In Table 1 we collect the obtained error estimates for the spatial semi-discretisation. We also marked whether these results are illustrated by numerical experiments.

We expect that further interesting problems, such as wave equations with new types of dynamic boundary conditions, or with time- and space-dependent coefficients, as well as semi-linear problems can be treated within this setting (or slight modifications of it) using the presented techniques, subject to the error analysis of the mentioned geometric and approximation errors.

Since the matrix–vector formulation of these second order problems coincides with the ODE system for wave problems with standard boundary conditions, the convergence proofs for the full discretisation are straightforward. Some parts have already been covered in the literature, only the $L^2$ norm requires some simple modifications. We give these details, but for those parts which are not new we only give detailed references, following Mansour (2015); Hochbruck & Pažur (2015); Hochbruck et al. (2018); Kovács & Lubich (2018). We strongly believe, and the previous references also strengthen, that these techniques extend to time discretisations of more general, e.g. semi- or quasi-linear, problems.

The paper is organised as follows. In Section 2 we introduce the abstract framework, its assumptions, norms and bilinear forms. The second part of this section focuses on the four exemplary wave equations with dynamic boundary conditions considered in this paper. We then show how they fit into the abstract framework, i.e. we give the suitable Hilbert spaces, the bilinear forms and the classical conditions on the coefficients that ensure the abstract assumptions to be satisfied, and also guarantee well-posedness. Section 3 describes the bulk–surface finite element method and the strategy for dealing with the approximation of a smooth domain with possibly curved boundary. Section 4 presents the main results of this paper. It is split into four parts, one for each example from Section 2, giving the semi-discrete formulation, and the semi-discrete error estimates in the $L^2$ norm. The central part of our error analysis is done in Section 5. We use energy estimates to show the stability results. The semi-discrete stability bound is then applied to the error equation, the appearing defect is shown to be bounded by geometric and approximation errors (i.e. errors in the interpolation and the Ritz map). Section 6 is again split into four parts, and is devoted to bounding the errors from the abstract result. Here we give the proofs of the theorems of Section 4. In Section 7 we turn to time discretisations. Stability estimates and approximation results of standard time integrators, such as Gauss–Runge–Kutta methods, translate from standard wave equations (with usual boundary conditions) to the case of wave equations with dynamic

<table>
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<tr>
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<td>acoustic</td>
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<td>Theorem 4.4</td>
<td>✓</td>
</tr>
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Table 1. Overview of $L^2$ convergence rates for linear finite elements with mesh width $h$ shown in this article.
boundary conditions. The required modifications, compared to the literature, are presented in detail.

In Section 8 we present various numerical experiments – to all four problems – illustrating our theoretical results. The fractional convergence rates are observed in the presented numerical experiments, cf. the last column of Table 1.

2. Analysis of wave equations with dynamic boundary conditions

In this section, we present an abstract setting for wave equations, similar to the ones in Kovács & Lubich (2017) and Hipp et al. (2018), and then consider different examples of wave equations with dynamic boundary conditions fitting into this abstract framework.

2.1 Abstract framework

Let $V$ and $H$ two real Hilbert spaces with norms $\| \cdot \|_V$ and $| \cdot |$, the latter norm induced by the inner product $m(\cdot, \cdot)$ on $H$, such that $V$ is densely and continuously embedded in $H$ (i.e. $|u| \leq c \|u\|_V$). Furthermore, we identify $H$ with its dual $H^*$ which defines the Gelfand triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$.

As a consequence of this identification, the duality $\langle \cdot, \cdot \rangle_V: V^* \times V \rightarrow \mathbb{R}$ coincides with $m(\cdot, \cdot)$ on $H \times V$.

The general abstract wave equation, which covers all examples in this paper, reads: Find $u: [0, T] \rightarrow V$ such that

$$\langle \ddot{u}(t), v \rangle_V + b(\dot{u}(t), v) + a(u(t), v) = \langle f(t), v \rangle_V, \quad \forall v \in V, \quad (2.1a)$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1, \quad (2.1b)$$

where $f: [0, T] \rightarrow V^*$ is a given function, and where $a: V \times V \rightarrow \mathbb{R}$ and $b: V \times V \rightarrow \mathbb{R}$ are continuous bilinear forms such that $b + \rho m$ is monotone for some $\rho \geq 0$, i.e.,

$$b(v, v) + \rho |v|^2 \geq 0 \quad \text{for every} \quad v \in V, \quad (2.2)$$

and $a$ is coercive with an $\alpha > 0$:

$$a(v, v) \geq \alpha \|v\|^2_V \quad \text{for every} \quad v \in V. \quad (2.3)$$

The above variational equation (2.1) can be written as the evolution equation in $V^*$

$$\ddot{u}(t) + B\dot{u}(t) + Au(t) = f(t), \quad (2.4)$$

where $A, B \in \mathcal{L}(V, V^*)$ are induced by the bilinear forms $a$ and $b$ via

$$\langle Aw, v \rangle_V = a(w, v), \quad \text{and} \quad \langle Bw, v \rangle_V = b(w, v), \quad w, v \in V. \quad (2.5)$$

Note that, due to our assumptions, $A$ is an isomorphism by the Lax–Milgram theorem and $a$ is an inner product on $V$ such that

$$\|v\|^2 = a(v, v)$$
defines an equivalent norm, satisfying
\[ \sqrt{\alpha} \|v\|_V \leq \|v\| \leq \|A\|^{1/2}_{V', V} \|v\|_V, \quad v \in V. \]

The abstract wave equation (2.1) is well-posed in different settings. The following theorem collects a weak and a strong well-posedness result which are shown by using semigroup theory and, for the weak result, the theory of Sobolev towers. For the proof, we refer to (Hipp, 2017, Theorem 4.3 and 4.13) and note that the strong result is shown in Showalter (1994).

**Theorem 2.1** Let the above assumptions be fulfilled and let the initial values \( u_0 \in V, u_1 \in H \) and source term satisfy \( f \in C^1([0, T]; V^*) + C([0, T]; H) \). Then there exists a unique solution \( u \) of (2.1) such that
\[ u \in C^1([0, T]; H) \cap C([0, T]; V) \quad \text{and} \quad \dot{u} + Bu \in C^1([0, T]; V^*). \] (2.6)

If furthermore \( u_0, u_1 \in V \) such that \( Au_0 + Bu_1 \in H \) and \( f \in C^1([0, T]; H) \) or \( (f, Bf) \in C([0, T]; V \times H) \), then there exists a unique solution \( u \) of (2.1) such that
\[ u \in C^2([0, T]; H) \cap C^1([0, T]; V) \quad \text{and} \quad Au + Bu \in C([0, T]; H). \] (2.7)

### 2.2 Wave equations with dynamic boundary conditions

While the error analysis provided in this paper is done for the abstract wave equation (2.1), we will discuss the numerical solution and the convergence behaviour of four exemplary wave equations with dynamic boundary conditions in detail. In the following, we will introduce these examples and show how the corresponding variational formulations can be written as an abstract wave equation of the form of (2.1).

Let us briefly introduce some notations. Let the bulk \( \Omega \subset \mathbb{R}^d \) (\( d = 2 \) or \( 3 \)) be a bounded domain, with (at least) \( C^2 \) boundary \( \Gamma = \partial \Omega \), which is referred to as the surface. Further, let \( n \) denote the unit outward normal vector to \( \Gamma \). Then the surface gradient on \( \Gamma \), of a function \( u : \Gamma \to \mathbb{R} \), is denoted by \( \nabla_{\Gamma} u \), and is given by \( \nabla_{\Gamma} u = \nabla u - (\nabla u \cdot n)n \), while the Laplace–Beltrami operator on \( \Gamma \) is given by \( \Delta_{\Gamma} u = \nabla_{\Gamma} \cdot \nabla_{\Gamma} u \). Moreover, \( yu \) denotes the trace of \( u \) on \( \Gamma \), and \( \dot{\gamma}_h u \) denotes the normal derivative of \( u \) on \( \Gamma \). Finally, temporal derivatives are denoted by \( \dot{\cdot} = d/dt \).

#### 2.2.1 Purely second-order dynamic boundary conditions.

For mathematical models of wave phenomena, the main region of interest is (often) given by the volume of the transmission medium that propagates the waves. This transmission medium therefore defines the domain for the wave equation while boundary conditions are used to effectively model the behaviour of the wave at the border to its surrounding. If these effective models capture oscillations of the surrounding structure or waves propagating along its surface, then we call it a dynamic boundary condition.

Here we consider the prototype of such a situation where the boundary condition is another wave equation on which the normal derivative of the bulk function acts as a force. Depending on the authors, such boundary conditions have been called oscillatory or kinetic, cf. Gal & Tebou (2017) or Vitillaro (2013). We begin with an example of a wave equation endowed with dynamic boundary conditions which only contains second-order terms modelling local oscillations and propagation of waves along the boundary: Find the solution \( u : [0, T] \times \overline{\Omega} \to \mathbb{R} \)
\[ \begin{align*}
\ddot{u} &= \Delta u + f_\Omega \quad \text{in} \quad \Omega, \\
\beta \ddot{u} &= \beta \Delta_{\Gamma} u - \kappa u - \partial_n u + f_{\Gamma} \quad \text{on} \quad \Gamma,
\end{align*} \] (2.8)
where the constants $\mu$ and $\kappa$ are positive, $\beta$ is non-negative and $f_\Omega : [0,T] \times \Omega \to \mathbb{R}$ and $f_\Gamma : [0,T] \times \Gamma \to \mathbb{R}$ are given functions. Here we do not consider problems with tip masses, i.e. where $\kappa = \beta = 0$, see e.g. Andrews et al. (1996), however we expect them to be treatable with our techniques, although with more technicalities.

The variational formulation of (2.8) can be cast as the abstract wave equation (2.1) in the Hilbert spaces
\[
H = L^2(\Omega) \times L^2(\Gamma) \quad \text{and} \quad V = \{ v \in H^1(\Omega) \mid \beta \gamma v \in H^1(\Gamma) \},
\]
with the embedding $v \mapsto (v, \gamma v)$, (2.9)

see (Hipp, 2017, Corollary 6.7).

The inner products on $H$ and $V$ are given by
\[
m(w,v) = \int_\Omega w v \, dx + \mu \int_\Gamma (\gamma w)(\gamma v) \, d\sigma \quad \text{and} \quad a(w,v) = \int_\Omega \nabla w \cdot \nabla v + \beta \int_\Gamma \nabla w \cdot \nabla \gamma v \, d\sigma + \kappa \int_\Gamma (\gamma w)(\gamma v) \, d\sigma.
\]

where for brevity we abbreviate the pairs $(v, \gamma v)$ by their first component $v$, and write $\nabla \gamma v$ instead of $\nabla \gamma \Gamma v$. We will employ these notations throughout the paper. Furthermore, since there is no velocity term, we have $b = 0$ and the right-hand side function $f$ is understood as
\[
(f, v)_V = \int_\Omega f \gamma v \, dx + \int_\Gamma f \gamma (\gamma v) \, d\sigma.
\]

2.2.2 Advection dynamic boundary conditions. Waves propagating through a medium in motion are subject to advection effects, which lead to terms containing first-order time derivatives, cf. (Campos, 2007, wave eq. W4). The following example accounts for advective and (weak) damping effects in the bulk and on the surface: We seek the solution $u : [0,T] \times \overline{\Omega} \to \mathbb{R}$ of
\[
\begin{align*}
\ddot{u} &= \Delta u - (\alpha_\Omega + v_\Omega \cdot \nabla) \dot{u} + f_\Omega & \text{in } \Omega, \quad (2.11a) \\
\mu \ddot{u} &= \beta \Delta_\Gamma u - (\alpha_\Gamma + v_\Gamma \cdot \nabla_\Gamma) \dot{u} - \kappa u - \partial_n u + f_\Gamma & \text{on } \Gamma. \quad (2.11b)
\end{align*}
\]

Here $\mu, \beta, \kappa > 0$, $\alpha_\Omega, \alpha_\Gamma \geq 0$ are constants and $v_\Omega \in L^\infty(\Omega; \mathbb{R}^d)$, $v_\Gamma \in L^\infty(\Gamma; \mathbb{R}^d)$ are given vector fields with $\text{div} \, v_\Omega \in L^\infty(\Omega)$, $\text{div}_\Gamma \, v_\Gamma \in L^\infty(\Gamma)$ such that
\[
\alpha_\Omega - \frac{1}{2} \text{div} \, v_\Omega \geq 0 \quad \text{in } \Omega \quad \text{and} \quad \alpha_\Gamma + \frac{1}{2} \left( v_\Omega \cdot n - \text{div}_\Gamma \, v_\Gamma \right) \geq 0 \quad \text{on } \Gamma.
\]

These last assumptions guarantee that $b$ is monotone, cf. (Hipp, 2017, Lemma 6.3). Note that for undamped models, i.e. if $\alpha_\Omega = \alpha_\Gamma = 0$, the first condition implies that $v_\Omega$ has no sources and the second one that any flow of $v_\Omega$ over $\Gamma$ is compensated by $v_\Gamma$.

This problem can also be written in the abstract form (2.1), using the same Hilbert spaces $H$ and $V$ from (2.9), and the duality and bilinear form from (2.10), while $b$ is now given by
\[
b(w,v) = \int_\Omega (\alpha_\Omega w + v_\Omega \cdot \nabla w) v \, dx + \int_\Gamma (\alpha_\Gamma \gamma v + v_\Gamma \cdot \nabla_\Gamma w) \gamma v \, d\sigma.
\]

(2.13)
2.2.3 **Strongly damped dynamic boundary conditions.** Strong damping is of great relevance in engineering due as it increases the robustness of systems against perturbations. Boundary conditions involving strong damping are particularly interesting for applications in wave–structure interactions, cf. Graber & Lasiecka (2014) and Nicaise (2017). We seek \( u: [0, T] \times \overline{\Omega} \rightarrow \mathbb{R} \) such that

\[
\ddot{u} = d_\Omega \Delta u + \Delta u + f_\Omega \quad \text{in} \quad \Omega, \\
\mu \ddot{u} = d_\Gamma \Delta \Gamma u + \beta \Delta \Gamma u - \kappa u - \partial_n u - d_\Omega \partial_n \dot{u} + f_\Gamma \quad \text{on} \quad \Gamma,
\]

(2.14)

with the same constants as in (2.8), except again \( \beta \) is assumed to be positive, and additionally with the damping coefficients \( d_\Omega, d_\Gamma > 0 \).

