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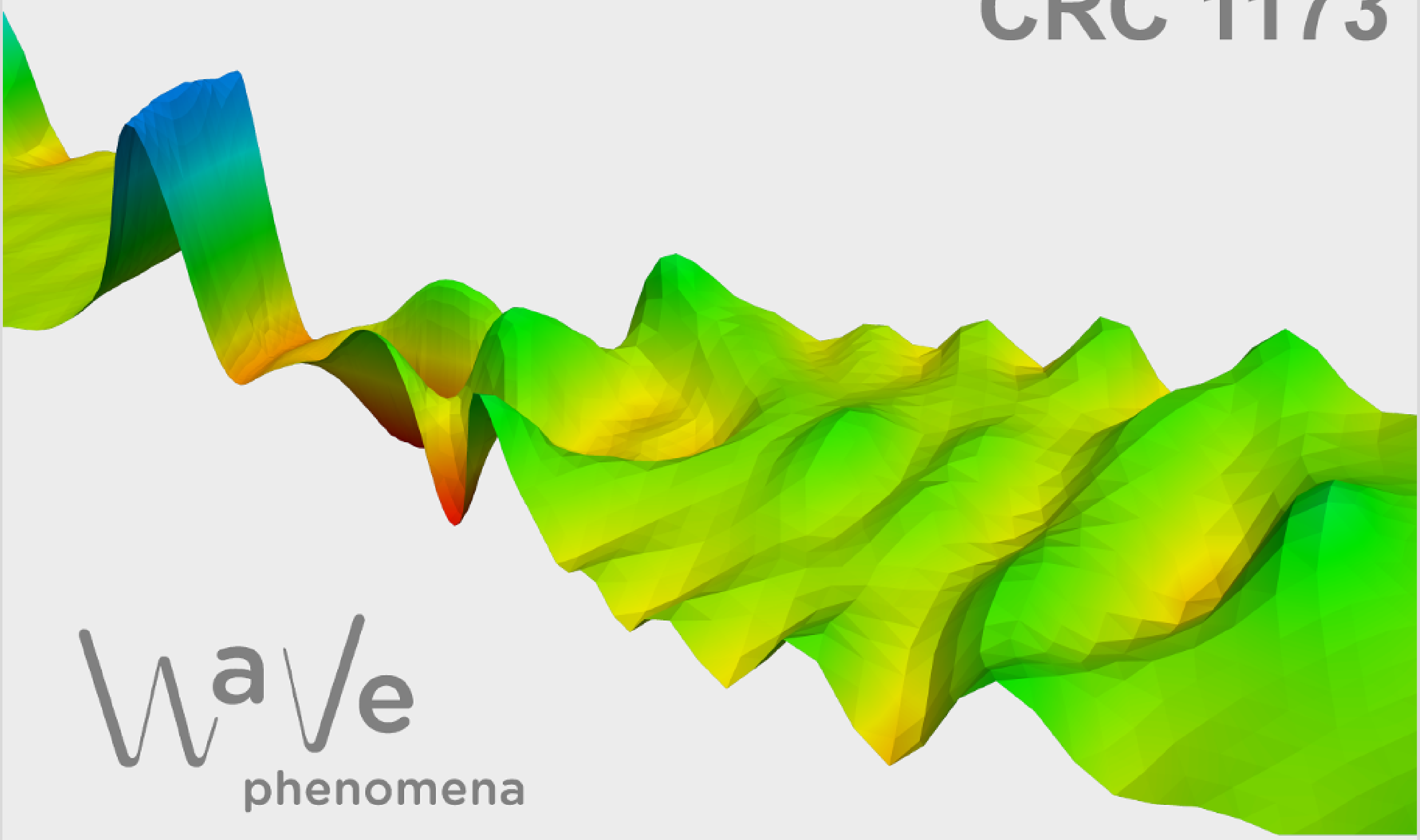

Wave phenomena

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A model for the periodic water wave problem and its long wave amplitude equations

Roman Bauer, Patrick Cummings and Guido Schneider

Abstract. We are interested in the validity of the KdV and of the long wave NLS approximation for the water wave problem over a periodic bottom. Approximation estimates are non-trivial, since solutions of order $\mathcal{O}(\varepsilon^2)$, resp. $\mathcal{O}(\varepsilon)$, have to be controlled on an $\mathcal{O}(1/\varepsilon^3)$, resp. $\mathcal{O}(1/\varepsilon^2)$, time scale. In contrast to the spatially homogeneous case, in the periodic case new quadratic resonances occur and make a more involved analysis necessary. For a phenomenological model we present some results and explain the underlying ideas. The focus is on results which are robust in the sense that they hold under very weak non-resonance conditions without a detailed discussion of the resonances. This robustness is achieved by working in spaces of analytic functions. We explain that, if analyticity is dropped, the KdV and the long wave NLS approximation make wrong predictions in case of unstable resonances and suitably chosen periodic boundary conditions. Finally we outline, how, we think, the presented ideas can be transferred to the water wave problem.

Mathematics Subject Classification (2000). Primary 76B15; Secondary 35Q53, 35Q55.

Keywords. KdV approximation, NLS approximation, error estimates.

