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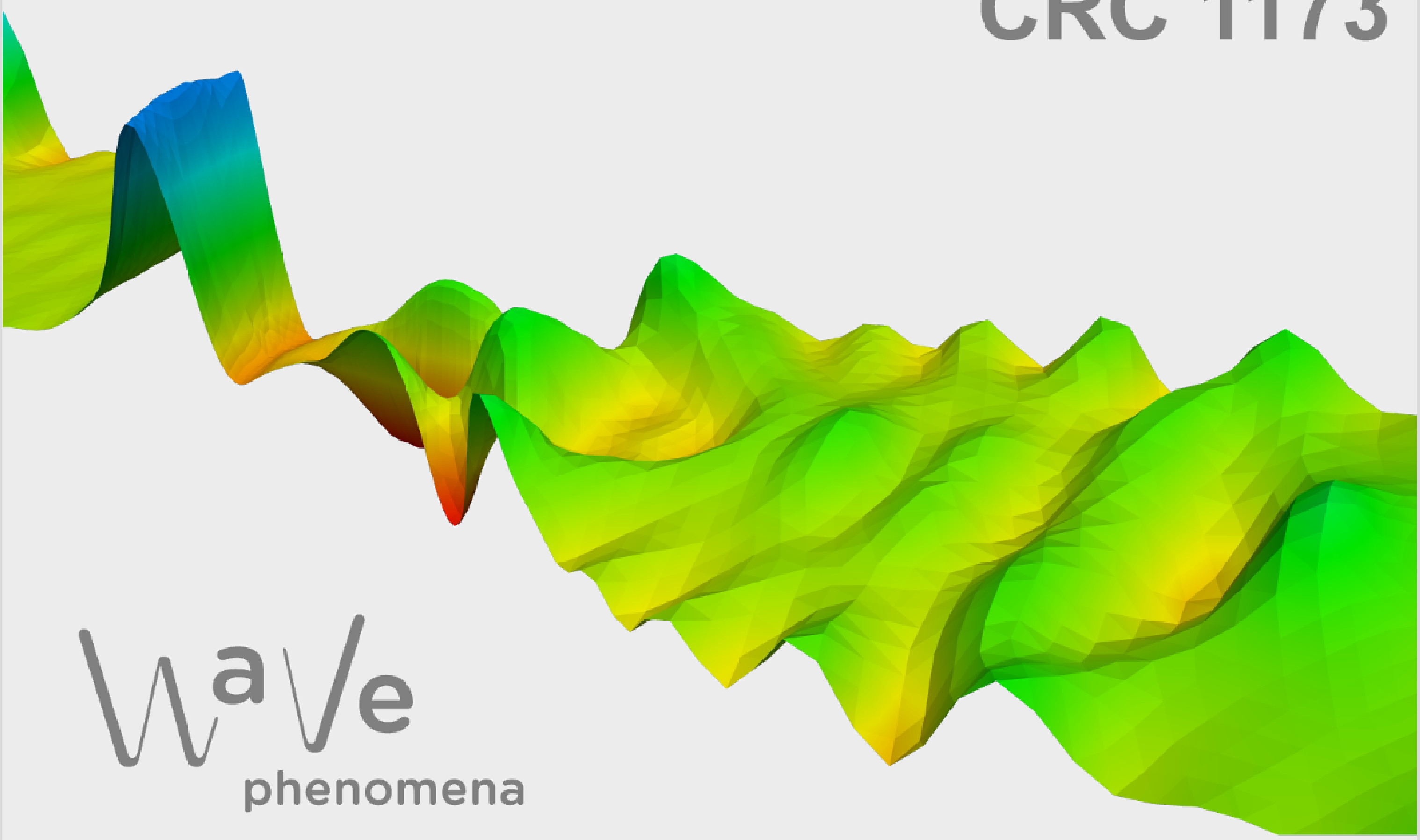

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ERROR ANALYSIS OF AN ENERGY PRESERVING ADI SPLITTING SCHEME FOR THE MAXWELL EQUATIONS

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Abstract. We investigate an alternating direction implicit (ADI) scheme for the time-integration of the Maxwell equations with currents, charges and conductivity. This method is unconditionally stable, numerically efficient, and preserves the norm of the solution exactly in absence of the external current and the conductivity. We prove that the semidiscretization in time converges in a space similar to H^{-1} with order two to the solution of the Maxwell system.

1. Introduction

The time-dependent Maxwell equations describe the interaction and propagation of electric and magnetic fields. They are a cornerstone of classical physics, and solving these equations numerically is a crucial task in a plethora of applications. For problems on cuboids the classical Yee scheme proposed in [23] is very popular among engineers (cf. [22]) due to its simplicity. It is well-known, however, that the Yee scheme is unstable if the step-size does not satisfy a CFL condition. If a fine spatial discretization is required to capture small wavelengths of the solution, then this CFL condition imposes a huge number of time-steps with a tiny step-size, which is computationally inefficient. Other classical time integrators such as, e.g., the Crank-Nicolson scheme, are unconditionally stable but implicit. Hence, the advantage of larger and fewer time-steps comes at the price of solving a large linear system in each time-step so that the total runtime of these methods is sometimes even larger than for the Yee scheme.

Alternating direction implicit (ADI) methods offer a very attractive alternative. This class of methods is based on the idea to split the right-hand side of the Maxwell equations into two parts by a suitable decomposition of the curl operators. Then a splitting method is applied: every time-step consists of a sequence of sub-steps in each of which only one part of the problem is propagated. The decomposition is done in such a way that the method is both unconditionally stable and efficient. The efficiency is due to the fact that in the sub-steps only a number of small linear systems instead of one large linear system have to be solved, see Subsection 3.5. The first ADI method for the Maxwell equations (without currents, conductivity and charges) has been proposed independently in [20] and [24]. The convergence of the semidiscretization with this

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ADI method has been analyzed in [14] on $\mathbb{R}^3$ and on a cuboid Q with perfectly conducting boundary conditions. For the Maxwell system with sources, currents and conductivity, second order convergence in a weak sense has been shown in [9]. Under stronger regularity assumptions on the coefficients, the initial value and the inhomogeneity second-order convergence in L^2 has been proved in [10]. For a detailed elaboration of the analysis we refer to [8].

It is well-known that the solution of the Maxwell equations has constant energy in absence of sources, currents and conductivity. The ADI method considered in [8, 9, 10, 14, 20, 24], however, is based on the Peaceman-Rachford scheme and hence does not conserve the energy of the numerical solution. Unconditionally stable ADI schemes which do preserve energy have been proposed and investigated, e.g., in [4, 5, 12, 13, 17, 18]. The error of different energy-conserving ADI schemes with space discretization by finite differences on the Yee grid has been analyzed in [5]. It was shown that one of the schemes converges with order two in space and time if the solution belongs to $C^3([0,T];[C^3(Q)]^6)$. Similar error bounds have been derived in [13], and in [4] and [12] for the two-dimensional case. However, on a cuboid Q the solutions of the Maxwell system only belong to $H^\alpha(Q)$ for $\alpha < 3$, in general, cf. Paragraph 4d in [6].

We extend one of the unconditionally stable, norm-conserving ADI schemes from [5] to the Maxwell system (2.1) containing conductivity, currents, and charges. We prove an error bound of order two for this system. In contrast to earlier papers, we work on a level of regularity that is covered by the available existence theory for (2.1), recalled in Section 2. In particular, we only make assumptions on the given current and the initial fields, but *not* on the solution itself. Our data belong to a suitable subspace of H^2 and the error is measured in H^{-1}, roughly speaking. The proof of our main convergence Theorem 4.1 is based on a new and quite sophisticated error recursion. It involves triple products of the two (first order) operators A and B forming the ADI splitting, cf. Section 3. Since we can only guarantee that the solution belong to H^2, we thus have to measure the error in H^{-1}. For the non-conservative ADI system from [20, 24], the error formulas of [8, 9, 10, 14] only contain double products of A and B. In these papers we were thus able to establish second convergence in L^2, assuming one more degree of initial regularity. To treat conductivity, currents, and charges, we make use of the functional analytic framework developed in our recent works [8, 10]. Besides the solvability of the Maxwell system in certain subspaces of H^1 and H^2, see Section 2, we need mapping properties of the operators A, B and their adjoints in L^2 and a suitable subspace of H^1, as explained in Section 3. In Theorem 4.1 we apply these properties in a weak setting which requires great care. They also imply the stability of the schemes in L^2 and H^1 in Theorem 3.2.

Instead of the full discretization as in [4, 5, 12, 13], we analyze the semidiscretization in time. Although ADI methods are typically combined with a space discretization by finite differences on the Yee grid, it was shown in [15, 3] that discontinuous Galerkin methods or finite element methods with mass lumping are also compatible with the ADI approach. In order to make use of this flexibility it is advantageous to analyze the disretization in time and space independently. We expect that our results can be extended to such a full discretization.

2. The Maxwell system and auxiliary results

Let $Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+) \subseteq \mathbb{R}^3$ be a non-empty cuboid with (Lipschitz) boundary $\Gamma = \partial Q$ and outer unit normal $\nu(x)$ defined for almost all $x \in \partial Q$. Our goal is to approximate the electric and magnetic fields $\mathbf{E}(t,x) \in \mathbb{R}^3$ and $\mathbf{H}(t,x) \in \mathbb{R}^3$ which solve the Maxwell equations

$$\begin{aligned}
\partial_t \mathbf{E}(t) &= \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}(t) - \frac{1}{\varepsilon}(\sigma\mathbf{E}(t) + \mathbf{J}(t)) && \text{in } Q,\ t \in [0,T], && \text{(2.1a)}\\
\partial_t \mathbf{H}(t) &= -\frac{1}{\mu}\operatorname{curl}\mathbf{E}(t) && \text{in } Q,\ t \in [0,T], && \text{(2.1b)}\\
\operatorname{div}(\varepsilon\mathbf{E}(t)) &= \rho(t), \qquad \operatorname{div}(\mu\mathbf{H}(t)) = 0 && \text{in } Q,\ t \in [0,T], && \text{(2.1c)}\\
\mathbf{E}(t) \times \nu &= 0, \qquad \mu\mathbf{H}(t)\cdot\nu = 0 && \text{on } \partial Q,\ t \in [0,T], && \text{(2.1d)}\\
\mathbf{E}(0) &= \mathbf{E}_0, \qquad \mathbf{H}(0) = \mathbf{H}_0 && \text{in } Q. && \text{(2.1e)}
\end{aligned}$$

As we consider functions as points in function spaces, we often omit the spatial variable and write $\mathbf{E}(t)$ instead of $\mathbf{E}(t,x)$ and so on. The initial fields $\mathbf{E}_0(x) \in \mathbb{R}^3$ and $\mathbf{H}_0(x) \in \mathbb{R}^3$ in (2.1e), the current density $\mathbf{J}(t,x) \in \mathbb{R}^3$, the permittivity $\varepsilon(x)$, the permeability $\mu(x)$, and the conductivity $\sigma(x)$ are given. As in [9] we assume throughout that the material coefficients satisfy

$$\begin{aligned}
&\varepsilon, \mu, \sigma \in W^{1,\infty}(Q,\mathbb{R}) \cap W^{2,3}(Q,\mathbb{R}), && \text{(2.2a)}\\
&\varepsilon, \mu \geq \delta \quad \text{for a constant } \delta > 0, \qquad \sigma \geq 0. && \text{(2.2b)}
\end{aligned}$$

By Proposition 2.3 in [9], the charge density $\rho(t,x) \in \mathbb{R}$ depends on $\mathbf{E}$ and $\mathbf{J}$ via

$$\rho(t) = \operatorname{div}(\varepsilon\mathbf{E}(t)) = \operatorname{div}(\varepsilon\mathbf{E}_0) - \int_0^t \operatorname{div}(\sigma\mathbf{E}(s) + \mathbf{J}(s))\,\mathrm{d}s, \qquad t \geq 0. \quad \text{(2.3)}$$

The boundary conditions (2.1d) model a perfectly conducting boundary.

