Spectrum of a singular pertubed periodic thin waveguide
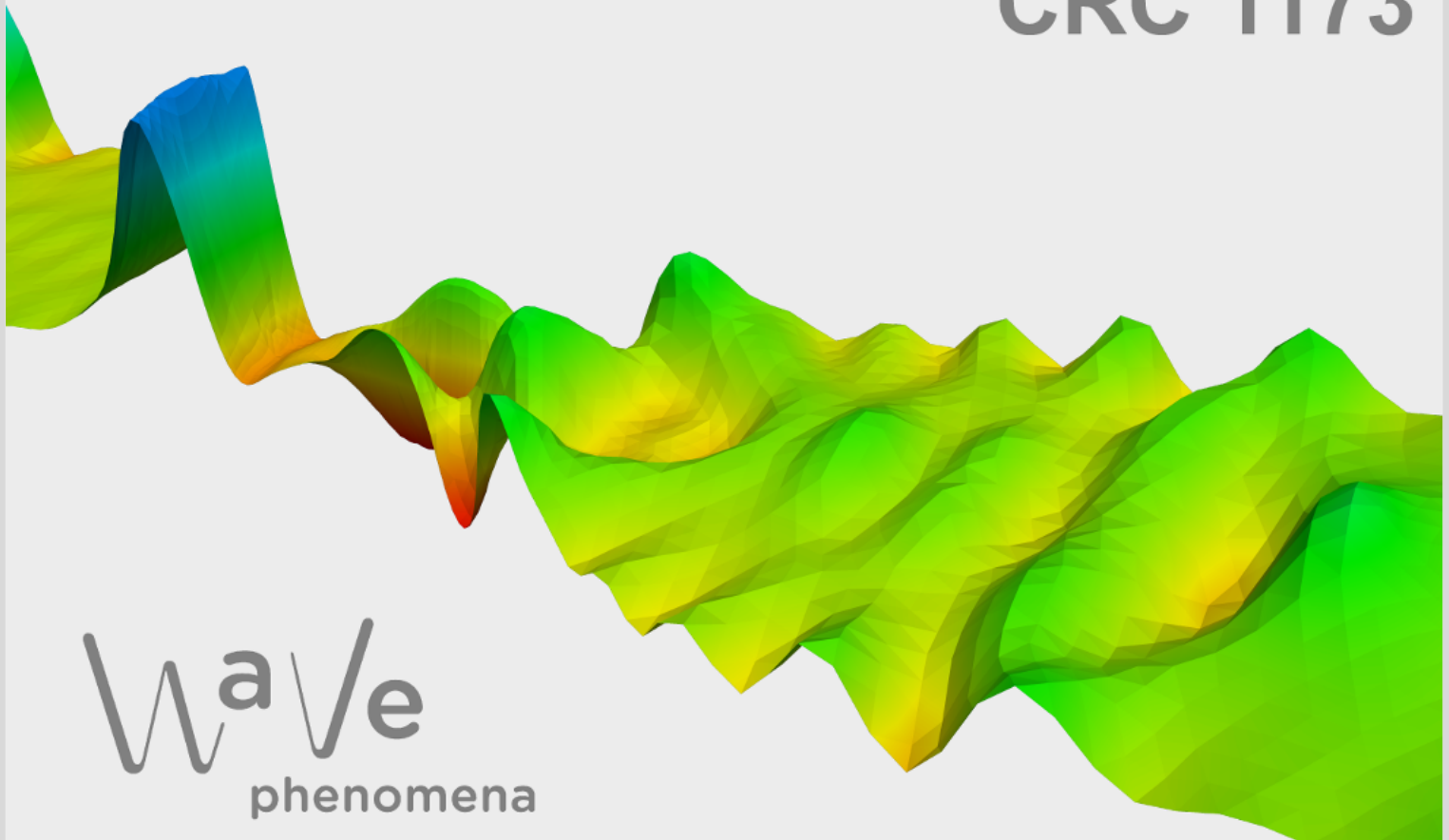
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Spectrum of a singularly perturbed periodic thin waveguide

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Abstract

We consider a family $\{\Omega^\varepsilon\}_{\varepsilon>0}$ of periodic domains in $\mathbb{R}^2$ with waveguide geometry and analyse spectral properties of the Neumann Laplacian $-\Delta_{\Omega^\varepsilon}$ on Ω^ε. The waveguide Ω^ε is a union of a thin straight strip of the width ε and a family of small protuberances with the so-called "room-and-passage" geometry. The protuberances are attached periodically, with a period ε, along the strip upper boundary. For $\varepsilon \to 0$ we prove a (kind of) resolvent convergence of $-\Delta_{\Omega^\varepsilon}$ to a certain ordinary differential operator. Also we demonstrate Hausdorff convergence of the spectrum. In particular, we conclude that if the sizes of "passages" are appropriately scaled the first spectral gap of $-\Delta_{\Omega^\varepsilon}$ is determined exclusively by geometric properties of the protuberances. The proofs are carried out using methods of homogenization theory.

Keywords: singularly perturbed domains, periodic waveguides, Neumann Laplacian, spectral gaps, homogenization

1. Introduction

In the paper we study the limiting behaviour as $\varepsilon \to 0$ of the Neumann Laplacian on a thin periodic domain $\Omega^\varepsilon \subset \mathbb{R}^2$ with waveguide geometry – see Figure 1. The domain Ω^ε is obtained from the straight unbounded strip $\Pi^\varepsilon = \mathbb{R} \times (0, \varepsilon)$ by attaching an array of small identical protuberances (counted by the parameter $j \in \mathbb{Z}$). Each protuberance consists of two parts (below $\simeq$ means that domains coincide up to a translation):

- *the "room" $B_j^\varepsilon \simeq \varepsilon B$*, where $B \subset \mathbb{R}^2$ is a fixed domain,

- *the "passage" $T_j^\varepsilon \simeq (0, d^\varepsilon) \times [0, h^\varepsilon]$* connecting the "room" B_j^ε with the strip Π^ε. Here $h^\varepsilon \to 0$, $d^\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0$.

The protuberances $T_j^\varepsilon \cup B_j^\varepsilon$, $j \in \mathbb{Z}$ are attached periodically, with a period ε, along the strip upper boundary.

Peculiar properties of Neumann spectral problems on domains perturbed by attaching small "room-and-passage" protuberances were observed for the first time by R. Courant and D. Hilbert [10, Volume I, Chapter VI, § 2.6]. Below we sketch the example considered in [10]. Let us perturb a bounded connected domain Ω to a domain Ω^ε by attaching a single "room-and-passage" protuberance. The domain Ω^ε differs from Ω only in a ball of the radius tending to zero as $\varepsilon \to 0$. One can easily show

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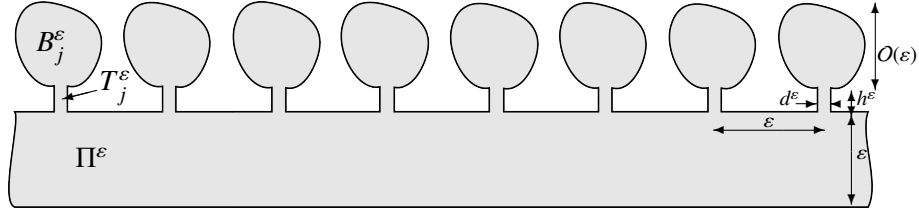
Fig. 1: Domain Ω^ε

(using, e.g., [31, Theorem 1.5]) that for each $k \in \mathbb{N}$ the k-th eigenvalue of *the Dirichlet Laplacian* on Ω^ε converges to the k-th eigenvalue of the Dirichlet Laplacian on Ω. In contrast, for the Neumann Laplacians (we denote them $-\Delta_{\Omega^\varepsilon}$ and $-\Delta_\Omega$) the continuity of eigenvalues does not hold in general: the first eigenvalues λ_1 and λ_1^ε of $-\Delta_\Omega$ and $-\Delta_{\Omega^\varepsilon}$ are zero, the second eigenvalue λ_2 of $-\Delta_\Omega$ is strictly positive, while the second eigenvalue λ_2^ε of $-\Delta_{\Omega^\varepsilon}$ tends to zero as $\varepsilon \to 0$ provided $h^\varepsilon = \varepsilon$, $d^\varepsilon = \varepsilon^\alpha$, $\alpha > 3$.

Later the aforementioned example was studied in [1] for more general geometry of "rooms" and "passages". The authors also inspected the case of *finitely* many attached "room-and-passage" domains proving that $\lim\limits_{\varepsilon \to 0} \lambda_k^\varepsilon = 0$ as $k = 2, \ldots, r+1$ and $\lim\limits_{\varepsilon \to 0} \lambda_k^\varepsilon = \lambda_{k-r}$ as $k \geq r+2$, where $r \in \mathbb{N}$ is the number of attached domains.