1. Introduction

The water wave problem with a free surface $\Gamma(t) = \{y = \eta(x,t) : x \in \mathbb{R}\}$ over an L-periodic bottom $\mathcal{B} = \{y = b(x) : b(x) = b(x+L),\ x \in \mathbb{R}\}$ is governed by the

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system of nonlinear PDEs

$$\begin{aligned} \partial_x^2\phi + \partial_y^2\phi &= 0, && \text{in} \quad \Omega(t), \\ \partial_{\vec{n}}\phi &= 0, && \text{on} \quad \mathcal{B}, \\ \partial_t\eta &= \partial_y\phi - (\partial_x\eta)\partial_x\phi, && \text{on} \quad \Gamma(t), \\ \partial_t\phi = -\frac{1}{2}((\partial_x\phi)^2 + (\partial_y\phi)^2) &+ \mu\partial_x\left(\frac{\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}}\right) - \eta, && \text{on} \quad \Gamma(t), \end{aligned}$$

for the flow potential ϕ and the elevation of the top surface η, where $\Omega(t) = \{(x,y) : b(x) < y < \eta(x,t)\}$, and where $\mu \geq 0$ is the surface tension parameter. It is well known that the water wave problem is completely described by the elevation η of the top surface and the horizontal velocity $w = \partial_x\phi|_\Gamma$ at the top surface $\Gamma(t)$.

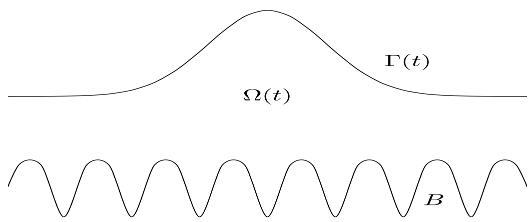

FIGURE 1. The water wave problem over a periodic bottom

We are interested in the qualitative behavior of the solutions:

- The linearized problem is solved by Bloch modes
$$\begin{pmatrix} \eta \\ w \end{pmatrix} = e^{ilx} f_n(l,x)^{i\omega_n(l)t},$$
with $n \in \mathbb{Z}/\{0\}$, $f_n(l,x) = f_n(l,x+L) \in \mathbb{C}^2$ and $l \in \left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$. Curves of eigenvalues $\omega_n(l)$ are sketched in Figure 2. They are ordered as $\omega_n(l) \leq \omega_{n+1}(l)$ with $\omega_{-n}(l) = -\omega_n(l)$. Due to the periodicity of the bottom, spectral gaps can occur.
- With the ansatz
$$\begin{pmatrix} \eta \\ w \end{pmatrix} = \varepsilon^2 A(\varepsilon(x-ct), \varepsilon^3 t) f_1(0,x) \tag{1.1}$$
a KdV equation
$$\partial_T A = \nu_1 \partial_X^3 A + \nu_2 \partial_X (A^2), \tag{1.2}$$
can be derived, with amplitude $A(X,T) \in \mathbb{R}$, with group velocity $c \in \mathbb{R}$, with $0 < \varepsilon \ll 1$ a small perturbation parameter, and with coefficients $\nu_1, \nu_2 \in \mathbb{R}$.
- With the ansatz
$$\begin{pmatrix} \eta \\ w \end{pmatrix} = \varepsilon A(\varepsilon(x-ct), \varepsilon^2 t) f_n(0,x) e^{i\omega_n(0)t} + \text{c.c.} \qquad (n \neq \pm 1) \tag{1.3}$$

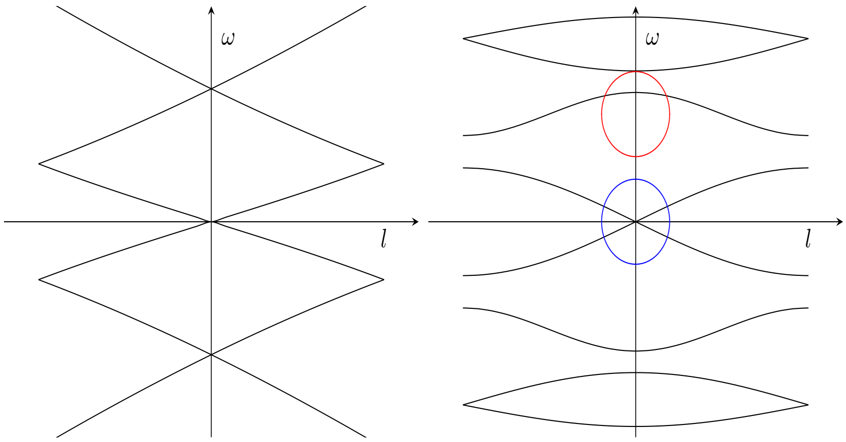

FIGURE 2. The panels show the curves of eigenvalues $l \mapsto \omega_n(l)$, $n \in \mathbb{Z}/\{0\}$ of the linearized water wave problem. The left panel shows the curves of eigenvalues in the homogeneous case, i.e., $b(x) = \text{const.}$, in case of positive surface tension. For $L = 2\pi$ the dispersion relation $\omega^2 = (k + \mu k^3)\tanh(k)$ in Fourier space transfers to Bloch space by setting $k = n + l$ with $n \in \mathbb{Z}$. In case of periodic bottom, spectral gaps, such as sketched in the right panel, can occur. The modes in the blue circle can be described by a KdV approximation. The modes in the red circle can be described by an NLS approximation.

an NLS equation

$$i\partial_T A = \nu_1 \partial_X^2 A + \nu_2 A\,|A|^2\,, \tag{1.4}$$

can be derived, with amplitude $A(X,T) \in \mathbb{C}$, with group velocity $c \in \mathbb{R}$, with $0 < \varepsilon \ll 1$ a small perturbation parameter, and with coefficients $\nu_1, \nu_2 \in \mathbb{R}$.