Before deriving numerical methods for the Maxwell system, we introduce notation and collect a number of basic results, cf. [9] and [10]. We use the standard Sobolev spaces $W^{k,p}(\Omega)$ for $k \in \mathbb{N}_0$, $p \in [1,\infty]$ and open subsets $\Omega \subseteq \mathbb{R}^m$, where we put $W^{0,p}(\Omega) = L^p(\Omega)$ and $H^k(\Omega) = W^{k,2}(\Omega)$. For $s \in (0,\infty) \setminus \mathbb{N}$ and an integer $k > s$, the Slobodeckij spaces $H^s(\Omega) = (L^2(\Omega), H^k(\Omega))_{s/k,2}$ are defined by real interpolation, see Section 7.32 in [1] or [19]. We set $H^{-s}(\Omega) = H_0^s(\Omega)^*$ for $s \geq 0$, where the subscript 0 always denotes the closure of test functions in the respective norm. For $s \in [0,1]$ we employ the spaces $H^s(\Gamma)$ at the boundary, see Section 2.5 of [21]. Moreover, $H^{-s}(\Gamma)$ is the dual space of $H^s(\Gamma)$. We write

$$\Gamma_j^\pm = \{x \in \overline{Q} \mid x_j = a_j^\pm\} \qquad \text{and} \qquad \Gamma_j = \Gamma_j^- \cup \Gamma_j^+ \qquad \text{for } j \in \{1,2,3\}.$$

The symbol c denotes a generic constant that may have different values at different occurrences, but may depend only on Q, δ, $\|\varepsilon\|_{W^{1,\infty}} + \|\varepsilon\|_{W^{2,3}}$, $\|\mu\|_{W^{1,\infty}} + \|\mu\|_{W^{2,3}}$, or $\|\sigma\|_{W^{1,\infty}} + \|\sigma\|_{W^{2,3}}$. We also note that operators like $f \mapsto \varepsilon f$ are bounded on $H^2(Q)$ and $H^1(Q)$ by Sobolev's embedding with a norm controlled by the constants from (2.2).

The intersection $X \cap Y$ of two real Banach spaces X and Y is endowed with the norm $\|z\|_X + \|z\|_Y$. If Y is continuously embedded into X, this is expressed by $Y \hookrightarrow X$. The notation $X \cong Y$ means that X and Y are isomorphic. The duality pairing between X and its dual X^* is denoted by $\langle x^*, x\rangle_{X^*,X}$ for $x \in X$ and $x^* \in X^*$. If X is a Hilbert space with inner product $(\cdot|\cdot)_X$, then a dense embedding $Y \hookrightarrow X$ implies that $X \hookrightarrow Y^*$, and $x \in X \cong X^*$ acts on Y via $\langle x, y\rangle_{Y^*,Y} = (x|y)_X$ for $y \in Y \hookrightarrow X$.

Let I be the identity operator and let $\mathcal{B}(X, Y)$ be the space of bounded linear operators from X to Y with the special case $\mathcal{B}(X) = \mathcal{B}(X, X)$. The domain $D(L)$ of a linear operator L is always equipped with the graph norm $\|\cdot\|_L$ of L. If $Y \hookrightarrow X$, then we define the part L_Y of L in Y by $D(L_Y) = \{y \in Y \cap D(L) \,|\, Ly \in Y\}$ and $L_Y y = Ly$ for all $y \in D(L_Y)$. The product LG of two operators L and G is defined on the domain $D(LG) = \{x \in D(G) \,|\, Gx \in D(L)\}$.

For a closed operator L in X and a number λ in the resolvent set of L, the extrapolation space $X_{-1} = X_{-1}^L$ of L is the completion of X with respect to the norm $\|x\|_{-1} = \|(\lambda I - L)^{-1}x\|_X$. There is a continuous extension $L_{-1} : X \to X_{-1}$ of $L : D(L) \to X$, and the resolvent operators of L_{-1} extend those of L. If L is the generator of a C_0–semigroup $T(\cdot)$ on X, then L_{-1} generates the semigroup $T_{-1}(\cdot)$ of extensions to X_{-1}. If X is reflexive, then X_{-1}^L can be identified with the dual space of $D(L^*)$. These results can be found, e.g., in Section V.1.3 in [2] or Section II.5a in [11].

The Maxwell system (2.1) is studied in the space $X = L^2(Q)^6$ with the weighted inner product

$$\big((u, v) \mid (\varphi, \psi)\big)_X = \int_Q \big(\varepsilon u \cdot \varphi + \mu v \cdot \psi\big)\, \mathrm{d}x$$

for $(u, v), (\varphi, \psi) \in X$. Here and below $v \cdot w$ is the Euclidean inner product in $\mathbb{R}^m$. All vectors in this article are column vectors, but in order to keep notation simple we write (u, v) instead of $(u^T, v^T)^T$, and so on. Vectors $(\cdot, \cdot)$ are not to be confused with the inner products $(\cdot \mid \cdot)$. The square of the norm $\|\cdot\|_X$ induced by the weighted inner product is twice the physical energy of the fields $(\mathbf{E}, \mathbf{H})$, and because of (2.2) it is equivalent to the usual L^2–norm. In addition to X we use the Hilbert spaces

$$H(\mathrm{curl}, Q) = \{u \in L^2(Q)^3 \mid \mathrm{curl}\, u \in L^2(Q)^3\}, \qquad \|u\|_{\mathrm{curl}}^2 = \|u\|_{L^2}^2 + \|\mathrm{curl}\, u\|_{L^2}^2\,,$$
$$H(\mathrm{div}, Q) = \{u \in L^2(Q)^3 \mid \mathrm{div}\, u \in L^2(Q)\}, \qquad \|u\|_{\mathrm{div}}^2 = \|u\|_{L^2}^2 + \|\mathrm{div}\, u\|_{L^2}^2\,.$$

According to Theorems 1 and 2 in Section IX.A.1.2 of [7], the spaces $H(\mathrm{curl}, Q)$ and $H(\mathrm{div}, Q)$ have the following properties. The set of restrictions to Q of test functions on $\mathbb{R}^3$ is a dense subspace of $H(\mathrm{curl}, Q)$ and $H(\mathrm{div}, Q)$. The tangential trace $u \mapsto (u \times \nu)|_\Gamma$ on $C(\overline{Q})^3 \cap H^1(Q)^3$ has a unique continuous extension $\mathrm{tr}_t : H(\mathrm{curl}, Q) \to H^{-1/2}(\Gamma)^3$ with kernel $H_0(\mathrm{curl}, Q)$. Similarly, the normal trace $u \mapsto (u \cdot \nu)|_\Gamma$ on $C(\overline{Q})^3 \cap H^1(Q)^3$ has a unique continuous extension $\mathrm{tr}_n : H(\mathrm{div}, Q) \to H^{-1/2}(\Gamma)$. By Section 2.4 and 2.5 of [21] the map $f \mapsto f|_\Gamma$ defined on $C(\overline{Q}) \cap H^1(Q)$ can be extended to a continuous and surjective trace operator $\mathrm{tr} : H^1(Q) \to H^{1/2}(\Gamma)$ with kernel $H_0^1(Q)$.

Some of the functions considered below have a different degree of regularity with respect to different spatial dimensions. Assume, for example, that $f \in L^2(Q)$ is a function with $\partial_1 f \in L^2(Q)$, and let $Q_1 = (a_2^-, a_2^+) \times (a_3^-, a_3^+)$.

Then $f \in H^1((a_1^-, a_1^+), L^2(Q_1)) \cong L^2(Q_1, H^1(a_1^-, a_1^+))$, and thus f has traces at the rectangles $\Gamma_1^\pm = \{a_1^\pm\} \times Q_1$ whose norms in $L^2(\Gamma_1^\pm)$ are bounded by $c(\|f\|_{L^2(Q)} + \|\partial_1 f\|_{L^2(Q)})$. This argument yields trace operators $\operatorname{tr}_{\Gamma_j^\pm}$ and $\operatorname{tr}_{\Gamma_j}$ for $j \in \{1,2,3\}$. If $f \in H^1(Q)$, then these trace operators coincide in $L^2(\Gamma_j^\pm)$, respectively $L^2(\Gamma_j)$, with the respective restrictions of $\operatorname{tr} f$. In order to keep the notation simple, we usually write $u_1 = 0$ on Γ_2 instead of $\operatorname{tr}_{\Gamma_2}(u_1) = 0$, and so on.

