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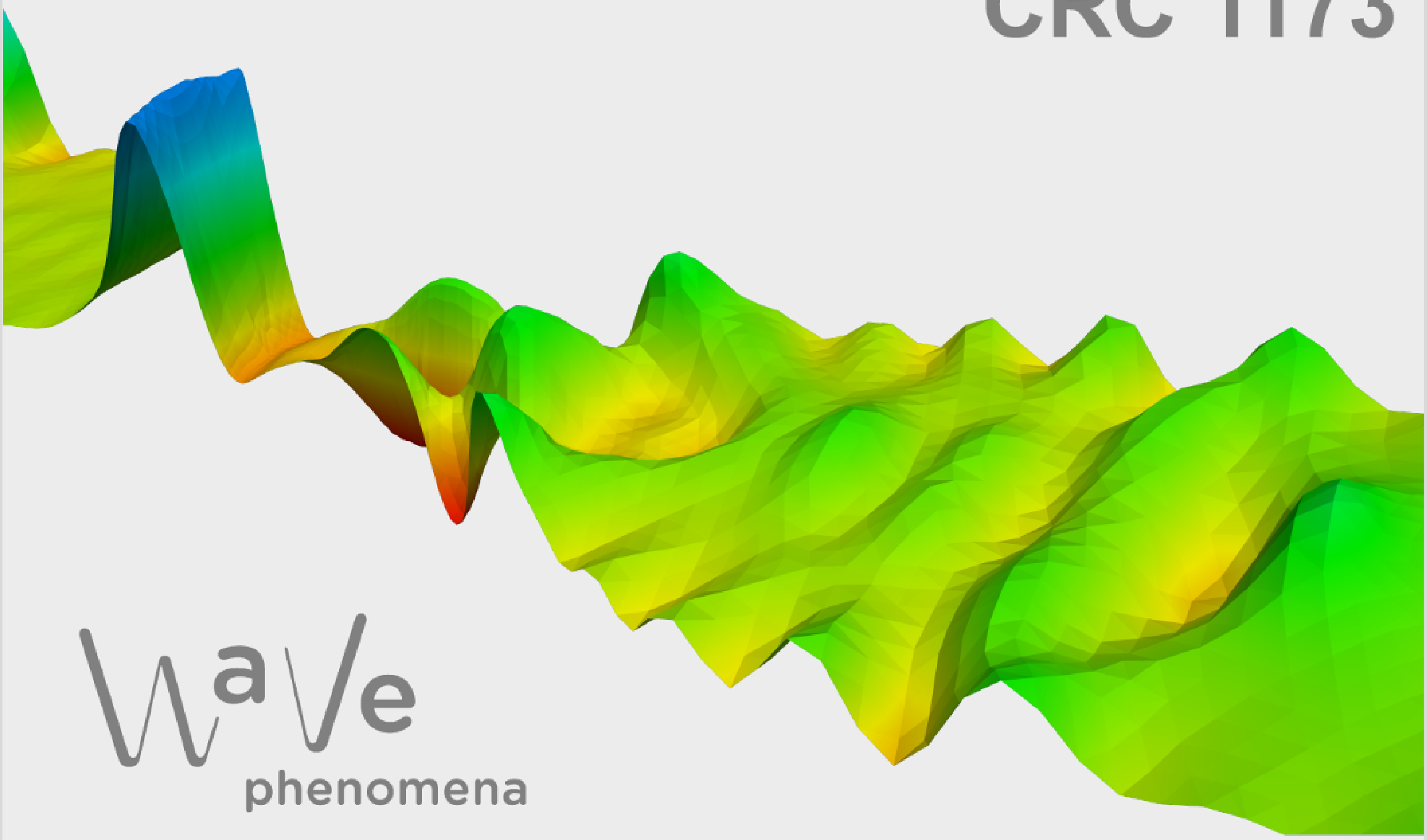

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UNCOUNTABLY MANY SOLUTIONS FOR NONLINEAR HELMHOLTZ AND CURL-CURL EQUATIONS WITH GENERAL NONLINEARITIES

RAINER MANDEL

Abstract. We obtain uncountably many solutions of nonlinear Helmholtz and curl-curl equations on the entire space using a fixed point approach. As an auxiliary tool a Limiting Absorption Principle for the curl-curl operator is proved.

1. Introduction and main results

The propagation of light in nonlinear media is governed by Maxwell's equations

$$\nabla\times\mathcal{E}+\partial_t\mathcal{B}=0,\qquad \operatorname{div}(D)=0,$$
$$\nabla\times\mathcal{H}-\partial_t\mathcal{D}=0,\qquad \operatorname{div}(B)=0$$

for the electric respectively magnetic field $\mathcal{E},\mathcal{H}:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3$, the displacement field $\mathcal{D}:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3$ and the magnetic induction $\mathcal{B}:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3$. Here, the effect of charges and currents is neglected. The nonlinearity of the medium is typically expressed through nonlinear material laws of the form $\mathcal{D}=\varepsilon(x)\mathcal{E}+\mathcal{P}$ and the linear relation $\mathcal{H}=\frac{1}{\mu}\mathcal{B}$ for the permittivity function $\varepsilon:\mathbb{R}^3\to\mathbb{R}$ and the magnetic permeability $\mu\in\mathbb{R}\setminus\{0\}$. In [2,23] it was shown that special solutions of the form $\mathcal{E}(x,t)=E(x)\cos(\omega t)$, $\mathcal{P}(x,t)=P(x,E)\cos(\omega t)$ can be approximately described by solutions of the nonlinear curl-curl equations

$$\nabla\times\nabla\times E+V(x)E=f(x,E)\qquad\text{in }\mathbb{R}^3 \tag{1}$$

where $V(x)=-\mu\omega^2\varepsilon(x)\leq 0$ and $f(x,E)=\mu\omega^2P(x,E)$. We stress that V is nonpositive and it becomes a negative constant in the simplest and most relevant case of the vacuum where $\varepsilon(x)\equiv\varepsilon_0>0$. We refer to Section 1.3 in [2] for further details concerning the modelling aspect of (1). One of our main results (Theorem 3) will provide a new existence result for solutions of this problem. A simplified version of (1) is the nonlinear Helmholtz equation

$$-\Delta u+V(x)u=f(x,u)\qquad\text{in }\mathbb{R}^n, \tag{2}$$

which in the two-dimensional case $n=2$ can be derived from (1) via the ansatz $E(x_1,x_2,x_3)=(0,0,u(x_1,x_2))$. In this paper we are interested in solutions of (1),(2) when the potential V is a negative constant and the nonlinearity is rather general. Our principal motivation is to show that for large classes of nonlinearities there are uncountably many solutions of these equations sharing the same decay rate $|x|^{\frac{1-n}{2}}$ as $|x|\to\infty$ but with a different farfield pattern.

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We first recall some facts about the nonlinear Helmholtz equation with constant potential

$$-\Delta u - \lambda u = f(x,u) \qquad \text{in } \mathbb{R}^n. \tag{3}$$

In 2004 Gutiérrez [17] set up a fixed point approach for this equation when $f(x,u) = |u|^2 u$, $n \in \{3,4\}$ and u is complex-valued. Using an L^p-version of the Limiting Absorption Principle for the Helmholtz operator (Theorem 6 in [17]) she found that small nontrivial solutions of (3) can be obtained via the Contraction Mapping Theorem (Banach's Fixed Point Theorem) on a small ball in $L^4(\mathbb{R}^n)$. Around ten years later Evéquoz and Weth started to write a series of papers [9–14] containing new methods to prove existence results for solutions of (3) that, in contrast to Gutiérrez' solutions, are large in suitable norms. Some of these results were extended by the author in [21,22]. In each of the aforementioned papers the nonlinearity has to satisfy quite specific conditions that allow to deal with slow decay rates of solutions at infinity.

In the case of power-type nonlinearities $f(x,u) = Q(x)|u|^{p-2}u$ one of the main results in [10] is that there is an unbounded sequence of solutions in $L^p(\mathbb{R}^n)$ provided $\frac{2(n+1)}{n-1} < p < \frac{2n}{n-2}$ and $Q \in L^\infty(\mathbb{R}^n)$ is positive and evanescent at infinity. If Q is $\mathbb{Z}^n$-periodic and positive (for negative Q see Theorem 1.3, 1.4 in [21]) the existence of one nontrivial solution is shown. These solutions are obtained using quite sophisticated dual variational methods and the solution at the mountain pass level of the dual functional is called a dual ground state of the equation. One of the drawbacks of this approach is that the assumption on p does not allow for cubic nonlinearities, which certainly are the most interesting ones for applications in physics. Moreover, sign-changing or non-monotone nonlinearities can not be treated. In addition to that, solutions have to be looked for in $L^p(\mathbb{R}^n)$. This is a problem given that solutions decay slowly at infinity so that a solution theory in $L^q(\mathbb{R}^n)$ with $q > p$ is more convenient a priori. For this reason we will not consider dual variational methods but rather revive Gutiérrez' fixed point approach [17].

Our refinement of Gutiérrez' method allows to discuss nonlinear Helmholtz equations with very general nonlinearities that improve existing results even in the case of power-type nonlinearities as we will see below. In our main result dealing with (3) we show that in the case $f(x,u) = Q(x)|u|^{p-2}u$ with $Q \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ we get solutions for all exponents

$$p > \max\Big\{2, \frac{2s(n^2+2n-1) - 2n(n+1)}{(n^2-1)s}\Big\}. \tag{4}$$

More generally, we can treat nonlinearities satisfying the following conditions:

(A) $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying for some $Q \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

$$\begin{aligned} |f(x,z)| &\le Q(x)|z|^{p-1} && (x \in \mathbb{R}^n, |z| \le 1) \\ |f(x,z_1) - f(x,z_2)| &\le Q(x)(|z_1| + |z_2|)^{p-2}|z_1 - z_2| && (x \in \mathbb{R}^n, |z_1|, |z_2| \le 1). \end{aligned} \tag{5}$$

where $s \in [1,\infty]$ and p as in (4).

We stress that only conditions near zero are needed since we are going to construct small solutions in $L^q(\mathbb{R}^n)$ which will turn out to be small also in $L^\infty(\mathbb{R}^n)$. Clearly, $|z| \le 1$ can be replaced by $|z| < z_0$ for any given $z_0 > 0$. We mention that in the case $s \le \frac{n+1}{2}$ all exponents $p > 2$ in the superlinear regime are allowed.

The fundamental tools of Gutiérrez' fixed point approach are an L^p-version of the Limiting Absorption Principle for the Helmholtz operator $-\Delta - \lambda$ ($\lambda > 0$) and results about the so-called Herglotz waves. As we will recall in Proposition 1, these functions are analytic solutions of the linear Helmholtz equation $-\Delta\phi - \lambda\phi = 0$ in $\mathbb{R}^n$. They are given by the formula

$$\widehat{h\,d\sigma_\lambda}(x) := \frac{1}{(2\pi)^{n/2}} \int_{S_\lambda^{n-1}} h(\xi) e^{-i\langle x,\xi\rangle}\, d\sigma_\lambda(\xi)$$

for complex-valued densities $h \in L^2(S_\lambda^{n-1};\mathbb{C})$. Here, σ_λ denotes the canonical surface measure of the sphere $S_\lambda^{n-1} = \{\xi \in \mathbb{R}^n : |\xi|^2 = \lambda\}$. In order to ensure the real-valuedness and good poinwise decay properties of $\widehat{h\,d\sigma_\lambda}$ at infinity, we will consider a smaller class of densities h belonging to the set

$$X_\lambda^\delta := \left\{ h \in C^m(S_\lambda^{n-1};\mathbb{C}) : h(\xi) = \overline{h(-\xi)}, \|h\|_{C^m} \le \delta \right\}$$

where $m := \lfloor \frac{n-1}{2} \rfloor + 1$. Here, our approach differs from [17] where L^2-densities are used. Theorem 1 shows that for all $h \in X_\lambda^\delta$ we find a strong solution of (3) that resembles $|x|^{\frac{1-n}{2}} u_h^\infty(x)$ at infinity where

$$u_h^\infty(x) := \lambda^{\frac{n-3}{4}} \sqrt{\frac{\pi}{2}}\, \mathrm{Re}\left(e^{i(\frac{n-3}{4}\pi - \sqrt{\lambda}|x|)} \left(\widehat{f(\cdot,u_h)}(-\sqrt{\lambda}\hat{x}) + i \cdot \frac{2\sqrt{\lambda}}{\pi} h(\sqrt{\lambda}\hat{x}) \right) \right), \qquad \hat{x} := \frac{x}{|x|}.$$

More precisely, we show the following.

Theorem 1. *Assume (A) and $\lambda > 0$. Then there are $\delta > 0$ and mutually different solutions $(u_h)_{h \in X_\lambda^\delta}$ of (3) that form a $W^{2,r}(\mathbb{R}^n)$-continuum and satisfy $\|u_h\|_{W^{2,r}(\mathbb{R}^n)} \to 0$ as $\|h\|_{C^m} \to 0$ for any given $r \in (\frac{2n}{n-1}, \infty)$ as well as*

$$\lim_{R\to\infty} \frac{1}{R} \int_{B_R} \left| u_h(x) - |x|^{\frac{1-n}{2}} u_h^\infty(\hat{x}) \right|^2 dx = 0.$$

If additionally $p > \frac{(3n-1)s-2n}{(n-1)s}$ holds, then $|u_h(x)| \le C_h(1+|x|)^{\frac{1-n}{2}}$ for all $x \in \mathbb{R}^n$.

Let us discuss in which way this theorem improves earlier results. Most importantly, Theorem 1 shows that nonlinear Helmholtz equations of the form (3) admit uncountably many solutions for a large class of nonlinearities which need not be odd, let alone of power-type. Its proof is short and elementary in the sense that it only uses the Contraction Mapping Theorem, elliptic regularity theory and mostly well-known results about the linear Helmholtz equation. Up to now such general nonlinearities have only been treated in the paper [12] by Evéquoz and Weth, but their additional requirement (f_0) on p.361 requires the nonlinearity to be supported in a bounded subset of $\mathbb{R}^n$, which is quite restrictive. In our approach such an assumption is not necessary. Given that applications often deal with power-type nonlinearities $f(x,z) = Q(x)|z|^{p-2}z$ let us comment on our improvements for this particular case in more detail. In the case $Q \in L^\infty(\mathbb{R}^n)$ we obtain solutions for all exponents $p > \frac{2(n^2+2n-1)}{n^2-1}$. This bound is smaller than $\frac{2(n+1)}{n-1}$ so that our range of exponents is larger than in all other nonradial approaches except for [12] where Q has compact support and exponents $2 < p < \frac{2n}{n-2}$ are allowed. Additionally, we need not require Q to be periodic nor evanescent (as in [9,10,21]) nor compactly supported (as in [12,13]) and the growth rate of the nonlinearity may be supercritical (i.e. $p \geq \frac{2n}{n-2}$) which is an entirely new feature. The latter fact is worth mentioning given that Evéquoz and Yesil [14] proved the nonexistence of dual ground states $u \in L^p(\mathbb{R}^n)$ for $n = 3$ in the critical case $p = \frac{2n}{n-2} = 6$ provided $f(x,u) = Q(x)u^5$ and $Q \in L^\infty(\mathbb{R}^3)$ is nonnegative and nontrivial. Since Theorem 1 yields solutions belonging to $L^p(\mathbb{R}^3)$ we conclude that dual ground states need not exist while other nontrivial solutions do. Finally we mention that in the physically most relevant case of a cubic nonlinearity $p = 4$ we obtain uncountably many solutions whenever $n \geq 3, s \in [1,\infty]$ or $n = 2, s \in [1,6)$.

Remark 1. *(a) The decay rate $|x|^{\frac{1-n}{2}}$ is best possible. This is a consequence of Theorem 3 in [20] where nontrivial solutions of the elliptic PDE $-\Delta u - \lambda u = W(x)u$ in $\mathbb{R}^n$ with $W \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$ and $\lambda > 0$ are shown to satisfy $u(x)|x|^{-\frac{1}{2}-\varepsilon} \notin L^2(\mathbb{R}^n)$ for all $\varepsilon > 0$. In particular, better decay rates than $|x|^{\frac{1-n}{2}}$ as $|x| \to \infty$ are excluded. Notice that in the setting of Theorem 1 the function $W(x) := f(x,u(x))/u(x)$ satisfies $W \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$ because of $Q \in L^s(\mathbb{R}^n)$ and*

$$|W(x)| \leq Q(x)|u(x)|^{p-2}, \quad u \in \bigcap_{r > \frac{2n}{n-1}} L^r(\mathbb{R}^n), \quad p > \frac{2s(n^2+2n-1) - 2n(n+1)}{(n^2-1)s},$$

see (4). It is remarkable that precisely this lower bound for p appears in this context. Up to now existence and optimal decay results for nonlinear Helmholtz equations for lower exponents p are only known in the radial setting [12,21]. Notice that for smaller p the (nonradial) counterexample of Ionescu and Jerison from Theorem 2.5 in [19] has to be taken into account: For any given $N \in \mathbb{N}$ there is $W \in L^q(\mathbb{R}^n)$ with $q > \frac{n+1}{2}$ and a solution of $-\Delta u - \lambda u = W(x)u$ in $\mathbb{R}^n$ with $|u(x)| \leq (1+|x|)^{-N}$ for all $x \in \mathbb{R}^n$.