Our goal is to prove error estimates between these approximations and true solutions of the water wave problem. Such estimates are a nontrivial task since for the KdV approximation we have to control solutions of order $\mathcal{O}(\varepsilon^2)$ on an $\mathcal{O}(1/\varepsilon^3)$-time scale, and for the NLS approximation we have to control solutions of order $\mathcal{O}(\varepsilon)$ on an $\mathcal{O}(1/\varepsilon^2)$-time scale.

A) In the homogeneous case, $b(x) = -1$, there are two fundamentally different approaches to prove KdV approximation results. For solutions to the KdV equation with analytic initial conditions a Cauchy-Kowalevskaya based approach can be chosen, see [14, 17]. Working in spaces of analytic functions gives some artificial smoothing which allows to gain the above explained missing order w.r.t. ε via the derivative in front of the nonlinear terms in the KdV equation. This 'analytic' approach is very robust and works without a detailed analysis of the underlying problem, but gives not optimal results.

For initial conditions in Sobolev spaces the underlying idea to gain such estimates is conceptually rather simple, namely the construction of a suitable chosen energy which include the terms of order $\mathcal{O}(\varepsilon^2)$ in the equation for the error, such that for the energy finally $\mathcal{O}(\varepsilon^3 t)$ growth rates occur. However, the method is less robust since for every single original system a different energy occurs and the major difficulty is the construction of this energy. Estimates that the formal KdV approximation and true solutions of the different formulations of the homogeneous water wave problem stay close together over the natural KdV time scale have been shown for instance in [7, 21, 22, 8] by using this approach.

B) The NLS approximation has been justified in various papers for a number of original systems, cf. [13, 15, 18]. If no quadratic terms are present in the original system a simple application of Gronwall's inequality allows to prove the validity of the NLS approximation. Quadratic terms can be eliminated by a near identity change of variables, if a non-resonance condition is satisfied. This non resonance condition has been weakened in a number of papers, cf. [19]. The NLS approximation has been justified for the water wave problem in case of infinite depth and no surface tension [24, 23], and in case of finite depth and no surface tension [10].

A+B) KdV approximation results in the spatially periodic case are only known for small perturbations of a flat bottom, cf. [3, 12, 4]. An NLS approximation result in a periodic medium has been obtained in [2]. However, due to $\omega_1(0) = 0$ in the spectral picture plotted in Figure 2 the approach from [2] does not transfer to the situation we are interested in.

It is the purpose of this paper to present for a phenomenological model, which has similar properties as the water wave problem, some approximation results and the underlying ideas of their proofs. One focus is on results which are robust in the sense that they hold under very weak non-resonance conditions without a detailed discussion of the resonances. This robustness is achieved by working in spaces of analytic functions. We explain that, if analyticity is dropped, the KdV approximation and the long wave NLS approximation make wrong predictions in case of unstable resonances. Finally we outline how, we think, the presented ideas can be transferred to the water wave problem.

2. The Boussinesq Klein-Gordon model

The model, which we consider, is a Boussinesq equation coupled with a Klein-Gordon equation, in the following called BKG system, namely

$$\partial_t^2 u = \alpha^2 \partial_x^2 u + \partial_t^2 \partial_x^2 u + \partial_x^2 (a_{uu} u^2 + 2 a_{uv} uv + a_{vv} v^2), \tag{2.1}$$

$$\partial_t^2 v = \partial_x^2 v - v + b_{uu} u^2 + 2 b_{uv} uv + b_{vv} v^2, \tag{2.2}$$

where $u = u(x,t), v = v(x,t), x, t \in \mathbb{R}$, and coefficients $\alpha > 0$, $a_{uu}, \ldots, b_{vv} \in \mathbb{R}$. The curves of eigenvalues are given by

$$\omega_u(k) = \frac{\alpha k}{\sqrt{1+k^2}} \qquad \text{and} \qquad \omega_v(k) = \sqrt{1+k^2}. \tag{2.3}$$

Hence the spectral picture of the water wave problem over a periodic bottom, which is qualitatively sketched in the right panel of Figure 2, and of the BKG system, see Figure 3, look qualitatively the same. Moreover, in both systems the nonlinear terms vanish for modes associated to $\omega_{\pm 1}$, resp. ω_u, at the wave numbers $l = 0$, resp. $k = 0$.

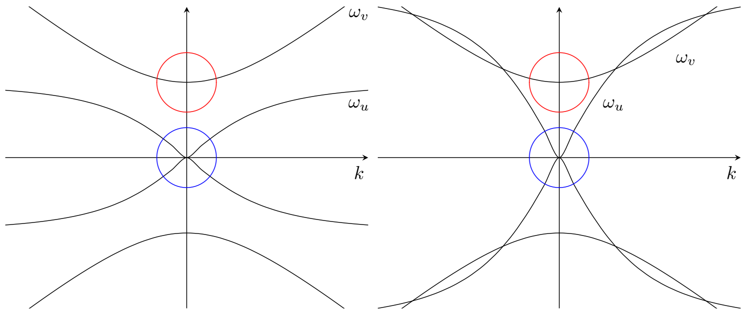

FIGURE 3. The curves of eigenvalues $\pm\omega_u$, $\pm\omega_v$ for the linearized BKG system plotted as a function over the Fourier wave numbers in case $\alpha^2 = 1$ (left) and $\alpha^2 = 5$ (right). The modes in the blue circles are described by the KdV approximation. The modes in the red circles are described by the NLS approximation.