The case, when the number of attached "room-and-passage" protuberances tends to infinity as $\varepsilon \to 0$, was studied in our previous paper [11] (still with a bounded Ω). We considered the operator

$$\mathcal{H}^\varepsilon = -(\rho^\varepsilon)^{-1}\Delta_{\Omega^\varepsilon},$$

where $-\Delta_{\Omega^\varepsilon}$ is the Neumann Laplacian on Ω^ε, the weight ρ^ε is equal to 1 everywhere except the union of the "rooms", where it is equal to the constant $\varrho^\varepsilon > 0$ satisfying $\lim\limits_{\varepsilon \to 0} \varrho^\varepsilon \varepsilon < \infty$. It was proved that the spectrum $\sigma(\mathcal{H}^\varepsilon)$ of the operator $\mathcal{H}^\varepsilon$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set $\sigma(\mathcal{H}) \cup \{q\}$, where $q = \lim\limits_{\varepsilon \to 0} \frac{d^\varepsilon}{h^\varepsilon \varrho^\varepsilon \varepsilon^2 |B|} \in [0, \infty]$ (by $|\cdot|$ we denote the Lebesgue measure of a domain) and $\mathcal{H}$ is the operator associated with the following spectral problem:

$$-\Delta u = \lambda u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = \mathcal{V}(\lambda)u \text{ on } \Gamma, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma,$$

where Γ is the perturbed part of $\partial\Omega$, n is the outward-pointing unit normal to $\partial\Omega$. If $\lim\limits_{\varepsilon \to 0} \varrho^\varepsilon \varepsilon = 0$ one has $\mathcal{V}(\lambda) \equiv 0$, otherwise $\mathcal{V}(\lambda)$ is either linear function ($q = \infty$) or rational function ($q < \infty$) with a pole at q, which is also a point of accumulation of eigenvalues of $\mathcal{H}$.

Note, that in the case $\varrho^\varepsilon = 1$ (i.e., $\mathcal{H}^\varepsilon$ is simply the Neumann Laplacian on Ω^ε) one has $\mathcal{V}(\lambda) \equiv 0$, i.e. $\mathcal{H}$ is the Neumann Laplacian on Ω. The case $\varrho^\varepsilon = 1$, $q = 0$ was also studied in [34, Chapter XII].

The results of [11] were extended in [12][1] to Ω, which is an unbounded straight strip of the *fixed* width $L > 0$. In this case Γ is its upper (or lower) boundary. It turns out that the form of the limit problem remains the same as in the case of a bounded domain, but the structure of its spectrum is essentially different: it is either the whole positive semi-axis or the set $[0, \infty) \setminus (q, \widehat{q})$ (this case occur

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[1]In fact, in [12] we deal with the most interesting case $q > 0$, $\lim\limits_{\varepsilon \to 0} \varrho^\varepsilon \varepsilon > 0$ only. The analysis for the rest cases can be carried out a similar way.

if $\lim_{\varepsilon \to 0} \varrho^\varepsilon \varepsilon > 0$ and $0 < q < \left(\frac{\pi}{2L}\right)^2$). The number $\widehat{q} \in (q, \infty)$ is a solutions to some transcendental equation involving q and L. In the last instance we are able to make the following useful conclusion: the spectrum of $\mathcal{H}^\varepsilon$ has a gap provided ε is small enough, the edges of this gap converge to q and $\widehat{q}$.

In the current paper we study the asymptotic behaviour of the "pure" Neumann Laplacian (i.e., $\varrho^\varepsilon = 1$), but now, in contrast to [12], the "basic" strip $\Omega = \Pi^\varepsilon$ also depends on ε. Since Ω^ε shrinks to $\mathbb{R}$ as $\varepsilon \to 0$ it is natural to expect that $-\Delta_{\Omega^\varepsilon}$ converges (in suitable sense) to some ordinary differential operator on the line. It turns out that the form of this operator depends on $q = \lim_{\varepsilon \to 0} \frac{d^\varepsilon}{h^\varepsilon \varepsilon^2 |B|} \in [0, \infty]$.

Boundary value and spectral problems on *thin domains with oscillating boundary* were studied in a lot works – see, e.g., [2, 18, 32, 33] and references therein. In these papers the authors deal with thin domains, whose boundary (or its part) has the form of a graphic of some smooth oscillating function (for example, $\Omega^\varepsilon = \left\{ \mathbf{x} \in \mathbb{R}^2 : -\varepsilon < x_2 < \varepsilon \varphi^\varepsilon(x_1) \right\}$, where $\varepsilon > 0$ is again a small parameter, $\varphi^\varepsilon(x) = \varphi(x, x\varepsilon^{-\alpha})$, $\alpha > 0$, $\varphi(x, y)$ is a smooth positive function, periodic with respect to y). More general geometries were treated in [3], where thin strip is perturbed by attaching small protuberances, ε^α-periodically along it boundary; the protuberances are obtained from a fixed bounded domain by ε^α-rescaling in x_1 direction and ε-rescaling in x_2 direction. In [22, 23], besides an oscillating external boundary, additional internal holes are allowed.

At first, we study the resolvent equation

$$-\Delta_{\Omega^\varepsilon} u^\varepsilon + \mu u^\varepsilon = f^\varepsilon, \ \mu > 0. \tag{1.1}$$

We prove (see Theorems 2.1-2.2) that under some natural assumptions on f^ε the solution u^ε to the problem (1.1) converges in a suitable sense to the solution of the following problem on the line:

$$-u''(x) + \mathcal{V}(\mu)u(x) = \mathcal{F}(\mu, x). \tag{1.2}$$

The functions $\mathcal{V}(\mu)$, $\mathcal{F}(\mu, x)$ are either linear ($q = \infty$) or rational ($q < \infty$) functions of μ. In the later case they both have one pole – at the point q.

Problem (1.2) can be associated with a resolvent equation for some self-adjoint operator $\mathcal{H}$ acting in $[L_2(\mathbb{R})]^2$. The spectrum of this operator has the form

$$\sigma(\mathcal{H}) = [0, \infty) \setminus (q, q + q|B|) \tag{1.3}$$

provided $q > 0$, otherwise $\sigma(\mathcal{H}) = [0, \infty)$.

Our second result concerns the spectral convergence in the most interesting case $q < \infty$. Periodicity of Ω^ε leads to the band structure of $\sigma(-\Delta_{\Omega^\varepsilon})$, i.e. $\sigma(-\Delta_{\Omega^\varepsilon})$ is a locally finite union of compact intervals called *bands*. In general they may overlap, otherwise we have a *gap* in the spectrum – a bounded open interval having an empty intersection with the spectrum but with ends belonging to it.

We prove (see Theorem 2.3) that the spectrum of $-\Delta_{\Omega^\varepsilon}$ converges as $\varepsilon \to 0$ to the spectrum of $\mathcal{H}$ in the Hausdorff sense. This means that $\sigma(-\Delta_{\Omega^\varepsilon})$ has a gap provided ε is small enough; when $\varepsilon \to 0$ this gap converges to the interval $(q, q + q|B|)$. Moreover, we show (see Lemma 2.1) that other gaps (if any) "escape" from any finite interval when $\varepsilon \to 0$.

Theorems 2.1-2.3 remain valid if Ω is a bounded strip, cf. Remark 2.3 below.

Note, that using the same ideas one can also construct a waveguide with several gaps. Namely, if we attach to Π^ε $\quad m \in \mathbb{N}$ different families of "room-and-passage" domains we will arrive at the same limit problem (1.2), but the functions $\mathcal{V}(\mu)$, $\mathcal{F}(\mu, x)$ will have m poles. The corresponding operator $\mathcal{H}$

will act in $[L_2(\mathbb{R})]^{m+1}$ and have m gaps. The proof relies on the same methods as in the case $m = 1$, but is more cumbersome.

The possibility to open up gaps in the spectrum of periodic differential operators is important from the point of view of applications, in particular, to the so-called photonic crystals – periodic nanostructures that have been attracting much attention in recent years. The characteristic property of photonic crystals is that the light waves at certain optical frequencies fail to propagate in them, which is caused by gaps in the spectrum of the Maxwell operator or related scalar operators. Pioneer mathematical results justifying the opening of spectral gaps for some $2D$ dielectric media were obtained in [16]. We refer to the overview [21] and the book [15] concerning mathematical problems arising in this field.

As we already mentioned, in general, the presence of gaps is not guaranteed – for instance, if Ω is a straight unbounded strip then the spectrum of the Laplace operator is a ray $[\Lambda, \infty)$, where $\Lambda = 0$ for the Neumann Laplacian and $\Lambda > 0$ for the Dirichlet Laplacian. There exist several approaches how to construct a periodic waveguide-like domain with non-void gaps in the spectrum of the Laplace operator on this domain subject to Neumann or Dirichlet boundary conditions. The simplest way is to consider the waveguide consisting of an array of identical compact domains connected by thin passages or windows – see, e.g., [5, 9, 29]. In this case the spectrum typically has small bands separated by relatively large gaps. The opposite picture (i.e., large bands versus small gaps) occurs under "small" perturbations of straight waveguides (see [4, 14, 25, 27, 28]) — either by a periodic nucleation of small holes or by a gentle periodic bending of the boundary. The waveguide Ω^ε constructed in the current paper falls into an intermediate case – the length of the first band is comparable with the length of the first gap. Moreover, in contrast to [12], both edges of this gap depends on geometric properties of the waveguide in a very simple fashion.

Thin periodic waveguides of constant width were treated in [35]. It was proved that the Dirichlet Laplacian on such a waveguide always has at least one gap provided the signed curvature of the boundary curve is smooth and non-constant and the waveguide is thin enough. The opening of spectral gaps for the Dirichlet Laplacian on the waveguide of the form $\left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < \varepsilon h(x_1) \right\}$, where $h(x)$ is a positive periodic function, was established in [17] under a suitable assumptions on h. The waveguide consisting of two parallel strips coupled through a period family of thin windows was studied in [7]. Finally, we mention the papers [8, 13, 19, 24, 26] where the same type problems were addressed for more general elliptic selfadjoint operators, and [6], where the Steklov spectral problem was considered.