(b) Theorem 1 yields a symmetry-breaking result: For any subgroup $\Gamma \subset O(n)$ such that $\Gamma \neq \{\mathrm{id}\}$ and any Γ-invariant nonlinearity f satisfying (A) one has uncountably many solutions that are not Γ-invariant. In particular, for $\Gamma = O(n)$, radial nonlinearities allow for nonradial solutions. We will prove this in Remark 2(a).

(c) In order to construct radial solutions, we can get the same conclusions as in Theorem 1 under weaker assumptions on p. This is due to an improved version of the Limiting Absorption Principle for the Helmholtz operator. In Remark 2(b) we comment on the necessary modifications of the proof and show that the admissible range of exponents for the existence of radial solutions is no longer given by (4), but

$$p > \max\Big\{2, \frac{s(2n^2+n-1) - 2n^2}{sn(n-1)}\Big\}. \tag{6}$$

Notice that the resulting radial version of Theorem 1 is not covered by earlier contributions from [21] (Theorem 1.2, Theorem 2.10) or [12] (Theorem 4). For instance, it provides solutions of the radial nonlinear Helmholtz equation $-\Delta u - u = Q(x)|u|^{p-2}u$ for any $Q \in L^s_{rad}(\mathbb{R}^n) \cap L^\infty_{rad}(\mathbb{R}^n)$ and p as in (6) whereas in the above-mentioned papers Q has to be bounded, differentiable and radially decreasing. On the other hand, our *restrictions on the exponent p do not appear in [12, 21] (where all $p > 2$ are allowed) so that* (6) *might be improved further.*

Next we discuss variants of these results for related semilinear elliptic PDEs from mathematical physics. First let us mention that a nonlinearity $f(\cdot,u)$ satisfying (A) may be without any major difficulty be replaced by a nonlocal right hand side such as $K * f(\cdot,u)$ where $K \in L^1(\mathbb{R}^n)$. Clearly, imposing more assumptions K may even lead to larger ranges of exponents than (4). In this way it is possible to obtain small solutions of nonlocal Helmholtz equations. Similarly, one may ask how our results are affected by changes in the linear operator. For instance, if the Helmholtz operator is perturbed to a periodic Schrödinger operator $-\Delta + V(x) - \lambda$ then it should be possible to adapt the proof in such a way that it provides small solutions of $-\Delta u + V(x)u - \lambda u = f(x,u)$ in $\mathbb{R}^n$ provided λ belongs to the essential spectrum of $-\Delta + V(x)$ and the band structure of this periodic Schrödinger operator is sufficiently nice. To be more precise, one would require (A1),(A2),(A3) from [22] to hold so that Herglotz-type waves, defined as suitable oscillatory integrals over the so-called Fermi surfaces associated with $-\Delta + V(x)$, exist and have the properties stated in Proposition 1 below. Since the technicalities (including a Limiting Absorption Principle for such operators) are quite involved and mostly carried out in [22], we prefer not to discuss this issue further.

We now turn our attention to a fourth order version of (3) given by

$$\Delta^2 u - \beta\Delta u + \alpha u = f(x,u) \quad \text{in } \mathbb{R}^n, \tag{7}$$

which we will briefly discuss for α, β satisfying

$$\text{(i)}\ \ \alpha < 0, \beta \in \mathbb{R} \qquad \text{or} \qquad \text{(ii)}\ \ \alpha > 0, \beta < -2\sqrt{\alpha}. \tag{8}$$

Under these assumptions dual variational methods were employed in [7] to prove the existence of one nontrivial solution when $f(x,z) = Q(x)|z|^{p-2}z$ where $\frac{2(n+1)}{n-1} < p < \frac{2n}{n-2}$ and Q is positive and $\mathbb{Z}^n$-periodic. Notice that in the case $\beta^2 - 4\alpha < 0$ classical variational methods such as constrained minimization apply and a number of papers revealed the existence of positive and sign-changing solutions $u \in H^4(\mathbb{R}^n)$ of (7) again for power-type nonlinearities. We refer to [5, 6, 8] for results in this direction. Our intention is to show that in the case (i) or (ii) uncountably many solutions of (7) exist for all nonlinearities f satisfying (A). The main observation is that in case (i) or (ii) there are analoga of the Herglotz waves given by densities $h \in Y^\delta$ where

$$\begin{aligned} &\text{In case (i):} \quad Y^\delta := X^\delta_\lambda, && \text{where} \quad \lambda = \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2} > 0, \\ &\text{In case (ii):} \quad Y^\delta := X^\delta_{\lambda_1} \times X^\delta_{\lambda_2} && \text{where} \quad \lambda_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2} > 0. \end{aligned} \tag{9}$$

The fixed point approach used in the proof of Theorem 1 may be rather easily adapted to (7) and we can prove the following result.

Theorem 2. *Assume (A) and (i) or (ii). Then there is $\delta > 0$ and mutually different solutions $(u_h)_{h \in Y^\delta}$ of* (7) *that form a $W^{4,r}(\mathbb{R}^n)$-continuum satisfying $\|u_h\|_{W^{4,r}(\mathbb{R}^n)} \to 0$ as $\|h\|_{C^m} \to 0$*

for any given $r \in (\frac{2n}{n-1}, \infty)$*. If additionally* $p > \frac{(3n-1)s-2n}{(n-1)s}$ *holds, then* $|u_h(x)| \le C_h(1+|x|)^{\frac{1-n}{2}}$ *for all* $x \in \mathbb{R}^n$*.*

As in Theorem 1 one can say more about the asymptotics of the constructed solutions; we refer to Section 5.2 in [7] for a related discussion. Further generalizations to more general higher order semilinear elliptic problems of the form $Lu = f(x,u)$ in $\mathbb{R}^n$ are possible provided the linear differential operator with constant coefficients L has a Fourier symbol $P(\xi)$ with the property that $\{\xi \in \mathbb{R}^n : P(\xi) = 0\}$ is a compact manifold with nonvanishing Gaussian curvature. Notice that this assumption makes the method of stationary phase work and provides pointwise decay of oscillatory integrals as demonstrated in the proof of Proposition 1. Moreover, one needs a Limiting Absorption Principle in order to make sense of the Fourier multiplier $1/P(\xi)$ as a mapping between Lebesgue spaces. At least in the case $P(\xi) = P_0(\xi)(|\xi|^2-\lambda_1)\cdot\ldots\cdot(|\xi|^2-\lambda_k)$ with $0<\lambda_1<\ldots<\lambda_k$ and P_0 positive this can be established as in Theorem 3.3 in [7]. With these tools our fixed point approach can be adapted to find nontrivial solutions of $Lu = f(x,u)$ in $\mathbb{R}^n$.

Finally, we discuss nonlinear curl-curl equations of the form

$$\nabla\times\nabla\times E - \lambda E = f(x,E) \quad \text{in } \mathbb{R}^3 \tag{10}$$

that describe the electric field $E:\mathbb{R}^3\to\mathbb{R}^3$ of an electromagnetic wave in a nonlinear medium. This equation has been studied in the past years on bounded domains in $\mathbb{R}^3$ [3,4] but also on the entire space $\mathbb{R}^3$, which is the situation we focus on. Up to our knowledge there is only one result for solutions of nonlinear curl-curl equations on $\mathbb{R}^3$ without symmetry assumption. In [23] Mederski proves the existence of a weak solution of (10) by variational methods when λ is replaced by a small nonnegative potential $V(x)$ that decays suitably fast to zero at infinity, see assumption (V) in [23]. In particular, constant functions $V(x) = \lambda$ can not be treated by this method so that our setting must be considered as entirely different from the one in [23].

In the cylindrically symmetric setting the existence of solutions can be proved using various approaches. Here, the electrical field is assumed to be of the form

$$E(x_1,x_2,x_3) = \frac{E_0(\sqrt{x_1^2+x_2^2},x_3)}{\sqrt{x_1^2+x_2^2}}\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix} \qquad \text{where } E_0:[0,\infty)\times\mathbb{R}\to\mathbb{R}. \tag{11}$$

Such functions are divergence-free so that $\nabla\times\nabla\times E = -\Delta E$ implies that one actually has to deal with the elliptic 3×3-system

$$-\Delta E - \lambda E = f(x,E) \quad \text{in } \mathbb{R}^3, \tag{12}$$

which may equally be expressed in terms of E_0 provided the nonlinearity $f(x,E)$ is compatible with this symmetry assumption, see page 3 in [2]. In this special case further results [2,18,26] are known but none of those applies in the case $\lambda>0$ and $f(x,E) = \pm|E|^{p-2}E$ that we are mainly interested in.

Given our earlier results for the nonlinear Helmholtz equation (3) it is not surprising that we obtain an existence result for (12) that is entirely analogous to the one from Theorem 1.

Since this result fills a gap in the literature, we state it in part (i) of our theorem even though its proof is a straightforward adaptation of the fixed point approach used in the proof of Theorem 1. The corresponding assumption on the nonlinearity is the following.

(A') $f:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3, (x,E)\mapsto f_0(\sqrt{x_1^2+x_2^2},x_3,|E|^2)E$ is a Carathéodory function satisfying (5) for some $Q\in L^s(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$.

Assumption (A') ensures that f is compatible with cylindrical symmetry. Indeed, for E as in (11) one can check that $f(\cdot,E)$ is of the form (11), too.

In the general non-symmetric case the construction of solutions is more difficult since the curl-curl operator satisfies a much weaker Limiting Absorption Principle as in the cylindrically symmetric setting, cf. Theorem 6. Moreover, well-known regularity results for elliptic problems are not available so that we have to consider a substantially smaller class of nonlinearities satisfying the following:

(B) $f:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ is a Carathéodory function satisfying for some $Q\in L^s(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ the estimate

$$\begin{aligned} |f(x,E)| &\le Q(x)|E|^{p-1}(1+|E|)^{\tilde p-p} && (x,E\in\mathbb{R}^3),\\ |f(x,E_1)-f(x,E_2)| &\le Q(x)(|E_1|+|E_2|)^{p-2}(1+|E_1|+|E_2|)^{\tilde p-p}|E_1-E_2| && (x,E_1,E_2\in\mathbb{R}^3) \end{aligned} \tag{13}$$

where $1\le s\le 2$ and $\tilde p\le 2\le p<\infty$.

Additionally, we will have to require that $\|Q\|_s+\|Q\|_\infty$ is small enough in order to obtain solutions of (10). In the cylindrically symmetric respectively non-symmetric setting the counterparts of the Herglotz waves (introduced in Section 2) are parametrized by functions $h\in Z^\delta_{cyl}$ respectively $h\in Z$ where

$$\begin{aligned} Z &:= \Big\{h\in C^2(S^2_\lambda;\mathbb{C}^3): h(\xi)=\overline{h(-\xi)},\ \langle h(\xi),\xi\rangle=0\ \forall\xi\in S^2_\lambda\Big\},\\ Z^\delta_{cyl} &:= \Big\{h\in Z: \|h\|_{C^2}<\delta \text{ and } \operatorname{Re}(h),\operatorname{Im}(h) \text{ satisfy (11)}\Big\}. \end{aligned}$$

Notice that both sets are nonempty. With these definitions we can formulate our main result for the nonlinear curl-curl equation (10).

Theorem 3.

(i) Assume (A') and $\lambda>0$. Then there is $\delta>0$ and a family $(E_h)_{h\in Z^\delta_{cyl}}$ of mutually different cylindrically symmetric solutions of (10) that form a $W^{2,r}(\mathbb{R}^3;\mathbb{R}^3)$-continuum and satisfy $\|E_h\|_{W^{2,r}(\mathbb{R}^3;\mathbb{R}^3)}\to 0$ as $\|h\|_{C^2}\to 0$ for any given $r\in(3,\infty)$. If additionally $p>\frac{4s-3}{s}$ holds, then $|E_h(x)|\le C_h(1+|x|)^{-1}$.

(ii) Assume (B) and $\lambda>0, 3<q<\frac{3s}{(2s-3)_+}$. If $\|Q\|_s+\|Q\|_\infty$ is sufficiently small then there is a family $(E_h)_{h\in Z}$ of mutually different weak solutions of (10) lying in $H_{loc}(\mathrm{curl};\mathbb{R}^3)\cap L^q(\mathbb{R}^3;\mathbb{R}^3)$. Moreover:

(a) If additionally $(p,s)\ne(2,2)$ holds, then $E_h\in L^r(\mathbb{R}^3;\mathbb{R}^3)$ for all $r\in(3,q)$.

(b) If additionally $\tilde p<2<p$ holds, then $E_h\in L^r(\mathbb{R}^3;\mathbb{R}^3)$ for all $r\in(q,\frac{3s(p-1)}{(2s-3)_+})$.

As an application we obtain uncountably many distinct weak solutions of the curl-curl equation (10) with saturated nonlinearities of the form

$$f(x,E) = \frac{\delta |E|^2 \Gamma(x) E}{1 + P(x)|E|^2}$$

where $\inf P > 0$, $\Gamma \in L^s(\mathbb{R}^3;\mathbb{R}^{3\times 3}) \cap L^\infty(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ and $\delta > 0$ is sufficiently small.

The paper is organized as follows. In Section 2 we review the Limiting Absorption Principles and properties of Herglotz-type waves that we will need for the proofs of our results. In Section 3, Section 4, Section 5 we then prove Theorem 1, Theorem 2 and Theorem 3. Since the proofs of Theorem 1, Theorem 2 and Theorem 3 (i) are almost identical, we carry out the first in detail and only comment on the modifications when it comes to the latter. Finally, in Appendix A we review the method of stationary phase and prove Proposition 2. In Appendix B we prove our Limiting Absorption Principle for the curl-curl operator (Theorem 6). In Appendix C we review some resolvent estimates for the Helmholtz operator due to Ruiz and Vega.

In the following C will denote a generic constant that can change from line to line and $\frac{1}{r_+}$ stands for $\frac{1}{r}$ if $r > 0$ and for ∞ if $r \le 0$. The symbol $\mathcal{F}f = \hat{f}$ represents the Fourier transform of (the tempered distribution) $f \in L^q(\mathbb{R}^n)$ and $\mathcal{F}_1, \mathcal{F}_{n-1}$ are the Fourier transforms in $\mathbb{R}^1, \mathbb{R}^{n-1}$, respectively. For $R > 0$ the symbol B_R denotes the open ball of radius R around the origin in $\mathbb{R}^n$ and $\langle \cdot,\cdot \rangle$ is the inner product in $\mathbb{R}^n$ extended by bilinearity to $\mathbb{C}^n$. $L^q(\mathbb{R}^n), L^{q,w}(\mathbb{R}^n)$ denote the classical respectively weak Lebesgue spaces on $\mathbb{R}^n$ equipped with the standard norms $\|\cdot\|_q, \|\cdot\|_{q,w}$.

2. Herglotz waves and Limiting absorption principles

In this section we review some partly well-known results on Herglotz waves and Limiting Absoprtion Principles for the linear differential operators we are interested in. A classical Herglotz wave associated with the Helmholtz operator $-\Delta - \lambda$ is defined via the formula

$$\mathcal{F}(h\, d\sigma_\lambda)(x) := \widehat{h\, d\sigma_\lambda}(x) := \frac{1}{(2\pi)^{n/2}} \int_{S^{n-1}_\lambda} h(\xi) e^{-i\langle x,\xi\rangle}\, d\sigma_\lambda(\xi)$$

where $h \in L^2(S^{n-1}_\lambda;\mathbb{C})$ and σ_λ denotes the canonical surface measure of $S^{n-1}_\lambda = \{\xi \in \mathbb{R}^n : |\xi|^2 = \lambda\}$. Herglotz waves are analytic functions that solve the linear Helmholtz equation $-\Delta\phi - \lambda\phi = 0$. Their pointwise decay properties are well-understood for smooth densities h and result from an application of the method of stationary phase. Unfortunately, we could not find a quantitative version of this result telling how smooth the density h needs to be in order to ensure that $\widehat{h\, d\sigma_\lambda}(x)$ decays like $|x|^{\frac{1-n}{2}}$ as $|x| \to \infty$ in the pointwise sense. In our first auxiliary result we provide such an estimate and its proof will be given in Appendix A. For notational convenience we introduce the quantity

$$m_h(x) := e^{i(\frac{n-1}{4}\pi - \sqrt{\lambda}|x|)} h(\sqrt{\lambda}\hat{x}) + e^{-i(\frac{n-1}{4}\pi - \sqrt{\lambda}|x|)} h(-\sqrt{\lambda}\hat{x}) \tag{14}$$

so that our claim is the following.