The weak formulation of this problem again fits into the framework of (2.1), by using the same spaces as before (2.9), and using the duality and bilinear form defined in (2.10), and

\[
b(w, v) = d_\Omega \int_\Omega \nabla w \cdot \nabla v \, dx + d_\Gamma \int_\Gamma \nabla w \cdot \nabla v \, d\sigma.
\]

(2.15)

Note that we have to apply Green’s formula in the bulk twice and then insert the boundary condition for \( \partial_n u + d_\Omega \partial_n \dot{u} \) to derive the variational formulation.

2.2.4 **Acoustic boundary conditions.** The wave equation with acoustic boundary condition models the propagation of sound waves in a fluid at rest filling a tank \( \Omega \), whose walls \( \Gamma \), are subject to small oscillations in normal direction and elastic effects in tangential direction. The model is described by the following system: Seek the acoustic velocity potential \( u: [0, T] \times \overline{\Omega} \rightarrow \mathbb{R} \) and the displacement of \( \Gamma \) in normal direction \( \delta: [0, T] \times \Gamma \rightarrow \mathbb{R} \) such that

\[
\ddot{u} = -a_\Omega u + c_\Omega \Delta u + f_\Omega \quad \text{in} \quad \Omega, \\
\mu \ddot{\delta} = -k_\Gamma \delta + c_\Gamma \Delta \Gamma \delta - c_\Omega \dot{u} + f_\Gamma \quad \text{on} \quad \Gamma,
\]

(2.16a)

(2.16b)

(2.16c)

where we assume that \( c_\Gamma, c_\Omega, \mu, a_\Omega, k_\Gamma > 0 \) are constants. This model was first proposed in Beale & Rosencrans (1974) and its analytical properties continue to be a topic of research. See, e.g., Gal et al. (2003) for a comparison with Wentzell boundary conditions, Mugnolo (2006) for a spectral analysis using operator matrices and Frota et al. (2011) for well-posedness analysis of a non-linear version.

For problems with acoustic boundary conditions, we denote functions in the bulk by **Roman letters**, functions on the surface by **Greek letters**, and functions in the **bulk–surface product space** are labelled with \( ^- \), and usually denoting the vector with the same latter as the bulk function, e.g. \( \vec{w} = (w, \omega) \).

The variational formulation of (2.16) is obtained by testing the bulk and surface equations separately by \( v \in H^1(\Omega) \) and \( \psi \in H^1(\Gamma) \), using Green’s formula on the surface and the bulk, and finally add up the equations, cf. (Hipp, 2017, Section 6.3). To write this as an abstract wave equation, we use the product spaces \( V = H^1(\Omega) \times H^1(\Gamma) \) and \( H = L^2(\Omega) \times L^2(\Gamma) \) and obtain the following problem: Find \( \vec{u} = (u, \delta): [0, T] \rightarrow V \) such that

\[
\langle \vec{u}, \vec{v} \rangle_V + b(\vec{u}, \vec{v}) + a(\vec{u}, \vec{v}) = \langle \vec{f}, \vec{v} \rangle_V, \quad \text{for every} \quad \vec{v} \in V,
\]
where the duality and the bilinear forms are given by, for functions \( \vec{w} = (w, \omega) \) and \( v = (v, \psi) \),

\[
\begin{align*}
    m(\vec{w}, \vec{v}) &= \int_{\Omega} wv \, dx + \int_{\Gamma} \mu \omega \psi \, d\sigma \text{,} \\
    b(\vec{w}, \vec{v}) &= c\gamma \int_{\Gamma} (\gamma w) \psi - \omega (\gamma v) \, d\sigma \text{,} \\
    a(\vec{w}, \vec{v}) &= \int_{\Omega} a_{\Omega} wv + c\gamma \nabla w \cdot \nabla v \, dx + \int_{\Gamma} k_{\Gamma} \omega \psi + c\gamma \nabla_{\Gamma} \omega \cdot \nabla_{\Gamma} \psi \, d\sigma \text{,}
\end{align*}
\]

(2.17a) (2.17b)

and the right-hand-side function for acoustic boundary conditions is understood as

\[
\langle \vec{f}, \vec{v} \rangle_{\Gamma} = \int_{\Omega} f_{\Omega} v \, dx + \int_{\Gamma} f_{\Gamma} \psi \, d\sigma \text{.}
\]

Note that \( a \) is coercive with \( \alpha = \min\{c_{\Omega}, a_{\Gamma}, c_{\gamma}, k_{\Gamma}\} \) and that \( b \) is skew-symmetric and therefore monotone.

3. Finite element method

For the numerical solution of the above examples we consider a linear finite element method. In the following, we will shortly recall the construction of the discrete domain, the finite element space and the lift operation from Elliott & Ranner (2013), (Kovács & Lubich, 2017, Section 3.2.1).

The domain \( \Omega \) is approximated by a triangulation \( \mathcal{T}_h \) with maximal mesh width \( h \). The union of all elements of \( \mathcal{T}_h \) defines the polyhedral domain \( \Omega_h \), whose boundary \( \Gamma_h := \partial \Omega_h \) is an interpolation of \( \Gamma \), i.e. the vertices of \( \Gamma_h \) are on \( \Gamma \). Analogously, we denote the outer unit normal vector of \( \Gamma_h \) by \( n_h \). We assume that \( h \) is sufficiently small to ensure that for every point \( x \in \Gamma_h \) there is a unique point \( p \in \Gamma \) such that \( x - p \) is orthogonal to the tangent space \( T_p \Gamma \) of \( \Gamma \) at \( p \). For convergence results, we consider a quasi-uniform family of such triangulations \( \mathcal{T}_h \) of \( \Omega_h \).

The finite element space \( S_h \subseteq H^1(\Omega) \) corresponding to \( \mathcal{T}_h \) is spanned by continuous, piecewise linear nodal basis functions on \( \Omega_h \), satisfying for each node \( (x_k)_{k=1}^N \)

\[
    \phi_j(x_k) = \delta_{jk}, \quad \text{for} \quad j, k = 1, \ldots, N.
\]

Then the finite element space is given as

\[
    S_h = \text{span}\{\phi_1, \ldots, \phi_N\}.
\]

We note here that the restrictions of the basis functions to the boundary \( \Gamma_h \) again form a surface finite element basis over the approximate boundary elements.

Following Dziuk (1988), we define the lift of functions \( v_h : \Gamma_h \rightarrow \mathbb{R} \) to \( v^\ell_h : \Gamma \rightarrow \mathbb{R} \) by setting \( v^\ell_h(p) = v_h(x) \) for \( p \in \Gamma_h \), where \( x \in \Gamma_h \) is the unique point on \( \Gamma_h \) with \( x - p \) orthogonal to the tangent space \( T_p \Gamma \). We further consider the lift of functions \( v_h : \Omega_h \rightarrow \mathbb{R} \) to \( v^\ell_h : \Omega \rightarrow \mathbb{R} \) by setting \( v^\ell_h(p) = v_h(x) \) if \( x \in \Omega_h \) and \( p \in \Omega \) are related as discussed in (Elliott & Ranner, 2013, Section 4.1.1). The unlift \( v^{-\ell} : \Gamma_h \rightarrow \mathbb{R} \) denotes a function whose lift is \( v \) on \( \Gamma \), and similarly for the bulk as well. Note that both definitions of the lift coincide on \( \Gamma \). Finally, the lifted finite element space is denoted by \( S^\ell_h \), and is given as \( S^\ell_h = \{ v^\ell_h \mid v_h \in S_h \} \).
4. Semi-discretisations of wave equations and \( L^2 \) error bounds

We will now introduce the finite element discretisation of the wave equations with dynamic boundary conditions from Section 2, and state our main results: convergence rates for the \( L^2 \) error of the (lifted) finite element approximation.

The finite element approximation of wave equations is based on its variational formulation with integrals over \( \Omega \) and \( \Gamma \) replaced by integrals over \( \Omega_h \) and \( \Gamma_h \), respectively. Using the finite element space for a Galerkin ansatz for this variational problem on the polygonal domain \( \Omega_h \) then yields the semi-discrete problem. Such finite element discretisations of a wave equations can be written as abstract differential equations in a finite dimensional Hilbert space \( V_h \) (specified below): Find the solution \( u_h : [0,T] \rightarrow V_h \) such that, for all \( v_h \in V_h \),

\[
m_h(\ddot{u}_h(t), v_h) + b_h(\dot{u}_h(t), v_h) + a_h(u_h(t), v_h) = m_h(f(t), v_h), \quad \text{for } t \in [0,T],
\]

\[
u_h(0) = \tilde{I}_h u_0, \quad \dot{u}_h(0) = \tilde{I}_h u_1.
\]

Where for a continuous function \( v \) on \( \Omega \) we denote by \( \tilde{I}_h v \in V_h \) the piecewise linear finite element interpolation.

In the rest of the section, we consider the finite element approximation of the examples from Sections 2.2.1-2.2.4. First, we give concrete definitions for the respective finite element approximation (4.1) and then we state the corresponding \( L^2 \) error estimates.

**Remark 4.1** The following error estimates require spatial regularity of the solution and its time derivatives. Constants in the error estimates will depend on the canonical norm of the space \( H^2(0,T;H^2(\Omega)) \), and similarly for \( \Gamma \), i.e.

\[
K_\Omega(u) = \|u\|_{H^2(0,T;H^2(\Omega))} + \|f\|_{L^2(0,T;H^2(\Omega))}, \quad K_\Gamma(\gamma u) = K_\Omega(u) + K_\Gamma(\gamma u).
\]

Note that by standard theory the estimate \( \max_{0 \leq t \leq T} \|u(t)\|_X \leq c \|u\|_{H^1(0,T;X)} \) holds, see, e.g. (Evans, 1998, Section 5.9.2). Therefore, we have

\[
\sup_{0 \leq t \leq T} \|u(t)\|_{H^2(\Omega)} + \sup_{0 \leq t \leq T} \|\gamma u(t)\|_{H^2(\Gamma)} \leq K(u, \gamma u).
\]

4.1 Purely second-order dynamic boundary conditions

The finite element approximation of (2.8) is given by (4.1) and seeks the numerical solution \( u_h \) in the space of piecewise linear finite elements

\[ V_h = S_h. \]

As described above, the semi-discrete problem is derived from the variational formulation by replacing the bilinear forms (2.10) with their discrete counterparts, i.e. for \( w_h, v_h \in V_h \),

\[
m_h(w_h, v_h) = \int_{\Omega_h} w_h v_h \, dx + \mu \int_{\Gamma_h} (\gamma_h w_h)(\gamma_h v_h) \, ds_h,
\]

\[
a_h(w_h, v_h) = \int_{\Omega_h} \nabla w_h \cdot \nabla v_h \, dx + \beta \int_{\Gamma_h} \nu_{\Gamma_h} w_h \cdot \nu_{\Gamma_h} v_h \, ds_h + \kappa \int_{\Gamma_h} (\gamma_h w_h)(\gamma_h v_h) \, ds_h,
\]

where \( \gamma_h \) denotes the trace operator onto \( \Gamma_h \), and \( \nu_{\Gamma_h} \) is the discrete tangential gradient (defined in a piecewise sense by \( \nu_{\Gamma_h} w_h = \nabla \tilde{w}_h - (\nabla \tilde{w}_h \cdot n_h) n_h \)). Finally, since \( b = 0 \), we also have \( b_h = 0 \).
Due to the following result, the lifted finite element approximation converges quadratically in the mesh size $h$ if measured in the a bulk–surface $L^2$ norm.

**Theorem 4.1 (Purely second-order wave equations)** Let $u$ be the solution of the wave equation with purely second-order dynamic boundary conditions (2.8).

If $\beta > 0$ and the solution $u \in C^2(0,T;H^1(\Omega)) \cap H^2(0,T;H^2(\Omega))$ with $\gamma u \in C^2(0,T;H^1(\Gamma)) \cap H^2(0,T;H^2(\Gamma))$, then there is an $h_0 > 0$ such that for all $h \leq h_0$ the error between the solution $u$ and the linear finite element solution $u_h$ of (4.1) (with (4.3)) satisfies the optimal second-order error estimate, for $0 \leq t \leq T$,

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} + \|\gamma (u_h(t) - u(t))\|_{L^2(\Gamma)} \leq C(u,T)h^2. \quad (4.4)$$

If $\beta = 0$ and the solution $u \in C^2(0,T;H^1(\Omega)) \cap H^2(0,T;H^2(\Omega))$ and $\gamma u \in C^2(0,T;L^2(\Gamma))$, then we have the optimal second-order error estimate, for $0 \leq t \leq T$,

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} + \|\gamma (u_h(t) - u(t))\|_{H^{-1/2}(\Gamma)} \leq C(u,T)h^2. \quad (4.5)$$

In both cases, the constant $C(u,T) > 0$ depends on $K(u, \gamma u)$ from (4.2) and grows linearly in the final time $T$, but it is independent of $h$ and $t$.

### 4.2 Wave equations with advective dynamic boundary conditions

For the semi-discretisation of the wave equation with advective dynamic boundary conditions, we again use the space $V_h = S_h$, together with the semi-discrete bilinear forms $m_h$ and $a_h$ from (4.3).

For the discrete counterpart of $b$ from (2.13), which accounts for the advective effects, we use the unlift of the vector fields and define, for $w_h, v_h \in V_h$,

$$b_h(w_h, v_h) = \int_{\Omega_h} (\alpha_\Omega w_h + v_\Omega \cdot \nabla w_h) v_h \, dx_h + \int_{\Gamma_h} (\alpha_\Gamma \gamma w_h + v_\Gamma \cdot \nabla_h w_h) \gamma v_h \, d\sigma_h. \quad (4.6)$$

**Remark 4.2** To obtain the discrete vector fields for the discrete bilinear form $b_h$ (4.6), the unlifted vector fields might be difficult to compute in practice. Alternatively, approximative vector fields can be used. If the discrete vector fields $v_\Omega$ and $v_\Gamma$ are sufficiently close (in terms of $h$) to $v_\Omega$ and $v_\Gamma$, then the following convergence estimate remains valid. We will return to this later on (see Remark 6.1), when the necessary tools are introduced.

For the finite element approximation of the wave equation with advective dynamic boundary conditions, we prove the following convergence results.