The paper is organized as follows. In Section 2 we set up the problem and formulate the main results. We prove resolvent convergence of $-\Delta_{\Omega^\varepsilon}$ in Sections 3 ($q < \infty$) and 4 ($q = \infty$). In Section 5 we prove Hausdorff convergence of the spectrum. Finally, in Section 6 we show that $-\Delta_{\Omega^\varepsilon}$ has at most one gap on finite intervals provided ε is small enough.

2. Setting of the problem and main results

Let $\varepsilon > 0$ be a small parameter, and $d^\varepsilon, h^\varepsilon$ be positive numbers depending on ε and satisfying

$$\lim_{\varepsilon \to 0} \frac{d^\varepsilon}{\varepsilon} = 0, \tag{2.1}$$

$$\lim_{\varepsilon \to 0} h^\varepsilon = 0, \tag{2.2}$$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln d^\varepsilon = 0. \tag{2.3}$$

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Condition (2.3) is rather technical, one needs it to have a better control on the behaviour of functions from $H^1(\Omega^\varepsilon)$ near the bottom and the top of the passages (see Lemma 3.1 below).

Hereinafter by $\mathbf{x} = (x_1, x_2)$ we denote the points in $\mathbb{R}^2$, by x we denote the points in $\mathbb{R}$. Further, we introduce the following sets (below $j \in \mathbb{Z}$):

- $\Pi^\varepsilon = \left\{ \mathbf{x} \in \mathbb{R}^2 : -\varepsilon < x_2 < 0 \right\}$,

- $T_j^\varepsilon = \left\{ \mathbf{x} \in \mathbb{R}^2 : |x_1 - x_j^\varepsilon| < \dfrac{d^\varepsilon}{2}, \ 0 \le x_2 \le h^\varepsilon \right\}$, where $x_j^\varepsilon = \varepsilon(j + 1/2)$,

- $B_j^\varepsilon = \left\{ \mathbf{x} \in \mathbb{R}^2 : \mathbf{x} - \widetilde{\mathbf{x}}_j^\varepsilon \in \varepsilon B \right\}$, where $\widetilde{\mathbf{x}}_j^\varepsilon = (x_j^\varepsilon, h^\varepsilon) \in \mathbb{R}^2$, $B \subset \mathbb{R}^2$ is an open bounded domain with Lipschitz boundary such that

$$B \subset \left\{ \mathbf{x} \in \mathbb{R}^2 : |x_1| < \frac{1}{2}, \ x_2 > 0 \right\}, \tag{2.4}$$

$$\exists R \in \left(0, \frac{1}{2}\right) : \ \left\{ \mathbf{x} \in \mathbb{R}^2 : |x_1| < R, \ x_2 = 0 \right\} \subset \partial B. \tag{2.5}$$

By virtue of the condition (2.4) the neighbouring "rooms" B_j^ε are pairwise disjoint. Condition (2.5) together with (2.1) imply the correct gluing of the j-th "room" and the j-th "passage", namely the upper face of T_j^ε is contained in ∂B_j^ε.

Finally, we define the waveguide Ω^ε as a union of the straight strip Π^ε and ε-periodically attached "room-and-passage" protuberances $B_j^\varepsilon \cup T_j^\varepsilon$ (see Figure 1):

$$\Omega^\varepsilon = \Pi^\varepsilon \cup \left(\bigcup_{j \in \mathbb{Z}} B_j^\varepsilon \cup T_j^\varepsilon \right).$$

We denote by $\mathcal{H}^\varepsilon$ the Neumann Laplacian in $L_2(\Omega^\varepsilon)$ – the self-adjoint and positive operator associated with the sesquilinear form $\mathfrak{h}^\varepsilon$,

$$\mathfrak{h}^\varepsilon[u, v] = \int_{\Omega^\varepsilon} \nabla u(\mathbf{x}) \cdot \overline{\nabla v(\mathbf{x})} \, d\mathbf{x}, \quad \mathrm{dom}(\mathfrak{h}^\varepsilon) = H^1(\Omega^\varepsilon)$$

(i.e., $(\mathcal{H}^\varepsilon u, v)_{L_2(\Omega^\varepsilon)} = \mathfrak{h}^\varepsilon[u, v]$, $\forall u \in \mathrm{dom}(\mathcal{H}^\varepsilon)$, $\forall v \in \mathrm{dom}(\mathfrak{h}^\varepsilon)$).

Our first goal is to describe the behaviour of the resolvent $(\mathcal{H}^\varepsilon + \mu I)^{-1}$, $\mu > 0$ as $\varepsilon \to 0$ under the assumption that the following limit q, either finite or infinite, exists:

$$\lim_{\varepsilon \to 0} \frac{d^\varepsilon}{h^\varepsilon \varepsilon^2 |B|} = q \in [0, \infty]. \tag{2.6}$$

Note, that if $q < \infty$ then (2.1) follows automatically from (2.2), (2.6).

Before to formulate the result we need to introduce auxiliary operators.

We define the operator $J_1^\varepsilon : L_2(\Pi^\varepsilon) \to L_2(\mathbb{R})$ by the formula

$$(J_1^\varepsilon u)(x) = \frac{1}{\sqrt{\varepsilon}} \int_{-\varepsilon}^0 u(\mathbf{x}) \, dx_2, \quad \text{where } \mathbf{x} = (x, x_2). \tag{2.7}$$

Also we introduce the operator $J_2^\varepsilon : L_2(\bigcup_{j \in \mathbb{Z}} B_j^\varepsilon) \to L_2(\mathbb{R})$ by

$$(J_2^\varepsilon u)(x) = \sum_{j \in \mathbb{Z}} \left(\frac{1}{\sqrt{\varepsilon|B_j^\varepsilon|}} \int_{B_j^\varepsilon} u(\mathbf{x}) \, d\mathbf{x} \right) \chi_j(x), \tag{2.8}$$

where $\chi_j(x)$ is the indicator function of the interval $\left(x_j^\varepsilon - \frac{\varepsilon}{2}, x_j^\varepsilon + \frac{\varepsilon}{2}\right)$.

Using the Cauchy-Schwarz inequality we get

$$\forall f \in L_2(\Omega^\varepsilon) : \quad \|J_1^\varepsilon f\|_{L_2(\mathbb{R})} \le \|f\|_{L_2(\Pi^\varepsilon)}, \quad \|J_2^\varepsilon f\|_{L_2(\mathbb{R})} \le \|f\|_{L_2(\bigcup_{j \in \mathbb{Z}} B_j^\varepsilon)}. \tag{2.9}$$

Moreover, it is easy to show that $J_1^\varepsilon u \in H^1(\mathbb{R})$ provided $u \in H^1(\Omega^\varepsilon)$ and the following Poincaré-type estimates are valid:

$$\|u\|_{L_2(\Pi^\varepsilon)}^2 \le \|J_1^\varepsilon u\|_{L_2(\mathbb{R})}^2 + C\varepsilon^2 \|\nabla u\|_{L_2(\Pi^\varepsilon)}^2, \tag{2.10}$$

$$\|u\|_{L_2(\bigcup_j B_j^\varepsilon)}^2 \le \|J_2^\varepsilon u\|_{L_2(\mathbb{R})}^2 + C\varepsilon^2 \|\nabla u\|_{L_2(\bigcup_j B_j^\varepsilon)}^2. \tag{2.11}$$

Hereinafter by $C, C_1, C_2, \ldots$ we denote generic constants which do not depend on ε. Inequalities (2.9)-(2.11) mean that $J_1^\varepsilon, J_2^\varepsilon$ are "almost" isometries (as $\varepsilon \ll 1$).

We are now in position to formulate the first result; it deals with the most interesting case $q < \infty$. Below, as usual, $\rightharpoonup$ denotes the weak convergence (in an appropriate space).

Theorem 2.1. *Let $q < \infty$. Let $\{f^\varepsilon\}_\varepsilon$ be a family of functions from $L_2(\Omega^\varepsilon)$ satisfying*

$$\|f^\varepsilon\|_{L_2(\Omega^\varepsilon)} \le C, \quad J_1^\varepsilon f^\varepsilon \rightharpoonup f_1 \text{ in } L_2(\mathbb{R}), \quad J_2^\varepsilon f^\varepsilon \rightharpoonup f_2 \text{ in } L_2(\mathbb{R}) \text{ as } \varepsilon \to 0, \tag{2.12}$$

where $f_1, f_2 \in L_2(\mathbb{R})$. We set $u^\varepsilon = (\mathcal{H}^\varepsilon + \mu I)^{-1} f^\varepsilon$, $\mu > 0$.