Proposition 1. *Let $n \in \mathbb{N}, n \geq 2$ and $m := \lfloor \frac{n-1}{2} \rfloor + 1$. Then for all $h \in C^m(S_\lambda^{n-1};\mathbb{C})$ the Herglotz wave $\widehat{h\,d\sigma_\lambda}$ is an analytic solution of $-\Delta\phi - \lambda\phi = 0$ in $\mathbb{R}^n$ and satisfies the estimate $|(\widehat{h\,d\sigma_\lambda})(x)| \leq C\|h\|_{C^m}(1+|x|)^{\frac{1-n}{2}}$ as well as*

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left| \widehat{h\,d\sigma_\lambda}(x) - \frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} m_h(x)\right|^2 dx = 0.$$

In particular, we have $\|\widehat{h\,d\sigma_\lambda}\|_r \leq C_r\|h\|_{C^m}$ for all $r > \frac{2n}{n-1}$.

While Herglotz waves solve the homogeneous Helmholtz equation, we also need to discuss the inhomogeneous equation. Since λ lies in the essential spectrum of $-\Delta$ it is a nontrivial task to solve $-\Delta u - \lambda u = f$ in $\mathbb{R}^n$. The method to find such solutions is to study the limit of solutions $u_\varepsilon := \mathcal{R}(\lambda + i\varepsilon)f \in H^2(\mathbb{R}^n;\mathbb{C})$ of $-\Delta u_\varepsilon - (\lambda + i\varepsilon)u_\varepsilon = f$ in a suitable topology. The complex-valued limit of these functions as $\varepsilon \to 0^+$ is denoted by $\mathcal{R}(\lambda + i0)f$ and we define $\mathfrak{R}_\lambda f := \mathrm{Re}(\mathcal{R}(\lambda + i0))f$ as its real part since we are interested in real-valued solutions. These operators have the following properties:

Theorem 4 (Theorem 6 [17], Theorem 2.1 [9])**.** *Let $n \in \mathbb{N}, n \geq 2$. The operator $\mathfrak{R}_\lambda : L^t(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is a bounded linear operator provided*

$$(15)\qquad \begin{aligned} &\frac{1}{t} > \frac{n+1}{2n}, \qquad \frac{1}{q} < \frac{n-1}{2n}, \qquad \frac{2}{n+1} \leq \frac{1}{t} - \frac{1}{q} \leq \frac{2}{n} \qquad (n \geq 3),\\ &\frac{1}{t} > \frac{n+1}{2n}, \qquad \frac{1}{q} < \frac{n-1}{2n}, \qquad \frac{2}{n+1} \leq \frac{1}{t} - \frac{1}{q} < \frac{2}{n} \qquad (n = 2). \end{aligned}$$

Moreover, for $f \in L^t(\mathbb{R}^n)$ the function $\mathfrak{R}_\lambda f \in W^{2,t}_{loc}(\mathbb{R}^n)$ is a real-valued strong solution of $-\Delta u - \lambda u = f$ in $\mathbb{R}^n$.

The last statement is actually not included in the references given above, but it is a consequence of elliptic regularity theory for distributional solutions. We refer to Proposition A.1 in [10] for a similar result. Next we discuss the asymptotic behaviour of the solutions $\mathfrak{R}_\lambda f$ that we will deduce from the following result.

Proposition 2. *Let $n \in \mathbb{N}, n \geq 2$ and assume $f \in L^{p'}(\mathbb{R}^n)$ for $\frac{2(n+1)}{n-1} \leq p \leq \frac{2n}{(n-4)_+}, (n,p) \neq (4,\infty)$. Then:*

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left| \mathcal{R}(\lambda + i0)f(x) - \sqrt{\frac{\pi}{2\lambda}}\left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} e^{i(\frac{n-3}{4}\pi - \sqrt{\lambda}|x|)}\widehat{f}(-\sqrt{\lambda}\hat{x})\right|^2 dx = 0.$$

Proof. The claim for $n \geq 3$ and $\lambda = 1$ is provided in Proposition 2.7 [10] so that the general case follows from rescaling via $\mathcal{R}_\lambda f(x) = \frac{1}{\lambda}\mathcal{R}_1(f(\lambda^{-1/2}\cdot))(\sqrt{\lambda}x)$. The proof in the case $n = 2$ is essentially the same. Indeed, repeating the proof of Proposition 2.7 [10] one finds that the claimed result holds true provided Proposition 2.6 in [10] (the Stein-Tomas Theorem) and the estimate

$$(16)\qquad \sup_{R\geq 1}\left(\frac{1}{R}\int_{B_R} |\mathcal{R}(\lambda + i0)f(x)|^2\,dx\right)^{1/2} \leq C\|f\|_{p'}$$

are valid in the case $n = 2$ under our assumptions on p. For the Stein-Tomas Theorem this is clear. The inequality (16) is due to Ruiz and Vega [24], but we could not find an accurate reference for it that covers our range of exponents and all space dimensions $n \geq 2$. We provide the estimate (16) and further details in Appendix C so that the proof is finished. □

Next we recall a Limiting Absorption Principle that we will need in the discussion of the fourth order problem (7). In Theorem 3.3 in [7] the following extension of Theorem 4 to linear differential operators of the form $\Delta^2 - \beta\Delta + \alpha$ was proved.

Theorem 5 (Theorem 3.3 [7])**.** *Let $n \in \mathbb{N}, n \geq 2$ and assume (i) or (ii). Then there is a bounded linear operator $\mathbf{R} : L^t(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for*

$$\begin{aligned} &\frac{1}{t} > \frac{n+1}{2n}, && \frac{1}{q} < \frac{n-1}{2n}, && \frac{2}{n+1} \leq \frac{1}{t} - \frac{1}{q} \leq \frac{4}{n} && \text{if } n \geq 5, \\ &\frac{1}{t} > \frac{n+1}{2n}, && \frac{1}{q} < \frac{n-1}{2n}, && \frac{2}{n+1} \leq \frac{1}{t} - \frac{1}{q} < 1 && \text{if } n = 4, \\ &\frac{1}{t} > \frac{n+1}{2n}, && \frac{1}{q} < \frac{n-1}{2n}, && \frac{2}{n+1} \leq \frac{1}{t} - \frac{1}{q} \leq 1 && \text{if } n = 2,3 \end{aligned}$$

such that for $f \in L^t(\mathbb{R}^n)$ the function $\mathbf{R}f$ belongs to $W^{4,t}_{loc}(\mathbb{R}^n)$ and is a real-valued strong solution of $\Delta^2 u - \beta\Delta u + \alpha u = 0$ in $\mathbb{R}^n$.

Finally we provide the tools for proving Theorem 3. As for the previous results we need a family of elements lying in the kernel of the linear operator which now is $E \mapsto \nabla \times \nabla \times E - \lambda E$. These are given by vectorial variants of the Herglotz waves

$$\widehat{h\, d\sigma_\lambda}(x) := \frac{1}{(2\pi)^{3/2}} \int_{S^2_\lambda} h(\xi) e^{-i\langle x,\xi\rangle}\, d\sigma_\lambda(\xi)$$

(the integral to be understood componentwise) where $h : S^2_\lambda \to \mathbb{C}^3$ is a tangential vectorfield field, i.e. $\langle h(\xi), \xi\rangle = 0$ for all $\xi \in S^2_\lambda$. These functions are real-valued whenever $h(\xi) = \overline{h(-\xi)}$. Applying the results from Proposition 1 in each component, we deduce the following properties.

Proposition 3. *For all $h \in Z$ the function $\widehat{h\, d\sigma_\lambda}$ is an analytic solution of $\nabla \times \nabla \times \phi - \lambda\phi = 0$ in $\mathbb{R}^3$ and satisfies the pointwise estimate $|\widehat{h\, d\sigma_\lambda}(x)| \leq \|h\|_{C^2}(1 + |x|)^{-1}$ for all $x \in \mathbb{R}^3$. In particular, $\|\widehat{h\, d\sigma_\lambda}\|_r \leq C_r \|h\|_{C^2}$ for all $r > 3$. If $h \in Z^\delta_{cyl}$, then $\widehat{h\, d\sigma_\lambda}$ is cylindrically symmetric.*

Having described the analoga of the Herglotz waves we finally discuss a Limiting Absorption Principle for the curl-curl operator. So let $\mathcal{R}(\lambda + i\varepsilon)$ denote the resolvent of $E \mapsto \nabla \times \nabla \times E - (\lambda + i\varepsilon)E$ which we will prove to exist in Proposition 6. As for the Helmholtz operator one is interested in the (complex-valued) limit $\mathcal{R}(\lambda + i0)G$ for $G \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. It turns out that G decomposes into two parts behaving quite differently. So we split G into a curl-free (gradient-like) part $G_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and a divergence-free remainder $G_2 : \mathbb{R}^3 \to \mathbb{R}^3$ of G which are defined via

$$G_1 := \mathcal{F}^{-1}\left(\langle \hat{G}(\xi), \frac{\xi}{|\xi|}\rangle \frac{\xi}{|\xi|}\right), \qquad G_2 := \mathcal{F}^{-1}\left(\hat{G}(\xi) - \langle \hat{G}(\xi), \frac{\xi}{|\xi|}\rangle \frac{\xi}{|\xi|}\right).$$

This splitting corresponds to a Helmholtz decomposition of a vector field in $\mathbb{R}^3$.

Theorem 6. *Let $\lambda > 0$ and assume that $t, q \in (1,\infty)$ satisfy (15). Then there is a bounded linear operator $R_\lambda : L^t(\mathbb{R}^3;\mathbb{R}^3) \cap L^q(\mathbb{R}^3;\mathbb{R}^3) \to L^q(\mathbb{R}^3;\mathbb{R}^3)$ such that $R_\lambda G \in H_{loc}(\mathrm{curl};\mathbb{R}^3)$ is a weak solution of $\nabla\times\nabla\times E - \lambda E = G$ provided $G \in L^t(\mathbb{R}^3;\mathbb{R}^3)\cap L^q(\mathbb{R}^3;\mathbb{R}^3)$. Moreover, we have*

$$\|R_\lambda G\|_q \le C(\|G_1\|_q + \|G_2\|_t) \le C(\|G\|_q + \|G\|_t)$$

and $R_\lambda G = -\frac{1}{\lambda}G_1 + \mathfrak{R}_\lambda G_2$ for $\mathfrak{R}_\lambda$ from Theorem 4 (applied componentwise). If $G \in L^t(\mathbb{R}^3;\mathbb{R}^3)$ is cylindrically symmetric then so is $R_\lambda G$ and $R_\lambda G \in W^{2,q}_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ is a strong solution satisfying $\|R_\lambda G\|_q \le C\|G\|_t$.

The proof of Theorem 6 will be given in Appendix B. With these technical preparations we have all the tools to prove our main results in the following sections.

3. Proof of Theorem 1

We prove Theorem 1 with the aid of Banach's Fixed Point Theorem following the approach by Gutiérrez [17]. We consider the map $T(\cdot,h) : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ given by

$$T(u,h) := \widehat{h\,d\sigma_\lambda} + \mathfrak{R}_\lambda(f(\cdot,\chi(u))) \tag{17}$$

where $h \in X^\delta_\lambda$ as in Proposition 1 and χ is a smooth function such that $|\chi(z)| \le \min\{|z|,1\}$ and $\chi(z) = z$ for $|z| \le \frac{1}{2}$. In view of the properties of the Herglotz waves and $\mathfrak{R}_\lambda$ mentioned earlier, a fixed point of $T(\cdot,h)$ is a strong solution of the equation $-\Delta u - \lambda u = f(x,\chi(u))$ in $\mathbb{R}^n$. Since that fixed point u will belong to a small ball in $L^q(\mathbb{R}^n)$, we will be able to show $\chi(u) = u$ so that a solution of (3) is found. The choice of the exponent q is delicate; it's the major technical issue in our approach. Our assumption (4) implies that the set

$$\Xi_{s,p} := \left\{ q \in \left(\frac{2n}{n-1}, \frac{2n}{(n-3)_+}\right) : \ q < \min\left\{\frac{s(n+1)(p-2)}{(2s-(n+1))_+}, \frac{2ns(p-1)}{(s(n+1)-2n)_+}\right\}\right\}$$

is non-empty, so we may choose some arbitrary but fixed $q \in \Xi_{s,p}$ throughout this section.

Proposition 4. *Assume (A) and $\lambda > 0, h \in X^\delta_\lambda$. Then the map $T(\cdot,h) : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ from (17) is well-defined and we have*

$$\begin{aligned} \|T(u,h)\|_q &\le C(\|u\|_q^\alpha + \|h\|_{C^m}), \\ \|T(u,h) - T(v,h)\|_q &\le C\|u-v\|_q(\|u\|_q + \|v\|_q)^{\alpha-1} \end{aligned} \tag{18}$$

for some $\alpha > 1$ and all $u, v \in L^q(\mathbb{R}^n)$.

Proof. Using (A), Hölder's inequality and $|\chi(u)| \le \min\{|u|,1\}$ we get for all $u \in L^q(\mathbb{R}^n)$

$$\begin{aligned} &\|f(\cdot,\chi(u))\|_t \le \||Q||\chi(u)|^{p-1}\|_t \le \|Q\|_{\tilde{s}}\|\chi(u)^{p-1}\|_{\frac{t\tilde{s}}{\tilde{s}-t}} \le \|Q\|_{\tilde{s}}\|u\|_q^{\frac{q}{t}-\frac{q}{\tilde{s}}} < \infty, \\ &\text{provided} \quad t^*(q,\tilde{s}) := \max\left\{1, \frac{\tilde{s}q}{\tilde{s}(p-1)+q}\right\} \le t \le \tilde{s},\ \tilde{s} \in [s,\infty]. \end{aligned} \tag{19}$$

In particular we find $f(\cdot,\chi(u)) \in L^t(\mathbb{R}^n)$ for all $t \in [t^*(q,s),\infty]$. For any such t we choose $\tilde{s} := \frac{tq}{(q-t(p-1))_+} \in [s,\infty]$ (largest possible) so that $t^*(q,\tilde{s}) \le t \le \tilde{s}$ holds. So the previous estimate gives for $\alpha_t := \frac{q}{t} - \frac{q}{\tilde{s}} = \min\{\frac{q}{t}, p-1\}$

$$\|f(\cdot,\chi(u))\|_t \le C\|u\|_q^{\alpha_t} < \infty \qquad \text{for all } t \in [t^*(q,s),\infty]. \tag{20}$$

Now we have to choose $t \in [t^*(q,s),\infty]$ in such a way that the mapping properties of $\mathfrak{R}_\lambda$ from Theorem 4 ensure $\mathfrak{R}_\lambda(f(\cdot,\chi(u))) \in L^q(\mathbb{R}^n)$. In view of (15) we have to require

$$\frac{nq}{n+2q} \le t < \frac{(n+1)q}{n+1+2q} \qquad \text{and} \qquad t < \frac{2n}{n+1}. \tag{21}$$

Since $q \in \Xi_{s,p}$ implies $q < \frac{2n}{(n-3)_+}$ and hence $\frac{nq}{n+2q} < \frac{2n}{n+1}$, we can find such t if and only if

$$t^*(q,s) < \frac{(n+1)q}{n+1+2q} \qquad \text{and} \qquad t^*(q,s) < \frac{2n}{n+1}. \tag{22}$$

These two inequalities hold due to $q \in \Xi_{s,p}$. From this, Proposition 1 and Theorem 4 we get

$$\begin{aligned}\|T(u,h)\|_q &\le \|\mathfrak{R}_\lambda(f(\cdot,\chi(u)))\|_q + \|\widehat{h\,d\sigma_\lambda}\|_q \\ &\le C(\|f(\cdot,\chi(u))\|_t + \|h\|_{C^m}) \\ &\overset{(20)}{\le} C(\|u\|_q^{\alpha_t} + \|h\|_{C^m}).\end{aligned}$$

Since t was chosen according to (21), we have $t < q$ and thus $\alpha_t = \min\{p-1, \frac{q}{t}\} > 1$. Moreover, from (A) and Hölder's inequality ($\frac{1}{t} = \frac{1}{\tilde{s}} + \frac{1}{q} + \frac{\alpha_t - 1}{q}$) we get

$$\begin{aligned}\|f(\cdot,\chi(u)) - f(\cdot,\chi(v))\|_t &\le C\Big\||Q||\chi(u)-\chi(v)|(|\chi(u)|+|\chi(v)|)^{p-2}\Big\|_t \\ &\le C\|Q\|_{\tilde{s}}\|\chi(u)-\chi(v)\|_q\|(|\chi(u)|+|\chi(v)|)^{p-2}\|_{\frac{q}{\alpha_t-1}} \\ &\le C\|u-v\|_q\|(|u|+|v|)^{\alpha_t-1}\|_{\frac{q}{\alpha_t-1}} \\ &\le C\|u-v\|_q\big(\|u\|_q + \|v\|_q\big)^{\alpha_t-1}.\end{aligned} \tag{23}$$

Here $p-2 \ge \alpha_t - 1 > 0$ was used. Hence we get

$$\begin{aligned}\|T(u,h) - T(v,h)\|_q &\le \|\mathfrak{R}_\lambda(f(\cdot,\chi(u))) - \mathfrak{R}_\lambda(f(\cdot,\chi(v)))\|_q \\ &\le C\|u-v\|_q(\|u\|_q + \|v\|_q)^{\alpha_t-1},\end{aligned}$$

which finishes the proof. □

Proof of Theorem 1:

Step 1: Existence of a solution continuum (u_h) in $L^q(\mathbb{R}^n)$: We apply Banach's Fixed Point Theorem to $T(\cdot,h)$ on a closed small ball around zero $\overline{B_\rho} \subset L^q(\mathbb{R}^n)$, $q \in \Xi_{s,p} \neq \emptyset$ and $h \in X^\delta$ with $\delta > 0$ sufficiently small. From (18) we get that $T(\cdot,h) : \overline{B_\rho} \to \overline{B_\rho}$ is a contraction provided $\rho,\delta > 0$ are chosen sufficiently small. It is even a uniform contraction since its Lipschitz constant is independent of h. Moreover, T is continuous with respect to the topology of $L^q(\mathbb{R}^n) \times C^m(S^{n-1}_\lambda;\mathbb{C})$, which follows from Proposition 1 and Proposition 4. Hence, Banach's

Fixed Point Theorem for continuous uniform contractions yields a continuum of (uniquely determined) fixed points $u_h \in \overline{B_\rho}$ of $T(\cdot,h)$ for $h \in X^\delta$ provided $\delta > 0$ is small enough.