Inserting the ansatz

$$\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t) = \varepsilon^2 A(\varepsilon(x-\alpha t), \varepsilon^3 t) \qquad \text{and} \qquad \varepsilon^2 \psi_v^{\mathrm{KdV}} = 0 \tag{2.4}$$

into (2.1)-(2.2) yields the KdV equation

$$\partial_T A = \nu_1 \partial_X^3 A + \nu_2 \partial_X (A^2), \tag{2.5}$$

with coefficients ν_1, $\nu_2 \in \mathbb{R}$, the long temporal variable $T = \varepsilon^3 t$, and the long spatial variable $X = \varepsilon(x - \alpha t)$.

Inserting the ansatz

$$\varepsilon \psi_u^{\mathrm{NLS}}(x,t) = \mathcal{O}(\varepsilon^2) \qquad \text{and} \qquad \varepsilon \psi_v^{\mathrm{NLS}} = \varepsilon A(\varepsilon x, \varepsilon^2 t) e^{it} + c.c. + \mathcal{O}(\varepsilon^2) \tag{2.6}$$

into (2.1)-(2.2) yields the NLS equation

$$i\partial_T A = \nu_1 \partial_X^2 A + |A|^2 A = 0, \tag{2.7}$$

with coefficients ν_1, $\nu_2 \in \mathbb{R}$, the long temporal variable $T = \varepsilon^2 t$, and the long spatial variable $X = \varepsilon x$. The ansatz is called long wave NLS approximation since we have $k_0 = 0$ for the wave number of the underlying carrier wave $e^{i(k_0 x + \omega_0 t)}$.

We are interested in the validity of the KdV approximation and long wave NLS approximation for the BKG system. For this phenomenological model we present some approximation results and explain the underlying ideas. Approximation estimates are non-trivial, since solutions of order $\mathcal{O}(\varepsilon^2)$, resp. $\mathcal{O}(\varepsilon)$, have to be controlled on an $\mathcal{O}(1/\varepsilon^3)$, resp. $\mathcal{O}(1/\varepsilon^2)$, time scale. That these approximation results are really subtle is explained in the next section when the resonance structure of the problem is discussed.

3. The resonance structure

The BKG system is written as

$$\partial_t U = \Lambda U + N(U, U),$$

where ΛU stands for the linear terms and where the nonlinear terms are represented by the symmetric bilinear mapping $N(U, U)$.

3.1. Resonances in the KdV case

The error $\varepsilon^\beta R = U - \varepsilon^2 \psi^{\mathrm{KdV}}$ made by the KdV approximation $\varepsilon^2 \psi^{\mathrm{KdV}}$ satisfies

$$\partial_t R = \Lambda R + 2\varepsilon^2 N(\psi^{\mathrm{KdV}}, R) + \mathcal{O}(\varepsilon^3),$$

were $\mathcal{O}(\varepsilon^3)$ contains the nonlinear terms w.r.t. R and the residual terms, i.e., the terms which do not cancel after inserting the KdV approximation into the BKG system. In order to obtain $\mathcal{O}(\varepsilon^3)$ for the residual terms in this equation, higher order terms have to be added to the KdV approximation $\varepsilon^2 \psi^{\mathrm{KdV}}$. This is standard and so we will concentrate on other aspects. Due to the term $2\varepsilon^2 N(\psi^{\mathrm{KdV}}, R)$ a simple application of Gronwall's inequality is not sufficient to obtain an $\mathcal{O}(1)$-bound for R on the long $\mathcal{O}(1/\varepsilon^3)$-time scale. The difficulty can be overcome by normal form transformations and energy estimates. In this section we will concentrate on the normal form transformations, i.e., near identity change of variables, and the resonances which prevent the elimination of the quadratic terms by normal form transformations. A term $\psi^{\mathrm{KdV}} R_j$ can be eliminated in the i-th equation with a near identity change of variables

$$R_i = \widetilde{R}_i + \varepsilon^2 M(\psi^{\mathrm{KdV}}, \widetilde{R}_j),$$

with M a suitably chosen bilinear mapping, if the non-resonance condition

$$\omega_i(k) \neq \omega_j(k)$$

is satisfied for all $k \in \mathbb{R}$. Herein, ω_j is the curve of eigenvalues corresponding to R_j. Hence, in the R_u-equation the term $2\psi^{\mathrm{KdV}} R_u$ cannot be eliminated. If only this term is resonant, it can be controlled with energy estimates. However, for $\alpha > 2$ there are $k_1, k_2 > 0$ with $\omega_u(k_j) = \omega_v(k_j)$ for $j = 1, 2$, see the right panel of Figure 3. Hence, the terms $2\psi^{\mathrm{KdV}}(0) R_v(k_j)$ for $j = 1, 2$ cannot be eliminated in the R_u-equation.

Similarly, in the R_v-equation the term $2\psi^{\mathrm{KdV}} R_v$ cannot be eliminated. If only this term is resonant, it can be controlled with energy estimates. The fact, that $\omega_u(k_j) = \omega_v(k_j)$ for $j = 1, 2$, implies now that the terms $2\psi^{\mathrm{KdV}}(0) R_u(k_j)$ for $j = 1, 2$ cannot be eliminated in the R_v-equation.