After these preparations we can introduce the Maxwell operator

$$M = \begin{pmatrix} -\frac{\sigma}{\varepsilon} I & \frac{1}{\varepsilon}\operatorname{curl} \\ -\frac{1}{\mu}\operatorname{curl} & 0 \end{pmatrix}, \qquad D(M) = H_0(\operatorname{curl}, Q) \times H(\operatorname{curl}, Q) \tag{2.4}$$

on X. The domain $D(M)$ includes the electric boundary condition, but neither the magnetic boundary conditions nor the divergence conditions in (2.1). In order to respect all conditions and to encode the regularity of the charge density $\rho = \operatorname{div}(\varepsilon u)$, we define the subspace

$$\begin{aligned} X_{\text{div}} &:= \{(u,v) \in X \mid \operatorname{div}(\mu v) = 0,\ \ \operatorname{tr}_n(\mu v) = 0,\ \ \operatorname{div}(\varepsilon u) \in L^2(Q)\} \\ &= \{(u,v) \in X \mid \operatorname{div}(\mu v) = 0,\ \ \operatorname{tr}_n v = 0,\ \ \operatorname{div} u \in L^2(Q)\}. \end{aligned}$$

All constraints in this definition are understood in $H^{-1}(Q)$ or $H^{-1/2}(\Gamma)$, respectively. As noted in (2.4) of [9], one can drop here ε and the second μ because of (2.2). Moreover, X_{div} is a Hilbert space with the norm given by

$$\|(u,v)\|_{X_{\text{div}}}^2 = \|(u,v)\|_X^2 + \|\operatorname{div}(\varepsilon u)\|_{L^2(Q)}^2 .$$

Let M_{div} be the part of M in X_{div}. We have seen in (2.5) of [9] that

$$D(M_{\text{div}}^k) = D(M^k) \cap X_{\text{div}}$$

for $k \in \mathbb{N}$. Proposition 2.2 in [9] yields the embedding and the traces

$$D(M_{\text{div}}) \hookrightarrow H^1(Q)^6 \qquad \text{and} \qquad H_i = E_j = E_k = 0 \qquad \text{on } \Gamma_i \tag{2.5}$$

for $(\mathbf{E}, \mathbf{H}) \in D(M_{\text{div}})$ and $(i,j,k) = \{(1,2,3), (2,1,3), (3,1,2)\}$. The norm of the embedding is controlled by the constants from (2.2). Proposition 2.3 in [9] shows that M generates a contraction semigroup $(\mathrm{e}^{tM})_{t\geq 0}$ on X whose restrictions $\mathrm{e}^{tM_{\text{div}}}$ form a linearly bounded C_0–semigroup on X_{div} with generator M_{div}; i.e., $\|\mathrm{e}^{tM_{\text{div}}}\|_{\mathcal{B}(X_{\text{div}})} \leq c(1+t)$. Henceforth, we will use the abbreviations

$$w := (\mathbf{E}, \mathbf{H}) \qquad \text{and} \qquad f(t) := -\big(\tfrac{1}{\varepsilon}\mathbf{J}(t), (0,0,0)\big).$$

If $w_0 = (\mathbf{E}_0, \mathbf{H}_0) \in D(M_{\text{div}})$ and $f \in C([0,\infty), D(M_{\text{div}})) + C^1([0,\infty), X_{\text{div}})$, the Maxwell system (2.1) is equivalent to the evolution equation

$$w'(t) = Mw(t) + f(t), \qquad\qquad w(0) = w_0 \tag{2.6}$$

in X_{div}, and there is a unique solution

$$w = (\mathbf{E}, \mathbf{H}) \in C^1([0,\infty), X_{\text{div}}) \cap C([0,\infty), D(M_{\text{div}}))$$

of (2.6) given by

$$w(t) = \mathrm{e}^{tM_{\text{div}}} w_0 + \int_0^t \mathrm{e}^{(t-s)M_{\text{div}}} f(s)\,\mathrm{d}s. \tag{2.7}$$

The charge density in (2.1c) is contained in $L^2(Q)$ and determined by (2.3).

In our error analysis we need a subspace of H^2 on which e^{tM} induces a C_0–semigroup. For the corresponding charge densities we use the space

$$H^1_{00}(Q) = \Big\{ f \in H^1(Q) \mid \operatorname{tr}_{\Gamma'} f \in H^{1/2}_0(\Gamma') \text{ for all faces } \Gamma' \text{ of } Q \Big\},$$

where we put $H^{1/2}_0(\Gamma') = \big(L^2(\Gamma'), H^1_0(\Gamma')\big)_{1/2,2}$. By interpolation, $H^{1/2}_0(\Gamma')$ is embedded into $H^{1/2}(\Gamma')$. On the other hand, Proposition 2.11 of [16] implies the embedding $H^\alpha_0(\Gamma') \hookrightarrow H^{1/2}_0(\Gamma')$ for $\alpha > 1/2$. We now define the smaller state space

$$X_2 = \{(u, v) \in D(M^2) \cap X_{\mathrm{div}} \mid \operatorname{div}(\varepsilon u) \in H^1_{00}(Q)\}$$

with the norm given by

$$\|(u, v)\|^2_{X_2} = \|(u, v)\|^2_{D(M^2)} + \|\operatorname{div}(\varepsilon u)\|^2_{H^1} + \sum_{\Gamma' \text{ face of } Q} \|\operatorname{div}(\varepsilon u)\|^2_{H^{1/2}_0(\Gamma')}.$$

Note that X_2 is a Hilbert space. It contains fields in $D(M^2_{\mathrm{div}})$ whose charge densities belong to H^1 and vanish on the edges of Q in a generalized sense.

Proposition 3.2 in [10] says that X_2 is continuously embedded into $H^2(Q)^6$, and the norm of the embedding is controlled by the constants from (2.2). The part M_2 of M in X_2 has the domain $D(M_2) = D(M^3) \cap X_2$. By Proposition 3.3 in [10] the restrictions e^{tM_2} of e^{tM} form a C_0–semigroup on X_2 generated by M_2 which is bounded by

$$\|e^{tM_2}\|_{\mathcal{B}(X_2)} \le c(1 + t^3), \qquad t \ge 0. \tag{2.8}$$

3. Unconditionally stable ADI methods

3.1. **Decomposition of the Maxwell operator.** ADI splitting methods are based on a decomposition of the Maxwell operator M defined in (2.4) into

$$A = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & \frac{1}{\varepsilon} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & -\frac{1}{\varepsilon} C_2 \\ -\frac{1}{\mu} C_1 & 0 \end{pmatrix}$$

with the operator-valued matrices

$$C_1 = \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \qquad \text{and} \qquad C_2 = \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}.$$

The domains of A and B are

$$D(A) = \{(u, v) \in X \mid (C_1 v, C_2 u) \in X,\ \operatorname{tr}_{\Gamma_2} u_1 = 0,\ \operatorname{tr}_{\Gamma_3} u_2 = 0,\ \operatorname{tr}_{\Gamma_1} u_3 = 0\},$$
$$D(B) = \{(u, v) \in X \mid (C_2 v, C_1 u) \in X,\ \operatorname{tr}_{\Gamma_3} u_1 = 0,\ \operatorname{tr}_{\Gamma_1} u_2 = 0,\ \operatorname{tr}_{\Gamma_2} u_3 = 0\}.$$

$D(A)$ contains one half of the electric boundary conditions in $D(M_{\mathrm{div}})$, and $D(B)$ contains the other half; cf. (2.5). These traces exist since they fit to the partial derivatives in $C_2 u$ for A and in $C_1 u$ for B. We note that

$$My = (A + B)y \quad \text{for all } y \in D(A) \cap D(B) \hookrightarrow D(M).$$

Since neither the divergence conditions nor the magnetic boundary condition for the magnetic field are included in $D(A)$ or $D(B)$, the operators A and B act on X and not on X_{div}.

By Proposition 3.1 in [9], their adjoints are given by

$$A^* = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & -\frac{1}{\varepsilon} C_1 \\ -\frac{1}{\mu} C_2 & 0 \end{pmatrix} \qquad \text{and} \qquad B^* = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & \frac{1}{\varepsilon} C_2 \\ \frac{1}{\mu} C_1 & 0 \end{pmatrix}$$

in X with $D(A^*) = D(A)$ and $D(B^*) = D(B)$. They further satisfy $M^* = A^* + B^*$ on $D(A^*) \cap D(B^*) \hookrightarrow D(M^*) = D(M)$. We often use that $D(M_{\mathrm{div}}) \hookrightarrow D(L)$ and that the resolvents $(I - \tau L)^{-1}$ are contractive on X for $L \in \{A, B, A^*, B^*\}$ and $\tau > 0$, see Proposition 3.1 in [9]. Moreover, Proposition 4.1 in [10] and the definition of X_2 yield the embedding

$$X_2 \hookrightarrow D(A^2) \cap D(AB) \cap D(BA) \cap D(B^2) \cap D(M_{\mathrm{div}}^2), \tag{3.1}$$

which is crucial for our main convergence result. The norm of these embeddings are controlled by the constants from (2.2) and by Q.

3.2. **Construction of ADI splitting methods.** For simplicity we first consider the case $f = 0$. The problem (2.6) then reduces to $w'(t) = Mw(t)$ with initial value $w(0) = w_0$ and solution $w(t) = e^{tM} w(0)$. By a suitable space discretization (e.g., by finite differences on the Yee grid), the unbounded operator M is replaced by a matrix $\tilde{M}$, which is typically so large that computing $e^{t\tilde{M}}$ directly is impossible. Splitting methods are based on the observation that solving "parts" of the abstract Cauchy problem numerically is much cheaper. On a short time interval of length $\tau > 0$, the solution on X can formally be approximated by

$$w(\tau) = e^{\tau M} w(0) \approx e^{\tau A/2} e^{\tau B} e^{\tau A/2} w(0). \tag{3.2}$$

The error originates from the fact that A and B do not commute. In this approximation the problem $w'(t) = Mw(t)$ is replaced by the two sub-problems $y'(t) = Ay(t)$ and $z'(t) = Bz(t)$ on intervals of length $\tau/2$ and τ, respectively. This is numerically very attractive, because each of the two sub-problem corresponds to three *decoupled* wave equations; cf. [17, 5]. For example, the sub-problem $y'(t) = Ay(t)$ with $y = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ is equivalent to the wave equations

$$\left\{ \begin{array}{rcl} \mathcal{E}_1' & = & -\frac{\sigma}{2\varepsilon}\mathcal{E}_1 + \frac{1}{\varepsilon}\partial_2 \mathcal{H}_3, \qquad \operatorname{tr}_{\Gamma_2} \mathcal{E}_1 = 0, \\ \mathcal{H}_3' & = & \frac{1}{\mu}\partial_2 \mathcal{E}_1 \end{array} \right. \tag{3.3a}$$

$$\left\{ \begin{array}{rcl} \mathcal{E}_2' & = & -\frac{\sigma}{2\varepsilon}\mathcal{E}_2 + \frac{1}{\varepsilon}\partial_3 \mathcal{H}_1, \qquad \operatorname{tr}_{\Gamma_3} \mathcal{E}_2 = 0, \\ \mathcal{H}_1' & = & \frac{1}{\mu}\partial_3 \mathcal{E}_2 \end{array} \right. \tag{3.3b}$$

$$\left\{ \begin{array}{rcl} \mathcal{E}_3' & = & -\frac{\sigma}{2\varepsilon}\mathcal{E}_3 + \frac{1}{\varepsilon}\partial_1 \mathcal{H}_2, \qquad \operatorname{tr}_{\Gamma_1} \mathcal{E}_3 = 0, \\ \mathcal{H}_2' & = & \frac{1}{\mu}\partial_1 \mathcal{E}_3. \end{array} \right. \tag{3.3c}$$

These three pairs of scalar-valued partial differential equations can be solved independently, whereas $w'(t) = Mw(t)$ is a coupled system of six differential equations. The second sub-problem $z'(t) = Bz(t)$ is equivalent to three similar wave equations; cf. [17, 5]. The approximation (3.2) is only useful if τ is sufficiently small, but iterating this procedure allows us to approximate the exact solution at time $t_n = n\tau$ for $n \in \mathbb{N}$ by

$$w(t_n) = e^{n\tau M} w(0) \approx \left(e^{\tau A/2} e^{\tau B} e^{\tau A/2}\right)^n w(0).$$

This procedure is called (exponential) Strang splitting method, and it is well-known that the time-symmetric Strang splitting $\left(e^{\tau A/2} e^{\tau B} e^{\tau A/2}\right)^n w(0)$ has a higher order than the non-symmetric Lie-Trotter splitting $\left(e^{\tau A} e^{\tau B}\right)^n w(0)$.