Step 2: The continuum property in $L^r(\mathbb{R}^n)$ for $r \in (q,\infty]$: As a fixed point of $T(\cdot,h)$ the function u_h solves $-\Delta u - \lambda u = f(x,\chi(u))$ in the strong sense on $\mathbb{R}^n$, see Theorem 4. Since $f(\cdot,\chi(u_h))$ is bounded, we even have $u_h \in W^{2,r}_{loc}(\mathbb{R}^n)$ for all $r \in [1,\infty)$. Fixing now $\tilde{q} \geq q$ such that $\tilde{q} > \frac{n}{2}$ we get from Theorem 8.17 in [15] (Moser iteration)

$$\begin{aligned}\|u_{h_1} - u_{h_2}\|_\infty &\leq \|u_{h_1} - u_{h_2}\|_q + \|f(\cdot,\chi(u_{h_1})) - f(\cdot,\chi(u_{h_2}))\|_{\tilde{q}} \\ &\leq \|u_{h_1} - u_{h_2}\|_q + \|Q\|_\infty \|(|\chi(u_{h_1})|^{p-2} + |\chi(u_{h_2})|^{p-2})(\chi(u_{h_1}) - \chi(u_{h_2}))\|_{\tilde{q}} \\ &\leq \|u_{h_1} - u_{h_2}\|_q + 2^{p-2+\frac{\tilde{q}-q}{q}} \|Q\|_\infty \||u_{h_1} - u_{h_2}|^{q/\tilde{q}}\|_{\tilde{q}} \\ &\leq C(\|u_{h_1} - u_{h_2}\|_q + \|u_{h_1} - u_{h_2}\|_q^{q/\tilde{q}})\end{aligned}$$

so that (u_h) is also a continuum in $L^\infty(\mathbb{R}^n)$ and hence in $L^r(\mathbb{R}^n)$ for all $r \in (q,\infty]$.

Step 3: The continuum property in $L^r(\mathbb{R}^n)$ for $r \in (\frac{2n}{n-1},q)$: From $u_h \in L^q(\mathbb{R}^n)$ we deduce $f(\cdot,\chi(u_h)) \in L^{t^*(q,s)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $t^*(q,s)$ defined in (19). We set

$$\tilde{q} := \max\Big\{r, \frac{(n+1)t^*(q,s)}{n+1-2t^*(q,s)}\Big\}$$

so that Theorem 4 gives $u_h \in L^{\tilde{q}}(\mathbb{R}^n)$ since the tuple of exponents $(t^*(q,s),\tilde{q})$ satisfies the inequalities (15). Notice that $q < \frac{s(n+1)(p-2)}{(2s-(n+1))_+}$ (because of $q \in \Xi_{s,p}$) implies $\tilde{q} < q$. Moreover, we have

$$\begin{aligned}\|u_{h_1} - u_{h_2}\|_{\tilde{q}} &\leq \|\mathcal{F}((h_1-h_2)\,d\sigma_\lambda)\|_{\tilde{q}} + \|\mathfrak{R}_\lambda(f(\cdot,\chi(u_{h_1})) - f(\cdot,\chi(u_{h_2})))\|_{\tilde{q}} \\ &\leq C\|h_1-h_2\|_{C^m} + C\|f(\cdot,\chi(u_{h_1})) - f(\cdot,\chi(u_{h_2})))\|_{t^*(q,s)} \\ &\overset{(23)}{\leq} C(\|h_1-h_2\|_{C^m} + \|u_{h_1} - u_{h_2}\|_q\big(\|u_{h_1}\|_q + \|u_{h_2}\|_q\big)^{p-2}) \\ &\leq C(\|h_1-h_2\|_{C^m} + \|u_{h_1} - u_{h_2}\|_q).\end{aligned}$$

Taking now $\tilde{q}$ as the new q and repeating the above arguments we get after finitely many steps $u_h \in L^r(\mathbb{R}^n)$ as well as

$$\|u_{h_1} - u_{h_2}\|_r \leq C(\|h_1-h_2\|_{C^m} + \|u_{h_1} - u_{h_2}\|_q).$$

Hence, (u_h) is a continuum in $L^r(\mathbb{R}^n)$ for all $r \in (\frac{2n}{n-1},q)$.

Step 4: The continuum property in $W^{2,r}(\mathbb{R}^n)$ for $r \in (\frac{2n}{n-1},\infty)$ and (3): From step 2 and step 3 we get

$$-\Delta u_h + u_h = (1+\lambda)u_h + f(\cdot,\chi(u_h)) \in L^r(\mathbb{R}^n)$$

because of $|f(x,\chi(u_h))| \leq \|Q\|_\infty |u_h| \in L^r(\mathbb{R}^n)$. Bessel potential estimates imply $u_h \in W^{2,r}(\mathbb{R}^n)$ and as above one finds $\|u_{h_1} - u_{h_2}\|_{W^{2,r}(\mathbb{R}^n)} \leq C\|u_{h_1} - u_{h_2}\|_r$ so that the continuum property is proved. In particular, $\|h\|_{C^m} \to 0$ implies $\|u_h\|_\infty = \|u_h - u_0\|_\infty \to 0$ and thus $\chi(u_h) = u_h$ is a $W^{2,r}(\mathbb{R}^n)$-solution of (3) for all $h \in X^\delta_\lambda$ provided $\delta > 0$ is sufficiently small.

Step 5: Asymptotics of u_h: From the previous steps we get $u_h \in L^r(\mathbb{R}^n)$ for all $r \in (\frac{2n}{n-1}, \infty]$ and this implies $|f(\cdot,u_h)| \le Q|\chi(u_h)|^{p-1} \in L^t(\mathbb{R}^n)$ for $t > \frac{2ns}{2n+s(n-1)(p-1)}$. In particular we have $f(\cdot,u_h) \in L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)$ because

$$p \overset{(4)}{>} \frac{2s(n^2+2n-1)-2n(n+1)}{(n^2-1)s} > \frac{s(2n^2+3n-1)-2n(n+1)}{(n^2-1)s}.$$

Hence, Proposition 2 yields

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left| \mathfrak{R}_\lambda(f(\cdot,u_h))(x) - \sqrt{\frac{\pi}{2\lambda}}\left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} \mathrm{Re}\left(e^{i(\frac{n-3}{4}\pi - \sqrt{\lambda}|x|)}\widehat{f(\cdot,u_h)}(-\sqrt{\lambda}\hat{x})\right)\right|^2 dx = 0.$$

Using $\overline{h(-\xi)} = h(\xi)$ and Proposition 1 we moreover get

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left| \widehat{h\,d\sigma_\lambda}(x) - \sqrt{\frac{2}{\pi}}\left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} \mathrm{Re}\left(e^{i(\frac{n-1}{4}\pi - \sqrt{\lambda}|x|)}h(\sqrt{\lambda}\hat{x})\right)\right|^2 dx = 0.$$

So $u_h = T(u_h,h) = \widehat{h\,d\sigma_\lambda} + \mathfrak{R}_\lambda(f(\cdot,u_h))$ and the above asymptotics imply (**??**). Finally, in the case $p > \frac{s(3n-1)-2n}{(n-1)s}$ we have $V := Q|u|^{p-2} \in L^t(\mathbb{R}^n)$ for some $t < \frac{2n}{n+1}$ because of $Q \in L^s(\mathbb{R}^n)$ and $u \in L^r(\mathbb{R}^n)$ for all $r > \frac{2n}{n-1}$. Hence, Lemma 2.9 in [10] yields the pointwise bounds if $n \ge 3$ and Lemma 2.3 in [9] in the case $n = 2$.

Step 6: Distinguishing u_{h_1}, u_{h_2}: From Step 2 we deduce that $h_1 \ne h_2$ implies $u_{h_1} \ne u_{h_2}$. Indeed, assuming $u_{h_1} = u_{h_2}$ we get $T(u_{h_1},h_1) = T(u_{h_2},h_2)$ and thus $\mathcal{F}\big((h_1-h_2)\,d\sigma_\lambda\big) = 0$. We show that this implies $h_1 = h_2$. To see this we use the scaling property

$$\mathcal{F}(h\,d\sigma_\lambda)(x) = \lambda^{\frac{n-1}{2}}\mathcal{F}\big(h(\sqrt{\lambda}\cdot)\,d\sigma_1\big)(\sqrt{\lambda}x).$$

From Corollary 4.6 in [1] we infer

$$\begin{aligned}
0 &= \lim_{R\to\infty}\frac{1}{R}\int_{B_R} |\mathcal{F}\big((h_1-h_2)\,d\sigma_\lambda\big)|^2\,dx \\
&= \lambda^{n-1}\lim_{R\to\infty}\frac{1}{R}\int_{B_R} |\mathcal{F}\big((h_1-h_2)(\sqrt{\lambda}\cdot)\,d\sigma_1\big)(\sqrt{\lambda}x)|^2\,dx \\
&= \lambda^{\frac{n-1}{2}}\cdot\lim_{R\to\infty}\frac{1}{\sqrt{\lambda}R}\int_{B_{\sqrt{\lambda}R}} |\mathcal{F}\big((h_1-h_2)(\sqrt{\lambda}\cdot)\,d\sigma_1\big)(x)|^2\,dx \\
&= 2(2\pi)^{n-1}\lambda^{\frac{n-1}{2}}\int_{S^{n-1}} |(h_1-h_2)(\sqrt{\lambda}x)|^2\,d\sigma_1(x),
\end{aligned}$$

which implies $h_1 = h_2$. □

Remark 2. *(a) Let us describe how Theorem 1 provides nonsymmetric solutions of symmetric nonlinear Helmholtz equations as mentioned in Remark 1(b). We assume $f(\gamma x,z) = f(x,z)$ for almost all $x \in \mathbb{R}^n$ and all $z \in \mathbb{R}, \gamma \in \Gamma$ where $\Gamma \subset O(n), \Gamma \ne \{\mathrm{id}\}$*

is a subgroup. Since $\Gamma \neq \{\mathrm{id}\}$ *we can find* $h \in X^\delta$ *and* $\gamma \in \Gamma$ *satisfying* $h \neq h \circ \gamma$ *and our claim is that this implies* $u_h \neq u_h \circ \gamma$*. Indeed, otherwise we would have*

$$\begin{aligned}
\widehat{h\, d\sigma_\lambda} + \mathfrak{R}_\lambda(f(\cdot, \chi(u_h))) &= T(u_h, h) = u_h = u_h \circ \gamma = T(u_h, h) \circ \gamma \\
&= \widehat{h\, d\sigma_\lambda} \circ \gamma + \mathfrak{R}_\lambda(f(\cdot, \chi(u_h))) \circ \gamma \\
&= \widehat{h \circ \gamma\, d\sigma_\lambda} + \mathfrak{R}_\lambda(f(\cdot, \chi(u_h)) \circ \gamma) \\
&= \widehat{h \circ \gamma\, d\sigma_\lambda} + \mathfrak{R}_\lambda(f(\cdot, \chi(u_h \circ \gamma))) \\
&= \widehat{h \circ \gamma\, d\sigma_\lambda} + \mathfrak{R}_\lambda(f(\cdot, \chi(u_h))).
\end{aligned}$$

From the second to the third line we used that $\mathfrak{R}_\lambda$ *is a convolution operator with a radially symmetric and hence* Γ*-symmetric kernel and from the third to the fourth line we used that* f *is* Γ*-invariant. So we conclude* $\widehat{h\, d\sigma_\lambda} = \widehat{h \circ \gamma\, d\sigma_\lambda}$*, which implies* $h = h \circ \gamma$ *as in Step 6 above, a contradiction. Hence,* $u_h \neq u_h \circ \gamma$ *so that* u_h *is not* Γ*-symmetric.*

(b) *In Remark 1(c) we claimed that Theorem 1 provides radial solutions assuming the weaker condition* (6) *instead of* (4)*. This is due to an improved version of the resolvent estimates from* (15) *where in both lines* $\frac{2}{n+1} \leq \frac{1}{t} - \frac{1}{q}$ *can be replaced by* $\frac{3n-1}{2n^2} < \frac{1}{t} - \frac{1}{q}$*. This was demonstrated in Remark 3.1 in [7]. Let us explain how these improved resolvent estimates allow to obtain radial solutions for a larger range of exponents. The only radially symmetric Herglotz waves* $\widehat{h\, d\sigma_\lambda}$ *are given by real-valued and constant densities* h*. So for* $h \in \mathbb{R}$ *we get that* $T(\cdot, h)$ *maps* $L^q_{rad}(\mathbb{R}^n)$ *into itself provided* $f(x,u) = f_0(|x|, u)$ *is radially symmetric. Here, the exponent* q *may be chosen from*

$$\Xi^{rad}_{s,p} := \left\{ q \in \left(\frac{2n}{n-1}, \frac{2n}{(n-3)_+} \right) : \ q < \min\left\{ \frac{2n^2 s(p-2)}{((3n-1)s - 2n^2)_+}, \frac{2ns(p-1)}{(s(n+1) - 2n)_+} \right\} \right\},$$

which now is nonempty due to (6)*. Replacing in Proposition 4 the first inequality in* (22) *by* $t^*(q,s) < \frac{2n^2 q}{2n^2 + (3n-1)q}$ *and redefining* $\tilde{q}$ *in the proof of Theorem 1 accordingly, we get the desired existence result again from Banach's Fixed Point Theorem.*

(c) *Under severe restrictions on the nonlinearity our result may also be proved using dual variational methods originally developed by Evéquoz and Weth [10]. To demonstrate this we consider the special case* $f(x,z) = |z|^{p-2} z$ *with* $\frac{2(n+1)}{n-1} < p < \frac{2n}{n-2}$*. The dual functional* $J_h : L^{p'}(\mathbb{R}^n) \to \mathbb{R}$ *is then given by*

$$J_h(v) := \frac{1}{p'} \int_{\mathbb{R}^n} |v|^{p'} - \int_{\mathbb{R}^n} v \cdot \mathcal{F}(h\, d\sigma_\lambda) - \frac{1}{2}\,\mathrm{p.v.} \int_{\mathbb{R}^n} \frac{|\hat{v}(\xi)|^2}{|\xi|^2 - \lambda}\, d\xi$$

and a local minimizer of J_h *lying in the interior of a small ball may be shown to exist for* $h \in X^\delta$ *for* $\delta > 0$ *sufficiently small using Ekeland's variational principle. Since every critical point* v_h *of* J_h *provides a fixed point of* $T(\cdot, h)$ *vai* $u_h := |v_h|^{p'-2} v_h$*, see Section 4 in [10], we rediscover the solutions found in Theorem 1.*

4. Proof of Theorem 2

In this section we discuss how the above approach needs to be modified in order to get solutions of the fourth order problem (7).

We first consider the case (i) in (8). From Theorem 5 we know that there is a resolvent-type operator $\mathbf{R}$ associated with $\Delta^2 - \beta\Delta + \alpha$ which is linear and bounded between the same (and even more) pairs of Lebesgue spaces as $\mathfrak{R}_\lambda$. So all estimates in Proposition 4 involving $\mathfrak{R}_\lambda$ equally hold for $\mathbf{R}$. As a replacement for the Herglotz wave of the Helmholtz operator we take again a Herglotz wave $\widehat{h\,d\sigma_\lambda}$ where now $h \in Y^\delta = X^\delta_\lambda$ and λ was defined in (9) in dependence of α, β. This definition of λ was made in such a way that $\widehat{h\,d\sigma_\lambda}$ satisfies the homogeneous equation $\Delta^2\phi - \beta\Delta\phi + \alpha\phi = 0$ because of $\lambda^2 - \beta\lambda + \alpha = 0$. As a consequence, also the Herglotz-wave part of the map

$$T(u,h) := \widehat{h\,d\sigma_\lambda} + \mathbf{R}(f(\cdot, \chi(u))),$$

may be estimated as in Proposition 4. So we can find a fixed point of $T(\cdot, h)$ in a small ball of $L^q(\mathbb{R}^n)$ for $q \in \Xi_{s,p}$ exactly as in the proof of Theorem 1. The qualitative properties can also be proved the same way, see also Section 5 in [7] where $u_h \in W^{4,r}(\mathbb{R}^n)$ for all $r \in (\frac{2n}{n-1}, \infty)$ as well as its pointwise decay rate was proved in the special case $f(x,z) = \Gamma(x)|z|^{p-2}z$, $\Gamma \in L^\infty(\mathbb{R}^n)$.