These resonances can be used to prove that in case of $2\pi/k_1$-periodicity, with $k_2 \notin k_1\mathbb{N}$, the KdV equation makes wrong predictions about the dynamics of the BKG system. In order to illustrate this, we make the ansatz

$$\begin{aligned} u &= \varepsilon^2 A(\varepsilon^2 t) + \varepsilon^n A_1(\varepsilon^2 t) e^{i\omega_u(k_1)t} e^{ik_1 x} + \varepsilon^n A_{-1}(\varepsilon^2 t) e^{-i\omega_u(-k_1)t} e^{-ik_1 x}, \\ v &= \varepsilon^n B_1(\varepsilon^2 t) e^{i\omega_v(k_1)t} e^{ik_1 x} + \varepsilon^n B_{-1}(\varepsilon^2 t) e^{i\omega_v(-k_1)t} e^{-ik_1 x}, \end{aligned}$$

to analyze the resonance at the wave number $k = k_1$. Equating the coefficients to zero at $\varepsilon^4 e^{0it} e^{0ix}$ in the u-equation and at $\varepsilon^{n+2} e^{i\omega_u(k_1)t} e^{ik_1 x}$ both in the u and v-equation yields, with $\tau = \varepsilon^2 t$, that

$$\partial_\tau^2 A = 0, \tag{3.1}$$

$$2i\omega_u(k_1)\partial_\tau A_1 = -2k_1^2 (a_{uu} A A_1 + a_{uv} A B_1), \tag{3.2}$$

$$2i\omega_v(k_1)\partial_\tau B_1 = 2(b_{uu} A A_1 + b_{uv} A B_1). \tag{3.3}$$

The first equation is the KdV equation, i.e., (2.5) restricted to the wave number $k = 0$. Hence, for instance on a $\mathcal{O}(\varepsilon^{-1/2})$-time scale w.r.t. τ, the variable A can be considered to be constant in time. The last two equations can be written as

$$\partial_\tau \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

with

$$M = \frac{1}{i\omega_u(k_1)} \begin{pmatrix} -a_{uu} k_1^2 A & -a_{uv} k_1^2 A \\ b_{uu} A & b_{uv} A \end{pmatrix}.$$

By choosing the coefficients a_{uu}, a_{uv}, b_{uu}, and b_{uv} in a suitable way the matrix M has eigenvalues with strictly positive real part. Hence, growth rates $e^{\beta\tau} = e^{\beta\varepsilon^2 t} = e^{\beta T/\varepsilon}$ with a $\beta > 0$ occur. These allow us to bring $\varepsilon^n A_1$ and $\varepsilon^n B_1$, which are initially of order $\mathcal{O}(\varepsilon^n)$, to an order $\mathcal{O}(\varepsilon^2)$ at a time $T = \mathcal{O}((n-2)\varepsilon|\ln(\varepsilon)|) \ll 1$. Therefore, we have that $v = \mathcal{O}(\varepsilon^2)$ far before the natural time scale of the KdV equation. Hence, in this situation the KdV approximation makes wrong predictions. These calculations can be transferred into a rigorous proof using analysis as presented in [20].

3.2. Resonances in the NLS case

The error $\varepsilon^\beta R = U - \varepsilon\psi^{\text{NLS}}$, made by the NLS approximation $\varepsilon\psi^{\text{NLS}}$, satisfies

$$\partial_t R = \Lambda R + 2\varepsilon N(\psi^{\text{NLS}}, R) + \mathcal{O}(\varepsilon^2),$$

where $\mathcal{O}(\varepsilon^2)$ contains the nonlinear terms w.r.t. R and the residual terms, i.e., the terms which do not cancel after inserting the NLS approximation into the BKG system. In order to obtain $\mathcal{O}(\varepsilon^2)$ for the residual terms in this equation, higher order terms have to be added to the NLS approximation $\varepsilon\psi^{\text{NLS}}$. This is standard and so we will concentrate on other aspects. Due to the term $2\varepsilon N(\psi^{\text{NLS}}, R)$ a simple application of Gronwall's inequality is not sufficient to obtain an $\mathcal{O}(1)$-bound for R on the long $\mathcal{O}(1/\varepsilon^2)$-time scale. A term $\psi^{\text{NLS}} R_j$ can be eliminated

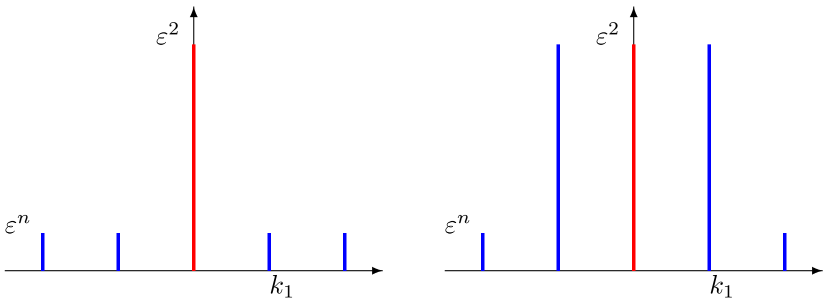

FIGURE 4. The mode distribution for $t = 0$ in the KdV case and the mode distribution for $t = \mathcal{O}(|\ln \varepsilon|/\varepsilon^2) \ll \mathcal{O}(1/\varepsilon^3)$. In the NLS case the magnitude ε^2 has to be replaced by ε and the second time is $t = \mathcal{O}(|\ln \varepsilon|/\varepsilon) \ll \mathcal{O}(1/\varepsilon^2)$. The KdV/NLS approximation is no longer valid in the right picture, since the modes at $\pm k_1$ are of the same order as the KdV/NLS modes at $k = 0$.

in the i-th equation by a near identity change of variables if the non-resonance condition

$$\omega_i(k) \neq 1 + \omega_j(k)$$

is satisfied for all $k \in \mathbb{R}$.