Then

$$J_1^\varepsilon u^\varepsilon \rightharpoonup u_1 \text{ in } H^1(\mathbb{R}) \text{ as } \varepsilon \to 0,$$

where the function u_1 belongs to $H^2(\mathbb{R})$ and is a solution of the problem

$$-u_1'' + \mu \left(1 + \frac{q|B|}{q + \mu} \right) u_1 = f_1 + \frac{q|B|^{1/2}}{q + \mu} f_2. \tag{2.13}$$

Moreover

$$J_2^\varepsilon u^\varepsilon \rightharpoonup u_2 = \frac{q|B|^{1/2}}{q + \mu} u_1 + \frac{1}{q + \mu} f_2 \text{ in } L_2(\mathbb{R}).$$

Remark 2.1. The typical example of a family $\{f^\varepsilon\}_\varepsilon$ satisfying (2.12) is as follows. Let $F = (f_1, f_2) \in [L_2(\mathbb{R})]^2$ be an arbitrary function. We introduce the function f^ε by

$$f^\varepsilon(\mathbf{x}) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}} f_1(x_1), & \mathbf{x} \in \Pi^\varepsilon, \\[2mm] 0, & \mathbf{x} \in T_j^\varepsilon, \\[2mm] \dfrac{1}{\sqrt{\varepsilon|B_j^\varepsilon|}} \displaystyle\int_{-\varepsilon/2 + x_j^\varepsilon}^{\varepsilon/2 + x_j^\varepsilon} f_2(x) \, dx, & \mathbf{x} \in B_j^\varepsilon. \end{cases} \tag{2.14}$$

It is easy to show that f^ε satisfies (2.12).

Theorem 2.1 can be rewritten by assigning to the problem (2.13) some self-adjoint and positive operator. Namely, we introduce the operator $\mathcal{H}$ acting in $[L_2(\mathbb{R})]^2$ by

$$\mathcal{H}U = \begin{pmatrix} -\mathrm{d}^2/\mathrm{d}x^2 + q|B| & -q|B|^{1/2} \\ -q|B|^{1/2} & q \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad U = (u_1, u_2), \quad \mathrm{dom}(\mathcal{H}) = H^2(\mathbb{R}) \times L_2(\mathbb{R}). \qquad (2.15)$$

It is straightforward to show that u_1 solves (2.13) and $u_2 = \frac{q|B|^{1/2}}{q+\mu} u_1 + \frac{1}{q+\mu} f_2$ iff

$$\mathcal{H}U + \mu U = F, \quad \text{where } U = (u_1, u_2), \ F = (f_1, f_2).$$

Thus Theorem 2.1 claims

$$J^\varepsilon (\mathcal{H}^\varepsilon + \mu I)^{-1} f^\varepsilon \rightharpoonup (\mathcal{H} + \mu I)^{-1} F \text{ as } \varepsilon \to 0 \quad \text{provided } J^\varepsilon f^\varepsilon \rightharpoonup F \text{ in } [L_2(\mathbb{R})]^2,$$

where $J^\varepsilon = (J_1^\varepsilon, J_2^\varepsilon) : H^1(\Omega^\varepsilon) \times L_2(\Omega^\varepsilon) \to H^1(\mathbb{R}) \times L_2(\mathbb{R})$.

In the case $q = \infty$ one has the following result.

Theorem 2.2. *Let $q = \infty$. Let $\{f^\varepsilon\}_\varepsilon$ be a family of functions from $\in L_2(\Omega^\varepsilon)$ satisfying (2.12). We set $u^\varepsilon = (\mathcal{H}^\varepsilon + \mu I)^{-1} f^\varepsilon$, $\mu > 0$. Then*

$$J_1^\varepsilon u^\varepsilon \rightharpoonup u_1 \text{ in } H^1(\mathbb{R}), \quad J_2^\varepsilon u^\varepsilon \rightharpoonup u_2 \text{ in } L_2(\mathbb{R}) \quad \text{as } \varepsilon \to 0,$$

where $u_1 \in H^2(\mathbb{R})$, $u_2 = |B|^{1/2} u_1$ and

$$-u_1'' + \mu (1 + |B|) u_1 = f_1 + |B|^{1/2} f_2. \qquad (2.16)$$

In what follows we consider the case $q > 0$ only. Our next goal is be to describe the behaviour of $\sigma(\mathcal{H}^\varepsilon)$ as $\varepsilon \to 0$.

Theorem 2.3. *Let $L > 0$ be an arbitrary number. Then*

$$\mathrm{dist}_H \left(\sigma(\mathcal{H}^\varepsilon) \cap [0, L], \ \sigma(\mathcal{H}) \cap [0, L] \right) \to 0 \text{ as } \varepsilon \to 0, \qquad (2.17)$$

where $\mathrm{dist}_H(\cdot, \cdot)$ stays for the Hausdorff distance between two sets. [2]

Remark 2.2. The claim of Theorem 2.3 is equivalent to the fulfilment of the following conditions:

(i) Let the family $\{\lambda^\varepsilon \in \sigma(\mathcal{H}^\varepsilon)\}_\varepsilon$ have a convergent subsequence, i.e. $\lambda^\varepsilon \to \lambda$ as $\varepsilon = \varepsilon_k \to 0$. Then $\lambda \in \sigma(\mathcal{H})$.

(ii) Let $\lambda \in \sigma(\mathcal{H})$. Then there exists a family $\{\lambda^\varepsilon \in \sigma(\mathcal{H}^\varepsilon)\}_\varepsilon$ such that $\lim\limits_{\varepsilon \to 0} \lambda^\varepsilon = \lambda$.

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[2] For two compact sets $X, Y \subset \mathbb{R}$ one has: $\mathrm{dist}_H(X, Y) = \max \left\{ \sup\limits_{x \in X} \inf\limits_{y \in Y} |x - y|; \sup\limits_{y \in Y} \inf\limits_{x \in X} |y - x| \right\}$.

It is easy to see that the spectrum of $\mathcal{H}$ has the following form:

$$\sigma(\mathcal{H}) = [0, \infty) \setminus (q, q + q|B|). \tag{2.18}$$

Indeed, for $\lambda \neq q$ the resolvent equation $\mathcal{H}U - \lambda U = F$ is equivalent to

$$-u_1'' - \rho(\lambda)u_1 = f_1 + \frac{q|B|^{1/2}}{q - \lambda}f_2, \quad u_2 = \frac{q|B|^{1/2}}{q - \lambda}u_1 + \frac{1}{q - \lambda}f_1, \quad \text{where } \rho(\lambda) = \lambda\left(1 + \frac{q|B|}{q - \lambda}\right),$$

whence, evidently, $\lambda \in \sigma(\mathcal{H}) \setminus \{q\}$ iff $\rho(\lambda) \in \sigma(-\frac{d^2}{dx^2}|_{L_2(\mathbb{R})}) = [0, \infty)$. But $\rho(\lambda) \in [0, \infty)$ iff $\lambda \in [0, \infty) \setminus [q, q + q|B|)$. Finally, $q \in \sigma(\mathcal{H})$ since $\sigma(\mathcal{H})$ is a closed set.

Remark 2.3. Theorems 2.1-2.3 (after a natural reformulation) remain valid if Ω is a *bounded* strip: $\Omega = (0, l) \times (-\varepsilon, 0)$, $l > 0$. In this case $\sigma(\mathcal{H}^\varepsilon)$ is purely discrete. The limit problems (2.13) and (2.16) are now considered on $(0, l)$ with Neumann conditions at the endpoints. If $q < \infty$ the spectrum of the corresponding limit operator $\mathcal{H}$ has the form

$$\sigma(\mathcal{H}) = \left\{\lambda_k^-, \ k = 1, 2, 3 \dots\right\} \cup \left\{\lambda_k^+, \ k = 1, 2, 3 \dots\right\} \cup \{q\},$$

where the point $\lambda_k^\pm$ belong to the discrete spectrum, q is the only point of the essential spectrum and

$$\lambda_1^- = 0, \quad \lambda_1^+ = q + q|B|, \quad \lambda_k^- \nearrow q, \quad \lambda_k^+ \nearrow \infty \quad \text{as } k \to \infty.$$

The proofs repeat word-by-word the proofs for the unbounded case.

From Theorem 2.3 and (2.18) we conclude that $\sigma(\mathcal{H}^\varepsilon)$ has at least one gap provided ε is small enough. Moreover, there is a gap converging to the interval $(q, q + q|B|)$ as $\varepsilon \to 0$. Unfortunately, Hausdorff convergence provides no information on the upper bound for the number of gaps, even within finite intervals: for example, the set $\sigma^\varepsilon := [0, L] \cap \left(\bigcup_{k \in \mathbb{N}} \left[\varepsilon k, \varepsilon(k + \frac{1}{2})\right]\right)$ converges to $[0, L]$ in the Hausdorff sense, but the number of gaps in σ^ε tends to infinity as $\varepsilon \to 0$. Nevertheless, for our problem one can say more, namely, the following lemma take place.

Lemma 2.1. *Within an arbitrary compact interval $[0, L]$ the spectrum of $\mathcal{H}^\varepsilon$ has at most one gap provided ε is small enough.*

Combining Theorem 2.3 and Lemma 2.1, we arrive at the main result of this work.

Theorem 2.4. *Let $L > 0$ be an arbitrary number. Then the spectrum of the operator $\mathcal{H}^\varepsilon$ in $[0, L]$ has the following structure for ε small enough:*

$$\sigma(\mathcal{H}^\varepsilon) \cap [0, L] = [0, L] \setminus (\alpha^\varepsilon, \beta^\varepsilon),$$

where the endpoints of the interval $(\alpha^\varepsilon, \beta^\varepsilon)$ satisfy

$$\lim_{\varepsilon \to 0} \alpha^\varepsilon = q, \quad \lim_{\varepsilon \to 0} \beta^\varepsilon = q + q|B|. \tag{2.19}$$

Theorems 2.1-2.3 as well as Lemma 2.1 will be proven in the next sections.