Unfortunately, computing the matrix exponentials $e^{\tau A/2}$ and $e^{\tau B}$ for the spatially discretized operators A and B is still too expensive in many applications, in spite of the decoupling. For this reason, we use the additional approximation $e^L \approx \gamma(L)$ by the Cayley transform

$$\gamma(L) = (I - \tfrac{1}{2}L)^{-1}(I + \tfrac{1}{2}L) = (I + \tfrac{1}{2}L)(I - \tfrac{1}{2}L)^{-1}$$

for $L \in \{\tau B, \frac{\tau}{2}A\}$. From Proposition 3.1 in [9] we recall that $\gamma(\tau L)$ is contractive on X for $L \in \{A, B, A^*, B^*\}$ and $\tau > 0$. From the perspective of numerical analysis, replacing e^L by $\gamma(L)$ corresponds to approximating the exact flow of a linear evolution equation by one step of the trapezoidal rule. Since this additional approximation is compatible with the decoupling, computing $\gamma(\frac{\tau}{2}A)$ and $\gamma(\tau B)$ for the spatially discretized operators A and B is much cheaper than computing $\gamma(\tau M)$. This will be explained in detail in Subsection 3.5.

To summarize, the semidiscretization in time of the Cauchy problem (2.6) in the special case $f = 0$ reads

$$w_{n+1} = \gamma(\tfrac{\tau}{2}A)\gamma(\tau B)\gamma(\tfrac{\tau}{2}A) w_n, \qquad n = 0, 1, 2, \ldots,$$

where $w_n \approx w(t_n) = w(n\tau)$ is the approximation after n steps with the chosen step-size $\tau > 0$. For $\sigma = 0$ this method coincides with the scheme EC-S-FDTDII-1 in [5] if finite differences on the Yee grid are used to discretize space. In the first step the initial value w_0 is available from (2.6).

Now we return to the general case $f \neq 0$. The solution of the additional sub-problem $y'(t) = f(t)$ is simply

$$y(t) = y_0 + \int_0^t f(s)\,\mathrm{d}s.$$

Since in general the integral cannot be computed exactly, we approximate

$$y(t) \approx y_0 + t f(s_\star)$$

for some $s_\star \in [0, t]$. This yields (for instance) the following algorithm to obtain the new approximation $w_{n+1} \approx w(t_{n+1})$ of the solution of (2.6) from the current approximation $w_n \approx w(t_n)$:

$$\begin{aligned}
&1. \qquad w_{n+1/5} = \gamma(\tfrac{\tau}{2}A) w_n \\
&2. \qquad w_{n+2/5} = w_{n+1/5} + \tfrac{\tau}{2} f(t_n) \\
&3. \qquad w_{n+3/5} = \gamma(\tau B) w_{n+2/5} \\
&4. \qquad w_{n+4/5} = w_{n+3/5} + \tfrac{\tau}{2} f(t_{n+1}) \\
&5. \qquad w_{n+1} = \gamma(\tfrac{\tau}{2}A) w_{n+4/5}
\end{aligned} \tag{3.4}$$

In steps 1. and 5. we thus approximate the flow of the sub-problem $y'(t) = Ay(t)$, in steps 2. und 4. the flow of $y'(t) = f(t)$, and in step 3. the flow of $y'(t) = By(t)$. Of course, the result of the five sub-steps can equivalently be expressed by

$$w_{n+1} = S_{\tau,t_n}(w_n) := \gamma(\tfrac{\tau}{2}A)\Big[\gamma(\tau B)\big[\gamma(\tfrac{\tau}{2}A)w_n + \tfrac{\tau}{2}f(t_n)\big] + \tfrac{\tau}{2}f(t_{n+1})\Big]. \quad (3.5)$$

The order in which the sub-problems are propagated ("$A \to f \to B \to f \to A$") in the ADI scheme is more or less arbitrary. For example, interchanging the roles of A and B does not alter the convergence order. Another ADI scheme with the same order of convergence is obtained by interchanging the sub-step 1. with 2. and 4. with 5. ("$f \to A \to B \to A \to f$"), which yields

$$w_{n+1} = \tfrac{\tau}{2}f(t_{n+1}) + \gamma(\tfrac{\tau}{2}A)\gamma(\tau B)\gamma(\tfrac{\tau}{2}A)\big[w_n + \tfrac{\tau}{2}f(t_n)\big]$$

instead of (3.5). Moreover, replacing both $f(t_n)$ and $f(t_{n+1})$ by $f(t_n + \tau/2)$ yields yet another ADI scheme with the same order of convergence. In Section 4.2 we will prove error bounds for the method (3.5), but other variants could be analyzed by the same techniques.

3.3. **The splitting scheme in H^1.** Our error analysis relies on the behavior of the split operators A and B in the subspace

$$Y = \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \text{ on } \Gamma \setminus \Gamma_j,\ \ v_j = 0 \text{ on } \Gamma_j \text{ for all } j \in \{1,2,3\}\}$$

of $H^1(Q)^6$, which is endowed with the weighted inner product

$$\big((u,v) \mid (\varphi,\psi)\big)_Y = \int_Q \Big(\varepsilon u\cdot\varphi + \mu v\cdot\psi + \varepsilon\sum_{j=1}^{3}\partial_j u\cdot\partial_j\varphi + \mu\sum_{j=1}^{3}\partial_j v\cdot\partial_j\psi\Big)\,\mathrm{d}x.$$

The induced norm $\|\cdot\|_Y$ is equivalent to the usual one on H^1 due to (2.2). The subspace Y is closed in $H^1(Q)^6$ due to the continuity of the traces. Assumption (2.2) yields that the space Y is invariant under maps like $(u,v) \mapsto (\varepsilon u, \mu v)$, and this will often be used henceforth. Moreover, our definitions imply the embedding

$$Y \hookrightarrow D(A) \cap D(B) \cap D(A^*) \cap D(B^*) \cap D(M) \cap D(M^*), \quad (3.6)$$

whose norm is controlled by the constants from (2.2). The parts of A, B, A^*, and B^* in Y are denoted by A_Y, B_Y, $(A^*)_Y$, and $(B^*)_Y$, respectively. By Lemma 3.2 in [9], we have

$$\begin{aligned}
D(A_Y) = D((A^*)_Y) &= \{(u,v) \in Y \mid (C_1 v, C_2 u) \in Y\}\\
&= \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \text{ on } \Gamma \setminus \Gamma_j,\ \ v_j = 0 \text{ on } \Gamma_j \text{ for } j \in \{1,2,3\},\\
&\qquad \partial_2 u_1, \partial_3 u_2, \partial_1 u_3, \partial_3 v_1, \partial_1 v_2, \partial_2 v_3 \in H^1(Q),\\
&\qquad \partial_3 v_1 = 0 \text{ on } \Gamma_3,\ \ \partial_1 v_2 = 0 \text{ on } \Gamma_1,\ \ \partial_2 v_3 = 0 \text{ on } \Gamma_2\},\\
D(B_Y) = D((B^*)_Y) &= \{(u,v) \in Y \mid (C_2 v, C_1 u) \in Y\}\\
&= \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \text{ on } \Gamma \setminus \Gamma_j,\ v_j = 0 \text{ on } \Gamma_j \text{ for } j \in \{1,2,3\},\\
&\qquad \partial_3 u_1, \partial_1 u_2, \partial_2 u_3, \partial_2 v_1, \partial_3 v_2, \partial_1 v_3 \in H^1(Q),\\
&\qquad \partial_2 v_1 = 0 \text{ on } \Gamma_2,\ \partial_3 v_2 = 0 \text{ on } \Gamma_3,\ \ \partial_1 v_3 = 0 \text{ on } \Gamma_1\}.
\end{aligned}$$

Proposition 3.6 in [9] states that for all $L \in \{A, B, A^*, B^*\}$ the part L_Y of L in Y generates a C_0–semigroup on Y bounded by $\mathrm{e}^{\kappa t}$ with

$$\kappa = \frac{3\,\|\nabla\sigma\|_{L^\infty}}{4\delta} + \frac{3\,\|\sigma\|_{L^\infty}\,\|\nabla\varepsilon\|_{L^\infty}}{4\delta^2} + \frac{\|\nabla\varepsilon\|_{L^\infty} + \|\nabla\mu\|_{L^\infty}}{2\delta^2}. \tag{3.7}$$

By the same proposition, the resolvent $(I-\tau L_Y)^{-1}$ is the restriction of $(I-\tau L)^{-1}$ to Y. This resolvent is bounded by

$$\left\|(I-\tau L_Y)^{-1}\right\|_{\mathcal{B}(Y)} \le \frac{1}{1-\tau\kappa}$$

for all $0 < \tau < \frac{1}{\kappa}$, and hence $\left\|(I-\tau L_Y)^{-1}\right\|_{\mathcal{B}(Y)} \le 2$ for all $0 < \tau \le \frac{1}{2\kappa}$. The Cayley transform $\gamma(\tau L_Y)$ coincides with the restriction of $\gamma(\tau L)$ to Y, and Proposition 3.6 in [9] provides the bound

$$\|\gamma(\tau L_Y)\|_{\mathcal{B}(Y)} \le \mathrm{e}^{3\kappa\tau} \tag{3.8}$$

for all $0 < \tau \le \tau_0$ with a constant $\tau_0 \in (0, (2\kappa)^{-1}]$ that depends only on κ.

The above results imply the following observation.

Remark 3.1. Let $w_0 \in Y$ and let $f(t) \in Y$ for all $t \ge 0$. Under the assumption (2.2), $w_{n+k/5}$ belongs to Y for all $n \in \mathbb{N}_0$ and all $k \in \{0, 1, 2, 3, 4\}$.