**Theorem 4.2 (Advective boundary conditions)** Let the solution of the wave equation with advective dynamic boundary conditions (2.11) have the regularity $u \in C^2(0,T;H^1(\Omega)) \cap H^2(0,T;H^2(\Omega))$ and $\gamma u \in C^2(0,T;H^1(\Gamma)) \cap H^2(0,T;H^2(\Gamma))$, then there is an $h_0 > 0$ such that for all $h \leq h_0$ the error between the solution $u$ and the linear finite element semi-discretisation $u_h$ of (4.1) (with (4.6)) satisfies the error estimate of order $3/2$, for $0 \leq t \leq T$,

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} + \|\gamma (u_h(t) - u(t))\|_{L^2(\Gamma)} \leq C(u,T)h^{3/2}, \quad (4.7)$$

and if $v_\Omega = 0$ (but $v_\Gamma$ not necessarily) we have the optimal-order error estimates

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} + \|\gamma (u_h(t) - u(t))\|_{L^2(\Gamma)} \leq C(u,T)h^2. \quad (4.8)$$

In both cases, the constant $C(u,T) > 0$ depends on $K(u, \gamma u)$ from (4.2) and grows linearly in the final time $T$, but it is independent of $h$ and $t$. 
\[ L^2 \text{ norm error analysis of wave equations with dynamic boundary conditions} \]

4.3 Wave equations with strongly damped dynamic boundary conditions

For the semi-discretisation of the wave equation with strongly damped dynamic boundary conditions, we again use the space \( V_h = S_h \), together with the semi-discrete bilinear forms \( m_h \) and \( a_h \) from (4.3).

The discrete bilinear form with strong damping corresponding to (2.15) reads, for \( w_h, v_h \in V_h \),

\[
b_h(w_h, v_h) = d_\Omega \int_{\Omega_h} \nabla w_h : \nabla v_h \, dx_h + d_T \int_{\Gamma_h} \nabla_{\Gamma_h} w_h : \nabla_{\Gamma_h} v_h \, d\sigma_h. \tag{4.9}
\]

**Theorem 4.3 (Strong damping)** Let the solution of the wave equation with strongly damped dynamic boundary conditions (2.14) have the regularity \( u \in C^2(0, T; H^1(\Omega)) \cap H^2(0, T; H^2(\Omega)) \) and \( \gamma u \in C^2(0, T; H^1(\Gamma)) \cap H^2(0, T; H^2(\Gamma)) \), then there is an \( h_0 > 0 \) such that for all \( h \leq h_0 \) the error between the solution \( u \) and the linear finite element semi-discretisation \( u_h \) of (4.1) (with (4.9)) satisfies the first-order error estimate, for \( 0 \leq t \leq T \),

\[
\| u_h'(t) - u(t) \|_{L^2(\Omega)} + \| \gamma (u_h'(t) - u(t)) \|_{L^2(\Gamma)} \leq C(u)h, \tag{4.10}
\]

and if \( \beta = d_T / d_\Omega \), i.e. the ratio of the diffusive and damping coefficients in the bilinear forms \( a \) and \( b \) coincide, then we have the optimal-order error estimate

\[
\| u_h'(t) - u(t) \|_{L^2(\Omega)} + \| \gamma (u_h'(t) - u(t)) \|_{L^2(\Gamma)} \leq C(u)h^2. \tag{4.11}
\]

In both cases, the constant \( C(u, T) > 0 \) depends on \( K(u, \gamma u) \) from (4.2) and grows linearly in the final time \( T \), but it is independent of \( h \) and \( t \).

**Remark 4.3** The strongly damped wave equation with strongly damped dynamic boundary conditions is governed by an analytic semigroup, cf. Graber & Lasiecka (2014). Since we treat (2.14) as a hyperbolic problem our estimate is probably suboptimal. We expect that the error of the finite element solution converges with \( \Theta(h^2) \) as shown in Larsson et al. (1991) for the strongly damped wave equation with homogeneous Dirichlet boundary conditions.

4.4 Wave equations with acoustic boundary conditions

The solution \( \bar{u} = (u, \delta) \) of (2.16) consists of two functions, one in the bulk and one on the surface. Therefore we introduce the boundary element space on the surface \( \Gamma_h \):

\[
S_h^\Gamma = \gamma_h S_h = \{ \gamma_h v_h \mid v_h \in S_h \},
\]

to approximate the surface function \( \delta : [0, T] \times \Gamma \to \mathbb{R} \). Hence we seek to approximate \( \bar{u}(t) \), which belongs to \( V = H^1(\Omega) \times H^1(\Gamma) \), in the bulk–surface finite element space

\[
V_h = S_h \times S_h^\Gamma. \tag{4.12}
\]

The finite element approximation of the wave equation with acoustic boundary conditions (2.16) now reads: Find \( \bar{u}_h = (u_h, \delta_h) : [0, T] \to V_h \) such that, for all \( \bar{v}_h \in V_h \),

\[
\begin{align*}
m_h(\bar{u}_h(t), \bar{v}_h) + b_h(\bar{u}_h(t), \bar{v}_h) + a_h(\bar{u}_h(t), \bar{v}_h) &= m_h(\bar{\bar{u}}_h(t), \bar{v}_h), \\
\bar{u}_h(0) &= \bar{I}_h \bar{u}_0, & \bar{\bar{u}}_h(0) &= \bar{I}_h \bar{u}_1.
\end{align*}
\tag{4.13}
\]
where the interpolation of any \( \tilde{v} = (v, \psi) \) is understood componentwise as \( \tilde{I}_h \tilde{v} = (\tilde{I}_h^2 v, \tilde{I}_h^3 \psi) \) and the discrete counterparts of the continuous bilinear forms (2.17) are given by, for \( h \) but it is independent of \( C \).

\[
\begin{align*}
    m_h(\tilde{w}_h, \tilde{v}_h) &= \int_{\Omega_h} w_h v_h \, dx_h + \int_{\Gamma_h} \mu \tilde{\omega}_h \psi_h \, d\sigma_h \\
    b_h(\tilde{w}, \tilde{v}_h) &= c_{\Omega} \int_{\Omega_h} \left( \gamma_h w_h \right) \psi_h - \omega_h \left( \gamma_h v_h \right) \, d\sigma_h, \\
    a_h(\tilde{w}, \tilde{v}_h) &= \int_{\Omega_h} a_{\Omega} w_h v_h + c_{\Omega} \nabla w_h \cdot \nabla v_h \, dx_h + \int_{\Gamma_h} k_{\Gamma} \tilde{\omega}_h \psi_h + c_{\Gamma} \nabla \tilde{\omega}_h \cdot \nabla \tilde{\omega}_h \psi_h \, d\sigma_h, \\
    m_h(I_h \tilde{f}, \tilde{v}_h) &= \int_{\Omega_h} \tilde{I}_h^2 f \Omega v_h \, dx_h + \int_{\Gamma_h} \tilde{I}_h^3 f \Gamma \psi_h \, d\sigma_h. 
\end{align*}
\]

Note that (4.13) is formally the same as (4.1).

As a consequence of the bulk–surface coupling our results show that the lifted finite element approximation only converges with \( O(h^{3/2}) \) instead of \( O(h^2) \) as one would generally expect.

**Theorem 4.4** (Acoustic boundary conditions) Let the solution of the wave equation with acoustic boundary conditions (2.16) have the regularity \( u \in C^2(0, T; H^1(\Omega)) \cap H^2(0, T; H^2(\Omega)) \) and \( \delta \in C^2(0, T; H^1(\Gamma)) \cap H^2(0, T; H^2(\Gamma)) \), then there is an \( h_0 > 0 \) such that for all \( h \leq h_0 \) the error between the solution \( u \) and the linear finite element semi-discretisation \( \tilde{u}_h = (u_h, \delta_h) \) of (4.13) (with (4.14)) satisfies the error estimate of order 3/2, for 0 \( \leq t \leq T \),

\[
||\tilde{u}_h(t) - u(t)||_{L^2(\Omega)} + ||\delta_h(t) - \delta(t)||_{L^2(\Omega)} \leq C(u, \delta, T) h^{3/2},
\]

where the constant \( C(u, \delta, T) > 0 \) depends on \( K(u, \delta) \) from (4.2), and grows linearly in the final time \( T \), but it is independent of \( h \) and \( t \).

## 5. Abstract error analysis

This section is devoted to the abstract error analysis of (4.1) with respect to (2.1). Our goal is to derive error bounds in terms of the \( H \) norm which can later be used to show convergence rates for concrete examples.

The section is structured as follows: First, we will show in Section 5.1 that the stability estimate derived by the usual energy technique cannot lead to the optimal convergence rate of \( O(h^2) \). Then, in Section 5.2, we prove an alternative stability estimate in a weak norm by an adapted energy technique. A discrete version of this estimate is shown in Section 5.3. The main part of the error analysis is then presented in Section 5.4 where we derive an abstract error bound in terms of Ritz projection errors, geometric errors and data errors. In Section 6 we then prove estimates for these errors separately for each of the four cases.

### 5.1 Stability estimate in the energy norm

Let us consider the purely second-order version of (2.1):

\[
\begin{align*}
    \langle \ddot{u}(t), v \rangle_V + a(u(t), v) &= \langle f(t), v \rangle_V \quad \forall v \in V, \\
    u(0) &= u_0, \quad \dot{u}(0) = u_1. 
\end{align*}
\]

The following result yields a stability estimate in the \( H \) norm for any sufficiently smooth solution.
PROPOSITION 5.1 Let \( u \in C^2([0,T];H) \) be a solution of (5.1). Then
\[
|u(t)|^2 \leq C (|\ddot{u}(t)|^2 + |u(t)|^2) \leq Ce \left( |u_1|^2 + \|u_0\|^2 + T \int_0^T |f(s)|^2 \, ds \right)
\]
(5.2)
for \( 0 \leq t \leq T \).

Proof. First note that due to \( H \simeq H^* \) and \( \ddot{u} \in H \), we have
\[
\langle \ddot{u}, v \rangle_V = m(\ddot{u}, v) \quad \text{for} \quad v \in V.
\]
To derive the stability estimate, we test (5.1) with \( v = \ddot{u}(t) \). Together with
\[
m(\ddot{u}(t), u(t)) = \frac{1}{2} \frac{d}{dt} |u(t)|^2 \quad \text{and} \quad a(\ddot{u}(t), u(t)) = \frac{1}{2} \frac{d}{dt} |u(t)|^2,
\]
we thus obtain, with the Cauchy–Schwarz and Young’s inequalities,
\[
\frac{1}{2} \frac{d}{dt} |\ddot{u}(t)|^2 + \frac{1}{2} \frac{d}{dt} |u(t)|^2 \leq |f(t)||\ddot{u}(t)| \leq \frac{T}{2} |f(t)|^2 + \frac{1}{2T} |\ddot{u}(t)|^2.
\]
Integrating both sides and then applying Gronwall’s inequality yields the classical stability bound
\[
|\ddot{u}(t)|^2 + |u(t)|^2 \leq e \left( |\ddot{u}(0)|^2 + |u(0)|^2 + T \int_0^T |f(s)|^2 \, ds \right).
\]
(5.3)

The first estimate of the claim is a consequence of \( V \overset{d}{\rightarrow} H \) and \( \|\cdot\|_V \sim \|\cdot\| \).

To obtain an error bound on the basis of (5.3), one would apply its discrete version to the error equation where the defect \( d_h \) would play the role of \( f \). Thus \( |d_h|_h \leq ch^2 \) is necessary for second-order convergence. However, as we will see later, the lower-order terms in the geometric estimates do not allow a second-order estimate in the discrete \( L^2 \) norm. Furthermore, this approach would require that the discrete initial value satisfies \( \|u^h_0(0) - u(0)\|_V \leq ch^2 \) which is only possible if we compute the Ritz map of \( u_0 \) and set it as a starting value for \( u_h \).

To avoid this additional computational effort and obtain second-order convergence, we use another stability estimate, which measures the error of the initial data and for the defect in weaker norms. A similar approach has been used in Baker & Bramble (1979) to show \( L^2 \) convergence rates for a finite element discretisation of the wave equation with homogeneous classical boundary conditions.

5.2 Stability estimate in weak norms

In addition to the canonical norm on \( V^* \), we define
\[
\|f\|_* = \sup_{v \neq 0} \frac{\langle f, v \rangle_V}{\|v\|} \quad \text{for} \quad f \in V^*.
\]
As a consequence of this definition and (2.5) we have, for \( f \in V^* \),
\[
\|f\|_* = \sup_{v \neq 0} \frac{\langle f, v \rangle_V}{\|v\|} = \sup_{v \neq 0} \frac{a(A^{-1}f, v)}{\|v\|} = \|A^{-1}f\|,
\]
where the last equality follows from the fact that the inner products are maximized by linear dependent elements. Since then
\[
\|f\|_V^2 = \|A^{-1}f\|^2 = a(A^{-1}f, A^{-1}f) = \langle AA^{-1}f, A^{-1}f \rangle_V = \langle f, A^{-1}f \rangle_V,
\]
the bilinear form \( \langle \cdot, A^{-1} \cdot \rangle_V \) is the inner product in \( V^* \) which induces \( \| \cdot \|_\ast \).

The weak stability estimate is a key step in proving the weak well-posedness result from Theorem 2.1. In the following lemma, we derive the same stability estimate in a different way using energy techniques.

**Proposition 5.2** Let \( u \) be a solution of (2.1) which satisfies (2.6). Then
\[
\|\dot{u}(t) + Bu(t)\|_V^2 + |u(t)|^2 \leq e^{\max\{0,2T\rho\}} \left( \|u_1 + Bu_0\|_V^2 + |u_0|^2 + T \int_0^T \|f(s)\|_V^2 ds \right),
\]
for \( 0 \leq t \leq T \).

We remark here, that for all four examples from Section 2.2 we have \( \rho = 0 \).