At the end of this section we introduce several notations, which further will be frequently used:

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- $Y_j^\varepsilon = \left\{ \mathbf{x} \in \Pi^\varepsilon : |x_1 - x_j^\varepsilon| < \frac{\varepsilon}{2} \right\}$,

- $S_j^\varepsilon = \left\{ \mathbf{x} \in \partial T_j^\varepsilon : x_2 = 0 \right\}$,

- $C_j^\varepsilon = \left\{ \mathbf{x} \in \partial T_j^\varepsilon : x_2 = h^\varepsilon \right\}$.

The notation $\langle u \rangle_D$ stays for the mean value of the function $u(\mathbf{x})$ in the domain D, i.e.

$$\langle u \rangle_D = \frac{1}{|D|} \int_D u(\mathbf{x}) \, d\mathbf{x}.$$

Also we keep the same notation if D is a segment (e.g., S_j^ε). In this case we integrate with respect to the natural coordinate on this segment, $|D|$ is its length.

3. Proof of Theorem 2.1

Let $\{f^\varepsilon\}_\varepsilon$ be a family of functions from $L_2(\Omega^\varepsilon)$ satisfying (2.12), $u^\varepsilon = (\mathcal{H}^\varepsilon + \mu I)^{-1} f^\varepsilon$, $\mu > 0$. One has the following standard estimates:

$$\|u^\varepsilon\|_{L_2(\Omega^\varepsilon)} \leq \frac{1}{\mu} \|f^\varepsilon\|_{L_2(\Omega^\varepsilon)}, \quad \|\nabla u^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2 \leq \|f^\varepsilon\|_{L_2(\Omega^\varepsilon)} \|u^\varepsilon\|_{L_2(\Omega^\varepsilon)},$$

whence, taking into account that $\|f^\varepsilon\|_{L_2(\Omega^\varepsilon)} \leq C$, we obtain

$$\|u^\varepsilon\|_{H^1(\Omega^\varepsilon)}^2 \leq C_1. \tag{3.1}$$

Recall that the operators J_j^ε, $j = 1, 2$ satisfy (2.9), moreover, changing the order of integration with respect to x_2 and differentiation with respect to x_1, one can easily prove that for $u \in H^1(\Omega^\varepsilon)$

$$\|(J_1^\varepsilon u^\varepsilon)'\|_{L_2(\mathbb{R})}^2 \leq \|\partial_{x_1} u^\varepsilon\|_{L_2(\Pi^\varepsilon)}^2 \leq \|\nabla u^\varepsilon\|_{L_2(\Omega^\varepsilon)}^2. \tag{3.2}$$

Then it follows from (2.9) (applied for u^ε), (3.2) that the families $\left\{J_1^\varepsilon u^\varepsilon\right\}_\varepsilon$ and $\left\{J_2^\varepsilon u^\varepsilon\right\}_\varepsilon$ are uniformly bounded in $H^1(\mathbb{R})$ and $L_2(\mathbb{R})$, respectively, and therefore there are a subsequence (for convenience, still indexed by ε) and $u_1 \in H^1(\mathbb{R})$, $u_2 \in L_2(\mathbb{R})$ such that

$$J_1^\varepsilon u^\varepsilon \rightharpoonup u_1 \text{ in } H^1(\mathbb{R}), \quad J_2^\varepsilon u^\varepsilon \rightharpoonup u_2 \text{ in } L_2(\mathbb{R}) \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

Now, let us write the variational formulation of the resolvent equation (1.1):

$$\int_{\Omega^\varepsilon} \left(\nabla u^\varepsilon \cdot \nabla w + \mu u^\varepsilon w \right) d\mathbf{x} = \int_{\Omega^\varepsilon} f^\varepsilon w \, d\mathbf{x}, \ \forall w \in H^1(\Omega^\varepsilon). \tag{3.4}$$

Our strategy will be to plug into (3.4) a specially chosen test-function $w = w^\varepsilon$ and then pass to the limit as $\varepsilon \to 0$ hoping to arrive at the equality $\mathcal{H}U + \mu U = F$ written in a weak form, where $\mathcal{H}$ is defined by (2.15), $U = (u_1, u_2)$ and $F = (f_1, f_2)$.

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We choose this test-function as follows (below, as usual, $\mathbf{x} = (x_1, x_2)$):

$$
w = w^\varepsilon(\mathbf{x}) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}}\left(w_1(x_1) + \sum_{j\in\mathbb{Z}}(w_1(x_j^\varepsilon) - w_1(x_1))\varphi_j^\varepsilon(\mathbf{x})\right), & \mathbf{x}\in\Pi^\varepsilon, \\[2ex] \dfrac{1}{h^\varepsilon\sqrt{\varepsilon}}\left(w_2(x_j^\varepsilon) - w_1(x_j^\varepsilon)\right)x_2 + \dfrac{w_1(x_j^\varepsilon)}{\sqrt{\varepsilon}}, & \mathbf{x}\in T_j^\varepsilon, \\[2ex] \dfrac{1}{\sqrt{\varepsilon}}w_2(x_j^\varepsilon), & \mathbf{x}\in B_j^\varepsilon. \end{cases} \tag{3.5}
$$

Here $w_1, w_2 \in C_0^1(\mathbb{R})$ are arbitrary functions, $\varphi_j^\varepsilon(\mathbf{x}) = \varphi\left(\frac{|x_1 - x_j^\varepsilon|}{d^\varepsilon}\right)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth functions satisfying $\varphi(t) = 1$ as $t \le 1$ and $\varphi(t) = 0$ as $t \ge 2$. Obviously $w^\varepsilon \in H^1(\Omega^\varepsilon)$ provided $d^\varepsilon \le \frac{\varepsilon}{4}$ (this holds for ε small enough, see (2.1)).

We denote:

$$
\mathcal{J}^\varepsilon = \left\{ j\in\mathbb{Z} : \ x_j^\varepsilon \in \operatorname{supp}(w_1) \cup \operatorname{supp}(w_2)\right\}.
$$

It is clear that

$$
\sum_{j\in\mathcal{J}^\varepsilon} 1 \le C\varepsilon^{-1}. \tag{3.6}
$$

Let us plug $w = w^\varepsilon(\mathbf{x})$ into (3.4). Since $\operatorname{supp}(\varphi_j^\varepsilon) \subset \overline{Y_j^\varepsilon}$ and $w^\varepsilon = \text{const.}$ in B_j^ε we obtain from (3.4):

$$
\underbrace{\varepsilon^{-\frac{1}{2}}\int_{\Pi^\varepsilon}\left(\nabla u^\varepsilon(\mathbf{x})\cdot\nabla w_1(x_1) + \mu u^\varepsilon(\mathbf{x})w_1(x_1)\right)d\mathbf{x}}_{I_1}
$$

$$
+ \underbrace{\varepsilon^{-\frac{1}{2}}\sum_{j\in\mathbb{Z}}\int_{Y_j^\varepsilon}\left(\nabla u^\varepsilon(\mathbf{x})\cdot\nabla\left((w_1(x_j^\varepsilon) - w_1(x_1))\varphi_j^\varepsilon(\mathbf{x})\right)\right) + \mu u^\varepsilon(w_1(x_j^\varepsilon) - w_1(x_1))\varphi_j^\varepsilon(\mathbf{x})\right)d\mathbf{x}}_{I_2}+
$$

$$
+ \underbrace{\sum_{j\in\mathbb{Z}}\int_{T_j^\varepsilon}\nabla u^\varepsilon(\mathbf{x})\cdot\nabla w^\varepsilon(\mathbf{x})d\mathbf{x}}_{I_3} + \mu\underbrace{\sum_{j\in\mathbb{Z}}\int_{T_j^\varepsilon}u^\varepsilon(\mathbf{x})w^\varepsilon(\mathbf{x})d\mathbf{x}}_{I_4} + \underbrace{\varepsilon^{-\frac{1}{2}}\mu\sum_{j\in\mathbb{Z}}w_2(x_j^\varepsilon)\int_{B_j^\varepsilon}u^\varepsilon(\mathbf{x})d\mathbf{x}}_{I_5} =
$$

$$
= \underbrace{\varepsilon^{-\frac{1}{2}}\int_{\Pi^\varepsilon}f^\varepsilon(\mathbf{x})w_1(x_1)d\mathbf{x}}_{I_6} + \underbrace{\varepsilon^{-\frac{1}{2}}\sum_{j\in\mathbb{Z}}\int_{Y_j^\varepsilon}f^\varepsilon(\mathbf{x})(w_1(x_j^\varepsilon) - w_1(x_1))\varphi_j^\varepsilon(\mathbf{x})d\mathbf{x}}_{I_7}
$$

$$
+ \underbrace{\sum_{j\in\mathbb{Z}}\int_{T_j^\varepsilon}f^\varepsilon(\mathbf{x})w^\varepsilon(\mathbf{x})d\mathbf{x}}_{I_8} + \underbrace{\varepsilon^{-\frac{1}{2}}\sum_{j\in\mathbb{Z}}w_2(x_j^\varepsilon)\int_{B_j^\varepsilon}f^\varepsilon(\mathbf{x})d\mathbf{x}}_{I_9}. \tag{3.7}
$$

Let us analyse step-by-step the terms I_j, $j = 1,\dots,9$.