The resonances found in Figure 5 can be used to prove that in case of $2\pi/k_1$-periodicity with $k_2 \notin k_1\mathbb{N}$ the NLS equation makes wrong predictions about the dynamics of the BKG system. In order to illustrate this we make the ansatz

$$\begin{aligned} u &= \varepsilon^n A_1(\varepsilon t) e^{-i\omega_u(k_1)t} e^{ik_1 x} + \text{c.c.}, \\ v &= \varepsilon B(\varepsilon t) e^{it} + \varepsilon^n B_1(\varepsilon t) e^{-i\omega_v(k_1)t} e^{ik_1 x} + \text{c.c.}, \end{aligned}$$

to analyze the resonance at the wave number $k = k_1$. Equating the coefficients to zero at $\varepsilon e^{it} e^{0ix}$ in the v-equation, at $\varepsilon^n e^{-i\omega_u(k_1)t} e^{ik_1 x}$ in the u-equation, and at $\varepsilon^n e^{-i\omega_v(k_1)t} e^{ik_1 x}$ in the v-equation, yields, with $\tau = \varepsilon^2 t$, that

$$2i\partial_\tau B = \mathcal{O}(\varepsilon), \tag{3.4}$$
$$-2i\omega_u(k_1)\partial_\tau A_1 = -2a_{vv} k_1^2 B B_1, \tag{3.5}$$
$$-2i\omega_v(k_1)\partial_\tau B_1 = 2b_{uv} \overline{B} A_1, \tag{3.6}$$

where we used $-\omega_u(k_1) = 1 - \omega_v(k_1)$. The first equation is the NLS equation, i.e., (2.7) restricted to the wave number $k = 0$. Hence, for instance on a $\mathcal{O}(\varepsilon^{-1/2})$-time scale w.r.t. τ, the variable B can be considered to be constant in time. The last two equations can be written as

$$\partial_\tau^2 A_1 = \Gamma A_1 \qquad \text{resp.} \qquad \partial_\tau^2 B_1 = \Gamma B_1,$$

with

$$\Gamma = \frac{|B|^2}{\omega_u(k_1)\omega_v(k_1)} a_{vv} b_{uv}.$$

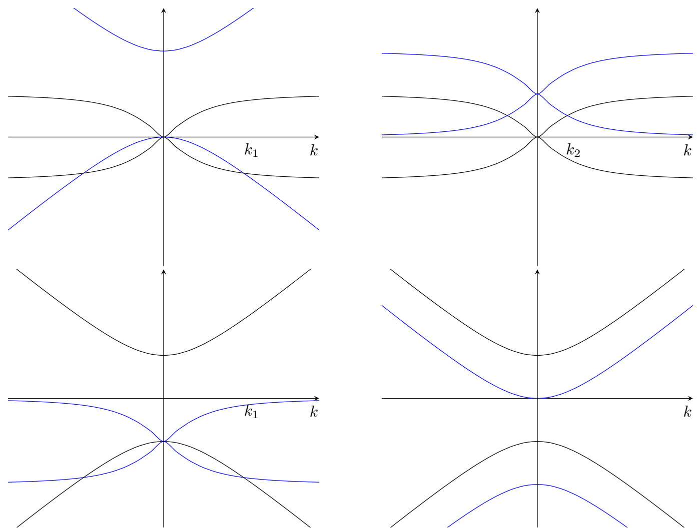

FIGURE 5. The intersection points of $k \mapsto \omega_i(k)$, and $k \mapsto \omega_v(0) \pm \omega_j(k)$ correspond to resonances. The associated nonlinear terms cannot be eliminated by near identity change of coordinates. The two graphs in the first line show that in the R_u-equation terms $\psi^{\mathrm{NLS}} R_v(k_1)$ and $\psi^{\mathrm{NLS}} R_u(k_2)$ for wave numbers k_1 and k_2 cannot be eliminated. For the same wave number k_1 the term $\psi^{\mathrm{NLS}} R_u(k_1)$ cannot be eliminated in the R_v-equation.

Since $\omega_u(k_1)\omega_v(k_1) > 0$, by choosing $a_{vv}b_{uv}$ positive we have growth rates $e^{\beta\tau} = e^{\beta\varepsilon t} = e^{\beta T/\varepsilon}$ with a $\beta > 0$. These allow us to bring $\varepsilon^n A_1$ and $\varepsilon^n B_1$, which are initially of order $\mathcal{O}(\varepsilon^n)$, to an order $\mathcal{O}(\varepsilon)$ at a time $T = \mathcal{O}((n-1)\varepsilon|\ln(\varepsilon)|) \ll 1$. Therefore, we have that $v = \mathcal{O}(\varepsilon)$ far before the natural scale of the NLS equation. Hence, in this situation the NLS approximation makes wrong predictions. These calculations can be transferred into a rigorous proof using analysis as presented in [20].

4. Validity in the non-oscillatory case

In this section we discuss the validity of the KdV approximation for the BKG system. There are essentially three different results which we would like to present. As in [1] the subsequent analysis is not only valid for the KdV limit, but also for the inviscid Burgers and the Whitham limit.