**Proof.** First observe that the left-hand side of (2.1) can be rewritten as
\[
\langle \dot{u}(t), v \rangle_V + b(\dot{u}(t), v) + a(u(t), v) = \langle (\dot{u}(t) + Bu(t))^*, v \rangle_V + \langle Au(t), v \rangle_V.
\]
Testing (2.1) with \( v = A^{-1}(\dot{u}(t) + Bu(t)) \), we therefore obtain
\[
\langle f(t), A^{-1}(\dot{u}(t) + Bu(t)) \rangle_V
= \langle (\dot{u}(t) + Bu(t))^*, A^{-1}(\dot{u}(t) + Bu(t)) \rangle_V + \langle Au(t), A^{-1}(\dot{u}(t) + Bu(t)) \rangle_V
= \frac{1}{2} \{ \|\dot{u}(t) + Bu(t)\|_V^2 + \frac{1}{2} [\rho] |u(t)|^2 \} + \langle Bu(t), u(t) \rangle_V,
\]
where we used that \( \langle Au, A^{-1} \dot{u} \rangle_V = m(u, \ddot{u}) \) and that \( \langle \cdot, A^{-1} \cdot \rangle_V \) is the inner product on \( V^* \) which induces \( \| \cdot \|_\ast \). Since the bilinear form \( b(\cdot, \cdot) + \rho m(\cdot, \cdot) \) is monotone by (2.2), we infer that
\[
\frac{1}{2} \{ \|\dot{u}(t) + Bu(t)\|_V^2 + \frac{1}{2} [\rho] |u(t)|^2 \}
\leq \frac{1}{2} \{ \|\dot{u}(t) + Bu(t)\|_V^2 + \frac{1}{2} [\rho] |u(t)|^2 \} + \langle Bu(t), u(t) \rangle_V + \rho |u(t)|^2
= \langle f(t), A^{-1}(\dot{u}(t) + Bu(t)) \rangle_V + \rho |u(t)|^2
\leq \frac{T}{2} \|f(t)\|_V^2 + \frac{1}{2} \max\{1/T, 2\rho \} \{ \|\dot{u}(t) + Bu(t)\|_V^2 + |u(t)|^2 \},
\]
where the last estimate is again shown by using Cauchy–Schwarz and Young’s inequalities. Finally, with Gronwall’s inequality, we obtain the stated stability bound. \( \square \)

We further note here that for wave equations with standard boundary conditions (but not in an abstract setting and without a velocity term) similar estimates have been shown, by choosing special test functions, started by the works of Dupont (1973), Baker (1976) and Baker & Bramble (1979). In the case of strong damping (with homogeneous Dirichlet boundary conditions) similar techniques were used by Larsson et al. (1991).

**5.3 Semi-discrete stability estimate in discrete weak norms**

This section is dedicated to the discrete weak norm stability estimate of the general semi-discrete problem (4.1). The proof is analogous to the continuous stability analysis of the previous section.
First note that the discrete bilinear forms \( m_h, a_h \) and \( b_h \) inherit the properties from their continuous counterparts. In particular, the two norms on \( V_h \)

\[
|v_h|^2_h = m_h(v_h, v_h) \quad \text{and} \quad \|v_h\|^2_h = a_h(v_h, v_h).
\]  

(5.5)

satisfy \( |v_h|_h \leq C\|v_h\|_h \) and there exists a constant \( \hat{\rho} \geq 0 \) such that \( b_h + \hat{\rho}m_h \) is monotone. For the error analysis, we use the lifts of discrete functions as introduced in Section 3. Due to (Kovács & Lubich, 2017, Lemma 3.9), its norm is equivalent to the discrete norm of the function itself:

\[
\|v_h\| \sim \|v_h\|_h \quad \text{and} \quad |v_h| \sim |v_h|_h \quad \text{uniformly in } h.
\]  

(5.6)

The abstract error analysis presented in the rest of this section applies if the semi-discretization satisfies these properties.

We define a discrete dual norm on the space \( V_h \)

\[
\|d_h\|_{*h} = \sup_{0 \neq v_h \in V_h} \frac{m_h(d_h, v_h)}{\|v_h\|_h}.
\]  

(5.7)

Hence, analogously to the continuous case, it follows that there exist constants \( C, c > 0 \) such that

\[
c\|v_h\|_{*h} \leq |v_h|_h \leq C\|v_h\|_h
\]

and that \( \|\cdot\|_{*h} \) is induced by the inner product \( m_h(A_h^{-1}, \cdot, \cdot) \) where the linear operators \( A_h, B_h : V_h \to V_h \) are given by

\[
a_h(w_h, v_h) = m_h(A_h w_h, v_h) \quad \text{and} \quad b_h(w_h, v_h) = m_h(B_h w_h, v_h).
\]  

(5.8)

We further introduce the following differences between the continuous and discrete bilinear forms:

For any \( w_h, v_h \in V_h \), we define

\[
\Delta m(w_h, v_h) = m(w^e_h, v^e_h) - m_h(w_h, v_h),
\]

\[
\Delta b(w_h, v_h) = b(w^e_h, v^e_h) - b_h(w_h, v_h).
\]

Furthermore, we will use the notation

\[
\|\Delta m(w_h, \cdot)\|_{*h} = \sup_{v_h \in V_h} \frac{\Delta m(w_h, v_h)}{\|v_h\|_h}.
\]

As in the proof of Proposition 5.2, the discrete weak stability estimate is shown by testing the semi-discrete problem with \( v_h = A_h^{-1}(u_h + B_h u_h) \). We obtain the following semi-discrete stability result.

**Proposition 5.3** Let \( u_h \) be a solution of the semi-discrete problem (4.1). Then

\[
\|u_h(t) + B_h u_h(t)\|_{*h}^2 + |u_h(t)|_h^2 \leq e^{\max\{1, 2T\hat{\rho}\}} \left( \|u_h(0) + B_h u_h(0)\|_{*h}^2 + |u_h(0)|_h^2 + T \int_0^T \|\bar{I}_h f(s)\|_h^2 ds \right),
\]  

(5.9)

for \( 0 \leq t \leq T \).

For the semi-discrete problems of our four examples, described in Section 4, we have \( \hat{\rho} = 0 \).
5.4 Error analysis and Ritz map

From (Kovács & Lubich, 2017, Section 3.4) we recall that the Ritz map \( R_h : V \to V_h^\ell \) (with boundary terms appearing in the bilinear form \( a \)) is defined as \( R_h u := (\bar{R}_h u)^\ell \in V_h^\ell \), where \( \bar{R}_h u \in V_h \) for \( u \in V \) is given via
\[
a_h(\bar{R}_h u, v_h) = a(u, v_h^\ell), \quad \text{for every } v_h \in V_h. \tag{5.10}
\]

To obtain an upper bound for the error \( u - u_h^\ell \), we split it into the error of the Ritz map and \( e_h = u_h - \bar{R}_h u \)
\[
|u(t) - u_h^\ell| \leq |u(t) - R_h u(t)| + \left| (\bar{R}_h u(t) - u_h(t))^\ell \right| 
\leq |u(t) - R_h u(t)| + C|e_h(t)|_h,
\]
where we used the norm equivalence \( (5.6) \) for the second inequality. In the rest of this section, we will show that \( |e_h(t)|_h \) is bounded from above by a combination of Ritz map, geometric and data errors, while the first term already is a Ritz map error.

For that purpose, we insert \( \bar{R}_h u \) into the semi-discrete problem \( (4.1) \) and define the defect (or semi-discrete residual) \( d_h : [0, T] \to V_h \), for all \( v_h \in V_h \), by
\[
m_h(\bar{R}_h \dot{u}, v_h) + b_h(\bar{R}_h \dot{u}, v_h) + a_h(\bar{R}_h u, v_h) = m_h(\bar{I}_h f, v_h) + m_h(d_h, v_h). \tag{5.11}
\]

Subtracting this from the semi-discrete weak problem \( (4.1) \) yields the error equation
\[
m_h(\dot{e}_h, v_h) + b_h(e_h, v_h) + a_h(e_h, v_h) = -m_h(d_h, v_h) \quad \forall v_h \in V_h.
\]

To obtain an upper bound for \( |e_h(t)|_h \), we apply the stability estimate from Proposition 5.3 to the error equation. Using Young’s inequality for products then gives
\[
|e_h(t)|_h \leq \|e_h(t) + B_h e_h(t)\|_{*,h} + |e_h(t)|_h 
\leq C e^{T\hat{\rho}} \left( \|\dot{e}_h(0) + B_h e_h(0)\|_{*,h}^2 + |e_h(0)|_h^2 + T \int_0^t \|d_h(s)\|_{*,h}^2 ds \right)^{1/2}. \tag{5.12}
\]

Since the errors at the initial time \( t = 0 \) are bounded by
\[
|e_h(0)|_h = |\bar{R}_h u_0 - \bar{I}_h u_0| \leq C|R_h u_0 - u_0| + C|I_h u_0 - u_0|
\]
and
\[
\|\dot{e}_h(0) + B_h e_h(0)\|_{*,h} \leq \|\bar{R}_h u_1 - \bar{I}_h u_1\|_{*,h} + \|B_h(\bar{R}_h u_0 - \bar{I}_h u_0)\|_{*,h} 
\leq C \|\bar{R}_h u_1 - \bar{I}_h u_1\|_{*,h} + \|B_h(\bar{R}_h u_0 - u_0^-)^\ell\|_{*,h} + \|B_h(u_0^- - \bar{I}_h u_0)\|_{*,h},
\]
where the last terms are further bounded from above since for \( w_h = \bar{R}_h u_0 - u_0^- \) and \( w_h = u_0^- - \bar{I}_h u_0 \)
\[
\|B_h w_h\|_{*,h} \leq \sup_{0 \neq v_h \in V_h} \Delta b(w_h, v_h) \|v_h\|_h + c \sup_{0 \neq v \in V} b(w_h^\ell, v) \|v\| \leq \|\Delta b(w_h, \cdot)\|_{*,h} + c \|B w_h^\ell\|_{*}.
\]
We have altogether shown
\[
|u(t) - u_0(t)| \leq |u(t) - R_h u(t)| + C e^{\beta T} \left( \varepsilon_0^2 + T \int_0^t ||d_h(s)||_{s,h}^2 ds \right)^{1/2},
\] (5.13a)
where
\[
\varepsilon_0 = |R_h u_1 - u_1| + |l_h u_1 - u_1| + |R_h u_0 - u_0| + |l_h u_0 - u_0|
+ ||\Delta b(R_h u_0 - u_0)\|_{s+h} + c\|B(R_h u_0 - u_0)\|_*
+ ||\Delta b(l_h u_0 - u_0)\|_{s+h} + c\|B(l_h u_0 - u_0)\|_*.
\] (5.13b)

To obtain convergence rates from this abstract estimate, it remains to study the defect \(d_h\) further.

**Lemma 5.1** The defect \(d_h\) defined by (5.11) satisfies the following estimate
\[
||d_h||_{s,h} \leq C \left( ||R_h \bar{u} - \bar{u}||_* + ||\Delta m(R_h \bar{u}, \cdot)||_{s,h}
+ ||f - l_h f||_* + ||\Delta m(l_h f, \cdot)||_{s,h} \right)
\] (5.14)
for \(0 \leq t \leq T\).

**Proof.** We subtract (2.1) with \(v = v_h^k\) from (5.11) to compute the defect
\[
m_h(d_h, v_h) = m_h(R_h \bar{u}, v_h) - m(\bar{u}, v_h^k)
+ b_h(R_h \bar{u}, v_h) - b(\bar{u}, v_h^k)
+ a_h(R_h u, v_h) - a(u, v_h^k)
+ m(f, v_h^k) - m_h(I_h f, v_h).
\] (5.15)
These pairs are then estimated separately. For the first pair, we have
\[
m_h(R_h \bar{u}, v_h) - m(\bar{u}, v_h^k) = m_h(R_h \bar{u}, v_h) - m(R_h \bar{u}, v_h^k) + m(R_h \bar{u} - \bar{u}, v_h^k)
\leq -\Delta m(R_h \bar{u}, v_h) + \|R_h \bar{u} - \bar{u}\|_* \|v_h^k\|,
\]
where we used that \(m(w, v) = \langle w, v \rangle_V \leq \|w\|_* \|v\| \leq C \|w\|_* \|v\|\). For the second pair, we have
\[
b_h(R_h \bar{u}, v_h) - b(\bar{u}, v_h^k) = b_h(R_h \bar{u}, v_h) - b(R_h \bar{u} - \bar{u}, v_h^k)
+ b(R_h \bar{u} - \bar{u}, v_h^k)\v
\leq -\Delta b(R_h \bar{u}, v_h) + \|B(R_h \bar{u} - \bar{u})\|_* \|v_h^k\|.
\]
The third pair vanishes by definition of the Ritz map, cf. (5.10).
For the fourth pair, we have
\[
m(f, v_h^k) - m_h(I_h f, v_h) = m(f - l_h f, v_h^k) + m(I_h f, v_h^k) - m_h(I_h f, v_h)
\leq \|f - l_h f\|_* \|v_h^k\| + \Delta m(I_h f, v_h)
\]
Since by (5.6)
\[ \|d_h\|_{*,h} = \sup_{v_h \in V_h} \frac{m_h(d_h, v_h)}{\|v_h\|_h} \leq C \sup_{v_h \in V_h} \frac{m_h(d_h, v_h)}{\|v_h\|}, \]
the claim follows upon combining the above estimates. \hfill \Box

6. Finite element error analysis

In this section, we prove the convergence results stated in Section 4. The main part of all these proofs was already done in Section 5 where we derived the abstract a priori estimate (5.13) and showed an upper bound for the defect in Lemma 5.1. To obtain convergence rates, it remains to estimate the error components in terms of the mesh width \( h \). This can be done by using approximation results from the literature and using the properties of the first-order term \( B \) which, as it will turn out, lead to the different convergence rates appearing in Theorem 4.1–4.4.

The results of the previous section can be also summarised as: By substituting the estimates (5.13b) and (5.14) into (5.13a), the \( L^2 \) error of the semi-discrete solution is bounded by
\[ |u(t) - u_h(t)| \leq \text{Ritz map errors} + \text{interpolation errors} + \text{geometric approximation errors}. \]
In the next section we show that these errors are indeed small.

6.1 Purely second-order wave equation

6.1.1 Geometric and interpolation errors. Before we turn to the proof of Theorem 4.1, we collect some approximation results of the geometry and the nodal interpolation.

**Lemma 6.1** (Kovács & Lubich, 2017, Lemma 3.9) For any \( v_h, w_h \in S_h \) we have the estimates
\[ |a(v_h, w_h) - a_h(v_h, w_h)| \leq Ch \| \nabla v_h \|_{L^2(B_h^*)} \| \nabla w_h \|_{L^2(B_h^*)}, \]
\[ + Ch^2 \left( \| \nabla v_h \|_{L^2(\Omega)} \| \nabla w_h \|_{L^2(\Omega)} + \beta \| \nabla v_h \|_{L^2(\Gamma)} \| \nabla w_h \|_{L^2(\Gamma)} + \kappa \| \nabla^2 v_h \|_{L^2(\Gamma)} \| \nabla^2 w_h \|_{L^2(\Gamma)} \right), \]
\[ |m(v_h, w_h) - m_h(v_h, w_h)| \leq Ch \| v_h \|_{L^2(B_h^*)} \| w_h \|_{L^2(B_h^*)}, \]
\[ + Ch^2 \left( \| v_h \|_{L^2(\Omega)} \| w_h \|_{L^2(\Omega)} + \mu \| \nabla^2 v_h \|_{L^2(\Gamma)} \| \nabla^2 w_h \|_{L^2(\Gamma)} \right), \]
where \( B_h^* \) denotes layer of lifted elements which have a boundary face.