(I_1) Using (3.3) and the definition of the operator J_1^ε we obtain:

$$I_1 = \varepsilon^{-\frac{1}{2}} \int\limits_{-\infty}^{\infty} \int\limits_{-\varepsilon}^{0} \left(\frac{\partial u^\varepsilon}{\partial x_1}(x_1, x_2) \frac{dw_1}{dx_1}(x_1) + \mu u^\varepsilon w_1 \right) dx_2 \, dx_1$$

$$= \int\limits_{\mathbb{R}} ((J_1^\varepsilon u^\varepsilon)' w_1' + \mu (J_1^\varepsilon u^\varepsilon) w_1) dx \underset{\varepsilon \to 0}{\to} \int\limits_{\mathbb{R}} (u_1' w_1' + \mu u_1 w_1) dx. \quad (3.8)$$

(I_2) One has the following properties (below φ_j^ε is regarded as a function of $x \in \mathbb{R}$):

$$\mathrm{supp}\left(\left(w_1(x_j^\varepsilon) - w_1 \right) \varphi_j^\varepsilon \right) \subset \left\{ x \in \mathbb{R} : |x - x_j^\varepsilon| < 2d^\varepsilon \right\}, \quad \left| \left(\left(w_1(x_j^\varepsilon) - w_1 \right) \varphi_j^\varepsilon \right)' \right| + \left| \left(w_1(x_j^\varepsilon) - w_1 \right) \varphi_j^\varepsilon \right| \le C.$$

Using them, (2.1), (3.1) and (3.6) we obtain easily:

$$|I_2| \le C\varepsilon^{-\frac{1}{2}} \|u^\varepsilon\|_{H^1(\Pi^\varepsilon)} \sqrt{\sum_{j \in \mathcal{J}^\varepsilon} \varepsilon d^\varepsilon} \le C_1 \sqrt{\frac{d^\varepsilon}{\varepsilon}} \underset{\varepsilon \to 0}{\to} 0. \quad (3.9)$$

(I_3) Integrating by parts and taking into account that $\Delta w^\varepsilon = 0$ in T_j^ε, we obtain:

$$I_3 = \sum_{j \in \mathbb{Z}} \int\limits_{x_j^\varepsilon - \frac{d^\varepsilon}{2}}^{x_j^\varepsilon + \frac{d^\varepsilon}{2}} \left(u^\varepsilon(x_1, h^\varepsilon) \frac{\partial w^\varepsilon}{\partial x_2}(x_1, h^\varepsilon) - u^\varepsilon(x_1, 0) \frac{\partial w^\varepsilon}{\partial x_2}(x_1, 0) \right) dx_1$$

$$= \frac{d^\varepsilon}{h^\varepsilon \sqrt{\varepsilon}} \sum_{j \in \mathbb{Z}} \left(\langle u^\varepsilon \rangle_{S_j^\varepsilon} - \langle u^\varepsilon \rangle_{C_j^\varepsilon} \right) (w_1(x_j^\varepsilon) - w_2(x_j^\varepsilon)). \quad (3.10)$$

Let us introduce the operator $Q^\varepsilon : C_0^1(\mathbb{R}) \to L_2(\mathbb{R})$ by

$$(Q^\varepsilon w)(x) = \sum_{j \in \mathbb{Z}} w(x_j^\varepsilon) \chi_j^\varepsilon(x)$$

(recall that $\chi_j(x)$ is the indicator function of the interval $\left(x_j^\varepsilon - \frac{\varepsilon}{2}, x_j^\varepsilon + \frac{\varepsilon}{2} \right)$). It is easy to show that

$$\forall w \in C^1(\mathbb{R}) : \quad Q^\varepsilon w \underset{\varepsilon \to 0}{\to} w \text{ in } L_2(\mathbb{R}). \quad (3.11)$$

With this operator one can rewrite (3.10) as

$$I_3 = \frac{d^\varepsilon}{h^\varepsilon \sqrt{\varepsilon}} \sum_{j \in \mathbb{Z}} \left(\langle u^\varepsilon \rangle_{Y_j^\varepsilon} - \langle u^\varepsilon \rangle_{B_j^\varepsilon} \right) (w_1(x_j^\varepsilon) - w_2(x_j^\varepsilon)) + \delta(\varepsilon)$$

$$= \frac{d^\varepsilon}{h^\varepsilon \varepsilon^2} \int\limits_{\mathbb{R}} \left(J_1^\varepsilon u^\varepsilon - |B|^{-1/2} J_2 u^\varepsilon \right) (Q^\varepsilon w_1 - Q^\varepsilon w_2) \, dx + \delta(\varepsilon), \quad (3.12)$$

where $\delta(\varepsilon) = \sum_{j \in \mathcal{J}^\varepsilon} \frac{d^\varepsilon}{h^\varepsilon \sqrt{\varepsilon}} \left(\langle u^\varepsilon \rangle_{S_j^\varepsilon} - \langle u^\varepsilon \rangle_{Y_j^\varepsilon} - \langle u^\varepsilon \rangle_{C_j^\varepsilon} + \langle u^\varepsilon \rangle_{B_j^\varepsilon} \right) \left(w_1(x_j^\varepsilon) - w_2(x_j^\varepsilon) \right)$. The last equality in
(3.12) follows simply from the definitions of the operators J_1^ε, J_2^ε, Q^ε.

To estimate the reminder $\delta(\varepsilon)$ we need an additional lemma.

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Lemma 3.1. *One has:*

$$\forall u \in H^1(Y_j^\varepsilon): \quad \left| \langle u \rangle_{S_j^\varepsilon} - \langle u \rangle_{Y_j^\varepsilon} \right| \leq C \sqrt{|\ln d^\varepsilon|} \|\nabla u\|_{L_2(Y_j^\varepsilon)}, \tag{3.13}$$

$$\forall u \in H^1(B_j^\varepsilon): \quad \left| \langle u \rangle_{C_j^\varepsilon} - \langle u \rangle_{B_j^\varepsilon} \right| \leq C \sqrt{|\ln d^\varepsilon|} \|\nabla u\|_{L_2(B_j^\varepsilon)}. \tag{3.14}$$

Proof. For an arbitrary $u \in H^1(Y_j^\varepsilon)$ one has the following estimate (see [11, Lemma 3.1]):

$$\left| \langle u \rangle_{S_j^\varepsilon} - \langle u \rangle_{\Gamma_j^\varepsilon} \right| \leq C_1 \sqrt{|\ln d^\varepsilon|} \|\nabla u\|_{L_2(Y_j^\varepsilon)}, \tag{3.15}$$

where $\Gamma_j^\varepsilon = \left\{ \mathbf{x} \in \partial Y_j^\varepsilon : x_2 = 0 \right\}$. Moreover, using the trace and Poincaré inequalities we obtain

$$\left| \langle u \rangle_{\Gamma_j^\varepsilon} - \langle u \rangle_{Y_j^\varepsilon} \right| = \left| \varepsilon^{-1} \int_{x_j^\varepsilon - \frac{\varepsilon}{2}}^{x_j^\varepsilon + \frac{\varepsilon}{2}} \left(u(x_1, 0) - \langle u \rangle_{Y_j^\varepsilon} \right) dx_1 \right| \leq \varepsilon^{-1/2} \sqrt{\int_{x_j^\varepsilon - \frac{\varepsilon}{2}}^{x_j^\varepsilon + \frac{\varepsilon}{2}} \left| u(x_1, 0) - \langle u \rangle_{Y_j^\varepsilon} \right|^2 dx_1}$$

$$\leq C\varepsilon^{-1/2} \sqrt{\varepsilon \|\nabla u\|_{L_2(Y_j^\varepsilon)}^2 + \varepsilon^{-1} \left\| u - \langle u \rangle_{Y_j^\varepsilon} \right\|_{L_2(Y_j^\varepsilon)}^2} \leq C_1 \|\nabla u\|_{L_2(Y_j^\varepsilon)}. \tag{3.16}$$

Then (3.13) follow from (3.15)-(3.16). The proof of estimate (3.14) is similar (instead of Γ_j^ε one should use the set $\left\{ \mathbf{x} \in \mathbb{R}^2 : x_2 = h^\varepsilon, |x_1 - x_j^\varepsilon| < R\varepsilon \right\}$, R is defined in (2.5)). $\qquad \square$

Remark 3.1. Using similar arguments (cf. [11, Lemma 3.1]) one can also prove the estimate

$$\forall u \in H^1(Y_j^\varepsilon): \quad \|u\|_{L_2(S_j^\varepsilon)}^2 \leq d^\varepsilon \varepsilon^{-2} \|u\|_{L_2(Y_j^\varepsilon)}^2 + C d^\varepsilon |\ln d^\varepsilon| \|\nabla u\|_{L_2(Y_j^\varepsilon)}^2. \tag{3.17}$$

We will use it later in the proof of Theorem 2.3.