4.1. Approach 1: using normal form transformations in the non-resonant case

In [6] the BKG system has been considered in case $\alpha = 1$ or more general in case without additional resonances, i.e., in case $\omega_u(k) \neq \omega_v(k)$ for all $k \in \mathbb{R}$. Then with normal form transformations and energy estimates the following result has been established.

Theorem 4.1. *Let $A \in C([0,T_0], H^8(\mathbb{R},\mathbb{R}))$ be a solution of the KdV equation* (2.5)*. Then there exist ε_0, $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions (u,v) of* (2.1)-(2.2) *with*

$$\sup_{t\in[0,T_0/\varepsilon^3]} \sup_{x\in\mathbb{R}} |(u,v)(x,t) - (\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t), 0)| \leq C\varepsilon^{7/2}.$$

Sketch of the proof. We write a true solution of (2.1)-(2.2) as approximation plus error, i.e., $u = \varepsilon^2\psi_u + \varepsilon^\beta R_u$ and $v = \varepsilon^4\psi_v + \varepsilon^\beta R_v$ with $\beta = 7/2$, where $(\varepsilon^2\psi_u, \varepsilon^4\psi_v)$ is an improved approximation which is formally $\mathcal{O}(\varepsilon^4)$ close to $(\varepsilon^2\psi_u^{\mathrm{KdV}}(x,t), 0)$. The error satisfies

$$\partial_t^2 R_u = \alpha^2\partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + 2\varepsilon^2\partial_x^2(a_{uu}\psi_u R_u + a_{uv}\psi_u R_v) + \mathcal{O}(\varepsilon^3), \quad (4.1)$$

$$\partial_t^2 R_v = \partial_x^2 R_v - R_v + 2\varepsilon^2 b_{uu}\psi_u R_u + 2\varepsilon^2 b_{uv}\psi_u R_v + \mathcal{O}(\varepsilon^3), \quad (4.2)$$

where we used the same symbols for the old and new variables. After elimination of the non-resonant terms the system decouples up to order $\mathcal{O}(\varepsilon^3)$, namely

$$\partial_t^2 R_u = \alpha^2\partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + 2\varepsilon^2\partial_x^2(a_{uu}\psi_u R_u) + \mathcal{O}(\varepsilon^3),$$

$$\partial_t^2 R_v = \partial_x^2 R_v - R_v + 2\varepsilon^2 b_{uv}\psi_u R_v + \mathcal{O}(\varepsilon^3).$$

Then multiplying the first equation with $\partial_t\partial_x^{-2}R_u$ and the second equation with $\partial_t R_v$ gives, after integration w.r.t. x, the energy estimates

$$\partial_t \int ((\partial_t\partial_x^{-1}R_u)^2 + \alpha^2(R_u)^2 + (\partial_t R_u)^2 - 2\varepsilon^2 a_{uu}\psi_u(R_u)^2$$
$$+(\partial_t R_v)^2 + (\partial_x R_v)^2 + (R_v)^2 - 2\varepsilon^2 b_{uv}\psi_u(R_v)^2)dx = \mathcal{O}(\varepsilon^3),$$

where we used integration by parts, $\partial_t\psi_u = \mathcal{O}(\varepsilon)$, and $\partial_x\psi_u = \mathcal{O}(\varepsilon)$. Hence the integral on the right hand side stays $\mathcal{O}(1)$-bounded on an $\mathcal{O}(1/\varepsilon^3)$-time scale. Since similar estimates can be obtained for the derivatives, the H^s-norm of the error stays $\mathcal{O}(1)$-bounded on the $\mathcal{O}(1/\varepsilon^3)$-time scale.

4.2. Approach 2: using the Hamiltonian

The second result is obtained when the lowest order part of the error equation can be written as Hamiltonian system. Then the ideas of [1] apply. There, a first justification result for the KdV approximation of a scalar dispersive PDE, posed in a spatially periodic medium of non-small contrast, has been obtained via some suitably chosen energy. Surprisingly this method also works in case of stable resonances. It is based on

$$\frac{d}{dt}H(R(t),t) = \nabla H \cdot \partial_t R(t) + \partial_t H = 0 + \mathcal{O}(\varepsilon^3), \quad (4.3)$$

since $\varepsilon^2\partial_t\psi_u = \mathcal{O}(\varepsilon^3)$ due to the long wave character of the KdV approximation w.r.t. time. The approximation result is as above. The sketch of the proof in this case is as follows. Without performing a normal form transform as before, we multiply the first equation of the system for the error (4.1)-(4.2) with $\partial_t\partial_x^{-2}R_u$

and the second equation with $\partial_t R_v$. This gives after integration w.r.t. x the energy estimates

$$\partial_t(b_{uu}E_u + a_{uv}E_v) = \varepsilon^2 s_1 + \mathcal{O}(\varepsilon^3),$$

with

$$\begin{aligned} E_u &= (\partial_t\partial_x^{-1}R_u)^2 + \alpha^2(R_u)^2 + (\partial_t R_u)^2 - 2\varepsilon^2 a_{uu}\psi_u(R_u)^2)dx, \\ E_v &= \int(\partial_t R_v)^2 + (\partial_x R_v)^2 + (R_v)^2 - 2\varepsilon^2 b_{uv}\psi_u(R_v)^2 dx, \\ s_1 &= 2a_{uv}b_{uu}\int(\partial_t R_u)\psi_u R_v + (\partial_t R_v)\psi_u R_u dx, \end{aligned}$$