In the case (ii) the proof is essentially the same. The only difference is that the fixed point operator now reads

$$T(u,h) := \widehat{h_1\,d\sigma_{\lambda_1}} + \widehat{h_2\,d\sigma_{\lambda_2}} + \mathbf{R}(f(\cdot, \chi(u))),$$

for $h = (h_1, h_2) \in Y^\delta = X^\delta_{\lambda_1} \times X^\delta_{\lambda_2}$. Besides that all arguments are identical and we conclude as above. □

5. Proof of Theorem 3

Theorem 3 will be proved via the Contraction Mapping Theorem on a small ball in $\mathbb{R}^3$ (part (i)) or on $\mathbb{R}^3$ (part (ii)). The reason for this is that the Limiting Absorption Principle in the latter case is much weaker and forces us to consider nonlinear curl-curl equations with nonlinearities that grow sublinearly at infinity. Notice that the growth of the nonlinearity with respect to E cannot be ignored by using a truncation as in our results proved above. In fact, an equivalent of local elliptic regularity theory and in particular (L^r, L^∞)-estimates for the curl-curl-operator are not known and may even be false.

We start with a few words on the proof of part (i), which is very similar to the proof of Theorem 1 and Theorem 2. So let f satisfy (A'). For $q \in \Xi_{s,p}$ defined in (4) (for $n = 3$) we consider the map $T(\cdot, h) : L^q_{cyl}(\mathbb{R}^3; \mathbb{R}^3) \to L^q_{cyl}(\mathbb{R}^3; \mathbb{R}^3)$ where

$$T(E,h) := \widehat{h\,d\sigma_\lambda} + R_\lambda(f(\cdot, \chi(|E|)E/|E|)). \tag{24}$$

Here, $L^q_{cyl}(\mathbb{R}^3; \mathbb{R}^3)$ denotes the Banach space of cylindrically symmetric functions lying in $L^q(\mathbb{R}^3; \mathbb{R}^3)$ and the function $\chi \in C^\infty(\mathbb{R})$ is chosen as before, i.e., it satisfies $|\chi(z)| \le \min\{|z|, 1\}$ as well as $\chi(z) = z$ provided $|z| \le \frac{1}{2}$. The map $T(\cdot, h)$ is well-defined for $h \in Z^\delta_{cyl}, \delta > 0$ since the nonlinearity is compatible with cylindrical symmetry by (A'). By Proposition 3 the functions $\widehat{h\,d\sigma_\lambda}$ are smooth cylindrically symmetric solutions of $\nabla \times \nabla \times E - \lambda E = 0$ and satisfy the same estimates as their scalar counterparts used in the proof of Theorem 1. Likewise, Theorem 6 implies that R_λ restricted to the space of cylindrically symmetric functions has the same $L^p - L^q$ mapping properties as $\mathfrak{R}_\lambda$. Moreover, not only f but also the function $(x,E) \mapsto f(x,\chi(|E|)E/|E|)$ satisfies (A') because $z \mapsto \chi(z)/z$ is smooth. So the operator T defined in (24) also satisfies the estimates from Proposition 4 and one obtains a unique fixed point E_h of $T(\cdot,h)$ on a small ball in $L^q_{cyl}(\mathbb{R}^3;\mathbb{R}^3)$ via the Contraction Mapping Theorem. As a cylindrically symmetric and hence divergence-free solution of (10) the vector field E_h even solves the elliptic system (12) so that elliptic regularity theory implies $\chi(E_h) = E_h$ provided the ball in $L^q_{cyl}(\mathbb{R}^3;\mathbb{R}^3)$ and $h \in Z^\delta_{cyl}, \delta > 0$ are chosen small enough. Also the estimate $|E_h(x)| \le C_h(1+|x|)^{\frac{1-n}{2}}$ is proved as in step 5 of the proof of Theorem 1 under the assumption $p > \frac{s(3n-1)-2n}{(n-1)s}$.

From now on we prove part (ii), so let f satisfy assumption (B). We fix an exponent q such that $3 < q < \frac{3s}{(2s-3)_+}$, which is possible due to $s \in [1,2]$. For $h \in Z$ we set

$$T(E,h) := \widehat{h\,d\sigma_\lambda} + R_\lambda(f(\cdot,E)). \tag{25}$$

We first verify that $T(\cdot,h) : L^q(\mathbb{R}^3;\mathbb{R}^3) \to L^q(\mathbb{R}^3;\mathbb{R}^3)$ is well-defined and Lipschitz continuous. In the proof of these estimates we use the number

$$\alpha_{p,\tilde p} := \sup_{z\in\mathbb{R}} |z|^{p-2}(1+|z|)^{\tilde p - p} = \frac{(p-2)^{p-2}(2-\tilde p)^{2-\tilde p}}{(p-\tilde p)^{p-\tilde p}} \qquad \text{where } 0^0 := 1 \text{ and } \tilde p \le 2 \le p.$$

Proposition 5. *Assume (B) and $\lambda > 0, h \in Z$. Then the map $T(\cdot,h) : L^q(\mathbb{R}^3;\mathbb{R}^3) \to L^q(\mathbb{R}^3;\mathbb{R}^3)$ from (25) is well-defined with*

$$\|T(E_1,h) - T(E_2,h)\|_q \le C\alpha_{p,\tilde p}(\|Q\|_s + \|Q\|_\infty)\|E_1 - E_2\|_q. \tag{26}$$

where C only depends on q and s.

Proof. By Proposition 3 the functions $\widehat{h\,d\sigma_\lambda}$ belong to $L^q(\mathbb{R}^3;\mathbb{R}^3)$ for all $h \in Z$. So the definition of T from (25) and the Limiting Absorption Principle for the curl-curl operator (Theorem 6) imply that $T(\cdot,h)$ is well-defined if we can show $f(\cdot,E) \in L^q(\mathbb{R}^3;\mathbb{R}^3) \cap L^t(\mathbb{R}^3;\mathbb{R}^3)$ for some $t \in (1,\infty)$ satisfying (15). To verify this we set $\tilde s := \max\{s, \frac{3}{2}\}$ for $s \in [1,2]$ as in assumption (B) and choose $t := \frac{\tilde s q}{\tilde s + q}$. This implies $1 < t < \frac32, \frac32 \le \tilde s \le 2$ so that (t,q) indeed satisfies (15). So we infer from assumption (B)

$$\begin{aligned}
\|f(\cdot,E)\|_q &\le \|Q|E|^{p-1}(1+|E|)^{\tilde p - p}\|_q \le \alpha_{p,\tilde p}\|Q|E|\|_q \le \alpha_{p,\tilde p}\|Q\|_\infty\|E\|_q < \infty,\\
\|f(\cdot,E)\|_t &\le \|Q|E|^{p-1}(1+|E|)^{\tilde p - p}\|_t \le \alpha_{p,\tilde p}\|Q|E|\|_t \le \alpha_{p,\tilde p}\|Q\|_{\tilde s}\|E\|_q < \infty.
\end{aligned}$$

This implies that $T(\cdot,h)$ is well-defined. Moreover, we have

$$\begin{aligned}
&\|f(\cdot,E_1) - f(\cdot,E_2)\|_q \le \alpha_{p,\tilde p}\|Q\|_\infty\|E_1 - E_2\|_q,\\
&\|f(\cdot,E_1) - f(\cdot,E_2)\|_t \le \alpha_{p,\tilde p}\|Q\|_{\tilde s}\|E_1 - E_2\|_q \le \alpha_{p,\tilde p}(\|Q\|_s + \|Q\|_\infty)\|E_1 - E_2\|_q.
\end{aligned}$$

Combining these estimates with Theorem 6 one gets (26) from the definition of $T(\cdot,h)$. □

Proof of Theorem 3 (ii): From the previous proposition we get that $T(\cdot,h)$ maps $L^q(\mathbb{R}^3;\mathbb{R}^3)$ into itself and it is a contraction provided $\alpha_{p,\tilde{p}}(\|Q\|_s+\|Q\|_\infty)$ is small enough, which is guaranteed by the assumptions of the Theorem 3. So for any given $h\in Z$ the Contraction Mapping Theorem and Theorem 6 provide a unique weak solution $E_h\in H_{loc}(\mathrm{curl};\mathbb{R}^3)\cap L^q(\mathbb{R}^3;\mathbb{R}^3)$ of (3). It remains to discuss the integrability properties of E_h. In this discussion we will w.l.o.g. assume that assumption (B) holds with $\tilde{p}\in(1,2]$ because otherwise we may simply increase $\tilde{p}$.

Proof of (ii)(a): Under the additional assumption $(p,s)\neq(2,2)$ we want to show $E_h\in L^r(\mathbb{R}^3;\mathbb{R}^3)$ for all $r\in(3,q)$. To achieve this iteratively we use $E_h\in L^q(\mathbb{R}^3;\mathbb{R}^3)$ and hence, by assumption (B),

$$f(\cdot,E_h)\in L^t(\mathbb{R}^3;\mathbb{R}^3)\cap L^{\tilde{q}}(\mathbb{R}^3;\mathbb{R}^3)\qquad\text{for all } t,\tilde{q}\in\Big[t^*(q,s),\frac{q}{\tilde{p}-1}\Big].$$

This follows as in (19), where also $t^*(q,s)$ is defined. In order to prove $E_h\in L^{\tilde{q}}(\mathbb{R}^3;\mathbb{R}^3)$ with $\tilde{q}<q$ we use $E_h=T(E_h,h)=\widehat{h\,d\sigma_\lambda}+R_\lambda(f(\cdot,E_h))$. In view of Proposition 3 and the mapping properties of R_λ from Theorem 6 we obtain $E_h\in L^{\tilde{q}}(\mathbb{R}^3;\mathbb{R}^3)$ with $\tilde{q}<q$ provided the pair $(t,\tilde{q})$ satisfies $1<t,\tilde{q}<\infty$ as well as the inequalities from (15) in the three-dimensional case $n=3$. In other words we require

$$1<t<\frac{3}{2},\qquad 3<\tilde{q}<q,\qquad \frac{1}{2}\le\frac{1}{t}-\frac{1}{\tilde{q}}\le\frac{2}{3},\qquad t^*(q,s)\le t,\tilde{q}\le\frac{q}{\tilde{p}-1}. \tag{27}$$

So we set $\tilde{q}:=\max\{r,\frac{2t}{2-t},t^*(q,s)\}=\max\{r,\frac{2t}{2-t}\}\ge r>3$. (Notice that our choice for t will ensure $t^*(q,s)<t<\frac{3}{2}<3<r$.) Plugging in the definition of $t^*(q,s)$ from (19) we obtain after some calculations that the inequalities (27) hold if

$$\max\Big\{1,\frac{3r}{3+2r},\frac{qs}{q+s(p-1)}\Big\}<t<\min\Big\{\frac{3}{2},\frac{2q}{q+2},\frac{q}{\tilde{p}-1}\Big\}.$$

Here non-strict inequalities in (27) were sharpened to strict inequalities for notational convenience. Such a choice for t is possible because of $1\le s\le 2\le p,(p,s)\neq(2,2)$ and $3<r<q<<\frac{3s}{(2s-3)_+}<\frac{3s(p-1)}{(2s-3)_+}$. For instance we may choose

$$t=t_q:=\frac{1}{2}\cdot\left(\max\Big\{1,\frac{3r}{3+2r},\frac{qs}{q+s(p-1)}\Big\}+\min\Big\{\frac{3}{2},\frac{2q}{q+2},\frac{q}{\tilde{p}-1}\Big\}\right)$$

and Theorem 6 implies $E_h\in L^{\tilde{q}}(\mathbb{R}^3;\mathbb{R}^3)$. In the case $\tilde{q}=r$ we are done. Otherwise, we may repeat this argument replacing q by $\tilde{q}$ so that $r\in(3,\tilde{q})$. The corresponding iteration yields $E_h\in L^r(\mathbb{R}^3;\mathbb{R}^3)$ after finitely many steps.

Proof of (ii)(b): Using $\tilde{p}<2<p$ we now prove $E_h\in L^r(\mathbb{R}^3;\mathbb{R}^3)$ for all $r\in(q,\frac{3s(p-1)}{(2s-3)_+})$. In view of (B) and $E_h\in L^q(\mathbb{R}^3;\mathbb{R}^3)$ we now have to choose $t,\tilde{q}\in[t^*(q,s),\frac{q}{\tilde{p}-1}]$ with $\tilde{q}>q$ such that the pair $(t,\tilde{q})$ satisfies $1<t,\tilde{q}<\infty$ as well as

$$1<t<\frac{3}{2},\qquad q<\tilde{q}<\infty,\qquad \frac{1}{2}\le\frac{1}{t}-\frac{1}{\tilde{q}}\le\frac{2}{3},\qquad t^*(q,s)\le t,\tilde{q}\le\frac{q}{\tilde{p}-1}.$$

So we set $\tilde q := \min\{\frac{3t}{3-2t}, \frac{q}{\tilde p-1}\}$ and due to $1<\tilde p<2$ it remains to choose t such that

$$\max\left\{1, \frac{3q}{3+2q}, \frac{qs}{q+s(p-1)}\right\} < t < \min\left\{\frac{3}{2}, \frac{2q}{q+2(\tilde p-1)}, \frac{q}{\tilde p-1}\right\}.$$

Such a choice is possible thanks to $q < \frac{3s(p-1)}{(2s-3)_+}$ and $1 \le s \le 2, 1<\tilde p<2$. For instance we may choose

$$t = t_q := \frac{1}{2}\cdot\left(\max\left\{1, \frac{3q}{3+2q}, \frac{qs}{q+s(p-1)}\right\} + \min\left\{\frac{3}{2}, \frac{2q}{q+2(\tilde p-1)}, \frac{q}{\tilde p-1}\right\}\right).$$

Then Theorem 6 implies $E_h \in L^{\tilde q}(\mathbb{R}^3;\mathbb{R}^3)$ and we have $\tilde q > q$. We may repeat this argument as long as $\tilde q < \frac{3s(p-1)}{(2s-3)_+}$ and thereby obtain $u \in L^r(\mathbb{R}^3;\mathbb{R}^3)$ for all $r \in (q, \frac{3s(p-1)}{(2s-3)_+})$, which finishes the proof. □

6. Appendix A: Proof of Proposition 1

In this section we give the proof of Proposition 1 which we repeat for convenience.

Proposition. *Let $n \in \mathbb{N}, n \ge 2$ and $m := \lfloor \frac{n-1}{2} \rfloor + 1$. Then for all $h \in C^m(S_\lambda^{n-1};\mathbb{C})$ the Herglotz wave $\widehat{h\,d\sigma_\lambda}$ is an analytic solution of $-\Delta\phi - \lambda\phi = 0$ in $\mathbb{R}^n$ and satisfies the estimate $|(\widehat{h\,d\sigma_\lambda})(x)| \le C\|h\|_{C^m}(1+|x|)^{\frac{1-n}{2}}$ as well as*

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left| \widehat{h\,d\sigma_\lambda}(x) - \frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} m_h(x)\right|^2 dx = 0.$$

In particular, we have $\|\widehat{h\,d\sigma_\lambda}\|_r \le C_r\|h\|_{C^m}$ for all $r > \frac{2n}{n-1}$.

The asymptotics of the functions $\widehat{h\,d\sigma_\lambda}$ are proved using the method of stationary phase, but typically it is assumed that h is smooth, see for instance Proposition 4,5,6 in Chapter VIII§2 of [25] or page 6-7 in [17]. In spirit, the above result is not new, but we could not find a reference for it covering all space dimensions $n \ge 2$ with an explicit value for m. For this reason, we decided to present a proof here.