**Lemma 6.2** (Elliott & Ranner, 2013, Lemma 6.3) For all \( v \in H^1(\Omega) \) the following estimate holds:
\[ \| v \|_{L^2(B_h^*)} \leq Ch^2 \| v \|_{H^1(\Omega)}. \] (6.1)

**Lemma 6.3** For \( v \in H^2(\Omega) \), such that \( \gamma v \in H^2(\Gamma) \), we denote by \( I_h v \in V_h \) the lift of the nodal finite element interpolation \( I_h v \in V_h \). Then the following estimates hold:

(i) Interpolation error in the bulk; see Bernardi (1989); Elliott & Ranner (2013):
\[ \| v - I_h v \|_{L^2(\Omega)} + h \| \nabla (v - I_h v) \|_{L^2(\Omega)} \leq Ch^2 \| v \|_{H^2(\Omega)}. \]
(ii) Interpolation error on the surface; see Dziuk (1988):
\[ \| \gamma (v - I_h v) \|_{L^2(\Gamma)} + h \| \nabla \gamma (v - I_h v) \|_{L^2(\Gamma)} \leq C h^2 \| v \|_{H^2(\Gamma)}. \]

6.1.2 Error estimates for the Ritz map. From (Kovács & Lubich, 2017, Lemma 3.13 and 3.15) we recall the following estimates for the error of the Ritz map. Note the weaker norm on \( \Gamma \) for \( \beta = 0 \) due to a lack of boundary regularity of solutions of the Poisson equation with Neumann boundary conditions.

**Lemma 6.4** The error of the Ritz map (5.10) corresponding to the bilinear form \( a \) from (2.10) satisfies the following second-order bounds:

For \( \beta = 0 \):
\[ \| v - R_h v \|_{L^2(\Omega)} + \| \gamma (v - R_h v) \|_{H^{-1/2}(\Gamma)} \leq C h^2 \| v \|_{H^2(\Omega)}, \]
where the constant \( C \) is independent of \( h \) and \( v \in H^2(\Omega) \).

For \( \beta > 0 \):
\[ \| v - R_h v \|_{L^2(\Omega)} + \| \gamma (v - R_h v) \|_{L^2(\Gamma)} \leq C h^2 \left( \| v \|_{H^2(\Omega)} + \| \gamma v \|_{H^2(\Gamma)} \right), \]
where the constant \( C \) is independent of \( h \) and \( v \in H^2(\Omega) \) with \( \gamma v \in H^2(\Gamma) \), but depends on \( \beta > 0 \).

6.1.3 Proof of Theorem 4.1. For the proof, we simply apply the abstract results from the Section 5 and use the estimates for the error of the Ritz map, the interpolation error and the geometric errors from above.

We start by considering the error of the Ritz map in the dual norm. For \( \beta = 0 \) and \( v \in H^2(\Omega) \), it follows from (Kovács & Lubich, 2017, Section 3.5.2) that
\[ \| v - R_h v \|_\ast \leq \left( \| v - R_h v \|_{L^2(\Omega)}^2 + \| v - R_h v \|_{H^{-1/2}(\Gamma)}^2 \right)^{1/2} \leq C h^2 \]
and, for \( \beta > 0 \) and \( v \in H^2(\Omega) \) with \( \gamma v \in H^2(\Gamma) \),
\[ \| v - R_h v \|_\ast \leq \| v - R_h v \| = \left( \| v - R_h v \|_{L^2(\Omega)}^2 + \| v - R_h v \|_{L^2(\Gamma)}^2 \right)^{1/2} \leq C h^2, \]
where we used Lemma 6.4 in the last inequality for both estimates.

Now we can further estimate the upper bound for the defect from Lemma 5.1: The Ritz map error for \( \bar{u} \in H^2(\Omega) \) is bounded due to our previous arguments, the first-order terms does not appear since \( b = 0 \) and hence \( B = 0 \), and the \( L^2 \) norm error estimate of the interpolation Lemma 6.3 yields
\[ \| f - I_h f \|_\ast \leq \| f - I_h f \| \leq C h^2. \]

The geometric errors can be bounded as follows: Combining Lemma 6.1 and 6.2 yields \( \Delta m (w_h, v_h) \leq C h^2 \| v_h \|_{H^1(\Omega)} \| v_h \|_{H^1(\Omega)}. \) Therefore we obtain with (5.6) that
\[ \| \Delta m (w_h, \cdot) \|_{\ast, h} = \sup_{\nu_h \in V_h} \frac{\Delta m (w_h, v_h)}{\| v_h \|_{h}} \leq C h^2 \| w_h \|_h. \] (6.3)

Since, first, the norm equivalence (5.6) and the interpolation estimate from Lemma 6.3 yield for \( f \in H^2(\Omega) \cap V \)
\[ \| I_h f \|_h \leq \| f - I_h f \| + \| f \| \leq C h + c, \]
and, second, \( \tilde{R}_h \in \mathcal{L}(V, V_h) \) and \( \bar{u} \in V \), the geometric error is bounded by
\[
\| \Delta m(\bar{R}_h \bar{u}, \cdot) \|_{+h} + \| \Delta m(\tilde{I}_h f, \cdot) \|_{+h} \leq \varepsilon_h^2.
\]

Altogether, we showed that under the given assumptions and, second, following geometric bound, shown in (Hipp, 2017, Lemma 7.3).

6.2 Geometric errors.

6.2.1 Geometric errors. The bilinear form containing the advective terms (2.13) and (4.6) satisfy the following geometric bound, shown in (Hipp, 2017, Lemma 7.3).

\[ \| \varepsilon \|_{L^2(\Omega)} + \| \varepsilon \|_{H^{-1/2}(\Gamma')} \leq \| u - R_h u \|_{L^2(\Omega)} + \| u - R_h u \|_{H^{-1/2}(\Gamma')} + C \| \varepsilon \|_h. \]

The errors of the Ritz map are bounded by (6.2a) and \( \| \varepsilon_h \|_h \) satisfies (5.12). Therefore, we obtain from the estimate for the defect the optimal-order convergence bound
\[
\| u(t) - u_h(t) \|_{L^2(\Omega)} + \| u(t) - u_h(t) \|_{H^{-1/2}(\Gamma')} \leq Ch^2.
\]

6.2 Advection boundary conditions

6.2.1 Geometric errors. The bilinear form containing the advective terms (2.13) and (4.6) satisfy the following geometric bound, shown in (Hipp, 2017, Lemma 7.3).

**Lemma 6.5** For sufficiently small \( h \leq h_0 \), and for any \( v_h, w_h \in S_h \) we have the estimate
\[
|b(w_h, v_h) - b_h(w_h, v_h)| \leq \left( \| \varepsilon_v \|_{L^2(B')} + \| \varepsilon_w \|_{L^2(B')} + \alpha \| w_h \|_{L^2(B')} + c \| v_h \|_{L^2(B')} \right)
\]
\[
+ ch^2 \left( \| \nabla v_h \|_{L^2(\Gamma')} + \| \nabla w_h \|_{L^2(\Gamma')} \right),
\]

where the constant \( c \) is independent of \( h \), but depends on the \( L^\infty \) norms of the coefficient functions \( \alpha, \alpha', \varepsilon_v \) and \( \varepsilon_w \).

We now give more details on the approximative vector fields discussed in Remark 4.2.

**Remark 6.1** One way to avoid computing the unlift of the vector fields is to use their interpolations \( v_{\Omega,h} = \tilde{I}_h v_{\Omega} \) and \( \bar{v}_{h} = \tilde{I}_h \bar{v}_{\Gamma} \) instead. Then the above geometric approximation estimate of Lemma 6.5 holds, using the interpolation error estimate, see the proof of (Hipp, 2017, Lemma 7.3). The quasi-monotonicity of \( b_h \) is shown using the assumptions which guarantee that the original bilinear form \( b \) is monotone (i.e. the conditions in (2.12)), by proving \( 0 \leq \bar{b} \leq ch \) below, meaning that the stability estimate for the semi-discrete equation (5.9) holds with an exponent which is almost zero.

First note that for two differentiable vector fields \( F : \Omega \to \mathbb{R}^d \) and \( G : \Omega' \to \mathbb{R}^d \), it follows by the chain rule that
\[
\text{div}(F \circ G) = \sum_{i=1,2} e_i^T D(F \circ G) e_i = \sum_{i=1,2} e_i^T (DF \circ G) DGe_i
\]
\[
= (\text{div} F) \circ G + \sum_{i=1,2} e_i^T ((DF \circ G)(DG - I)) e_i,
\]
where $e_i \in \mathbb{R}^d$ denotes unit vector along the $i$th coordinate axis. Now let $G_h : \Omega_h \to \Omega$ the smooth homomorphism such that the lift of a function $v_h : \Omega_h \to \mathbb{R}$ is given by $v_h = v \circ G_h$. Then we obtain that the divergence of the unlift of the vector field $v_{\Omega}$ is given by

$$\text{div} \ v_{\Omega}^{-\ell} = \text{div} \ (v_{\Omega} \circ G_h) = (\text{div} \ v_{\Omega})^{-\ell} + \sum_{i=1,2} e_i^T \left((Dv_{\Omega})^{-\ell}(DG_h - I)\right)e_i. \tag{6.4}$$

Recall that the bilinear form $b$ is monotone due to the conditions (2.12). However, the bulk condition

$$0 \leq \min_{x \in \Omega_h} \left(\alpha_{\Omega} - \frac{1}{2}\text{div} \ v_{\Omega}(x)\right) = \alpha_{\Omega} - \frac{1}{2}\max_{x \in \Omega} \text{div} \ v_{\Omega}(x), \tag{6.5}$$

is not necessarily satisfied for the interpolated vector fields (or analogously for the surface condition). Instead we have that $b_h + \hat{\rho}m_h$ is monotone for

$$\hat{\rho} = -\min_{x \in \Omega_h} \left(\alpha_{\Omega} - \frac{1}{2}\text{div} \ (\tilde{I}_h v_{\Omega})(x)\right) = -\alpha_{\Omega} + \frac{1}{2}\max_{x \in \Omega_h} \text{div} \ (\tilde{I}_h v_{\Omega})(x).$$

If $\hat{\rho}$ is negative then the semi-discrete bilinear form $b_h$ is monotone, and $b_h + \hat{\rho}m_h \geq 0$ holds with $\hat{\rho} = 0$. Hence we can assume $\hat{\rho} \geq 0$ and by (6.5) we have

$$\hat{\rho} \leq \hat{\rho} + \alpha_{\Omega} - \frac{1}{2}\max_{x \in \Omega} \text{div} \ v_{\Omega}(x)$$

$$\leq -\alpha_{\Omega} + \frac{1}{2}\max_{x \in \Omega_h} \text{div} \ (\tilde{I}_h v_{\Omega})(x) + \alpha_{\Omega} - \frac{1}{2}\max_{x \in \Omega} \text{div} \ v_{\Omega}(x)$$

$$= \frac{1}{2}\max_{x \in \Omega_h} \text{div} \ (\tilde{I}_h v_{\Omega})(x) - \frac{1}{2}\max_{x \in \Omega} \text{div} \ v_{\Omega}(x)$$

$$\leq \frac{1}{2}\max_{x \in \Omega_h} \left|\text{div} \ (\tilde{I}_h v_{\Omega})(x) - (\text{div} \ v_{\Omega})^{-\ell}(x)\right|$$

$$\leq \|\text{div} \ (\tilde{I}_h v_{\Omega}) - (\text{div} \ v_{\Omega})^{-\ell}\|_{L^\infty(\Omega_h)}.$$ 

Using (6.4) and the interpolation error estimate yields

$$\|\text{div} \ (\tilde{I}_h v_{\Omega}) - (\text{div} \ v_{\Omega})^{-\ell}\|_{L^\infty(\Omega_h)} \leq \|\text{div} \ (\tilde{I}_h v_{\Omega} - v_{\Omega}^{-\ell})\|_{L^\infty(\Omega_h)} + C\|\text{div} v_{\Omega}\|^{-\ell}_{L^\infty(\Omega_h)}\|DG_h - I\|_{L^\infty(\Omega_h)}$$

$$\leq \|\tilde{I}_h v_{\Omega} - v_{\Omega}^{-\ell}\|_{H^1(\Omega_h)} + C\|\text{div} \ v_{\Omega}\|_{L^\infty(\Omega_h)}\|DG_h - I\|_{L^\infty(\Omega_h)}$$

$$\leq ch\|v_{\Omega}\|_{H^{2}}(\Omega_h).$$

Altogether we proved that $0 \leq \hat{\rho} \leq ch$ for interpolated bulk vector fields.

Analogously, the surface condition in (2.12) might also fail for the interpolated vector fields. Repeating the argument above for this case, we have that

$$\hat{\rho} = \max \left\{ -\min_{x \in \Gamma_h} \left(\alpha_{\Gamma} - \frac{1}{2}\text{div} \ (\tilde{I}_h v_{\Gamma})(x)\right), -\min_{x \in \Gamma_h} \left(\alpha_{\Gamma} - \frac{1}{2}\left(\text{div}_{\Gamma} (\tilde{I}_h v_{\Gamma})(x)\right)\right) \right\}.$$

also satisfies the bounds $0 \leq \hat{\rho} \leq ch$.

Therefore, with interpolated vector fields, Proposition 5.3 and hence Theorem 4.2 holds with a constants which grow like $e^{\rho T} = e^{chT}$ in the final time $T$. 
6.2.2 Proof of Theorem 4.2. We proceed analogously to the proof of Theorem 4.1.

First we prove the case with advection in the bulk and on the boundary (a), and then prove the result without bulk advection, i.e.
\[ v_\Omega = 0, \]
(b).

Note that the error estimate (6.2b) for the Ritz map still applies in both situations.

(a) In order to show
\[ \|d_h\|_{\ast, h} \leq C h^{3/2}, \]
we only need to consider the first-order terms from Lemma 5.1 containing \( B \). The other terms were already treated in the proof of Theorem 4.1.