3.4. **Unconditional stability of the ADI scheme.** In this subsection we will prove that the numerical semidiscretization with the ADI scheme can be bounded by the initial data and the inhomogenity without any CFL condition on the step-size. This unconditional stability is a major advantage of the ADI method over, say, the Yee scheme, which requires a sufficiently small step-size to be stable. Unconditional stability of full discretizations with ADI methods in time and spatial discretization on the Yee grid has been shown in [18, 5] and for the two-dimensional case in [4]. Using the notation

$$C(\tau) = \gamma(\tfrac{\tau}{2}A)\gamma(\tau B)\gamma(\tfrac{\tau}{2}A)$$

for $\tau > 0$ we can write the ADI scheme (3.5) as

$$w_{n+1} = S_{\tau,t_n}(w_n) = C(\tau)w_n + \tfrac{\tau}{2}\gamma(\tfrac{\tau}{2}A)\Big[\gamma(\tau B)f(t_n) + f(t_{n+1})\Big]$$

for $n \in \mathbb{N}$. By induction, the approximation at $t_n = n\tau$ is given by

$$w_n = C(\tau)^n w_0 + \frac{\tau}{2}\sum_{k=1}^{n} C(\tau)^{n-k}\gamma(\tfrac{\tau}{2}A)\big(\gamma(\tau B)f(t_{k-1}) + f(t_k)\big). \tag{3.9}$$

Proposition 3.1 in [9] and (3.8) yield the bounds

$$\|C(\tau)\|_{\mathcal{B}(X)} \le 1 \quad \text{and} \quad \|C(\tau)\|_{\mathcal{B}(Y)} \le e^{6\kappa\tau} \tag{3.10}$$

under the restriction $\tau \le \tau_0$ for the second estimate. Together with the representation (3.9) this shows immediately that the ADI scheme (3.5) is unconditionally stable in X and Y. This is summarized in the following theorem. Corresponding results for a different ADI scheme have been shown in in [9, Theorem 4.2] and [10, Theorem 5.1].

Theorem 3.2. *Let* (2.2) *be true,* $n \in \mathbb{N}$, $\tau \in (0, 1]$ *with* $T \ge n\tau$, *and* w_n *be the approximations from* (3.5).

(a) *Let* $w_0 \in X$ *and* $f \in C([0,T],X)$. *We then have*

$$\|w_n\|_X \le \|w_0\|_X + T \max_{t\in[0,T]} \|f(t)\|_X.$$

(b) *Let* $0 < \tau \le \tau_0$, $w_0 \in Y$ *and* $f \in C([0,T],Y)$. *We then have*

$$\|w_n\|_{H^1} \le c\mathrm{e}^{6\kappa T}\big(\|w_0\|_{H^1} + T \max_{t\in[0,T]} \|f(t)\|_{H^1}\big).$$

The constants $c > 0$ *only depend on the constants from* (2.2).

Remark 3.3. If $\sigma = 0$ and $\mathbf{J} = 0$, then the inequality in Theorem 3.2(a) is actually an equality since then the operators A and B are skew-adjoint in X by Lemma 4.3 of [14], and hence their Cayley transforms are unitary in X. The scheme thus preserves the energy in this conservative case.

3.5. **Efficient evaluation of the Cayley transforms.** As mentioned before the Cayley transforms in the ADI method (3.5) involve implicit steps. In this subsection we explain how to evaluate the Cayley transforms in an efficient way. The case $\sigma = 0$ with space discretization on the Yee grid has been discussed in [5, Section 4], [18, Section 3.3] and for two space dimensions in [4, Section 2.2] and [12, Section 2].

Let $w_0 \in Y$ and $f(t) \in Y$ for all $t \ge 0$ so that $(\mathbf{E}_{n+k/5}, \mathbf{H}_{n+k/5})$ belong to Y for all $n \in \mathbb{N}$ and $k \in \{0,1,2,3,4\}$ by Remark 3.1. The first substep in (3.4) can be rewritten as

$$\big(1 + \tfrac{\sigma\tau}{8\varepsilon}\big)\mathbf{E}_{n+1/5} - \tfrac{\tau}{4\varepsilon}C_1\mathbf{H}_{n+1/5} = \big(1 - \tfrac{\sigma\tau}{8\varepsilon}\big)\mathbf{E}_n + \tfrac{\tau}{4\varepsilon}C_1\mathbf{H}_n, \tag{3.11a}$$

$$-\tfrac{\tau}{4\mu}C_2\mathbf{E}_{n+1/5} + \mathbf{H}_{n+1/5} = \tfrac{\tau}{4\mu}C_2\mathbf{E}_n + \mathbf{H}_n \tag{3.11b}$$

in $L^2(Q)^3$. These six equations for $\mathbf{E}_{n+1/5} = (\mathbf{E}^1_{n+1/5}, \mathbf{E}^2_{n+1/5}, \mathbf{E}^3_{n+1/5})$ and $\mathbf{H}_{n+1/5} = (\mathbf{H}^1_{n+1/5}, \mathbf{H}^2_{n+1/5}, \mathbf{H}^3_{n+1/5})$ decouple into three pairs of equations corresponding to (3.3a), (3.3b), and (3.3c). For example, the first component of (3.11a) and the third component of (3.11b) yield

$$\begin{pmatrix} (1 + \frac{\sigma\tau}{8\varepsilon})I & -\frac{\tau}{4\varepsilon}\partial_2 \\ -\frac{\tau}{4\mu}\partial_2 & I \end{pmatrix} \begin{pmatrix} \mathbf{E}^1_{n+1/5} \\ \mathbf{H}^3_{n+1/5} \end{pmatrix} = \begin{pmatrix} (1 - \frac{\sigma\tau}{8\varepsilon})I & \frac{\tau}{4\varepsilon}\partial_2 \\ \frac{\tau}{4\mu}\partial_2 & I \end{pmatrix} \begin{pmatrix} \mathbf{E}^1_n \\ \mathbf{H}^3_n \end{pmatrix}$$

which is nothing else than the trapezoidal rule with step-size $\tau/2$ applied to (3.3a).

Now we define for $\lambda \in \{\varepsilon, \mu\}$ the operators

$$\begin{aligned} D^{(1)}_\lambda &= C_1 \tfrac{1}{\lambda} C_2 = \begin{pmatrix} \partial_2\frac{1}{\lambda}\partial_2 & 0 & 0 \\ 0 & \partial_3\frac{1}{\lambda}\partial_3 & 0 \\ 0 & 0 & \partial_1\frac{1}{\lambda}\partial_1 \end{pmatrix} \quad \text{and} \\ D^{(2)}_\lambda &= C_2 \tfrac{1}{\lambda} C_1 = \begin{pmatrix} \partial_3\frac{1}{\lambda}\partial_3 & 0 & 0 \\ 0 & \partial_1\frac{1}{\lambda}\partial_1 & 0 \\ 0 & 0 & \partial_2\frac{1}{\lambda}\partial_2 \end{pmatrix} \end{aligned} \tag{3.12}$$

on the domains $D(\partial_{22})\times D(\partial_{33})\times D(\partial_{11})$ and $D(\partial_{33})\times D(\partial_{11})\times D(\partial_{22})$, respectively, where $D(\partial_{kk})$ is the set of $g \in L^2(Q)$ with $\partial_k g, \partial_{kk} g \in L^2(Q)$ and $g = 0$ on Γ_k.

We observe that for $(u,v) \in Y$ the fields $D_\lambda^{(j)} u$ and $D_\lambda^{(j)} v$ belong to $H^{-1}(Q)^3$. Inserting (3.11b) into (3.11a) yields

$$\big((1+\tfrac{\sigma\tau}{8\varepsilon})I - \tfrac{\tau^2}{16\varepsilon}D_\mu^{(1)}\big)\mathbf{E}_{n+1/5} = \big((1-\tfrac{\sigma\tau}{8\varepsilon})I + \tfrac{\tau^2}{16\varepsilon}D_\mu^{(1)}\big)\mathbf{E}_n + \tfrac{\tau}{2\varepsilon}C_1\mathbf{H}_n, \tag{3.13a}$$
$$\mathbf{H}_{n+1/5} = \tfrac{\tau}{4\mu}C_2(\mathbf{E}_n + \mathbf{E}_{n+1/5}) + \mathbf{H}_n, \tag{3.13b}$$

in $H^{-1}(Q)^3$. Since $D_\mu^{(1)}$ is diagonal, (3.13a) corresponds to three *decoupled*, scalar valued elliptic problems on a three-dimensional domain. In each of these problems partial derivatives in only *one* spatial direction occur such that the other two directions are uncoupled, too. Solving such problems is relatively cheap: After a space discretization with the Yee grid, for example, each component of (3.13a) reduces to a sequence of small linear problems with a tri-diagonal matrix (cf. [5]), which can be solved with linear complexity.

Similarly, the third sub-step in (3.4) can be rewritten as

$$\begin{aligned}\big(1+\tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_{n+3/5} + \tfrac{\tau}{2\varepsilon}C_2\mathbf{H}_{n+3/5} &= \big(1-\tfrac{\sigma\tau}{4\varepsilon}\big)\mathbf{E}_{n+2/5} - \tfrac{\tau}{2\varepsilon}C_2\mathbf{H}_{n+2/5}\\ \tfrac{\tau}{2\mu}C_1\mathbf{E}_{n+3/5} + \mathbf{H}_{n+3/5} &= -\tfrac{\tau}{2\mu}C_1\mathbf{E}_{n+2/5} + \mathbf{H}_{n+2/5}.\end{aligned}$$

Proceeding as before we then obtain the essentially one-dimensional problem

$$\begin{aligned}\big((1+\tfrac{\sigma\tau}{4\varepsilon})I - \tfrac{\tau^2}{4\varepsilon}D_\mu^{(2)}\big)\mathbf{E}_{n+3/5} &= \big((1-\tfrac{\sigma\tau}{4\varepsilon})I + \tfrac{\tau^2}{4\varepsilon}D_\mu^{(2)}\big)\mathbf{E}_{n+2/5} - \tfrac{\tau}{\varepsilon}C_2\mathbf{H}_{n+2/5},\\ \mathbf{H}_{n+3/5} &= -\tfrac{\tau}{2\mu}C_1(\mathbf{E}_{n+2/5} + \mathbf{E}_{n+3/5}) + \mathbf{H}_{n+2/5}.\end{aligned}$$

in $H^{-1}(Q)^3$. The last sub-step in (3.4) can be treated as the first one.