Proof of Proposition 1: We consider the Herglotz wave

$$\widehat{h\,d\sigma_\lambda}(x) = \frac{1}{(2\pi)^{n/2}}\int_{S_\lambda^{n-1}} h(\xi)e^{-i\langle x,\xi\rangle}\,d\sigma_\lambda(\xi).$$

To investigate its pointwise behaviour as $|x|\to\infty$ let $Q = Q_x \in O(n)$ satisfy $Q^T x = |x|e_n$, so

$$\widehat{h\,d\sigma_\lambda}(x) = \frac{1}{(2\pi)^{n/2}}\int_{S_\lambda^{n-1}} h(Q\xi)e^{-i|x|\xi_n}\,d\sigma_\lambda(\xi).$$

Now we choose $\eta_1,\ldots\eta_{2n} \in C^\infty(\mathbb{R}^n)$ such that $\eta_1+\ldots+\eta_{2n} = 1$ on S_λ^{n-1} and, for $k=1,\ldots,n$,

$$\operatorname{supp}(\eta_{2k-1}) \subset \{\xi = (\xi_1,\ldots,\xi_n)\in\mathbb{R}^n : \xi_k > +\sqrt{\lambda}\delta\},$$
$$\operatorname{supp}(\eta_{2k}) \subset \{\xi = (\xi_1,\ldots,\xi_n)\in\mathbb{R}^n : \xi_k < -\sqrt{\lambda}\delta\},$$

where $\delta \in (0, \frac{1}{\sqrt{n}})$ is fixed. Notice that such a partition of unity exists since the open sets on the right hand side cover the sphere S^{n-1}_λ. We define $h_j(\xi) := (2\pi)^{-n/2} h(Q\xi)\eta_j(\xi)$ so that we have to investigate the integrals

$$I_j := \int_{S^{n-1}_\lambda} h_j(\xi) e^{-i|x|\xi_n} \, d\sigma_\lambda(\xi) \qquad (j = 1, \dots, 2n).$$

We first estimate the oscillatory integrals $I_1, \dots, I_{2n-2}$ where the resonant poles $\pm e_n$ are cut out by the choice of $\eta_1, \dots, \eta_{2n-2}$. To estimate I_j we use the local parametrization given by

$$\begin{aligned}
&\text{If } j = 2k-1: \quad \psi_j(\xi') := \sqrt{\lambda}(\xi_1, \dots, \xi_{k-1}, +\sqrt{1-|\xi'|^2}, \xi_k, \dots, \xi_{n-1}) \in S^{n-1}_\lambda \\
&\text{If } j = 2k \quad\;\;\, : \quad \psi_j(\xi') := \sqrt{\lambda}(\xi_1, \dots, \xi_{k-1}, -\sqrt{1-|\xi'|^2}, \xi_k, \dots, \xi_{n-1}) \in S^{n-1}_\lambda
\end{aligned}$$

for $\xi' := (\xi_1, \dots, \xi_{n-1})$ belonging to the unit ball $B \subset \mathbb{R}^{n-1}$. The function

$$H_j(\xi') := \lambda^{\frac{n-1}{2}} h_j(\psi_j(\xi'))(1 - |\xi'|^2)^{-1/2}$$

then satisfies $\operatorname{supp}(H_j) \subset B$ so that integration by parts yields for all $|x| \geq 1$ and $j = 1, \dots, 2n-2$

$$\begin{aligned}
|I_j| &= \left| \int_B H_j(\xi') e^{-i\sqrt{\lambda}|x|\xi_{n-1}} \, d\xi' \right| \\
&= \left| \int_{\mathbb{R}^{n-1}} H_j(\xi') \frac{\partial^m}{\partial(\xi_{n-1})^m} \left(e^{-i\sqrt{\lambda}|x|\xi_{n-1}} \right) d\xi' \right| \cdot |\sqrt{\lambda} x|^{-m} \\
&= \left| \int_{\mathbb{R}^{n-1}} \left(\frac{\partial^m}{\partial(\xi_{n-1})^m} H_j(\xi') \right) e^{-i\sqrt{\lambda}|x|\xi_{n-1}} \, d\xi' \right| \cdot |\sqrt{\lambda} x|^{-m} \\
&= \int_B |\nabla^m H_j(\xi')| \, d\xi' \cdot |\sqrt{\lambda} x|^{-m} \\
&\leq C \|h\|_{C^m} \cdot |x|^{-m}
\end{aligned}$$

Since the estimate for $|x| \leq 1$ is trivial and $m = \lfloor \frac{n-1}{2} \rfloor + 1 \geq \frac{n}{2}$, we conclude

$$(28) \qquad |I_j| \leq C \|h\|_{C^m} (1 + |x|)^{\frac{1-n}{2} - \alpha} \quad \text{for all } x \in \mathbb{R}^n, \; j = 1, \dots, 2n-2, \; \alpha \in (0, \frac{1}{4}).$$

Next we analyze the integrals I_{2n-1}, I_{2n-2}. With ψ_j, H_j as above we define

$$H_j^*(\eta) := H_j(\eta\sqrt{2 - |\eta|^2}) \cdot (2 - 2|\eta|^2)(2 - |\eta|^2)^{\frac{n-3}{2}} \qquad (j = 2n-1, 2n).$$

Again the supports of H_j, H_j^* are contained in the interior of the unit ball B so that neither function is singular. Performing twice a change of coordinates we get

$$\begin{aligned}
I_{2n-1} &= \int_{\mathbb{R}^{n-1}} H_{2n-1}(\xi') e^{-i\sqrt{\lambda}|x|\sqrt{1-|\xi'|^2}} \, d\xi' = e^{-i\sqrt{\lambda}|x|} \int_{\mathbb{R}^{n-1}} H_{2n-1}^*(\eta) e^{i\sqrt{\lambda}|x||\eta|^2} \, d\eta, \\
I_{2n} &= \int_{\mathbb{R}^{n-1}} H_{2n}(\xi') e^{+i\sqrt{\lambda}|x|\sqrt{1-|\xi'|^2}} \, d\xi' = e^{+i\sqrt{\lambda}|x|} \int_{\mathbb{R}^{n-1}} H_{2n}^*(\eta) e^{-i\sqrt{\lambda}|x||\eta|^2} \, d\eta.
\end{aligned}$$

The integrals may be estimated using Proposition 11 in [22] for $s := m - 2\alpha$. Notice that $\alpha \in (0, \frac{1}{4})$ ensures $s \geq \frac{n+1}{2} - 2\alpha > \frac{n-1}{2}$ so that the estimate from this proposition is valid. Using

$$H^*_{2n-1}(0) = 2^{\frac{n-1}{2}} H_{2n-1}(0) = (2\lambda)^{\frac{n-1}{2}} h_{2n-1}(+\sqrt{\lambda}e_n) = \frac{1}{\sqrt{2\pi}} \left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} h(+\sqrt{\lambda}\hat{x}),$$
$$H^*_{2n}(0) = 2^{\frac{n-1}{2}} H_{2n}(0) = (2\lambda)^{\frac{n-1}{2}} h_{2n}(-\sqrt{\lambda}e_n) = \frac{1}{\sqrt{2\pi}} \left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} h(-\sqrt{\lambda}\hat{x})$$

as well as $\|H_j\|_{H^m(\mathbb{R}^{n-1})} \leq C\|h\|_{C^m}$ for $j \in \{2n-1, 2n\}$ we deduce from the first inequality in Proposition 11 [22]

$$\left| I_{2n-1} - e^{i(\frac{n-1}{4}\pi - \sqrt{\lambda}|x|)} \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} h(\sqrt{\lambda}\hat{x}) \right| \leq C\|h\|_{C^m} |x|^{\frac{1-n}{2}-\alpha},$$
$$\left| I_{2n} - e^{-i(\frac{n-1}{4}\pi - \sqrt{\lambda}|x|)} \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} h(-\sqrt{\lambda}\hat{x}) \right| \leq C\|h\|_{C^m} |x|^{\frac{1-n}{2}-\alpha}. \tag{29}$$

Combining (28),(29) and $\widehat{h\,d\sigma_\lambda} = I_1 + \ldots + I_{2n}$ we find for $|x| \geq 1$

$$\left| \widehat{h\,d\sigma_\lambda}(x) - \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} m_h(x) \right| \leq C\|h\|_{C^m} |x|^{\frac{1-n}{2}-\alpha},$$

where m_h was introduced in (14). In view of the estimate $|\widehat{h\,d\sigma_\lambda}(x)| \leq C\|h\|_\infty$ for $|x| \leq 1$ we derive the weaker statements

$$|\widehat{h\,d\sigma_\lambda}(x)| \leq C\|h\|_{C^m} (1 + |x|)^{\frac{1-n}{2}}$$

as well as

$$\begin{aligned}
&\frac{1}{R} \int_{B_R} \left| \widehat{h\,d\sigma_\lambda}(x) - \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\lambda}}{|x|}\right)^{\frac{n-1}{2}} m_h(x) \right|^2 dx \\
&\leq C\|h\|_\infty^2 \cdot \frac{1}{R} \int_{B_1} (1 + |x|^{1-n})\,dx + C\|h\|_{C^m}^2 \cdot \frac{1}{R} \int_{B_R \setminus B_1} |x|^{1-n-2\alpha}\,dx \\
&\leq C\Big(\frac{1}{R} + \frac{1}{R^{2\alpha}}\Big) \|h\|_{C^m}^2 \quad \to 0 \qquad \text{as } R \to \infty.
\end{aligned}$$

This finishes the proof of Proposition 1. □

7. Appendix B: Proof of Theorem 6

In this section we prove the Limiting Absorption Principle from Theorem 6. We recall the statement for the convenience of the reader.

Theorem. *Let $\lambda > 0$ and assume that $t, q \in (1, \infty)$ satisfy (15). Then there is a bounded linear operator $R_\lambda : L^t(\mathbb{R}^3;\mathbb{R}^3) \cap L^q(\mathbb{R}^3;\mathbb{R}^3) \to L^q(\mathbb{R}^3;\mathbb{R}^3)$ such that $R_\lambda G \in H_{loc}(\mathrm{curl};\mathbb{R}^3)$ is a weak solution of $\nabla \times \nabla \times E - \lambda E = G$ provided $G \in L^t(\mathbb{R}^3;\mathbb{R}^3) \cap L^q(\mathbb{R}^3;\mathbb{R}^3)$. Moreover, we have*

$$\|R_\lambda G\|_q \le C(\|G_1\|_q + \|G_2\|_t) \le C(\|G\|_q + \|G\|_t)$$

and $R_\lambda G = -\frac{1}{\lambda} G_1 + \mathfrak{R}_\lambda G_2$ for $\mathfrak{R}_\lambda$ from Theorem 4 (applied componentwise). If $G \in L^t(\mathbb{R}^3;\mathbb{R}^3)$ is cylindrically symmetric then so is $R_\lambda G$ and $R_\lambda G \in W^{2,q}_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ is a strong solution satisfying $\|R_\lambda G\|_q \le C\|G\|_t$.

To prove this, we first show that the domain of the selfadjoint realization of the curl-curl operator $LE := \nabla \times \nabla \times E$ is given by

$$\begin{aligned}\mathcal{D} &:= \{E \in L^2(\mathbb{R}^3;\mathbb{R}^3) : LE \in L^2(\mathbb{R}^3;\mathbb{R}^3)\} \\ &:= \{E \in L^2(\mathbb{R}^3;\mathbb{R}^3) : \xi \mapsto |\xi|^2 \hat{E}(\xi) - \langle \hat{E}(\xi), \xi\rangle \xi \in L^2(\mathbb{R}^3;\mathbb{R}^3)\}\end{aligned}$$

and that its spectrum is $[0, \infty)$. Using the Helmholtz decomposition

$$\hat{G}_1(\xi) := \langle \hat{G}(\xi), \frac{\xi}{|\xi|}\rangle \frac{\xi}{|\xi|}, \qquad \hat{G}_2(\xi) := \hat{G}(\xi) - \hat{G}_1(\xi) \tag{30}$$

of a vector field $G \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ we get the following.

Proposition 6. *The curl-curl operator $L : \mathcal{D} \to L^2(\mathbb{R}^3;\mathbb{R}^3)$ is selfadjoint with spectrum $\sigma(L) = [0, \infty)$. For $\mu \in \mathbb{C} \setminus [0, \infty)$ the resolvent is given by*

$$(L - \mu)^{-1} G = -\frac{1}{\mu} G_1 + \mathcal{R}(\mu) G_2$$

where $\mathcal{R}(\mu)$ is (in each component) the operator defined at the beginning of Section 2.

Proof. The curl-curl-operator L is symmetric when defined on the Schwartz functions in $\mathbb{R}^3$. So we have to show that all $E \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ in the domain of its adjoint actually belong to $\mathcal{D}$. So assume that $E \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ satisfies for all $F \in \mathcal{D}$ the inequality

$$\left| \int_{\mathbb{R}^3} \langle E, LF\rangle \right| \le C\|F\|_2.$$

Using Plancherel's identity on both sides this can be rewritten as

$$\left| \int_{\mathbb{R}^3} \langle |\xi|^2 \hat{E}(\xi) - \langle \hat{E}(\xi), \xi\rangle \xi, \overline{\hat{F}(\xi)}\rangle \, d\xi \right| = \left| \int_{\mathbb{R}^3} \langle \hat{E}(\xi), |\xi|^2 \overline{\hat{F}(\xi)} - \langle \overline{\hat{F}(\xi)}, \xi\rangle \xi\rangle \, d\xi \right| \le C\|\hat{F}\|_2.$$

Since this holds for all $F \in \mathcal{D}$, which is dense in $L^2(\mathbb{R}^3;\mathbb{R}^3)$, we infer that $\xi \mapsto |\xi|^2 \hat{E}(\xi) - \langle \hat{E}(\xi), \xi\rangle \xi$ is square-integrable, which precisely means $E \in \mathcal{D}$.

To prove the second claim let $\mu \in \mathbb{C} \setminus [0, \infty)$ and $G \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Then $\nabla \times \nabla \times E - \mu E = G$ is equivalent to

$$|\xi|^2 \hat{E}(\xi) - \langle \xi, \hat{E}(\xi)\rangle \xi - \mu \hat{E}(\xi) = \hat{G}(\xi).$$

Decomposing E into E_1, E_2 as in (30) and multiplying the above equation with $\xi/|\xi| \in \mathbb{R}^3$ we find

$$\hat{E}_1(\xi) = -\frac{1}{\mu} \langle \hat{G}(\xi), \frac{\xi}{|\xi|}\rangle \frac{\xi}{|\xi|} = -\frac{1}{\mu} \hat{G}_1(\xi).$$

This implies

$$\hat{E}_2(\xi)(|\xi|^2 - \mu) = \hat{G}(\xi) + \mu\hat{E}_1(\xi) = \hat{G}(\xi) - \langle \hat{G}(\xi), \frac{\xi}{|\xi|}\rangle\frac{\xi}{|\xi|} = \hat{G}_2(\xi).$$

So we have

$$E = E_1 + E_2 = -\frac{1}{\mu}G_1 + \mathcal{F}^{-1}\left(\frac{\hat{G}_2(\cdot)}{|\cdot|^2 - \mu}\right) = -\frac{1}{\mu}G_1 + \mathcal{R}(\mu)G_2.$$

Since the right hand side defines a bounded linear operator provided $\mu \in \mathbb{C} \setminus [0,\infty)$, this proves that $\mathbb{C} \setminus [0,\infty)$ belongs to the resolvent set of the curl-curl operator.

By the closedness of the spectrum it therefore suffices to show that for all $\mu \in (0,\infty)$ there is a Weyl sequence of the curl-curl operator. Indeed, as in the case of Laplacian one may consider the sequence

$$F_n(x) := c_n\chi(x/n)F(x) \qquad \text{where } F(x) := \begin{pmatrix} \cos(\sqrt{\mu}x_3) \\ 0 \\ 0 \end{pmatrix}$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$ is identically one on the cuboid $W := [-1,1]^3 \subset \mathbb{R}^3$ and zero outside $2W$. The factor $c_n > 0$ is a normalizing constant ensuring $\|F_n\|_2 = 1$. Using

$$\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} \cos^2(n\sqrt{\mu}x_3)\,dx_3\,dx_2\,dx_1 = 4a^3 + o(1) \qquad (n \to \infty),$$
$$\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} \sin^2(n\sqrt{\mu}x_3)\,dx_3\,dx_2\,dx_1 = 4a^3 + o(1) \qquad (n \to \infty)$$

for $a = 1$ we first get

$$c_n = \frac{1}{\|\chi(\cdot/n)F\|_2} = \frac{1}{n^{3/2}\|\chi F(n\cdot)\|_2} \le \frac{1}{n^{3/2}\|F(n\cdot)\|_{L^2(W)}} \le \frac{1}{n^{3/2}(2 + o(1))}.$$

Using this as well as the above asymptotics for $a = 2$ we find

$$\begin{aligned}
\|LF_n - \mu F_n\|_2 &= \| - \Delta F_n - \mu F_n\|_2 \\
&= c_n \cdot \left\|\chi(x/n)(-\Delta F - \mu F) - \frac{2}{n}\nabla\chi(x/n)\cdot\nabla F(x) - \frac{1}{n^2}\Delta\chi(x/n)F(x)\right\|_2 \\
&\le Cn^{-3/2}\cdot\Big(0 + 2n^{1/2}\|\nabla\chi\cdot\nabla F(n\cdot)\|_2 + n^{-1/2}\|(\Delta\chi)F(n\cdot)\|_2\Big) \\
&\le C\Big(n^{-1}\||\nabla F(n\cdot)|\|_{L^2(2W)} + n^{-2}\||F(n\cdot)|\|_{L^2(2W)}\Big) \\
&\le C\big(\sqrt{\mu}n^{-1} + n^{-2}\big)\big(\sqrt{4\cdot 2^3} + o(1)\big) \\
&= O(n^{-1}) \qquad \text{as } n \to \infty
\end{aligned}$$

so that a Weyl sequence at the level μ is found. □

Proof of Theorem 6: In order to construct R_λ consider a Schwartz function $G \in \mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ so that $E^\varepsilon := (L-\lambda-i\varepsilon)^{-1}G$ is well-defined and Proposition 6 implies

$$E^\varepsilon = -\frac{1}{\lambda+i\varepsilon}G_1 + \mathcal{R}(\lambda+i\varepsilon)G_2 \in \mathcal{D}+i\mathcal{D}.$$