To estimate
\[ \|B(R_h \dot{u} - \dot{u})\|_{\ast}, \]
for \( b \) defined in (2.13), we use
\[ b(w, v) \leq c |w||v|, \quad w \in H, v \in V, \]
which follows from integration by parts and the assumptions (2.12) on the coefficient functions. Therefore, we have by definition of the dual norm and (6.2b),
\[ \|B(R_h \dot{u} - \dot{u})\|_{\ast} \leq c |R_h \dot{u} - \dot{u}| \leq C h^2. \] (6.6)

For an upper bound for \( \Delta b \), we use the geometric estimates stated in Lemma 6.5. Applying Lemma 6.2 to further estimate the boundary layer norms for \( v_h \) then yields
\[ |\Delta b(w_h, v_h)| \leq C h^{3/2} |w_h|_{H^1(\Omega)} |v_h|_{H^1(\Omega)} + C h^2 |w_h|_{H^1(\Gamma)} |v_h|_{H^1(\Gamma)} \]
\[ \leq C h^{3/2} |w_h|_{H^1} |v_h|_{H^1}. \] (6.7)

Therefore, the geometric error for \( b \) converges with
\[ \|\Delta b(R_h \dot{u}, \cdot)\|_{\ast} \leq C h^{3/2}. \]

 Altogether, we obtain for a sufficiently small \( h \leq h_0 \)
\[ \|d_h\|_{\ast, h} \leq C h^{3/2}, \]
with a constant \( C \) independent of \( h \) but depending on Sobolev norms of the solution \( u \) (and also its time derivatives).

For the errors in the initial data (5.13b) we now have \( \varepsilon_0 \leq C h^{3/2} \), by similar arguments as above: using the bound (6.6) and the geometric estimate (6.7), and Ritz map and interpolation error estimates.

We again recall that the error \( u - u_h \) was estimated in terms of the defect and errors in the initial data (5.13). The combination of this estimate with the above results yields the convergence bound:
\[ |u(t) - u_h(t)| \leq C h^{3/2}. \]

(b) If there is no advection in the bulk, i.e., \( v_\Omega = 0 \), clearly the first term in the right-hand side of the estimate in Lemma 6.5 vanishes, hence, using Lemma 6.2, we have \( O(h^2) \) estimate in (6.7). Then, by the same techniques as before we then obtain the defect estimate \( \|d_h\|_{\ast, h} \leq C h^2 \) and initial data error \( \varepsilon_0 \leq C h^2 \), and hence the optimal-order convergence bound:
\[ |u(t) - u_h(t)| \leq C h^2. \]

\( \square \)
6.3 Strongly damped boundary conditions

6.3.1 Geometric errors. Since the bilinear forms $b$ and $b_h$ defined in (2.15) and (4.9) contain the same terms as $a$ and $a_h$, the following geometric estimate is an immediate consequence of Lemma 6.1.

**Lemma 6.6** For sufficiently small $h \leq h_0$, and for any $v_h, w_h \in S_h$ we have the estimates

$$|b(w_h^f, v_h^b) - b_h(w_h, v_h)| \leq ch\|\nabla w_h^f\|_{L^2(\Omega)}\|\nabla v_h^b\|_{L^2(\Omega)} + ch^2\|\nabla w_h^f\|_{L^2(\Gamma)}\|\nabla v_h^b\|_{L^2(\Gamma)},$$

where the constant $c$ is independent of $h$, but depends on $d_\Omega$ and $d_\Gamma$.

6.3.2 Proof of Theorem 4.3. We proceed analogously to the previous proofs.

We prove the general case and the case with coefficients satisfying $\beta = d_\Gamma / d_\Omega$ separately.

(a) Again, note that the error estimate (6.2b) for the Ritz map still applies in this situation and that it is only left to consider the first-order terms from Lemma 5.1 to prove a defect estimate.

Using $B \in \mathcal{Z}(V, V^*)$ and by the $\| \cdot \|$ norm error estimate for the Ritz map (Kovács & Lubich, 2017, Lemma 5.1) we find that for $\dot{u} \in H^2(\Omega) \cap V$

$$\|B(R_h\dot{u} - \dot{u})\|_*> c\|R_h\dot{u} - \dot{u}\| \leq ch.$$

In addition, the geometric error of $b$ is bounded as $O(h)$ such that altogether, for a sufficiently small $h \leq h_0$, we have

$$\|d_h\|_{*,h} \leq Ch,$$

with a constant $C$ independent of $h$ but depending on Sobolev norms of the solution $u$ (and also its time derivatives).

For the errors in the initial data (5.13b) we have $\epsilon_0 \leq ch$, by similar arguments as for the defect above, and using Ritz map and interpolation error estimates.

Again recalling the error estimate (5.13), and combining it with the above inequalities we obtain the stated, convergence bound

$$|u(t) - u_h^i(t)| \leq Ch.$$

(b) In case the ratio of the diffusion and damping coefficients coincide

$$\beta = d_\Gamma / d_\Omega,$$

the bilinear forms $a$ and $b$ (and the semi-discrete counterparts) also coincide up to a constant and some lower order terms (which are related to $m$), using the definitions (2.10) and (2.15), we have

$$b(w, v) = d_\Omega \left( \int_\Omega \nabla w \cdot \nabla v \, dx + \frac{d_\Gamma}{d_\Omega} \int_\Gamma \nabla w \cdot \nabla v \, d\sigma \right)$$

$$= d_\Omega \left( a(w, v) \right) - d_\Omega \kappa \int_\Gamma (\gamma w)(\gamma v) \, d\sigma.$$

In the proof of Lemma 5.1, in particular in (5.15), not only the pair for $a$ vanishes due to the definition of the Ritz map (5.10), but the pair for $b$ as well up to a mass term on the boundary, using the identity from above. Therefore, the critical term from part (a) does not appear at all, but instead we have to bound the boundary mass pair, similarly as we have done for (6.3), and obtain

$$\left| \int_{\Gamma_h} (\gamma \tilde{R}_h \dot{u})(\gamma v_h) \, d\sigma - \int_{\Gamma}(\gamma \dot{u})(\gamma v_h^b) \, d\sigma \right| \leq ch^2.$$
The rest of the proof is finished as part (a) and yields a defect estimate $|d_\varepsilon| + \varepsilon \leq C\varepsilon^2$ and initial data error bound $\varepsilon_0 \leq Ch^2$, and hence an optimal-order error estimate:

$$|u(t) - u_0(t)| \leq Ch^2.$$ 

6.4 Acoustic boundary conditions

6.4.1 Geometric and interpolation errors. In this section, we treat the geometric errors in the bilinear forms of the equation with acoustic boundary conditions. Although, the following estimates are a straightforward generalisation of the results from Dziuk & Elliott (2013) and Elliott & Ranner (2013), we present the proofs to avoid any confusion due to the vector valued functions.

**Lemma 6.7** For sufficiently small $h < h_0$, and for any $\tilde{w}_h = (w_h, \omega_h), \tilde{v}_h = (v_h, \psi_h) \in \tilde{S}_h$ we have the estimates

$$|m(\tilde{w}_h^c, \tilde{v}_h^c) - m_h(\tilde{w}_h, \tilde{v}_h)| \leq ch^2 \|w_h\|_{L^2(\Omega)} \|v_h\|_{L^2(\Gamma)} + ch^2 \|\omega_h\|_{L^2(\Gamma)} \|\psi_h\|_{L^2(\Gamma)},$$

$$|a(\tilde{w}_h^c, \tilde{v}_h^c) - a_h(\tilde{w}_h, \tilde{v}_h)| \leq ch^2 \|w_h\|_{H^1(\Omega)} \|v_h\|_{H^1(\Gamma)} + ch^2 \|\omega_h\|_{H^1(\Gamma)} \|\psi_h\|_{H^1(\Gamma)},$$

$$|b(\tilde{w}_h^c, \tilde{v}_h^c) - b_h(\tilde{w}_h, \tilde{v}_h)| \leq ch^2 \|\nabla w_h\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Gamma)} + ch^2 \|\omega_h\|_{H^1(\Gamma)} \|\psi_h\|_{H^1(\Gamma)},$$

with constants independent of $h$, but depending on $c_{T}, c_{\Omega}, \mu_{T}, a_{\Omega}$ and $k_{T}$.

**Proof.** The first and the second estimate can be shown in the same way as Lemma 6.1. For the last inequality, using (Dziuk & Elliott, 2013, Lemma 5.5, (5.13)) and using that the lift, see Section 3, satisfies $(v_h, \psi_h) = (v_h^c, \psi_h^c)$ for all $v_h \in V_h$, we therefore obtain

$$|b(\tilde{w}_h^c, \tilde{v}_h^c) - b_h(\tilde{w}_h, \tilde{v}_h)| \leq c_{T} \mid \int_{\Gamma} |\nabla \psi_h^c| d\sigma - \int_{\Gamma} (\nabla \psi_h) d\sigma \mid$$

$$\leq c_{T} \mid \int_{\Gamma} |\nabla \psi_h| d\sigma - \int_{\Gamma} \psi_h d\sigma \mid + c_{T} \mid \int_{\Gamma} (\nabla \psi_h) d\sigma - \int_{\Gamma} \psi_h d\sigma \mid$$

$$\leq ch^2 \|\nabla \psi_h\|_{L^2(\Omega)} \|\psi_h\|_{L^2(\Gamma)} + ch^2 \|\omega_h\|_{H^1(\Gamma)} \|\psi_h\|_{H^1(\Gamma)}.$$ 

For the numerical discretisation of the wave equation with acoustic boundary conditions, we need two interpolation operators. In addition to the bulk interpolation $\tilde{I}_h^\Omega : H^2(\Omega) \rightarrow S_h$ which was defined in Lemma 6.3, we introduce the nodal interpolation of surface functions

$$\tilde{I}_h^\Gamma : H^2(\Gamma) \rightarrow S_h^\Gamma.$$ 

As the mesh of $\Gamma_h$ is given by the boundary nodes of $\tilde{S}_h$, the following identity-via-traces holds for $v \in H^2(\Omega)$ with $\gamma(v) \in H^2(\Gamma)$

$$\tilde{I}_h^\Gamma (\gamma(v)) = \gamma(\tilde{I}_h^\Omega v).$$
Using the abbreviations

\[ H^k = H^k(\Omega) \times H^k(\Gamma) \quad \text{for} \quad k \geq 1, \quad \text{and} \quad L^2 = L^2(\Omega) \times L^2(\Gamma) \]

we define the bulk–surface interpolation operator \( \tilde{I}_h : H^2 \to V_h \) componentwise as \( \tilde{I}_h(v, \psi) = (\tilde{I}_h^\Omega v, \tilde{I}_h^\Gamma \psi) \).

Similarly as the estimates of Lemma 6.3, or by (Elliott & Ranner, 2013, Proposition 5.4), the lifted interpolation \( I_h \tilde{v} = (\tilde{I}_h \tilde{v})^\ell \) has the following approximation property.

**Lemma 6.8** Let \( \tilde{v} \in H^2 \). Then the interpolation error of \( I_h \tilde{v} \) is bounded by

\[ \| \tilde{v} - I_h \tilde{v} \|_{L^2} + h \| \tilde{v} - I_h \tilde{v} \|_{H^1} \leq Ch^2 \| \tilde{v} \|_{H^2}. \]

**6.4.2 Error estimates for the Ritz map.** We recall the definition of the Ritz map from (5.10) in the notation for acoustic boundary conditions: For \( \tilde{w} = (w, \omega) \in V \) we define \( \tilde{R}_h \tilde{w} \in V_h \) by

\[ a_h(\tilde{R}_h \tilde{w}, \tilde{v}_h) = a(\tilde{w}, \tilde{v}_h^\ell), \quad \text{for all} \quad \tilde{v}_h = (v_h, \psi_h) \in V_h \]

and set \( \tilde{R}_h \tilde{w} = (\tilde{R}_h^\Omega w, \tilde{R}_h^\Gamma \omega) \).

It is crucial to note that \( a \) and \( a_h \) defined in (4.1a) and (4.14c) do not couple bulk variables with surface variables. Therefore the Ritz map is given by a component-wise application of Ritz maps in the bulk and on the surface, i.e. we have

\[ \tilde{R}_h \tilde{w} = (\tilde{R}_h^\Omega w, \tilde{R}_h^\Gamma \omega). \]

Accordingly, we define the lift of components as \( R_h^\Omega w = (\tilde{R}_h^\Omega w)^\ell \) and \( R_h^\Gamma \omega = (\tilde{R}_h^\Gamma \omega)^\ell \), and hence \( \tilde{R}_h \tilde{w} = (R_h^\Omega w, R_h^\Gamma \omega) \).

The second-order error estimate for the Ritz map thus follows from a combination of existing results, cf. (Brenner & Scott, 2008, Section 5.4) for the bulk, and Lubich & Mansour (2015) for the surface Ritz map.

**Lemma 6.9** The error of the Ritz map (6.8), with the bilinear forms (4.1a) and (4.14c), on a smooth domain satisfies the following bounds, for \( h \leq h_0 \) with \( h_0 \) sufficiently small,

\[ \| \tilde{w} - R_h \tilde{w} \| \leq Ch\| \tilde{w} \|_{H^2}, \]

\[ | \tilde{w} - R_h \tilde{w} | \leq C h^2 \| \tilde{w} \|_{H^2}, \]

where the constants \( C > 0 \) are independent of \( h \) and \( \tilde{w} \in H^2 \).

**6.4.3 Proof of Theorem 4.4.** The structure of the proof is the same as in the previous sections. However, due to \( b \) containing the bulk–surface coupling, we present the complete analysis.

We start by estimating the defect. The Ritz map error estimate for acoustic boundary conditions Lemma 6.9, and interpolation estimates of Lemma 6.8 yield for \( \tilde{u}, \tilde{f} \in H^2 \)

\[ \| R_h \tilde{u} - \tilde{u} \|_* \leq C | R_h \tilde{u} - \tilde{u} | \leq ch^2, \]

\[ \| I_h \tilde{f} - \tilde{f} \|_* \leq C | I_h \tilde{f} - \tilde{f} | \leq ch^2. \]
The term $\|B(R_h \tilde{u} - \tilde{u})\|_*$ is estimated directly, using (2.17b), Lemma 6.9, and the following version of the trace inequality, for a function $w \in H^1(\Omega)$ and for an arbitrary but fixed $\varepsilon > 0$ we have the trace inequality

$$||\gamma w||_{L^2(\Gamma)} \leq \varepsilon \|\nabla w\|_{L^2(\Omega)} + c \frac{1}{\varepsilon} \|w\|_{L^2(\Omega)}.$$  \hspace{1cm} (6.11)

The proof of this trace inequality uses an $\varepsilon$-Young’s inequality instead of the standard one, cf. (Evans, 1998, (1) in Section 5.5), but otherwise it is the same as usual.