4. Error analysis

4.1. **Auxiliary results.** Let L generate a C_0–semigroup on a Banach space E bounded by $\|e^{tL}\| \le Ne^{\alpha t}$ for $t \ge 0$ for some constants $N \ge 1$ and $\alpha \ge 0$. We make use of the standard ϕ–functions

$$\phi_j(L)w = \int_0^1 \frac{\theta^{j-1}}{(j-1)!} e^{(1-\theta)L} w \,\mathrm{d}\theta, \quad j \in \mathbb{N}, \qquad \phi_0(L) = e^L. \tag{4.1}$$

Later we will insert here the operators $L \in \{\tau M, \tau M_{\mathrm{div}}, \tau M_2\}$ for $\tau \in (0,1]$. Let $j \in \mathbb{N}_0$. The operator $\phi_j(L) : E \to E$ is bounded by

$$\|\phi_j(L)w\| \le \frac{Ne^\alpha}{j!} \tag{4.2}$$

and maps into $D(L^{j-1})$. The recurrence relation

$$L\phi_{j+1}(L) = \phi_j(L) - \tfrac{1}{j!}\, I \tag{4.3}$$

follows from (4.1) via integration by parts. This recursion yields the finite Taylor expansion

$$e^L w = \phi_0(L)w = \sum_{k=0}^{m-1} \frac{1}{k!} L^k w + L^m \phi_m(L) w \tag{4.4}$$

for $m \in \mathbb{N}$ and $w \in D(L^{m-1})$.

We next derive similar expansions for the Cayley transform of L, assuming that $\alpha < 2$. These formulas will be used for $L = \frac{\tau}{2}A$, $L = \tau B$, and their restrictions to Y for sufficiently small $\tau > 0$. We first note the identity

$$(I - \tfrac{1}{2}L)^{-1} = I + \tfrac{1}{2}L(I - \tfrac{1}{2}L)^{-1}.$$

For all $w \in D(L)$ it follows that

$$\begin{aligned}\gamma(L)w &= (I + \tfrac{1}{2}L)(I - \tfrac{1}{2}L)^{-1}w = (I + \tfrac{1}{2}L)\big(I + \tfrac{1}{2}L(I - \tfrac{1}{2}L)^{-1}\big)w \\ &= w + \tfrac{1}{2}L(I + \gamma(L))w.\end{aligned}$$

Substituting this expression once again yields

$$\begin{aligned}\gamma(L)w &= w + \tfrac{1}{2}L(I + \gamma(L))w = w + \tfrac{1}{2}Lw + \tfrac{1}{2}L\left[I + \tfrac{1}{2}L(I + \gamma(L))\right]w \\ &= w + Lw + \tfrac{1}{4}L^2(I + \gamma(L))w\end{aligned}$$

for $w \in D(L^2)$. For $w \in D(L^3)$, one similarly obtains the formula

$$\begin{aligned}\gamma(L)w &= w + Lw + \tfrac{1}{4}L^2(I + \gamma(L))w \\ &= w + Lw + \tfrac{1}{4}L^2w + \tfrac{1}{4}L^2\left[I + \tfrac{1}{2}L(I + \gamma(L))\right]w \\ &= w + Lw + \tfrac{1}{2}L^2w + \tfrac{1}{8}L^3(I + \gamma(L))w.\end{aligned}$$

For $j, k \in \mathbb{N}_0$ with $j \le k$ we define the shorthand notation

$$F(j,k,L)w = \begin{cases} \frac{1}{j!}L^jw & \text{if } j < k, \\ \frac{1}{2^k}(I + \gamma(L))L^kw & \text{if } j = k > 0, \\ \gamma(L)w & \text{if } j = k = 0. \end{cases} \tag{4.5}$$

In this notation, the three expansions of the Cayley transform read

$$F(0,0,L)w = \sum_{j=0}^{k} F(j,k,L)w \tag{4.6}$$

for $w \in D(L^k)$ and $k \in \{1, 2, 3\}$. This equation is valid in X_{-1}^L if $w \in D(L^{k-1})$.

4.2. **Convergence of the ADI scheme.** Our main result establishes the second order convergence of the ADI scheme in Y^*. According to (3.7), the number $\kappa \ge 0$ only depends on the constants from (2.2), and we have $\kappa = 0$ in the case of constant coefficients.

Theorem 4.1. *Let $T \ge 1$ and $0 < \tau \le \min\{1, \tau_0\}$. Assume that $w_0 \in X_2$, that the material coefficients have the regularity (2.2), and that*

$$f \in \mathcal{F} := C\left([0,T], X_2\right) \cap C^1([0,T], D(M_{\mathrm{div}})) \cap W^{2,1}\left([0,T], X\right).$$

Let $w = (\boldsymbol{E}, \boldsymbol{H})$ be the solution of the Maxwell system (2.1), and let w_n be the approximation computed with the ADI method (3.4) or equivalently (3.9). Then, the error is bounded by

$$|(w_n - w(n\tau) \,|\, y)_X| \le c\tau^2T^5e^{6\kappa T}\big(\, \|w_0\|_{X_2} + \|f\|_{\mathcal{F}}\big)\|y\|_Y$$

for all $n\tau \le T$ and all $y \in Y$. The constant $c > 0$ only depends on the constants from (2.2) and on Q.

Remark 4.2. We can replace the factor T^5 by T^2 if $\sigma = 0$ or $\sigma \geq \sigma_0$ for a constant $\sigma_0 > 0$, using Remark 3.4 in [10].

Proof. 1) Let $n \in \mathbb{N}$ with $n\tau \leq T$ and set $t_k = k\tau$ for $k \in \{0, \dots, n\}$. The assumptions imply the existence of a unique solution $w \in C^1([0, T], X_{\mathrm{div}})$ of (2.1) given by Duhamel's formula

$$w(t_n) = \phi_0(t_n M) w_0 + \int_0^{t_n} \phi_0((t_n - s)M) f(s) \,\mathrm{d}s$$

in X, see (2.7). Here and below we often use the operators M, A or B instead of M_{div}, M_2, A_Y or B_Y to simplify notation. The representation (3.9) of w_n thus implies the expression

$$\begin{aligned} e_n := w_n - w(t_n) &= C(\tau)^n w_0 - \phi_0(t_n M) w_0 \\ &\quad + \frac{\tau}{2} \sum_{k=1}^{n} C(\tau)^{n-k} \gamma(\tfrac{\tau}{2} A) \big(f(t_k) + \gamma(\tau B) f(t_{k-1}) \big) \\ &\quad - \int_0^{t_n} \phi_0((t_n - s)M) f(s) \,\mathrm{d}s \end{aligned} \tag{4.7}$$

for the error. Employing the Taylor expansion

$$f(t_{k-1} + s) = f(t_{k-1}) + s f'(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1}+s} (t_{k-1} + s - r) f''(r) \,\mathrm{d}r \tag{4.8}$$

for $s \in [0, \tau]$, we write the integral in (4.7) as

$$\begin{aligned} &\int_0^{t_n} \phi_0((t_n - s)M) f(s) \,\mathrm{d}s \\ &\quad = \sum_{k=1}^{n} \phi_0(t_{n-k} M) \int_0^{\tau} \phi_0((\tau - s)M) f(t_{k-1} + s) \,\mathrm{d}s \\ &\quad = \sum_{k=1}^{n} \phi_0(t_{n-k} M) \big[\tau \phi_1(\tau M) f(t_{k-1}) + \tau^2 \phi_2(\tau M) f'(t_{k-1}) + R_k(\tau) \big], \end{aligned} \tag{4.9}$$

with remainder term

$$R_k(\tau) = \int_0^{\tau} e^{(\tau - s)M} \Big(\int_{t_{k-1}}^{t_{k-1}+s} (t_{k-1} + s - r) f''(r) \,\mathrm{d}r \Big) \,\mathrm{d}s$$

for $k \in \{1, \dots, n\}$. Similarly, the sum in (4.7) becomes

$$\begin{aligned} &\frac{\tau}{2} \sum_{k=1}^{n} C(\tau)^{n-k} \gamma(\tfrac{\tau}{2} A) \big(f(t_k) + \gamma(\tau B) f(t_{k-1}) \big) \\ &\quad = \sum_{k=1}^{n} C(\tau)^{n-k} \gamma(\tfrac{\tau}{2} A) \big[\tfrac{\tau}{2} f(t_{k-1}) + \tfrac{\tau^2}{2} f'(t_{k-1}) + r_k(\tau) + \gamma(\tau B) \tfrac{\tau}{2} f(t_{k-1}) \big] \end{aligned} \tag{4.10}$$

with

$$r_k(\tau) = \frac{\tau}{2} \int_{t_{k-1}}^{t_k} (k\tau - r) f''(r) \,\mathrm{d}r$$

if one inserts (4.8) for $s = \tau$. By means of (4.9) and (4.10), we deduce from (4.7) the basic error formula

$$e_n =: \Sigma_1(\tau) + \Sigma_2(\tau) + \Sigma_3(\tau) + \Sigma_4(\tau) \tag{4.11}$$

with

$$\Sigma_1(\tau) = \sum_{m=0}^{n-1} C(\tau)^m \big(C(\tau) - \phi_0(\tau M)\big)\phi_0((n-1-m)\tau M)w_0,$$

$$\Sigma_2(\tau) = \tau \sum_{k=1}^{n} \big[\tfrac{1}{2}C(\tau)^{n-k}\gamma(\tfrac{\tau}{2}A)\big(I + \gamma(\tau B)\big) - \phi_0((n-k)\tau M)\phi_1(\tau M)\big] f(t_{k-1}),$$

$$\Sigma_3(\tau) = \tau^2 \sum_{k=1}^{n} \big[\tfrac{1}{2}C(\tau)^{n-k}\gamma(\tfrac{\tau}{2}A) - \phi_0((n-k)\tau M)\phi_2(\tau M)\big] f'(t_{k-1}),$$

$$\Sigma_4(\tau) = \sum_{k=1}^{n} \big[C(\tau)^{n-k}\gamma(\tfrac{\tau}{2}A)r_k(\tau) - \phi_0((n-k)\tau M)R_k(\tau)\big].$$

2) We first treat the term $\Sigma_1(\tau)$. For every $y \in Y$, we have the weak formulation

$$(\Sigma_1(\tau)\,|\,y)_X = \sum_{m=0}^{n-1} \big([C(\tau)-\phi_0(\tau M)]\phi_0((n-1-m)\tau M)w_0 \,\big|\, (C(\tau)^m)^* y\big)_X. \tag{4.12}$$

Set $z_m = \phi_0((n-1-m)\tau M)w_0 \in X_2$ and $y_m = (C(\tau)^m)^* y \in Y$ for $m \in \{0, 1, \dots, n-1\}$. The estimates (2.8) and (3.10) yield the bounds

$$\|z_m\|_{X_2} \le cT^3 \|w_0\|_{X_2} \qquad \text{and} \qquad \|y_m\|_Y \le e^{6\kappa T} \|y\|_Y \tag{4.13}$$

for all m. From (3.6) and (3.1) we recall the embeddings

$$Y \hookrightarrow D(A^*) \cap D(B^*) \cap D(M^*),$$
$$X_2 \hookrightarrow D(A^2) \cap D(AB) \cap D(BA) \cap D(B^2) \cap D(M_{\mathrm{div}}^2).$$

The extrapolation spaces $X_{-1}^A \cong D(A^*)^*$, $X_{-1}^B \cong D(B^*)^*$, and $X_{-1}^M \cong D(M^*)^*$ are thus embedded in Y^*, cf. Section 2. For instance, for $w \in X$ an element $A_{-1}w \in X_{-1}^A$ acts on $y \in Y$ via

$$\langle A_{-1}w, y\rangle_{Y^* \times Y} = (w \,|\, A^* y)_X.$$

Let Γ denote either $\gamma(\frac{\tau}{2}A)$ ot $\gamma(\tau B)$ in $\mathcal{B}(X)$. This operator can be extended to a map $\tilde{\Gamma} \in \mathcal{B}(Y^*)$ by setting

$$\langle \tilde{\Gamma} y^*, y\rangle_{Y^* \times Y} = \langle y^*, \Gamma^* y\rangle_{Y^* \times Y}$$

for $y^* \in Y^*$ and the adjoint Γ^* in X, since the restriction $\Gamma^*|_Y$ belongs to $\mathcal{B}(Y)$. These facts are used below mostly without further comments.