By the Limiting Absorption Principle for the Helmholtz operator from Theorem 4 we get

$$(L-\lambda-i0)^{-1}G := \lim_{\varepsilon\to 0^+} E^\varepsilon = -\frac{1}{\lambda}G_1 + \mathcal{R}(\lambda+i0)G_2.$$

Here, both limits exist in $L^q(\mathbb{R}^3;\mathbb{R}^3)$. Indeed, the Riesz transform maps $L^r(\mathbb{R}^3)$ into itself for all $r \in (1,\infty)$ and thus $\|G_1\|_r \le C\|G\|_r$ for $r \in \{t,q\}$, hence $\|G_2\|_t \le C\|G\|_t < \infty$ and $\|G_1\|_q \le C\|G\|_q < \infty$. Taking the real part of this we obtain that

$$R_\lambda G := -\frac{1}{\lambda}G_1 + \mathfrak{R}_\lambda G_2$$

defines a bounded linear operator from $L^t(\mathbb{R}^3;\mathbb{R}^3)\cap L^q(\mathbb{R}^3;\mathbb{R}^3)$ to $L^q(\mathbb{R}^3;\mathbb{R}^3)$ with

$$\|R_\lambda G\|_q \le \frac{1}{\lambda}\|G_1\|_q + \|\mathfrak{R}_\lambda G_2\|_q \le C(\|G_1\|_q + \|G_2\|_t).$$

Next we prove that $R_\lambda G$ is a weak solution lying in $H_{loc}(\mathrm{curl};\mathbb{R}^3)$. We will use that $E^\varepsilon \in L^q(\mathbb{R}^3;\mathbb{R}^3)$ implies $E^\varepsilon \in L^{q'}_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ due to $q > 2 > q'$. Testing the equation for $E^\varepsilon \in \mathcal{D}+i\mathcal{D}$ with $\overline{E^\varepsilon}\phi^2 \in H(\mathrm{curl};\mathbb{C}^3)$ we get

$$\int_{\mathbb{R}^3} \langle \nabla\times E^\varepsilon, \nabla\times(\overline{E^\varepsilon}\phi^2)\rangle - \lambda|E^\varepsilon|^2\phi^2 = \int_{\mathbb{R}^3}\langle G,\overline{E^\varepsilon}\rangle\phi^2 \qquad \text{for all } \phi\in C_0^\infty(\mathbb{R}^3). \tag{31}$$

So for any given compact set $K\subset\mathbb{R}^3$ we may choose a nonnegative test function ϕ such that $\phi|_K \equiv 1$, set $K' := \mathrm{supp}(\phi)$. Then we get

$$\int_{\mathbb{R}^3}\langle G,\overline{E^\varepsilon}\rangle\phi^2 \le \|G\|_{L^{q'}(K';\mathbb{R}^3)}\|E^\varepsilon\|_{L^q(K';\mathbb{C}^3)}\|\phi\|_\infty^2$$

as well as

$$\begin{aligned}\int_{\mathbb{R}^3}\langle \nabla\times E^\varepsilon, \nabla\times(\overline{E^\varepsilon}\phi^2)\rangle - \lambda|E^\varepsilon|^2\phi &= \int_{\mathbb{R}^3}|\nabla\times(E^\varepsilon\phi)|^2 - |\nabla\phi\times E^\varepsilon|^2 - \lambda|E^\varepsilon|^2\phi^2\\ &\ge \int_{\mathbb{R}^3}|\nabla\times(E^\varepsilon\phi)|^2 - \big(|\nabla\phi|^2+\lambda\phi^2\big)|E^\varepsilon|^2\\ &\ge \int_{\mathbb{R}^3}|\nabla\times(E^\varepsilon\phi)|^2 - \||\nabla\phi|^2+\lambda\phi^2\|_{L^{\frac{q}{q-2}}(K';\mathbb{R}^3)}\|E^\varepsilon\|^2_{L^q(K';\mathbb{R}^3)}.\end{aligned}$$

Here we used $|a\times b| \le |a||b|$ for $a,b\in\mathbb{C}^3$ and $q > \frac{2n}{n-1} > 2$. Combining the previous two inequalities with (31) we get

$$\begin{aligned}\int_K |\nabla\times E^\varepsilon|^2 &\le \int_{\mathbb{R}^3}|\nabla\times(E^\varepsilon\phi)|^2\\ &\le C\left(\|E^\varepsilon\|^2_{L^q(K';\mathbb{C}^3)} + \|G\|_{L^{q'}(K';\mathbb{R}^3)}\|E^\varepsilon\|_{L^q(K';\mathbb{C}^3)}\right)\\ &\le C\left(\|E^\varepsilon\|^2_{L^q(\mathbb{R}^3;\mathbb{C}^3)} + \|G\|_{L^{q'}(K';\mathbb{R}^3)}\|E^\varepsilon\|_{L^q(\mathbb{R}^3;\mathbb{C}^3)}\right)\\ &\le C < \infty\end{aligned}$$

because the functions E^ε are equibounded in $L^q(\mathbb{R}^3;\mathbb{C}^3)$. The latter fact comes from the proof of Gutiérrez' Limiting Absorption Principle, see Theorem 6 in [17]. Since K was arbitrary, we conclude that (E^ε) is bounded in $H_{loc}(\text{curl};\mathbb{C}^3)$ and therefore the weak limit of its real part $R_\lambda G$ also belongs to that space.

Finally we show that R_λ leaves the space of cylindrically symmetric solutions invariant. If $F \in \mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ is cylindrically symmetric, then there is $F_0 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$F(x_1,x_2,x_3) = F_0(\sqrt{x_1^2+x_2^2},x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \qquad \text{for all } (x_1,x_2,x_3) \in \mathbb{R}^3.$$

It suffices to show that there is $\tilde{F}_0 : [0,\infty)\times\mathbb{R} \to \mathbb{C}$ such that $\overline{\tilde{F}_0(r,-z)} = -\tilde{F}_0(r,z)$ for all $r \geq 0, z \in \mathbb{R}$ and

$$\hat{F}(\xi_1,\xi_2,\xi_3) = \tilde{F}_0(\sqrt{\xi_1^2+\xi_2^2},\xi_3)\begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{pmatrix} \qquad \text{for all } (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3. \tag{32}$$

Clearly the third component of $\hat{F}$ vanishes identically. Using the symmetry of F and the definition of the Fourier transform we moreover obtain after some calculations that for all $\theta \in [0,2\pi)$ we have

$$\left\langle \hat{F}(\xi_1,\xi_2,\xi_3), \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix}\right\rangle = \rho_\theta(\sqrt{\xi_1^2+\xi_2^2},\xi_3)$$

for some function $\rho_\theta : [0,\infty)\times\mathbb{R} \to \mathbb{C}$, i.e., for every given $\theta \in [0,2\pi)$ the left hand side is invariant under all rotations with respect to the (ξ_1,ξ_2)-variable. Using this for $\theta = 0$ we get

$$\begin{aligned}
&\left\langle \hat{F}(\xi_1,\xi_2,\xi_3), \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix}\right\rangle \\
&= \rho_0(\sqrt{\xi_1^2+\xi_2^2},\xi_3) \quad = \quad \left\langle \hat{F}(0,\sqrt{\xi_1^2+\xi_2^2},\xi_3), \begin{pmatrix} 0 \\ \sqrt{\xi_1^2+\xi_2^2} \\ 0 \end{pmatrix}\right\rangle \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} F_0(\sqrt{x_1^2+x_2^2},x_3)\left\langle \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{\xi_1^2+\xi_2^2} \\ 0 \end{pmatrix}\right\rangle e^{-i(x_2\sqrt{\xi_1^2+\xi_2^2}+x_3\xi_3)}\,dx \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} F_0(\sqrt{x_1^2+x_2^2},x_3)x_1\,dx_1\right)\sqrt{\xi_1^2+\xi_2^2}e^{-i(x_2\sqrt{\xi_1^2+\xi_2^2}+x_3\xi_3)}\,d(x_2,x_3) \\
&= 0
\end{aligned}$$

because $x_1 \mapsto F_0(\sqrt{x_1^2+x_2^2},x_3)x_1$ is odd for all fixed $(x_2,x_3)\in\mathbb{R}^2$. This shows

$$\hat{F}(\xi_1,\xi_2,\xi_3) = \frac{\rho_{\pi/2}(\sqrt{\xi_1^2+\xi_2^2},\xi_3)}{\xi_1^2+\xi_2^2}\begin{pmatrix}-\xi_2\\ \xi_1\\ 0\end{pmatrix} \qquad \text{for all } (\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$$

so that $\hat{F}$ can be written in the form (32). Finally, since F is real-valued we get $\overline{\hat{F}(-\xi)} = \hat{F}(\xi)$ and hence $\overline{\tilde{F}_0(r,-z)} = -\tilde{F}_0(r,z)$ for all $r\geq 0, z\in\mathbb{R}$. This finishes the proof. □

8. Appendix C: The Ruiz-Vega resolvent estimates

In this section we review the resolvent estimates by Ruiz and Vega that are essentially contained in Theorem 3.1 in [24]. Since this theorem does not exactly provide the estimates we need, we decided to reformulate their results in the way we apply them in the proof of Proposition 2. We even show a bit more.

Theorem 7. *Let $n\in\mathbb{N}, n\geq 2$ and assume $f\in L^q(\mathbb{R}^n)$ for $\frac{1}{n+1}\leq\frac{1}{q}-\frac{1}{2}\leq\min\{\frac{1}{2},\frac{2}{n}\}$ with $(n,q)\neq(4,1)$. Then there is a $C>0$ such that for all $\varepsilon\neq 0$ we have*

$$\sup_{R\geq 1}\left(\frac{1}{R}\int_{B_R}|\mathcal{R}(\lambda+i\varepsilon)f(x)|^2\,dx\right)^{1/2}\leq C\|f\|_q.$$

If moreover $\frac{1}{n+1}\leq\frac{1}{q}-\frac{1}{2}\leq\frac{1}{n}, (q,n)\neq(1,2)$ holds, then

$$\sup_{R\geq 1}\left(\frac{1}{R}\int_{B_R}|\nabla\mathcal{R}(\lambda+i\varepsilon)f(x)|^2\,dx\right)^{1/2}\leq C\|f\|_q.$$

We emphasize that this result implies that the inequality (16) from the proof of Proposition 2 holds for $q:=p'$ because $\frac{2(n+1)}{n-1}\leq p\leq\frac{2n}{(n-4)_+}$, $(n,p)\neq(4,\infty)$ is equivalent to $\frac{1}{n+1}\leq\frac{1}{q}-\frac{1}{2}\leq\min\{\frac{1}{2},\frac{2}{n}\}$, $(n,q)\neq(4,1)$. It seems that our statement dealing with the two-dimensional case $n=2$ does not appear in the literature. In the case $n\geq 3$ our Theorem 7 is covered by Gutiérrez' Theorem 7 in [17] except for the endpoint case $n=3, q=1$. The proof by Gutiérrez is not carried out in detail but it is referred to the paper of Ruiz and Vega (Theorem 3.1 in [24]) where a closely related but different result is proved. So we believe that an updated and self-contained version of these resolvent estimates might be useful even though our proof below mainly reformulates the arguments of Ruiz and Vega.

Proof of Theorem 7: It suffices to prove the estimates for Schwartz functions $f\in\mathcal{S}(\mathbb{R}^n)$ and, via rescaling, for $\lambda=1$. Then we have

$$\mathcal{R}(1+i\varepsilon)f = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2-1-i\varepsilon}\hat{f}(\xi)\right).$$

We introduce the splitting $\mathcal{R}(1+i\varepsilon)f = v^\varepsilon + w^\varepsilon$ where

$$v^\varepsilon := \mathcal{F}^{-1}\left(\frac{\phi(\xi)}{|\xi|^2-1-i\varepsilon}\hat{f}(\xi)\right), \qquad w^\varepsilon := \mathcal{F}^{-1}\left(\frac{1-\phi(\xi)}{|\xi|^2-1-i\varepsilon}\hat{f}(\xi)\right)$$

and $\phi \in C_0^\infty(\mathbb{R}^n)$ is a test function satisfying $\mathrm{supp}(\phi) \subset B_{1+\delta} \setminus B_{1-\delta}$, $\delta := 1 - \frac{1}{\sqrt{1+\frac{1}{2n}}}$ as well as $\phi \equiv 1$ on a neighbourhood of the unit sphere. Then we have $w^\varepsilon = G^\varepsilon * f$ where

$$|G^\varepsilon(z)| \le C \begin{cases} \min\{|z|^{2-n}, |z|^{-1-n}\} & , n \ge 3 \\ \min\{|\log(z)|, |z|^{-3}\} & , n = 2 \end{cases} \qquad \text{and} \qquad |\nabla G^\varepsilon(z)| \le C \min\{|z|^{1-n}, |z|^{-1-n}\}$$

for some $C > 0$ independent of ε, see page 8-9 in [7] for related estimates. These bounds imply $G^\varepsilon \in L^{\frac{2q}{3q-2}}(\mathbb{R}^n)$ whenever $0 \le \frac{1}{q} - \frac{1}{2} \le \min\{\frac{1}{2}, \frac{2}{n}\}$ with $\frac{1}{q} - \frac{1}{2} < \frac{2}{n}$ and the corresponding norms are uniformly bounded from above with respect to ε. For such q we get

$$\sup_{R\ge1} \left(\frac{1}{R} \int_{B_R} |w^\varepsilon(x)|^2 \, dx \right)^{1/2} \le \|w^\varepsilon\|_2 = \|G^\varepsilon * f\|_2 \le \|G^\varepsilon\|_{\frac{2q}{3q-2}} \|f\|_q \le C\|f\|_q.$$

It remains to prove this estimate for $n \ge 5$ and $\frac{1}{q} - \frac{1}{2} = \frac{2}{n}$. In this case we use $G^\varepsilon \in L^{\frac{2q}{3q-2},w}(\mathbb{R}^n)$ with uniformly bounded norms so that Young's convolution inequality for classical and weak Lebesgue spaces (see Theorem 1.4.25 in [16] for the latter) yields

$$\sup_{R\ge1} \left(\frac{1}{R} \int_{B_R} |w^\varepsilon(x)|^2 \, dx \right)^{1/2} \le \|G^\varepsilon * f\|_2 \le \|G^\varepsilon\|_{\frac{2q}{3q-2},w} \|f\|_q \le C\|f\|_q.$$

(Notice that we also have $G^\varepsilon \in L^{\frac{2q}{3q-2},w}(\mathbb{R}^n)$ in the case $\frac{1}{q} - \frac{1}{2} = \frac{2}{n}$ and $n = 4$, but Theorem 1.4.25 [16] does not apply since each of the exponents $2, \frac{2q}{3q-2}, q$ has to be different from 1 or ∞.) The same way, if $0 \le \frac{1}{q} - \frac{1}{2} < \frac{1}{n}$ then we have $|\nabla G^\varepsilon| \in L^{\frac{2q}{3q-2}}(\mathbb{R}^n)$ and $\frac{1}{q} - \frac{1}{2} = \frac{1}{n}$ implies $|\nabla G^\varepsilon| \in L^{\frac{2q}{3q-2},w}(\mathbb{R}^n)$ with uniformly bounded norms. In the former case we get

$$\sup_{R\ge1} \left(\frac{1}{R} \int_{B_R} |\nabla w^\varepsilon(x)|^2 \, dx \right)^{1/2} \le \||\nabla w^\varepsilon|\|_2 = \||\nabla G^\varepsilon| * f\|_2 \le \||\nabla G^\varepsilon|\|_{\frac{2q}{3q-2}} \|f\|_q \le C\|f\|_q.$$

and in the latter case we have under the additional assumption $(q,n) \ne (1,2)$ (for the same reason as above)

$$\sup_{R\ge1} \left(\frac{1}{R} \int_{B_R} |\nabla w^\varepsilon(x)|^2 \, dx \right)^{1/2} \le \||\nabla G^\varepsilon| * f\|_2 \le \||\nabla G^\varepsilon|\|_{\frac{2q}{3q-2},w} \|f\|_q \le C\|f\|_q.$$

So it remains to show that the estimate

$$\sup_{R\ge1} \left(\frac{1}{R} \int_{B_R} |v^\varepsilon(x)|^2 + |\nabla v^\varepsilon(x)|^2 \, dx \right)^{1/2} \le C\|f\|_q \tag{33}$$

holds whenever $\frac{1}{q} - \frac{1}{2} \ge \frac{1}{n+1}$.