By choosing $\varepsilon = h^{1/2} > 0$ in (6.11), for $\tilde{v} = (v, \psi) \in V = H^1$, we obtain

$$b(R_h \tilde{u} - \tilde{u}, \tilde{v}) \leq c \int_{\Gamma} (\gamma(R_h^\Omega \tilde{u} - \tilde{u})) \psi - (R_h^\Gamma \delta - \delta)(\gamma v) \, d\sigma$$

$$\leq c \|\gamma (R_h^\Omega \tilde{u} - \tilde{u})\|_{L^2(\Gamma)} \|\psi\|_{L^2(\Gamma)} + c \|R_h^\Gamma \delta - \delta\|_{L^2(\Gamma)} \|\gamma v\|_{L^2(\Gamma)}$$

$$\leq c \left( h^{1/2} \|\gamma (R_h^\Omega \tilde{u} - \tilde{u})\|_{L^2(\Gamma)} + \frac{c}{h^{1/2}} \|R_h^\Omega \tilde{u} - \tilde{u}\|_{L^2(\Omega)} \|\psi\|_{L^2(\Gamma)} + c \|R_h^\Gamma \delta - \delta\|_{L^2(\Gamma)} \|\gamma v\|_{L^2(\Gamma)} \right)$$

$$\leq c \left( h^{1/2} \|\tilde{u}\|_{H^2(\Omega)} + c h^{2/2} \|\tilde{u}\|_{H^2(\Omega)} \|\psi\|_{L^2(\Gamma)} + c \|R_h^\Gamma \delta - \delta\|_{L^2(\Gamma)} \|\gamma v\|_{L^2(\Gamma)} \right)$$

$$\leq c \left( h^{1/2} \|\tilde{u}\|_{H^2(\Omega)} + c h^{2/2} \|\tilde{u}\|_{H^2(\Omega)} \|\psi\|_{L^2(\Gamma)} \right).$$

By the definition of the dual norm we have

$$\|B(R_h \tilde{u} - \tilde{u})\|_* \leq ch^{3/2}.$$  \hspace{1cm} (6.12)

For the convergence of the geometric errors note that by Lemma 6.7 we have the bound

$$\Delta b(\tilde{w}_h, \tilde{v}_h) \leq c h^2 \|\tilde{w}_h\|\tilde{v}_h\|_V.$$  \hspace{1cm} (6.13)

Therefore, it follows as in the proof of Theorem 4.1 that for $\tilde{u}, \tilde{v} \in V$ and $\tilde{f} \in H^2 \cap V$

$$\|\Delta m(\tilde{R}_h \tilde{u}, \cdot )\|_{*, h} + \|\Delta b(\tilde{R}_h \tilde{u}, \cdot )\|_{*, h} + \|\Delta m(\tilde{I}_h \tilde{f}, \cdot )\|_{*, h} \leq ch^2.$$  

Altogether, we obtain the defect estimate, for a sufficiently small $h \leq h_0$,

$$\|d_h\|_{*, h} \leq Ch^{3/2},$$

with a constant $C$ independent of $h$, but depending on Sobolev norms of the solution $u$ (and also its time derivatives).

For the errors in the initial data (5.13b) we again have $e_0 \leq ch^{3/2}$, by similar arguments used above to prove (6.12) and (6.13), together with Ritz map and interpolation error estimates.

We again recall that the error $u - u_h$ was estimated in terms of the defect and initial data error (5.13), the combination of this estimate with the above results yields the convergence bound:

$$|u(t) - u_h(t)| \leq Ch^{3/2}.$$  

7. Time discretisations

7.1 Matrix–vector formulation

We collect the nodal values of \( u_h(t) = \sum_{j=1}^{N} u_j(t) \phi_j(\cdot) \in V_h \) the solution of the semi-discrete problem (4.1) into the vector \( \mathbf{u}(t) = (u_1(t), \ldots, u_N(t)) \in \mathbb{R}^N \), and we define the matrices corresponding to the bilinear forms \( m_h, a_h \) and \( b_h \), respectively, and the load vector:

\[
\begin{align*}
M_{kj} &= m_h(\phi_j, \phi_k), \\
A_{kj} &= a_h(\phi_j, \phi_k), \\
B_{kj} &= b_h(\phi_j, \phi_k), \\
\mathbf{b}_k &= m_h(I_h f(\cdot, t), \phi_k),
\end{align*}
\]

(7.1)

where \( \phi_j \) (\( j = 1, \ldots, N \)) are the basis functions of \( V_h \). In the case of acoustic boundary conditions (4.14) all functions in \( V_h \) are vector valued, see (4.12). All matrices inherit their properties from their corresponding bilinear form, therefore, both matrices \( M \) and \( A \) are symmetric and positive definite, while the matrix \( B + \tilde{\rho} M \) (with \( \tilde{\rho} \geq 0 \)) is positive semi-definite, but can be non-symmetric.

Then the semi-discrete problem (4.1) is equivalent to the following matrix–vector formulation:

\[
\mathbf{M} \acute{\mathbf{u}}(t) + \mathbf{B} \mathbf{u}(t) + \mathbf{A} \mathbf{u}(t) = \mathbf{b}(t),
\]

(7.2)

with initial values \( \mathbf{u}(0) = \mathbf{u}^0 \) and \( \dot{\mathbf{u}}(0) = \mathbf{v}^0 \), where \( \mathbf{u}^0 \) and \( \mathbf{v}^0 \) collect the nodal values of the interpolations of \( u_0 \) and \( \dot{u}_1 \).

The above second order system of ordinary differential equations can be written as the first order system, by introducing the new variable \( \mathbf{v}(t) = \mathbf{M} \mathbf{u}(t) + \mathbf{B} \mathbf{u}(t) \),

\[
\dot{\mathbf{v}}(t) = -\mathbf{A} \mathbf{u}(t) + \mathbf{b}(t),
\]

\[
\dot{\mathbf{u}}(t) = \mathbf{M}^{-1} \mathbf{v}(t) - \mathbf{M}^{-1} \mathbf{B} \mathbf{u}(t).
\]

(7.3)

Using the variable \( \mathbf{y}(t) = (\mathbf{v}(t), \mathbf{u}(t))^T \in \mathbb{R}^{2N} \), the block matrices

\[
\mathbf{J} = \begin{pmatrix} 0 & \text{Id}_N \\ -\text{Id}_N & 0 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \mathbf{M}^{-1} & 0 \\ 0 & \mathbf{A} \end{pmatrix}
\]

and \( \hat{\mathbf{H}} = \begin{pmatrix} 0 & -\mathbf{M}^{-1} \mathbf{B} \\ 0 & 0 \end{pmatrix} \),

and the load vector \( \mathbf{b}(t) = (\mathbf{b}(t), 0)^T \), the ODE system (7.3) is equivalent to the first order ODE system

\[
\dot{\mathbf{y}}(t) = \mathbf{J}^{-1}(\mathbf{H} + \hat{\mathbf{H}}) \mathbf{y}(t) + \mathbf{b}(t),
\]

(7.4)

with initial value \( \mathbf{y}(0) = (\mathbf{M} \mathbf{v}^0 + \mathbf{B} \mathbf{u}^0, \mathbf{u}^0)^T \). The system is written in this form, since for \( \mathbf{B} = 0 \) it is Hamiltonian. This (further) geometric structure will be used to show stability of the full discretisation.

We further introduce the matrix

\[
\mathbf{S} = \begin{pmatrix} \mathbf{A}^{-1} & 0 \\ 0 & \mathbf{M} \end{pmatrix},
\]

(7.5)

and the corresponding induced norm, for arbitrary \( \mathbf{y} = (\mathbf{v}, \mathbf{u})^T \),

\[
\| \mathbf{y} \|_S = \mathbf{y}^T \mathbf{S} \mathbf{y} = \| \mathbf{v} \|^2_{\mathbf{A}^{-1}} + \| \mathbf{u} \|^2_\mathbf{M} = \| \mathbf{v}_h \|^2_{\mathbf{S}_h} + \| \mathbf{u}_h \|^2_{\mathbf{S}_h}.
\]

(7.6)
Along the proof of the stability bounds we need the following properties. The operator $J^{-1}H$ is skew-symmetric with respect to the $S$ inner product, direct computation shows:

\[
y^T SJ^{-1}Hy = \begin{pmatrix} A^{-1}v \\ Mu \end{pmatrix}^T \begin{pmatrix} 0 & -Id_N \\ Id_N & 0 \end{pmatrix} \begin{pmatrix} M^{-1}v \\ Au \end{pmatrix} \\
= -v^T A^{-1}Au + u^T MM^{-1}v \\
= 0.
\] (7.7)

While for $\hat{H}$, using the quasi monotonicity of the bilinear form $b_h$, we have the inequality

\[
y^T SJ^{-1}\hat{H}y = -u^T MM^{-1}Bu = -u^T (B + \hat{\rho}Id_N)u + \hat{\rho}u^TM^{-1}u \leq 0 + c\|u\|_M^2 \leq c\|y\|_S^2.
\] (7.8)

### 7.2 Implicit Runge–Kutta methods

The first-order system of ordinary differential equations (7.4) is discretised in time using an $s$-stage implicit Runge–Kutta method. For a fixed time step size $\tau > 0$, the method determines the approximations $y^n$ and the internal stages $Y^{ni}$, for $n\tau \leq T$, by

\[
Y^{ni} = y^n + \tau \sum_{j=1}^{n} a_{ij}Y^{nj}, \quad i = 1, \ldots, s, \quad (7.9a)
\]

\[
y^{n+1} = y^n + \tau \sum_{j=1}^{n} b_j Y^{nj} \quad (7.9b)
\]

where the internal stages satisfy

\[
\dot{Y}^{nj} = J^{-1}(H + \hat{H})Y^{nj} + b^nj, \quad j = 1, \ldots, s, \quad (7.9c)
\]

with $b^nj = b_j(t_n + c_j\tau)$, and where $\dot{Y}^{nj}$ is not a time derivative, only a suggestive notation. The method is determined by its coefficient matrix $A = (a_{ij})_{i,j=1}^s$, weights $b = (b_j)_{j=1}^s$ and nodes $c = (c_i)_{i=1}^s$.

In the following, we assume the Runge–Kutta method (7.9) to be algebraically stable, i.e. the coefficients $b_j \geq 0$ and the matrix with entries

\[b_ia_ij + b_ja_ii - b_ib_j \] is positive semi-definite.

We also assume that the coefficient matrix is invertible $A^{-1} = (w_{ij})_{i,j=1}^s$. Furthermore, the Runge–Kutta method is coercive, that is, there exists a positive definite diagonal matrix $\mathcal{D} \in \mathbb{R}^{s \times s}$ and $\alpha > 0$ such that

\[w^T \mathcal{D}A^{-1}w \geq \alpha w^T \mathcal{D}w \quad \text{for all} \quad w \in \mathbb{R}^s. \] (7.10)

The diagonal matrix is explicitly given by $\mathcal{D} = \text{diag}(b)(\text{diag}(c)^{-1} - Id_s)$, see Hairer & Wanner (1996).

The Runge–Kutta methods based on Gauss (and also those on Radau IA and Radau IIA) collocation nodes are known to be algebraically stable and coercive, and to have a non-singular coefficient matrix. For more details on these concepts and such methods we refer to (Hairer & Wanner, 1996, Chapter IV) and the references therein.
7.3 Convergence of the full discretisation with Gauss–Runge–Kutta methods

7.3.1 Stability. We first show stability for implicit Runge–Kutta methods applied to the ODE system (7.4). The proof of the stability bound is a straightforward simplification of the corresponding results of (Mansour, 2015, Lemma 4.1), or (Hochbruck & Pažur, 2015, Section 3), (Hochbruck et al., 2018, Lemma 5.1–5.2), (Kovács & Lubich, 2018, Lemma 4.3), where more general problems than (7.4) are considered.

**Lemma 7.1 (Stability)** The error \( e^n = y^n - y(t_n) \) between the numerical solution obtained by an \( s \)-stage implicit Runge–Kutta method and \( y(t_n) \) the exact solution of (7.4) satisfies the bound, for \( n \tau \leq T \),

\[
\|e^n\|_S \leq C \left( \|e^0\|_S^2 + \sum_{k=0}^{n-1} \sum_{j=1}^s \|D^{kj}\|_S^2 + \sum_{k=1}^n \|d^k\|_S^2 \right)^{1/2},
\]

where the defects \( D^{kj} \) and \( d^k \) are obtained by substituting the nodal values of the exact solution into the method (7.9). The constant \( C > 0 \) is independent of \( h, \tau \) and \( n \).

**Proof.** The proof is based on energy techniques for algebraically stable Runge–Kutta methods, using Lady Windermere’s fan (Hairer & Wanner, 1996, II.3 and I.7), and it is a straightforward simplification of the proof of Lemma 4.1 in Mansour (2015). A clear difference is the use of a different norm, induced by \( S \) here, which is due to the required weaker norms in the final estimates. Furthermore, the estimate (2.19) in Mansour (2015), analogous to (7.7) here, is slightly different, but it is used in the same way. The additional \( H \) is treated analogously as \( H \) but using the bound (7.8). \( \square \)

7.3.2 Convergence. Via the above stability bound and the defect bound for \( d_n \) (defined in (5.11)) from Lemma 5.1 (after using the suitable estimates from Section 6), we obtain the following fully discrete convergence estimates with the stage order \( s \) in time. The theorem holds for the general case as long as a defect bound in the dual norm is known. The result is stated simultaneously for all four exemplary cases considered in the paper.

**Theorem 7.1** Let the solution of the wave equation with dynamic boundary conditions be sufficiently regular in time \( u \in H^{s+2}(0, T; H^2(\Omega)) \) with \( y_u \in H^{s+2}(0, T; H^2(\Gamma)) \) (in addition to the regularity assumptions of Theorems 4.1–4.4). Then there is a \( \tau_0 > 0 \) and an \( h_0 > 0 \) such that, for \( \tau \leq \tau_0 \) and \( h \leq h_0 \), the error between the solution \( u(\cdot, t_n) \) and the fully discrete solution \( (u^h_n)^f \), obtained using first order finite elements and an \( s \)-stage Gauss–Runge–Kutta method, satisfies the following convergence estimates, for \( n \tau \leq T \),

\[
\|(u^h_n)^f - u(\cdot, t_n)\|_{L^2(\Omega)} + \|y((u^h_n)^f - u(\cdot, t_n))\|_{L^2(\Gamma)} \leq C(h^{k} + \tau^{\beta}),
\]

where the power \( k \) is given in Theorems 4.1, 4.2, 4.3 and 4.4, for the different problems, respectively. The constant \( C > 0 \) is independent of \( h, \tau \) and \( n \), but depends on the corresponding Sobolev norms of the exact solution \( u \) and on \( T \).

For purely second order problems with \( \beta = 0 \) the error on the boundary is measured in the \( H^{-1/2}(\Gamma) \) norm instead of the \( L^2(\Gamma) \) norm, cf. (4.5).

For problems with acoustic boundary conditions in the second term we have \( \delta \) on the boundary instead of \( y_u \), cf. (4.15).