We treat the components of the difference in (4.12) separately. Formula (4.6) implies the identity

$$C(\tau)z_m = \gamma(\tfrac{\tau}{2}A)\gamma(\tau B_Y)\gamma(\tfrac{\tau}{2}A_Y)z_m = F(0,0,\tfrac{\tau}{2}A)F(0,0,\tau B)F(0,0,\tfrac{\tau}{2}A)z_m$$
$$= \sum_{j_1=0}^{3} F(0,0,\tfrac{\tau}{2}A)F(0,0,\tau B)F(j_1,3,\tfrac{\tau}{2}A)z_m \tag{4.14}$$

$$
\begin{aligned}
&= \sum_{j_1=0}^{3}\sum_{j_2=0}^{3-j_1} F(0,0,\tfrac{\tau}{2}A)F(j_2,3-j_1,\tau B)F(j_1,3,\tfrac{\tau}{2}A)z_m \\
&= \sum_{j_1=0}^{3}\sum_{j_2=0}^{3-j_1}\sum_{j_3=0}^{3-j_1-j_2} F(j_3,3-j_1-j_2,\tfrac{\tau}{2}A)F(j_2,3-j_1,\tau B)F(j_1,3,\tfrac{\tau}{2}A)z_m \\
&= \sum_{k=0}^{3}\sum_{j_1+j_2+j_3=k} F(j_3,3-j_1-j_2,\tfrac{\tau}{2}A)F(j_2,3-j_1,\tau B)F(j_1,3,\tfrac{\tau}{2}A)z_m
\end{aligned}
$$

in Y^*. Actually, in the summands with $k=3$ the operator A or B applied last has to be extrapolated and the following Cayley transforms are extended to Y^*. But here and below we do not indicate this in the F–notation. The expansion (4.4) and the definition (4.5) yield

$$
\begin{aligned}
\phi_0(\tau M)z_m - \tau^3 M_{-1}M^2\phi_3(\tau M)z_m &= z_m + \tau M z_m + \tfrac{\tau^2}{2}M^2 z_m \qquad (4.15)\\
&= \sum_{k=0}^{2}\sum_{j_1+j_2+j_3=k} F(j_3,3-j_1-j_2,\tfrac{\tau}{2}A)F(j_2,3-j_1,\tau B)F(j_1,3,\tfrac{\tau}{2}A)z_m
\end{aligned}
$$

in Y^*. The new term $\tau^3 M_{-1}M^2\phi_3(\tau M)z_m$ on the right-hand side of (4.12) can be estimated by

$$
\begin{aligned}
\left|\left\langle \tau^3 M_{-1}M^2\phi_3(\tau M)z_m, y_m\right\rangle_{Y^*\times Y}\right| &= \tau^3\left|\left(\phi_3(\tau M)M^2 z_m \,\middle|\, M^* y_m\right)_X\right| \\
&\le c\tau^3 T^3 e^{6\kappa T}\left\|w_0\right\|_{X_2}, \qquad (4.16)
\end{aligned}
$$

using (4.2) and (4.13). The difference of (4.14) and (4.15) is given by the sum

$$
\begin{aligned}
&\sum_{j_1+j_2+j_3=3} F(j_3,3-j_1-j_2,\tfrac{\tau}{2}A)F(j_2,3-j_1,\tau B)F(j_1,3,\tfrac{\tau}{2}A)z_m \\
&= \Big(F(3,3,\tfrac{\tau}{2}A)F(0,3,\tau B)F(0,3,\tfrac{\tau}{2}A) + F(2,2,\tfrac{\tau}{2}A)F(0,2,\tau B)F(1,3,\tfrac{\tau}{2}A) \\
&\quad + F(2,2,\tfrac{\tau}{2}A)F(1,3,\tau B)F(0,3,\tfrac{\tau}{2}A) + F(1,1,\tfrac{\tau}{2}A)F(0,1,\tau B)F(2,3,\tfrac{\tau}{2}A) \\
&\quad + F(1,1,\tfrac{\tau}{2}A)F(1,2,\tau B)F(1,3,\tfrac{\tau}{2}A) + F(1,1,\tfrac{\tau}{2}A)F(2,3,\tau B)F(0,3,\tfrac{\tau}{2}A) \\
&\quad + F(0,0,\tfrac{\tau}{2}A)F(0,0,\tau B)F(3,3,\tfrac{\tau}{2}A) + F(0,0,\tfrac{\tau}{2}A)F(1,1,\tau B)F(2,3,\tfrac{\tau}{2}A) \\
&\quad + F(0,0,\tfrac{\tau}{2}A)F(2,2,\tau B)F(1,3,\tfrac{\tau}{2}A) + F(0,0,\tfrac{\tau}{2}A)F(3,3,\tau B)F(0,3,\tfrac{\tau}{2}A)\Big)z_m \\
&=: \sum_{l=1}^{10}\Sigma_{1,l}(\tau)z_m \qquad (4.17)
\end{aligned}
$$

on X_2. We first look at the term $\Sigma_{1,1}(\tau)z_m$ which reads

$$
\begin{aligned}
\Sigma_{1,1}(\tau)z_m &= F(3,3,\tfrac{\tau}{2}A)F(0,3,\tau B)F(0,3,\tfrac{\tau}{2}A)z_m \\
&= \frac{\tau^3}{64}A_{-1}(I+\gamma(\tfrac{\tau}{2}A))A^2 z_m
\end{aligned}
$$

according to definition (4.5). By means of (4.13), this term can be bounded by

$$
\begin{aligned}
\left|\langle \Sigma_{1,1}(\tau)z_m, y_m\rangle_{Y^*\times Y}\right| &= c\tau^3\left|\left(A^2 z_m \,\middle|\, (I+\gamma(\tfrac{\tau}{2}A^*))Ay_m\right)_X\right| \\
&\le c\tau^3 T^3 e^{6\kappa T}\left\|w_0\right\|_{X_2}\left\|y\right\|_Y .
\end{aligned}
$$

Here and below also the contractivity of the Cayley transforms on X is taken into account. The following five summands in (4.17) are estimated analogously, partly replacing the product A^2 by AB, BA or B^2. We next treat the term

$$\Sigma_{1,7}(\tau)z_m = \frac{\tau^3}{64}\gamma(\tfrac{\tau}{2}A)\gamma(\tau B)A_{-1}(I+\gamma(\tfrac{\tau}{2}A))A^2 z_m$$

omitting the tilde for the first two Cayley transforms. It is bounded by

$$\begin{aligned} |\langle \Sigma_{1,7}(\tau)z_m, y_m\rangle_{Y^*\times Y}| &= c\tau^3 \left|\big(A^2 z_m \,\big|\, (I+\gamma(\tfrac{\tau}{2}A^*))A\gamma(\tau(B^*)_Y)\gamma(\tfrac{\tau}{2}(A^*)_Y)y_m\big)_X\right| \\ &\le c\tau^3 T^3 e^{6\kappa T}\, \|w_0\|_{X_2}\, \|y\|_Y, \end{aligned}$$

where we also employ (3.8). The last tree summands in (4.17) can be controlled by the same technique. Combining these estimates with (4.12), (4.14), (4.15), (4.16) and (4.17), we arrive at the inequality

$$|(\Sigma_1(\tau)|y)_X| \le c\tau^2 T^4 e^{6\kappa T}\, \|w_0\|_{X_2}\, \|y\|_Y\,. \tag{4.18}$$

3) We next rewrite $\Sigma_2(\tau)$ as

$$\begin{aligned} (\Sigma_2(\tau)\,|\,y)_X &= \tau\sum_{k=1}^{n}\big(\big[\tfrac12\gamma(\tfrac{\tau}{2}A)\big(I+\gamma(\tau B)\big)-\phi_1(\tau M)\big]f(t_{k-1})\,\big|\,y_{n-k}\big)_X \\ &\quad +\tau\sum_{k=1}^{n}\big(\big[C(\tau)^{n-k}-\phi_0((n-k)\tau M)\big]\phi_1(\tau M)f(t_{k-1})\,\big|\,y\big)_X \\ &= \tau\sum_{k=1}^{n}\big(f(t_{k-1})\,\big|\,\tfrac12[\gamma(\tau B^*)-\tau B^*-I]\gamma(\tfrac{\tau}{2}(A^*)_Y)y_{n-k}\big)_X \\ &\quad +\tau\sum_{k=1}^{n}\big(\big[f(t_{k-1})\,\big|\,\big[(I+\tfrac{\tau}{2}B^*)\gamma(\tfrac{\tau}{2}(A^*)_Y)-\phi_1(\tau M^*)\big]y_{n-k}\big)_X \\ &\quad +\tau\sum_{k=1}^{n}\big(\big[(C(\tau))^{n-k}-\phi_0((n-k)\tau M)\big]\phi_1(\tau M)f(t_{k-1})\,\big|\,y\big)_X \\ &=: \Sigma_{2,1}(\tau)+\Sigma_{2,2}(\tau)+\Sigma_{2,3}(\tau). \end{aligned} \tag{4.19}$$

We expand the last summand by the telecoping sum

$$\tau\sum_{k=1}^{n}\sum_{m=0}^{n-k-1}\big(C(\tau)^m\big[C(\tau)-\phi_0(\tau M)\big]\phi_0((n-k-1-m)\tau M)\phi_1(\tau M)f(t_{k-1})\,\big|\,y\big)_X$$