where we used integration by parts, $\partial_t\psi_u = \mathcal{O}(\varepsilon)$, and $\partial_x\psi_u = \mathcal{O}(\varepsilon)$. Hence in case of the same sign of a_{uv} and b_{uu} the term s_1 can be written as time-derivative plus some small error, i.e.,

$$\partial_t\int\psi_u R_u R_v dx + \mathcal{O}(\varepsilon),$$

again due to $\partial_t\psi_u = \mathcal{O}(\varepsilon)$. Therefore, the time derivative term can be included into the energy on the right hand side. Then we have

$$\partial_t(b_{uu}E_u + a_{uv}E_v + 2\varepsilon^2 a_{uv}b_{uu}\int\psi_u R_u R_v dx) = \mathcal{O}(\varepsilon^3),$$

and so the modified energy stays $\mathcal{O}(1)$-bounded on an $\mathcal{O}(1/\varepsilon^3)$-time scale. Since similar estimates can be obtained for the derivatives, the H^s-norm of the error stays $\mathcal{O}(1)$-bounded on the $\mathcal{O}(1/\varepsilon^3)$-time scale.

4.3. Approach 3: handling unstable resonances

The third approach also works in case of unstable resonances. In order to explain the underlying idea we go back to the amplitude system (3.2)-(3.3) describing the unstable resonances. In order to have an $\mathcal{O}(1)$-bound for A_1 on an $\mathcal{O}(1/\varepsilon^3)$-time scale w.r.t. t we need that A_1 is exponentially small initially, i.e., $A_1(0) = e^{-r/\varepsilon}$ for an $r > 0$. Since $e^{\beta\varepsilon^2 t}e^{-r/\varepsilon} \leq 1$ for $t \leq r/(\beta\varepsilon^3)$ the exponential smallness for $t = 0$ allows to come at least to the correct time-scale. This idea has to be combined with energy estimates for the wave numbers close to $k = 0$. With this respect the approach is more involved than the one used in [14, 17] for the water wave problem over a flat bottom. There, functions exponentially decaying w.r.t. the Fourier wave numbers for $|k| \to \infty$ were used for a local existence and uniqueness proof.

Our third approximation result is as follows.

Theorem 4.2. *Let A be a solution of the KdV equation* (2.5) *with*

$$\sup_{T\in[0,T_0]}\int|\widehat{A}(K,T)|e^{r|K|}dK < \infty$$

for an $r > 0$. Then there exist $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$, $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of (2.1)-(2.2) *with*

$$\sup_{t\in[0,T_1/\varepsilon^3]}\sup_{x\in\mathbb{R}}|(u,v)(x,t) - (\varepsilon^2\psi_u^{\mathrm{KdV}}(x,t), 0)| \leq C\varepsilon^{7/2}.$$

A detailed proof will be given in a forthcoming paper.

5. Validity in the oscillatory case

A spectral picture, similar to the one for the BKG system, occurs for the Klein-Gordon-Zakharov (KGZ) system. A long wave NLS approximation result for the KGZ system can be found in [16]. A NLS approximation result for wave packets with carrier wave number $k_0 > 0$ for systems including the BKG system can be found in [9, 5, 11]. However, none of these results apply in the situation of long wave NLS approximations with unstable resonances.

In order to explain the underlying idea we again go back the amplitude system (3.5)-(3.6) describing the unstable resonances. In order to have an $\mathcal{O}(1)$-bound for A_1 on an $\mathcal{O}(1/\varepsilon^2)$-time scale w.r.t. t, we need that A_1 is exponentially small initially, i.e., $A_1(0) = e^{-r/\varepsilon}$ for a $r > 0$. Since $e^{\beta\varepsilon t}e^{-r/\varepsilon} \leq 1$ for $t \leq r/(\beta\varepsilon^2)$ the exponential smallness for $t = 0$ allows us to come at least to the correct time-scale. This idea has to be combined with energy estimates for the wave numbers close to $k = 0$.

Theorem 5.1. *Let A be a solution of the NLS equation* (2.7) *with*

$$\sup_{T\in[0,T_0]} \int |\widehat{A}(K,T)|e^{r|K|}dK < \infty$$

for an $r > 0$. Then there exist $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$, $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of (2.1)-(2.2) *with*

$$\sup_{t\in[0,T_1/\varepsilon^2]} \sup_{x\in\mathbb{R}} |(u, v)(x, t) - (0, \varepsilon\psi_u^{\mathrm{NLS}}(x, t))| \leq C\varepsilon^{3/2}.$$

A detailed proof will be given in a forthcoming paper.

6. How to transfer the ideas to the water wave problem?

In [20] a counterexample has been constructed showing that the NLS approximation makes wrong predictions about the dynamics of the water wave problem with surface tension and periodic boundary conditions, if the surface tension and the periodicity is suitably chosen. Since the water wave problem with a flat bottom is a special case of the periodic bottom case this counter example transfers to the periodic water wave problem. Since the construction of this counter example is robust under small perturbations of the ground b, a counter example can be constructed for a slightly periodic bottom, too. Therefore, it is the goal of future research to prove theorems similar to Theorem 4.2 and Theorem 5.1 for the water wave problem with a periodic bottom. This will be done by controlling the spatially periodic case first, then by handling the case $|l| > 0$ by some perturbation analysis with the help of exponential weights in Bloch space, and finally to use these exponential weights to control the resonances.

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