Using Lemma 3.1 and taking into account (2.3), (2.6), (3.1) and (3.6) we get:

$$|\delta(\varepsilon)| \leq \frac{d^\varepsilon}{h^\varepsilon \sqrt{\varepsilon}} \sqrt{\sum_{j \in \mathcal{J}^\varepsilon} \left(\left| \langle u^\varepsilon \rangle_{S_j^\varepsilon} - \langle u^\varepsilon \rangle_{Y_j^\varepsilon} \right|^2 + \left| \langle u^\varepsilon \rangle_{B_j^\varepsilon} - \langle u^\varepsilon \rangle_{C_j^\varepsilon} \right|^2 \right)}$$

$$\times \sqrt{\sum_{j \in \mathcal{J}^\varepsilon} \left(\max_{x \in \mathbb{R}} |w_1(x)|^2 + \max_{x \in \mathbb{R}} |w_2(x)|^2 \right)} \leq C \sqrt{\varepsilon^2 |\ln d^\varepsilon|} \frac{d^\varepsilon}{h^\varepsilon \varepsilon^2} \|\nabla u^\varepsilon\|_{L_2(\Omega^\varepsilon)} \underset{\varepsilon \to 0}{\to} 0. \tag{3.18}$$

Combining (3.12) and (3.18) and taking into account (2.6), (3.3), (3.11), we obtain:

$$I_3 \underset{\varepsilon \to 0}{\to} q|B| \int_{\mathbb{R}} \left(u_1 - |B|^{-1/2} u_2 \right) (w_1 - w_2) \, dx. \tag{3.19}$$

(I_4) Taking into account that $\max_{\mathbf{x} \in T_j^\varepsilon} |w^\varepsilon(\mathbf{x})| \leq C\varepsilon^{-1/2}$ and using (2.2), (2.6) and (3.6) we estimate:

$$|I_4| \leq C\|u\|_{L_2(\cup_j T_j^\varepsilon)} \sqrt{\sum_{j \in \mathcal{J}^\varepsilon} |T_j^\varepsilon| \varepsilon^{-1}} \leq C_1 h^\varepsilon \sqrt{\frac{d^\varepsilon}{h^\varepsilon \varepsilon^2}} \underset{\varepsilon \to 0}{\to} 0. \tag{3.20}$$

(I_5) Using (3.3), (3.11) we arrive at

$$I_5 = \mu|B|^{1/2} \int_{\mathbb{R}} (J_2^{\varepsilon} u^{\varepsilon})(Q^{\varepsilon} w_2)\,\mathrm{d}x \underset{\varepsilon\to 0}{\to} \mu|B|^{1/2} \int_{\mathbb{R}} u_2 w_2\,\mathrm{d}x. \tag{3.21}$$

Here the first equality follows simply from the definitions for the operators J_2^{ε} and Q^{ε}.

(I_6)-(I_9) By virtue of arguments similar to those ones in (I_1), (I_2), (I_4), (I_5) and taking into account (2.12) we obtain the following asymptotic behavior for the terms in the right-hand-side of (3.7):

$$I_6^{\varepsilon} \underset{\varepsilon\to 0}{\to} \int_{\mathbb{R}} f_1 w_1\,\mathrm{d}x, \quad I_7^{\varepsilon} \underset{\varepsilon\to 0}{\to} 0, \quad I_8^{\varepsilon} \underset{\varepsilon\to 0}{\to} 0, \quad I_9 \underset{\varepsilon\to 0}{\to} \mu|B|^{1/2} \int_{\mathbb{R}} f_2 w_2\,\mathrm{d}x. \tag{3.22}$$

Finally, combining (3.7)-(3.9), (3.19)-(3.22), we arrive at the equality

$$\int_{\mathbb{R}} u_1' w_1'\,\mathrm{d}x + q|B| \int_{\mathbb{R}} \left(u_1 - |B|^{-1/2} u_2\right)(w_1 - w_2)\,\mathrm{d}x + \mu \int_{\mathbb{R}} \left(u_1 w_1 + |B|^{1/2} u_2 w_2\right)\mathrm{d}x$$

$$= \int_{\mathbb{R}} \left(f_1 w_1 + |B|^{1/2} f_2 w_2\right)\mathrm{d}x, \quad (3.23)$$

which is valid for an arbitrary $w_1, w_2 \in C_0^1(\mathbb{R})$ (and therefore, by the density arguments, for an arbitrary $w_1 \in H^1(\mathbb{R})$ and $w_2 \in L_2(\mathbb{R})$).

Taking $w_1 \equiv 0$ in (3.23) we get $\int_{\mathbb{R}} \left(-q|B|u_1 + q|B|^{1/2} u_2 + \mu|B|^{1/2} u_2 - |B|^{1/2} f_2\right) w_2\,\mathrm{d}x$, $\forall w_2 \in L_2(\mathbb{R})$,

whence

$$u_2 = \frac{q|B|^{1/2}}{q + \mu} u_1 + \frac{1}{q + \mu} f_2. \tag{3.24}$$

Then, taking $w_2 \equiv 0$ in (3.23) and using (3.24), we arrive at

$$\int_{\mathbb{R}} u_1' w_1'\,\mathrm{d}x + \mu \int_{\mathbb{R}} \left(1 + \frac{q|B|}{q + \mu}\right) u_1 w_1\,\mathrm{d}x = \int_{\mathbb{R}} \left(f_1 + \frac{q|B|^{1/2}}{q + \mu} f_2\right) w_1\,\mathrm{d}x, \quad \forall w_1 \in H^1(\mathbb{R}),$$

whence, u_1 belongs to $H^2(\mathbb{R})$ and is a solution to the problem (2.13).

Finally, since the problem (2.13) has the unique solution and u_2 is uniquely determined by u_1 via (3.24), then (3.3) hold for the whole sequence u^{ε}. Theorem 2.1 is proved.

4. Proof of Theorem 2.2

Via the same arguments as in the proof of Theorem 2.1 we conclude that there is a subsequence (for convenience, still indexed by ε) and $u_1 \in H^1(\mathbb{R})$, $u_2 \in L_2(\mathbb{R})$ such that (3.3) holds.

For an arbitrary $w \in H^1(\Omega^{\varepsilon})$ one has the equality (3.4). We plug into this equality the function $w = w^{\varepsilon}(\mathbf{x})$ defined by (3.5), but with $w_1(x) = w_2(x)$. In this case the terms I_3 and I_8 (see (3.7)) are

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equal to zero, while the behaviour of the rest terms is independent of whether q is finite or not. Thus, passing to the limit in (3.4) we arrive at the equality

$$\int_{\mathbb{R}} u_1' w_1' \, dx + \mu \int_{\mathbb{R}} \left(u_1 + |B|^{1/2} u_2 \right) w_1 \, dx = \int_{\mathbb{R}} \left(f_1 + |B|^{1/2} f_2 \right) w_1 \, dx, \qquad (4.1)$$

which is valid for an arbitrary $w_1 \in H^1(\mathbb{R})$.

It remains to show that $u_2 = |B|^{1/2} u_1$ (then, evidently, (4.1) will imply $u_1 \in H^2(\mathbb{R})$ and (2.16)). One has, using the definitions of the operators J_1^ε and J_2^ε:

$$\left\| J_2^\varepsilon u^\varepsilon - |B|^{1/2} J_1^\varepsilon u^\varepsilon \right\|_{L_2(\mathbb{R})}^2 = \sum_{j \in \mathbb{Z}} \int_{x_j^\varepsilon - \varepsilon/2}^{x_j^\varepsilon + \varepsilon/2} \left| |B|^{1/2} \varepsilon^{1/2} \langle u^\varepsilon \rangle_{B_j^\varepsilon} - |B|^{1/2} \varepsilon^{-1/2} \int_{-\varepsilon}^0 u^\varepsilon(x_1, x_2) \, dx_2 \right|^2 dx_1$$

$$= \varepsilon^{-1} |B| \sum_{j \in \mathbb{Z}} \int_{x_j^\varepsilon - \varepsilon/2}^{x_j^\varepsilon + \varepsilon/2} \left| \int_{-\varepsilon}^0 \left(\langle u^\varepsilon \rangle_{B_j^\varepsilon} - u^\varepsilon(x_1, x_2) \right) dx_2 \right|^2 dx_1 \le |B| \sum_{j \in \mathbb{Z}} \left\| \langle u^\varepsilon \rangle_{B_j^\varepsilon} - u^\varepsilon \right\|_{L_2(Y_j^\varepsilon)}^2$$

$$\le 4|B| \sum_{j \in \mathbb{Z}} \left(\left\| \langle u^\varepsilon \rangle_{Y_j^\varepsilon} - u^\varepsilon \right\|_{L_2(Y_j^\varepsilon)}^2 + \varepsilon^2 \left(\left| \langle u^\varepsilon \rangle_{S_j^\varepsilon} - \langle u^\varepsilon \rangle_{Y_j^\varepsilon} \right|^2 + \left| \langle u^\varepsilon \rangle_{C_j^\varepsilon} - \langle u^\varepsilon \rangle_{S_j^\varepsilon} \right|^2 + \left| \langle u^\varepsilon \rangle_{B_j^\varepsilon} - \langle u^\varepsilon \rangle_{C_j^\varepsilon} \right|^2 \right) \right).$$

$$(4.2)$$

The following simple estimate holds (cf. [11, Lemma 3.2]):

$$\forall u \in H^1(T_j^\varepsilon) : \left| \langle u \rangle_{C_j^\varepsilon} - \langle u \rangle_{S_j^\varepsilon} \right|^2 \le C h^\varepsilon (d^\varepsilon)^{-1} \|\nabla u\|_{L_2(T_j^\varepsilon)}^2. \qquad (4.3)$$