Using the factor τ to bound the second sum, we can estimate this term as in step 2) by

$$|\Sigma_{2,3}(\tau)| \le c\tau^2 T^5 e^{6\kappa T}\, \|f\|_{C([0,T],X_2)}\, \|y\|_Y. \tag{4.20}$$

Inserting the identity $\gamma(\tau B^*)-\tau B^*-I=\frac{\tau^2}{2}(B^*)^2(I-\frac{\tau}{2}B^*)^{-1}$ on Y, we infer

$$\begin{aligned} |\Sigma_{2,1}(\tau)| &= \tau^3\Big|\sum_{k=1}^{n}\big(Bf(t_{k-1})\,\big|\,\tfrac14 B^*(I-\tfrac{\tau}{2}(B^*)_Y)^{-1}\gamma(\tfrac{\tau}{2}(A^*)_Y)y_{n-k}\big)_X\Big| \\ &\le c\tau^2 T e^{6\kappa T}\, \|f\|_{C([0,T],X_2)}\, \|y\|_Y. \end{aligned} \tag{4.21}$$

by means of (4.13). With the abbreviation

$$\chi_k(\tau) = (I - \tfrac{\tau}{4}(B^*)_Y)^{-1}(I - \tfrac{\tau}{4}(A^*)_Y)^{-1} y_{n-k} \in D((B^*)_Y),$$

we reformulate the summand $\Sigma_{2,2}(\tau)$ as

$$\begin{aligned}\Sigma_{2,2}(\tau) = \tau \sum_{k=1}^{n} \big\langle f(t_{k-1}), (I + \tfrac{\tau}{2}(B^*)_{-1})(I + \tfrac{\tau}{4}A^*)(I - \tfrac{\tau}{4}(B^*)_Y)\chi_k(\tau)\big\rangle_{D(B)\times X_{-1}^{B^*}} \\ - \tau \sum_{k=1}^{n} \big(f(t_{k-1}) \,\big|\, \phi_1(\tau M)^*(I - \tfrac{\tau}{4}A^*)(I - \tfrac{\tau}{4}(B^*)_Y)\chi_k(\tau)\big)_X.\end{aligned}$$

Recall that $M^* = A^* + B^*$ on Y, and that (4.3) yields $\phi_1(\tau M)^* = I + \tau M^*\phi_2(\tau M)^*$ and $\phi_2(\tau M)^* = \frac{1}{2}I + \tau M^*\phi_3(\tau M)^*$. We then calculate

$$\begin{aligned}\Sigma_{2,2}(\tau) &= \tau \sum_{k=1}^{n} \big\langle f(t_{k-1}), (I + \tfrac{\tau}{2}(B^*)_{-1}) \cdot \\ &\qquad\qquad \cdot [I + \tfrac{\tau}{4}(A^* - B^*) - \tfrac{\tau^2}{16}A^*B^*]\chi_k(\tau)\big\rangle_{D(B)\times X_{-1}^{B^*}} \\ &\quad - \tau \sum_{k=1}^{n} \big(f(t_{k-1}) \,\big|\, (I + \tau M^*\phi_2(\tau M)^*) \cdot \\ &\qquad\qquad \cdot [I - \tfrac{\tau}{4}(A^* + B^*) + \tfrac{\tau^2}{16}A^*B^*]\chi_k(\tau)\big)_X \\ &= \tau \sum_{k=1}^{n} \big\langle f(t_{k-1}), \big[I + \tfrac{\tau}{4}(A^* - B^*) - \tfrac{\tau^2}{16}A^*B^* + \tfrac{\tau}{2}B^* \\ &\qquad + \tfrac{\tau^2}{8}(B^*)_{-1}A^* - \tfrac{\tau^2}{8}B^*B^* - \tfrac{\tau^3}{32}(B^*)_{-1}A^*B^*\big]\chi_k(\tau)\big\rangle_{D(B)\times X_{-1}^{B^*}} \\ &\quad - \tau \sum_{k=1}^{n} \big(f(t_{k-1}) \,\big|\, (I - \tfrac{\tau}{4}M^* + \tfrac{\tau^2}{16}A^*B^*)\chi_k(\tau)\big)_X \\ &\quad - \tau \sum_{k=1}^{n} \big(f(t_{k-1}) \,\big|\, \tau M^*\phi_2(\tau M)^*(I + \tfrac{\tau^2}{16}A^*B^*)\chi_k(\tau)\big)_X \\ &\quad + \tau \sum_{k=1}^{n} \big\langle f(t_{k-1}), \tfrac{\tau^2}{4}M^*_{-1}M^*\phi_2(\tau M)^*\chi_k(\tau)\big\rangle_{D(M)\times X_{-1}^{M^*}} \\ &= \tau \sum_{k=1}^{n} \big\langle f(t_{k-1}), \big[-\tfrac{\tau^2}{8}A^*B^* + \tfrac{\tau^2}{8}(B^*)_{-1}A^* - \tfrac{\tau^2}{8}B^*B^* \\ &\qquad - \tfrac{\tau^3}{32}(B^*)_{-1}A^*B^*\big]\chi_k(\tau)\big\rangle_{D(B)\times X_{-1}^{B^*}} \\ &\quad + \tau \sum_{k=1}^{n} \big(f(t_{k-1}) \,\big|\, \tfrac{\tau}{2}M^* - \tau M^*\phi_2(\tau M)^*\chi_k(\tau)\big)_X \\ &\quad - \tau \sum_{k=1}^{n} \big(f(t_{k-1}) \,\big|\, \tfrac{\tau^3}{16}M^*\phi_2(\tau M)^*A^*B^*\chi_k(\tau)\big)_X\end{aligned}$$

$$
\begin{aligned}
&+ \tau \sum_{k=1}^{n} \big\langle f(t_{k-1}), \tfrac{\tau^2}{4}(M^*)_{-1} M^* \phi_2(\tau M)^* \chi_k(\tau) \big\rangle_{D(M)\times X_{-1}^{M^*}} \\
&= \tau^3 \sum_{k=1}^{n} \big(\tfrac{1}{8}[-BA + AB - B^2] f(t_{k-1}) \,\big|\, \chi_k(\tau) \big)_X \\
&\quad - \tau^4 \sum_{k=1}^{n} \big(\tfrac{1}{32} AB f(t_{k-1}) \,\big|\, B^* \chi_k(\tau) \big)_X \\
&\quad - \tau^3 \sum_{k=1}^{n} \big(M^2 f(t_{k-1}) \,\big|\, \phi_3(\tau M)^* \chi_k(\tau) \big)_X \\
&\quad - \tau^4 \sum_{k=1}^{n} \big(-\tfrac{1}{16} AM \phi_2(\tau M) f(t_{k-1}) \,\big|\, B^* \chi_k(\tau) \big)_X \\
&\quad + \tau^3 \sum_{k=1}^{n} \big(\tfrac{1}{4} M^2 f(t_{k-1}) \,\big|\, \phi_2(\tau M)^* \chi_k(\tau) \big)_X .
\end{aligned}
$$

Here we also use the embedding $D(M_{\mathrm{div}}) \hookrightarrow Y$ from (2.5). By means of (4.13), it follows that

$$
|\Sigma_{2,2}(\tau)| \le c\tau^2 T e^{6\kappa T} \big\| f \big\|_{C([0,T],X_2)} \, \|y\|_Y .
$$

Together with the estimates (4.20) and (4.21), we have shown that

$$
|(\Sigma_2(\tau)|y)_X| \le c\tau^2 T^5 e^{6\kappa T} \big\| f \big\|_{C([0,T],X_2)} \, \|y\|_Y . \tag{4.22}
$$

4) In a similar way, we compute

$$
\begin{aligned}
\Sigma_3(\tau) &= \tau^2 \sum_{k=1}^{n} C(\tau)^{n-k} [\tfrac{1}{2}\gamma(\tfrac{\tau}{2}A) - \phi_2(\tau M)] f'(t_{k-1}) \\
&\qquad + \tau^2 \sum_{k=1}^{n} [C(\tau)^{n-k} - \phi_0(t_{n-k}M)] \phi_2(\tau M) f'(t_{k-1}) \\
&=: \Sigma_{3,1}(\tau) + \Sigma_{3,2}(\tau).
\end{aligned}
$$

The second term can be treated as in step 2), now expanding the difference $C(\tau) - \phi_0(\tau M)$ up to second order $k = 2$ instead of $k = 3$, cf. (4.14)–(4.17). In this way one obtains the bound

$$
|(\Sigma_{3,2}(\tau) \,|\, y)_X| \le c\tau^2 T^5 e^{6\kappa T} \, \|f\|_{C^1([0,T],D(M_{\mathrm{div}}))} \, \|y\|_Y . \tag{4.23}
$$

Using (4.3) as in step 3), we rewrite $\Sigma_{3,1}(\tau)$ as

$$
\begin{aligned}
\Sigma_{3,1}(\tau) &= \tau^2 \sum_{k=1}^{n} C(\tau)^{n-k} (I - \tfrac{\tau}{4}A)^{-1} \big[\tfrac{1}{2}(I + \tfrac{\tau}{4}A) \\
&\qquad - (I - \tfrac{\tau}{4}A)(\tfrac{1}{2}I + \tau M \phi_3(\tau M)) \big] f'(t_{k-1}) \\
&= \tau^3 \sum_{k=1}^{n} C(\tau)^{n-k} \big[\tfrac{1}{4}(I - \tfrac{\tau}{4}A)^{-1} A - \phi_3(\tau M) M \big] f'(t_{k-1})
\end{aligned}
$$

partly in Y_*. This sum can be estimated even in X by $c\tau^2 T\|f\|_{C^1([0,T],D(M_{\mathrm{div}}))}$ since the Cayley transforms and e^{tM} are contractions on X and $D(M_{\mathrm{div}}) \hookrightarrow Y$ by (2.5). Combined with (4.23), we infer the inequality

$$|(\Sigma_3(\tau)\,|\,y)_X| \le c\tau^2 T^5 e^{6\kappa T} \|f\|_{C^1([0,T],D(M_{\mathrm{div}}))}\, \|y\|_Y. \tag{4.24}$$

The terms $r_k(\tau)$ and $R_k(\tau)$ are bounded in X by $c\tau^2 \int_{(k-1)\tau}^{k\tau} \|f''(s)\|_X \,\mathrm{d}s$, so that $\Sigma_4(\tau)$ is controlled by

$$\|\Sigma_4(\tau)\|_X \le c\tau^2\, \|f\|_{W^{2,1}([0,T],X)}\,. \tag{4.25}$$

The assertion is now a consequence of the error formula (4.11) and the inequalities (4.18), (4.22), (4.24) and (4.25). □

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