**Proof.** Similarly as in Section 5.4, the fully discrete error between the solution \( u(\cdot, t_n) \) and the fully discrete solution \( u^h_n \) (with nodal values \( u^n \), the second component of \( y^n \)) is decomposed as, with \( e_{h}(t_n) = \)
The stability bound from Lemma 7.1 yields the equality (5.11).

The error of the Ritz map has been estimated before in Lemma 6.4 as $O(h^2)$.

In order to bound the second term we use the stability bound of Lemma 7.1. This part of the proof is analogous to the proof of (Mansour, 2015, Theorem 5.2), however due to some differences we carry it out below.

The vector $e_h(t)$ collecting the nodal values of the semi-discrete error $e_h(t)$, from Section 5.4, satisfies the ODE

$$M \dot{e}_h(t) + B e_h(t) + A e_h(t) = -\Phi(t),$$

with the vector $r(t) \in \mathbb{R}^N$ collecting the nodal values of the semi-discrete residual $r_h(t) \in V_h$ satisfying the equality (5.11).

The error equation is again rewritten as a first order ODE system, collecting the two errors $e(t) = (e_v(t), e_u(t))^T$ (here $e_v$ denotes the error in the $v$ component, cf. (7.3)), which satisfies an ODE system similar to (7.4), with $r(t) = (r(t), 0)^T$,

$$\dot{e}(t) = J^{-1}(H + \tilde{H})e(t) - r(t).$$

The fully discrete errors $e^n$ and $E^n_j$ then satisfy the error equations for the Runge–Kutta method:

$$E^{n+1} = e^n + \tau \sum_{j=1}^{s} b_j J^{-1}(H + \tilde{H})E^{n,j} - \left( \tau \sum_{j=1}^{s} a_{ij} r^{n,j} + D^{n,j} \right),$$

$$e^{n+1} = e^n + \tau \sum_{j=1}^{s} b_j J^{-1}(H + \tilde{H})E^{n,j} - \left( \tau \sum_{j=1}^{s} b_j r^{n,j} + d^{n+1} \right).$$

The stability bound from Lemma 7.1 yields

$$\|e^n\|_S \leq C \left( \|e^0\|_S^2 + \tau^2 \sum_{k=0}^{s-1} \|r^{k+1}\|_S^2 + \sum_{k=0}^{s-1} \sum_{j=1}^{s} \|D^{k,j}\|_S^2 + \sum_{k=1}^{s} \|d^k\|_S^2 \right)^{1/2}.$$  

The defect terms on the right-hand side are estimated separately. The defects related to time discretisations, using Taylor expansion, satisfy

$$d^{n+1} = \tau \int_{t_n}^{t_{n+1}} K((t-t_n)/\tau)\tilde{y}^{(s+1)}(t) \, dt,$$

$$D^{n,j} = \tau \int_{t_n}^{t_{n+1}} K_j((t-t_n)/\tau)\tilde{y}^{(s+1)}(t) \, dt,$$

with bounded Peano kernels $K$ and $K_j$, see (Gautschi, 2011, Section 3.2.6), and where the vector $\tilde{y}(t) = (\tilde{v}(t), \tilde{u}(t))^T$ is the nodal vector corresponding to

$$R_h \dot{u}(\cdot,t) + BR_h \dot{u}(\cdot,t) = \sum_{j=1}^{N} \tilde{v}_j(t) \phi_j \quad \text{and} \quad R_h \dot{u}(\cdot,t) = \sum_{j=1}^{N} \tilde{u}_j(t) \phi_j.$$
Therefore, by (7.6), norm equivalences and Lemma 6.4, we obtain

\[
\|d^{r+1}\|_S + \sum_{j=1}^{n} \|D^{rj}\|_S \leq c \tau \int_{t_n}^{t_{n+1}} |R_hu^{(s+2)}(\cdot, t) + BR_hu^{(s+1)}(\cdot, t)|_S + |R_hu^{(s+1)}(\cdot, t)|_S \, dt
\]

\[
\leq c \tau \int_{t_n}^{t_{n+1}} |R_hu^{(s+1)}(\cdot, t)| + \|R_hu^{(s+1)}(\cdot, t)\|_S + |R_hu^{(s+2)}(\cdot, t)|_S \, dt
\]

\[
\leq c \tau \sum_{j=1}^{n} \int_{t_n}^{t_{n+1}} \left( \|u^{(s+i)}(t)\|_{H^2(\Omega)}^2 + \|\gamma(u^{(s+i)}(t))\|_{H^2(\Gamma)}^2 \right)^{1/2} \, dt.
\]

For the semi-discrete residual we have, since \( r(t) = (\bar{r}(t), 0)^T \) and by (7.6),

\[
\tau \sum_{k=1}^{n-1} \sum_{j=1}^{s} \|r^{kj}\|_S = \tau \sum_{k=1}^{n-1} \sum_{j=1}^{s} \|\bar{r}^{kj}\|_{A+1} = \tau \sum_{k=1}^{n-1} \sum_{j=1}^{s} \|d_h(t_n + c_j \tau)\|_{A+1},
\]

which was estimated in Lemma 5.1 and Section 6 as \( O(h^k) \) with the appropriate \( k \) from Theorems 4.1, 4.2, 4.3, or 4.4.

Therefore, by (7.6) and the above bounds, we have

\[
|e_h|_h \leq \|e^o\|_S \leq C(h^k + \tau^s).
\]

This, together with Ritz map error estimates, finishes the proof.

As in Lubich & Ostermann (1995) and Mansour (2015); Kovács & Lubich (2018), under stronger regularity assumptions temporal convergence with the classical order \( p \) can also be shown. For Gauss–Runge–Kutta methods the classical order \( p = 2s \), see, e.g. Hairer & Wanner (1996).

We assume that, for the nodal values of the Ritz map of the exact solution \( \bar{y}(t) \) (see (7.16))

\[
\|J^{-1}([\bar{H} + \tilde{H}]^{k-1}\cdots J^{-1}([\bar{H} + \tilde{H}]^{k-l} \bar{y}(\cdot, t))\|_S \leq C_0,
\]

\[
\|J^{-1}([\bar{H} + \tilde{H}]^{s}J^{-1}([\bar{H} + \tilde{H}]^{k-1}\cdots J^{-1}([\bar{H} + \tilde{H}]^{k-l} \bar{y}(\cdot, t))\|_S \leq C_0,
\]

for \( s \geq 2 \) with a \( C_0 > 0 \), for all \( 0 \leq k_l \leq s-1 \) and \( l \geq s+1 \) with \( k_1 + \cdots + k_l + l \leq 2s + 1 \), where negative powers of operators are understood as the identity operator.

**Theorem 7.2** Let the solution of the wave equation with dynamic boundary conditions satisfy the regularity conditions of Theorem 7.1 and additionally those in (7.17). Then there is a \( \tau_0 > 0 \) and an \( h_0 > 0 \) such that, for \( \tau \leq \tau_0 \) and \( h \leq h_0 \), the error between the solution \( u(\cdot, t_n) \) and the fully discrete solution \( (u^n_h)^\ell \), obtained using first order finite elements and an \( s \)-stage Gauss–Runge–Kutta method, satisfies the following convergence estimates of classical order \( p = 2s \), for \( n \tau \leq T \),

\[
\|((u^n_h)^\ell - u(\cdot, t_n))\|_{L^2(\Omega)} + \|\gamma((u^n_h)^\ell - u(\cdot, t_n))\|_{L^2(\Gamma)} \leq C(h^k + \tau^s),
\]

where the power \( k \) is the same as in Theorem 7.1. The constant \( C > 0 \) is independent of \( h, \tau \), and \( n \), but depends on the solution \( u \), on \( C_0 \) from (7.17), and on \( T \).

Along the same remarks from Theorem 7.1 for \( \beta = 0 \), and for problems with acoustic boundary conditions.

**Proof.** The proof of this theorem directly follows the proof of (Lubich & Ostermann, 1995, Theorem 1) (parabolic problems), (Mansour, 2015, Theorem 5.3) (wave equations), where one additional order is gained by studying the modified error equations with the modified solution \( \tilde{Y}^nj = \tilde{Y}^nj + D^{nj} \), using the stability bound from Lemma 7.1 and the proof of Theorem 7.1. The process can be iterated until convergence with classical order is achieved.
8. Numerical experiments

In this section, we report on numerical experiments which illustrate that the proven spatial and temporal convergence rates of Theorem 4.1–4.4 and 7.2 are indeed observed (with the exception of strongly damped problems).

We implemented a finite element discretization of the wave equation with dynamic boundary conditions in FEniCS, cf. Alnæs et al. (2015), while for problems with acoustic boundary conditions we have used a Matlab implementation based on the P2Q2Iso2D code provided by Bartels et al. (2006). The triangulation of the domain was computed using the DistMesh package by Persson & Strang (2004).

For all our numerical experiments we have used bulk–surface finite elements using piecewise linear basis functions as a space discretisation and the Gauss–Runge–Kutta method with one node (s = 1) and of order two, i.e. the implicit midpoint rule, for time integration. For each test problem the numerical solutions were computed for a sequence of time step sizes \( \tau_j = \tau_{k-1}/2 \) with \( \tau_0 = 2^{-5} \) and a sequence of meshes with mesh widths \( h_j \approx h_{k-1}/2 \) with \( h_0 \approx 0.33 \).

In the case of numerical experiments when the exact solution is not known, the errors shown in the figures are obtained by comparing the numerical solution with a reference solution, which is computed using quadratic isoparametric elements on the finest mesh (for FEniCS simulation on the second finest mesh) and using the smallest time step size from above. Otherwise the exact and numerical solutions are compared. In the figures we plotted the errors at time \( T = 1 \), while for acoustic boundary conditions at \( T = 0.2 \).

All tests were carried out on the two dimensional unit disc and its boundary:

\[
\Omega = \{ x \in \mathbb{R}^2 \mid |x| < 1 \}, \quad \text{and} \quad \Gamma = \partial \Omega = \{ x \in \mathbb{R}^2 \mid |x| = 1 \}.
\]

8.1 Purely second-order dynamic boundary conditions – Theorem 4.1

For our first test, we consider the wave equation with purely second-order dynamic boundary condition (2.8) with \( \mu = \beta = 1 \) and \( \kappa = 1 \) with \( f_\Omega = f_\Gamma = 0 \). The initial values are

\[
u(x, 0) = e^{-20((x_1-1)^2 + x_2^2)}, \quad \text{and} \quad \dot{u}(x, 0) = 0,
\]

such that the solution shows a surface wave travelling along \( \Gamma \) due to the dynamic boundary condition.

The logarithmic plots show the errors, in the \( L^2 \) and \( H^1 \) norms, against the mesh width \( (h_j) \) in Figure 1, and the error in the \( L^2 \) norm against the time step size \( (\tau_j) \) in Figure 2. As shown by Figures 1 and 2 the \( O(h^2) \) spatial and \( O(\tau^2) \) temporal convergence rates, respectively, are in agreement with the theoretical convergence results. Note the reference lines for first and second order (dotted), and for order 3/2 (dashed). The the energy norms (i.e. \( H^1 \) norms) are plotted in order to allow easy comparison with the results of Hipp (2017).

The lines marked with different symbols correspond to different time steps and mesh refinements.

In Figure 1 (in both plots) we can observe two regions: a region where the spatial discretisation error dominates, matching to the \( O(h^2) \) order of convergence of our theoretical results (the error curves are parallel to a reference line), and a region, with small time step sizes, where the temporal discretization error dominates (the error curves flatten out). In Figure 2, the same description applies with reversed roles.
FIG. 1. Spatial convergence plots for purely second order problems (Theorem 4.1)

FIG. 2. Temporal convergence plots for purely second order problems (Theorem 7.2)
8.2 Advective dynamic boundary conditions – Theorem 4.2

For the tests with advection terms, we consider the wave equation with advective dynamic boundary condition (2.11) with $\mu = \beta = 1$ and $\kappa = 1$ with $f_\Omega = f_\Gamma = 0$, and constant advection $\mathbf{v}_\Omega(x,t) = (2, 0)^T$, $\mathbf{v}_\Gamma(x,t) = (0, 0)^T$. The initial values are the ones in (8.1).

The plots in Figure 3 show the same convergence plots as before, but for the above problem with advective dynamic boundary conditions with bulk diffusion ($\mathbf{v}_\Omega \neq 0$). The $O(h^{3/2})$ spatial convergence rates are in agreement with our theoretical results (note the reference lines).

8.3 Strongly damped dynamic boundary conditions – Theorem 4.3

For the tests with strong damping, we consider the wave equation with strongly damped dynamic boundary condition (2.14) with $\mu = \beta = 1$ and $\kappa = 1$ with $f_\Omega = f_\Gamma = 0$, and with damping coefficients $d_\Omega = 0.1$, $d_\Gamma = 0.2$. The initial values are the same as in (8.1).

The plots in Figure 5 show the same convergence plots as before, but for the above problem with strongly damped dynamic boundary conditions. We note here that for the case $\beta \neq d_\Gamma / d_\Omega$ the convergence order of Theorem 4.3 is not observed. The expected optimal second-order convergence rate is illustrated by our numerical experiment, cf. the remark after Theorem 4.3.
Fig. 4. Spatial convergence plots for problems with advective dynamic boundary conditions (Theorem 4.2)

Fig. 5. Spatial convergence plots for a problem with strongly damped dynamic boundary conditions (Theorem 4.3)
8.4 Acoustic boundary conditions – Theorem 4.4

Finally, we consider the wave equation with acoustic boundary condition (2.16) with unit constants, and with $f_{\Omega}$ and $f_{\Gamma}$ choose such that the exact solution is $u(x,t) = \sin(2\pi t)(x_1^2 + x_2^2)^{k/2}$ and $\delta(x,t) = k(2\pi)^{-1}\cos(2\pi t)(x_1^2 + x_2^2)^{k/2}$ for $k = 1.2$. The initial values are the interpolations of the exact initial data.

The plots in Figure 6 show the same convergence plots as before, but for the above problem with acoustic boundary conditions. The $O(h^{3/2})$ spatial convergence rates are in agreement with our theoretical results.

![Figure 6](image-url)
9. Conclusions

Albeit the fact that the solutions of wave equations with dynamic boundary conditions have better regularity and stability properties (on the boundary) than classical Neumann or Dirichlet boundary conditions, our results show that in some cases one actually cannot expect $O(h^2)$ convergence in the $L^2$ norm. As our numerical tests show, these reduced convergence rates can actually be observed in simulations and are not due to a crude error analysis. Note however that also in these cases we observed $O(h^2)$ convergence for many configurations we numerically tested.

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