To prove (33) we split v^ε into $2n$ different pieces using a suitable partition of unity. In view of $\mathrm{supp}(\hat{v^\varepsilon}) \subset B_{1+\delta} \setminus B_{1-\delta}$ we consider the covering $\{O_{1,+}, O_{1,-}, \dots, O_{n,+}, O_{n,-}\}$ of $B_{1+\delta} \setminus B_{1-\delta}$ given by the open sets

$$O_{j,\pm} := \left\{ \xi \in B_{1+\delta} \setminus B_{1-\delta} : \pm\xi_j > \frac{1}{\sqrt{2n}}|\xi| \right\}$$

Let $\{\eta_{1,+},\eta_{1,-},\ldots,\eta_{n,+},\eta_{n,-}\}$ be an associated partition of unity so that

$$v=\sum_{j=1}^{n}(v^\varepsilon_{j,+}+v^\varepsilon_{j,-})\quad\text{where}\quad v^\varepsilon_{j,\pm}:=\mathcal{F}^{-1}\left(\frac{\eta_{j,\pm}(\xi)\phi(\xi)}{|\xi|^2-1-i\varepsilon}\hat{f}(\xi)\right)$$

The reason for this is that we want to make use of the following inequalities:

$$\xi\in\operatorname{supp}(\hat{v}^\varepsilon_{j,\pm})\quad\Longrightarrow\quad\begin{cases}|\xi_j|\le|\xi|\le 1+\delta,\\ \pm\xi_j>\frac{1}{\sqrt{2n}}|\xi|>\max\left\{\frac{1-\delta}{\sqrt{2n}},\frac{1}{\sqrt{2n}}|\xi_j'|\right\},\\ |\xi_j'|\le\frac{\sqrt{|\xi_j'|^2+|\xi_j|^2}}{\sqrt{1+\frac{1}{2n}}}\le\frac{1+\delta}{\sqrt{1+\frac{1}{2n}}}=1-\delta^2.\end{cases}\tag{34}$$

Here we used the notation $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$ and $\xi_j':=(\xi_1,\ldots,\xi_{j-1},\xi_{j+1},\ldots,\xi_n)$. Since the estimates for $v_{j,\pm}$ are the same for all $j\in\{1,\ldots,n\}$ up to textual modifications we only consider $j=1$. Moreover, the estimates for $v_{1,+},v_{1,-}$ only differ at one point (which we will mention), so that we only carry out the estimates for $v^\varepsilon_{1,+}$. To simplify the notation we write v^ε,η,ξ' instead of $v^\varepsilon_{1,+},\eta_{1,+},\xi_1'=(\xi_2,\ldots,\xi_n)$.

Using $B_R\subset[-R,R]\times\mathbb{R}^{n-1}$ we get

$$\begin{aligned}\frac{1}{R}\int_{B_R}|v^\varepsilon(x)|^2+|\nabla v^\varepsilon(x)|^2\,dx&\le\frac{1}{R}\int_0^R\Big(\int_{\mathbb{R}^{n-1}}|v^\varepsilon(x_1,x')|^2+|\nabla v^\varepsilon(x_1,x')|^2\,dx'\Big)\,dx_1\\&=\frac{1}{R}\int_0^R\|(A^\varepsilon*f)(x_1,\cdot)\|^2_{L^2(\mathbb{R}^{n-1})}\,dx_1\end{aligned}\tag{35}$$

where $A^\varepsilon:=\mathcal{F}^{-1}\left(\frac{\eta(\xi)\phi(\xi)(1+|\xi|^2)^{1/2}}{|\xi|^2-1-i\varepsilon}\right)$. We first provide some estimates related to this function. We have

$$\begin{aligned}\Psi^\varepsilon_t(\xi')&:=e^{-i\sqrt{1-|\xi'|^2}t}\mathcal{F}^{-1}_{\xi_1}(\hat{A}^\varepsilon(\xi_1,\xi'))(t)\\&=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{\eta(\xi_1,\xi')\phi(\xi_1,\xi')\sqrt{1+|\xi_1|^2+|\xi'|^2}}{\xi_1^2+|\xi'|^2-1-i\varepsilon}e^{i(\xi_1-\sqrt{1-|\xi'|^2})t}\,d\xi_1\\&=\int_{\mathbb{R}}\psi^\varepsilon(\xi_1,\xi')\cdot\frac{e^{i\xi_1 t}}{\xi_1}\,d\xi_1\end{aligned}\tag{36}$$

where

$$\psi^\varepsilon(\xi_1,\xi'):=\frac{\eta(\xi_1+\sqrt{1-|\xi'|^2},\xi')\phi(\xi_1+\sqrt{1-|\xi'|^2},\xi')\sqrt{2+2\xi_1\sqrt{1-|\xi'|^2}+|\xi_1|^2}}{\sqrt{2\pi}(\xi_1+2\sqrt{1-|\xi'|^2}-i\varepsilon/\xi_1)}.$$

In (36) we used the coordinate transformation $\xi_1\mapsto\xi_1+\sqrt{1-|\xi'|^2}$, which is well-defined because of $|\xi'|\le 1-\delta^2$ by (34). Moreover,

$$\xi_1+2\sqrt{1-|\xi'|^2}>\xi_1\overset{(34)}{\ge}\frac{1-\delta}{\sqrt{2n}}>0\qquad\text{for all }\xi\in\operatorname{supp}(\psi^\varepsilon),\varepsilon\in\mathbb{R}.$$

(In the estimates for $v_{1,-}$ the coordinate transformation to be used is $\xi_1\mapsto\xi_1-\sqrt{1-|\xi'|^2}$ and the above estimate has to be replaced by $\xi_1-2\sqrt{1-|\xi'|^2}<\xi_1<-\frac{1-\delta}{\sqrt{2n}}<0$.) So we conclude that ψ^ε is smooth for every fixed $\varepsilon \in \mathbb{R}$. Based on these properties of ψ^ε we now provide some estimates for Ψ_t^ε.

From (34) we infer $\eta(s,\xi') = 0$ whenever $|\xi'| \geq 1-\delta^2, s \in \mathbb{R}$ so that

$$\operatorname{supp}(\Psi_t^\varepsilon) \subset B_{1-\delta^2} \qquad \text{for all } t \in \mathbb{R}, \varepsilon \neq 0. \tag{37}$$

Moreover, we have for all $m \in \mathbb{N}_0$ and $\xi' \in \mathbb{R}^{n-1}, |\xi'| \leq 1-\delta^2$

$$\begin{aligned}
|\nabla^m \Psi_t^\varepsilon(\xi')| &\overset{(36)}{=} \left| e^{i\sqrt{1-|\xi'|^2}t} \mathcal{F}_1\left(\nabla_{\xi'}^m(\psi^\varepsilon(\cdot,\xi')) \cdot \mathrm{p.\,v.}\left(\frac{1}{\cdot}\right)\right)(t)\right| \\
&\leq \left\| \mathcal{F}_1(\nabla_{\xi'}^m(\psi^\varepsilon(\cdot,\xi'))) * \mathcal{F}_1\left(\mathrm{p.\,v.}\left(\frac{1}{\cdot}\right)\right)\right\|_{L^\infty(\mathbb{R})} \\
&\leq \|\mathcal{F}_1(\nabla_{\xi'}^m(\psi^\varepsilon(\cdot,\xi')))\|_{L^1(\mathbb{R})} \|i\pi \operatorname{sign}(\cdot)\|_{L^\infty(\mathbb{R})} \\
&\leq \pi \|\mathcal{F}_1(\nabla_{\xi'}^m(\psi^\varepsilon(\cdot,\xi'))) \cdot (1+|\cdot|^2)(1+|\cdot|^2)^{-1}\|_{L^1(\mathbb{R})} \\
&\leq \pi \|\mathcal{F}_1(\nabla_{\xi'}^m(-\partial_{\xi_1\xi_1}+1)(\psi^\varepsilon(\cdot,\xi')))\|_{L^\infty(\mathbb{R})} \|(1+|\cdot|^2)^{-1}\|_{L^1(\mathbb{R})} \\
&\leq C\|\nabla_{\xi'}^m(-\partial_{\xi_1\xi_1}+1)(\psi^\varepsilon(\cdot,\xi'))\|_{L^1(\mathbb{R})} \\
&\leq C_m.
\end{aligned} \tag{38}$$

From (37) and (38) we conclude that for all $m \in \mathbb{N}_0$ there is a $C_m > 0$ such that

$$|\nabla^m \Psi_t^\varepsilon(\xi')| \leq C_m(1+|\xi'|)^{-m} \qquad \text{for all } \xi' \in \mathbb{R}^{n-1}, t \in \mathbb{R}, \varepsilon \neq 0. \tag{39}$$

In view of (35) we now use (39) in order to estimate the term

$$\begin{aligned}
\|(A^\varepsilon * f)(x_1,\cdot)\|_{L^2(\mathbb{R}^{n-1})}^2 &= \int_{\mathbb{R}^{n-1}} (A^\varepsilon * f)(x_1,y')(\bar{A}^\varepsilon * \bar{f})(x_1,y')\,dy' \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{n-1}} f(z_1,y') S_{x_1,z_1,\tilde{z}_1}^\varepsilon (f(\tilde{z}_1,\cdot))(y')\,dy'\Big)\,dz_1\,d\tilde{z}_1
\end{aligned}$$

for any given $x_1, z_1, \tilde{z}_1 \in \mathbb{R}, \varepsilon \neq 0$ where $S_{x_1,z_1,\tilde{z}_1}^\varepsilon : \mathcal{S}(\mathbb{R}^{n-1}) \to \mathcal{S}(\mathbb{R}^{n-1})$ is given by

$$S_{x_1,z_1,\tilde{z}_1}^\varepsilon g := \bar{A}^\varepsilon(x_1-\tilde{z}_1,\cdot) * A^\varepsilon(x_1-z_1,\cdot) * \bar{g}.$$

The (L^2,L^2)-bound for $S_{x_1,z_1,\tilde{z}_1}^\varepsilon$ results from

$$\begin{aligned}
\|S_{x_1,z_1,\tilde{z}_1}^\varepsilon g\|_{L^2(\mathbb{R}^{n-1})} &= \|\mathcal{F}_{n-1}(\bar{A}^\varepsilon(x_1-\tilde{z}_1,\cdot) * A^\varepsilon(x_1-z_1,\cdot)) \cdot \hat{\bar{g}}\|_{L^2(\mathbb{R}^{n-1})} \\
&\leq \|\mathcal{F}_{n-1}\left(\bar{A}^\varepsilon(x_1-\tilde{z}_1,\cdot) * A^\varepsilon(x_1-z_1,\cdot)\right)\|_{L^\infty(\mathbb{R}^{n-1})} \|g\|_{L^2(\mathbb{R}^{n-1})} \\
&\leq \sup_{t\in\mathbb{R}} \|\mathcal{F}_{n-1}(\bar{A}^\varepsilon(t,\cdot))\|_{L^\infty(\mathbb{R}^{n-1})}^2 \|g\|_{L^2(\mathbb{R}^{n-1})} \\
&\leq \sup_{t\in\mathbb{R},\xi'\in\mathbb{R}^{n-1}} |\mathcal{F}_1^{-1}(\hat{A}^\varepsilon(\cdot,\xi'))(t)|^2 \cdot \|g\|_{L^2(\mathbb{R}^{n-1})} \\
&= \sup_{t\in\mathbb{R},\xi'\in\mathbb{R}^{n-1}} |\Psi_t^\varepsilon(\xi')|^2 \cdot \|g\|_{L^2(\mathbb{R}^{n-1})} \\
&\overset{(39)}{\leq} C\|g\|_{L^2(\mathbb{R}^{n-1})}.
\end{aligned} \tag{40}$$

The (L^1,L^∞)-bound is obtained via the method of stationary phase.

$$\|S_{x_1,z_1,\tilde{z}_1}^\varepsilon g\|_\infty \leq \left\|\bar{A}^\varepsilon(x_1-\tilde{z}_1,\cdot) * A^\varepsilon(x_1-z_1,\cdot)\right\|_\infty \|g\|_1$$

$$= \left\| \mathcal{F}_{n-1}^{-1}\left(\overline{\mathcal{F}_1^{-1}(\hat{A}^\varepsilon(\cdot,\xi'))(x_1-\tilde{z}_1)} \cdot \mathcal{F}_1^{-1}(\hat{A}^\varepsilon(\cdot,\xi'))(x_1-z_1)\right)\right\|_\infty \|g\|_1$$

$$\overset{(39)}{=} \left\| \mathcal{F}_{n-1}^{-1}\left(e^{-i\sqrt{1-|\xi'|^2}(x_1-\tilde{z}_1)}\overline{\Psi^\varepsilon_{x_1-\tilde{z}_1}(\xi')} \cdot e^{i\sqrt{1-|\xi'|^2}(x_1-z_1)}\Psi^\varepsilon_{x_1-z_1}(\xi')\right)\right\|_\infty \|g\|_1$$

$$= \left\| \mathcal{F}_{n-1}^{-1}\left(e^{i\sqrt{1-|\xi'|^2}(\tilde{z}_1-z_1)}\overline{\Psi^\varepsilon_{x_1-\tilde{z}_1}(\xi')}\Psi^\varepsilon_{x_1-z_1}(\xi')\right)\right\|_\infty \|g\|_1$$

$$= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \sup_{y'\in\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^{n-1}} e^{i(\langle y',\xi'\rangle + \sqrt{1-|\xi'|^2}(z_1-\tilde{z}_1))}\overline{\Psi^\varepsilon_{x_1-\tilde{z}_1}(\xi')}\Psi^\varepsilon_{x_1-z_1}(\xi')\,d\xi'\right| \|g\|_1.$$

In the last integral the smooth phase function $\Phi(\xi') := \langle y',\xi'\rangle + \sqrt{1-|\xi'|^2}(z_1-\tilde{z}_1)$ is stationary precisely at $\xi' = \frac{\operatorname{sign}(z_1-\tilde{z}_1)}{\sqrt{|y'|^2+(z_1-\tilde{z}_1)^2}}y'$ and the Hessian in that point

$$D^2\Phi(\xi') = \frac{\tilde{z}_1 - z_1}{\sqrt{1-|\xi'|^2}}\left(\mathrm{Id} + \frac{1}{1-|\xi'|^2}\xi'(\xi')^T\right) \in \mathbb{R}^{(n-1)\times(n-1)}$$

possesses the eigenvalues $1,\ldots,1,\frac{1}{1-|\xi'|^2}$, which are all uniformly bounded away from zero and infinity on the support of $\xi' \mapsto \overline{\Psi^\varepsilon_{x_1-\tilde{z}_1}(\xi')}\Psi^\varepsilon_{x_1-z_1}(\xi')$. Moreover, by (39) all derivatives of this function are square integrable with L^2-norms that are uniformly bounded with respect to $x_1,\tilde{z}_1,z_1,\varepsilon$. Hence, the Morse Lemma and the method of stationary phase yield

$$\|S^\varepsilon_{x_1,z_1,\tilde{z}_1}g\|_{L^\infty(\mathbb{R}^{n-1})} \leq C(1+|z_1-\tilde{z}_1|)^{\frac{1-n}{2}}\|g\|_{L^1(\mathbb{R}^{n-1})}. \tag{41}$$

Interpolating the (L^2,L^2)-estimate (40) and the (L^1,L^∞)-estimate (41) we get from the Riesz-Thorin Theorem

$$\|S_{x_1,z_1,\tilde{z}_1}g\|_{L^{p'}(\mathbb{R}^{n-1})} \leq C(1+|z_1-\tilde{z}_1|)^{(1-n)(\frac{1}{p}-\frac{1}{2})}\|g\|_{L^p(\mathbb{R}^{n-1})} \qquad \text{for all } p\in[1,2]. \tag{42}$$

With this estimate we are finally ready to conclude.

So assume $\frac{1}{q}-\frac{1}{2} \geq \frac{1}{n+1}$. Then $(1+|\cdot|)^{(1-n)(\frac{1}{q}-\frac{1}{2})}$ lies in $L^{\frac{q}{2(q-1)},w}(\mathbb{R}^n)$ so that Young's inequality for weak Lebesgue spaces implies

$$\begin{aligned}
&\frac{1}{R}\int_{B_R} |v^\varepsilon(x)|^2 + |\nabla v^\varepsilon(x)|^2\,dx \\
&\overset{(35)}{\leq} \sup_{x_1\in\mathbb{R}} \|(A^\varepsilon * f)(x_1,\cdot)\|^2_{L^2(\mathbb{R}^{n-1})} \\
&= \sup_{x_1\in\mathbb{R}} \int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^{n-1}} f(z_1,y')S^\varepsilon_{x_1,z_1,\tilde{z}_1}(f(\tilde{z}_1,\cdot))(y')\,dy'\Big)\,dz_1\,d\tilde{z}_1 \\
&\leq \sup_{x_1\in\mathbb{R}} \int_{\mathbb{R}}\int_{\mathbb{R}} \|f(z_1,\cdot)\|_{L^q(\mathbb{R}^{n-1})}\|S^\varepsilon_{x_1,z_1,\tilde{z}_1}(f(\tilde{z}_1,\cdot))\|_{L^{q'}(\mathbb{R}^{n-1})}\,dz_1\,d\tilde{z}_1 \\
&\overset{(42)}{\leq} C\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|z_1-\tilde{z}_1|)^{(1-n)(\frac{1}{p}-\frac{1}{2})}\|f(z_1,\cdot)\|_{L^q(\mathbb{R}^{n-1})}\|f(\tilde{z}_1,\cdot)\|_{L^q(\mathbb{R}^{n-1})}\,dz_1\,d\tilde{z}_1 \\
&\leq C\|f\|_q^2.
\end{aligned}$$

This finally shows (33) and the proof is finished. □

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