Then, using (3.13), (3.14), (4.3) and the Poincaré inequality, we obtain from (4.2):

$$\left\| J_2^\varepsilon u^\varepsilon - |B|^{1/2} J_1^\varepsilon u^\varepsilon \right\|_{L_2(\mathbb{R})}^2 \le C_1 \varepsilon^2 \|\nabla u^\varepsilon\|_{L_2(\bigcup_{j \in \mathbb{Z}} Y_j^\varepsilon)}^2$$

$$+ C_2 \varepsilon^2 |\ln d^\varepsilon| \left(\|\nabla u^\varepsilon\|_{L_2(\bigcup_{j \in \mathbb{Z}} (B_j^\varepsilon \cup Y_j^\varepsilon))}^2 \right) + C_3 \varepsilon^2 h^\varepsilon (d^\varepsilon)^{-1} \|\nabla u^\varepsilon\|_{L_2(\bigcup_{j \in \mathbb{Z}} T_j^\varepsilon)}^2 \to 0 \text{ as } \varepsilon \to 0 \quad (4.4)$$

(here the the right-hand-side tends to zero due to (2.3) and (2.6) (recall, that $q = \infty$)). Finally, in view the Rellich embedding theorem, the weak convergence of $J_1^\varepsilon u^\varepsilon$ to u_1 in $H^1(\mathbb{R})$ implies

$$\forall L > 0 : \quad \|J_1^\varepsilon u^\varepsilon - u_1\|_{L_2(-L,L)} \to 0 \text{ as } \varepsilon \to 0. \qquad (4.5)$$

From (4.2) and (4.5) we deduce $u_2 = |B|^{1/2} u_1$.

Since the problem (2.16) has the unique solution and u_2 is uniquely determined by u_1, then (3.3) hold for the whole sequence u^ε. Theorem 2.2 is proved.

5. Proof of Theorem 2.3

Recall, that we have to check the fulfilment of the properties (i)-(ii) (see Remark 2.2).

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5.1. Proof of the property (i)

Let $\lambda^\varepsilon \in \sigma(\mathcal{H}^\varepsilon)$ and $\lambda^\varepsilon \to \lambda$ as $\varepsilon = \varepsilon_k \to 0$. We have to show that $\lambda \in \sigma(\mathcal{H})$.

In what follows we will use the notation ε taking in mind ε_k. To simplify the presentation we suppose that ε takes values in the discrete set $\left\{\varepsilon : \varepsilon^{-1} \in \mathbb{N}\right\}$. The general case needs slight modifications.

We denote

- $\mathcal{N}^\varepsilon = \{1, 2, \ldots, \varepsilon^{-1}\}$,

- $\widetilde{\Pi}^\varepsilon = \left\{\mathbf{x} \in \mathbb{R}^2 : 0 < x_1 < 1, \ -\varepsilon < x_2 < 0\right\}$,

- $\widetilde{\Omega}^\varepsilon = \widetilde{\Pi}^\varepsilon \cup \left(\bigcup_{j \in \mathcal{N}^\varepsilon} \left(T_j^\varepsilon \cup B_j^\varepsilon\right) \right)$,

It is clear that the set $\widetilde{\Omega}^\varepsilon$ is a period cell for Ω^ε, namely

$$\Omega^\varepsilon = \bigcup_{k \in \mathbb{Z}} \overline{(\widetilde{\Omega}^\varepsilon + k)}, \quad (\widetilde{\Omega}^\varepsilon + k) \cap (\widetilde{\Omega}^\varepsilon + l) = \varnothing \ \text{for} \ k, l \in \mathbb{Z}, \ k \neq l.$$

Let $\varphi \in \mathbb{R}\backslash(2\pi\mathbb{Z})$. In the space $L_2(\widetilde{\Omega}^\varepsilon)$ we introduce the sesquilinear form $\mathfrak{h}^{\varphi,\varepsilon}$ by

$$\mathfrak{h}^{\varphi,\varepsilon}[u, v] = \int_{\widetilde{\Omega}^\varepsilon} \nabla u \cdot \overline{\nabla v} \, d\mathbf{x}, \quad \text{dom}(\mathfrak{h}^{\varphi,\varepsilon}) = \left\{u \in H^1(\widetilde{\Omega}^\varepsilon) : \ u(1, x_2) = \exp(i\varphi)u(0, x_2) \ \text{for} \ x_2 \in (-\varepsilon, 0)\right\}.$$

We denote by $\mathcal{H}^{\varphi,\varepsilon}$ the operator associated with this form. One has $\mathcal{H}^{\varphi,\varepsilon} u = -\Delta u$ in the generalized sense; the function $u \in \text{dom}(\mathcal{H}^{\varphi,\varepsilon})$ satisfies (in a sense of traces)

$$u(1, x_2) = \exp(i\varphi)u(0, x_2), \quad \frac{\partial u}{\partial x_1}(1, x_2) = \exp(i\varphi)\frac{\partial u}{\partial x_1}(0, x_2) \quad \text{for} \ x_2 \in (-\varepsilon, 0).$$

The spectrum of $\mathcal{H}^{\varphi,\varepsilon}$ is purely discrete. We denote by $\left\{\lambda_k^{\varphi,\varepsilon}\right\}_{k \in \mathbb{N}}$ the sequence of eigenvalues of $\mathcal{H}^{\varphi,\varepsilon}$ arranged in the ascending order and repeated according to their multiplicity. By $\left\{u_k^{\varphi,\varepsilon}\right\}_{k \in \mathbb{N}}$ we denote the corresponding sequence of eigenfunctions such that $(u_k^{\varphi,\varepsilon}, u_l^{\varphi,\varepsilon})_{L_2(\widetilde{\Omega}^\varepsilon)} = \delta_{kl}$.

Using Floquet-Bloch theory (see, e.g., [15, 20, 30]) we deduce the following relationship between the spectra of $\mathcal{H}^\varepsilon$ and $\mathcal{H}^{\varphi,\varepsilon}$:

$$\sigma(\mathcal{H}^\varepsilon) = \bigcup_{k \in \mathbb{N}} \left\{\lambda_k^{\varphi,\varepsilon} : \ \varphi \in \mathbb{R}\backslash(2\pi\mathbb{Z})\right\}. \tag{5.1}$$

For fixed $k \in \mathbb{N}$ the set $\left\{\lambda_k^{\varphi,\varepsilon} : \ \varphi \in \mathbb{R}\backslash(2\pi\mathbb{Z})\right\}$ is a compact interval.

Since $\lambda^\varepsilon \in \sigma(\mathcal{H}^\varepsilon)$ then in view of (5.1) there is $\varphi^\varepsilon \in \mathbb{R}\backslash(2\pi\mathbb{Z})$, $k^\varepsilon \in \mathbb{N}$ such that $\lambda^\varepsilon = \lambda_{k^\varepsilon}^{\varphi^\varepsilon,\varepsilon}$. By $u^\varepsilon = u_{k^\varepsilon}^{\varphi^\varepsilon,\varepsilon}$ we denote the corresponding eigenfunction. One has:

$$\|u^\varepsilon\|_{L_2(\widetilde{\Omega}^\varepsilon)} = 1 \quad (\text{and, consequently, } \|\nabla u^\varepsilon\|_{L_2(\widetilde{\Omega}^\varepsilon)}^2 = \lambda^\varepsilon). \tag{5.2}$$

One can extract a convergent subsequence (for convenience, still indexed by ε)

$$\varphi^\varepsilon \to \varphi \in \mathbb{R}\backslash(2\pi\mathbb{Z}). \tag{5.3}$$

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We define the operators $\widetilde{J}_1^\varepsilon : H^1(\widetilde{\Pi}^\varepsilon) \to H^1(0,1)$ and $\widetilde{J}_2^\varepsilon : L_2(\bigcup_{j \in \mathcal{N}^\varepsilon} B_j^\varepsilon) \to L_2(0,1)$ by (2.7)-(2.8) with $\mathcal{N}^\varepsilon$ instead of $\mathbb{Z}$. Via the same arguments as in the proof of Theorem 2.1, we conclude from (5.2) that the families $\left\{ J_1^\varepsilon u^\varepsilon \right\}_\varepsilon$ and $\left\{ J_2^\varepsilon u^\varepsilon \right\}_\varepsilon$ are uniformly bounded in $H^1(0,1)$ and $L_2(0,1)$, respectively, and therefore there exists a subsequence (again indexed by ε) and $u_1 \in H^1(0,1)$, $u_2 \in L_2(0,1)$ such that

$$\widetilde{J}_1^\varepsilon u^\varepsilon \rightharpoonup u_1 \text{ in } H^1(0,1), \tag{5.4}$$

$$\widetilde{J}_2^\varepsilon u^\varepsilon \rightharpoonup u_2 \text{ in } L_2(0,1). \tag{5.5}$$

Moreover, using the trace theorem, we obtain

$$(\widetilde{J}_1^\varepsilon u^\varepsilon)(0) \to u_1(0), \quad (\widetilde{J}_1^\varepsilon u^\varepsilon)(1) \to u_1(1). \tag{5.6}$$

It is clear that $(\widetilde{J}_1^\varepsilon u^\varepsilon)(1) = \exp(i\varphi^\varepsilon)(\widetilde{J}_1^\varepsilon u^\varepsilon)(0)$, whence, in view of (5.3) and (5.6),

$$u_1(1) = \exp(i\varphi)u_1(0). \tag{5.7}$$

We start from the case

$$u_1 \neq 0.$$

We need the following analogue of Theorem 2.1.

Lemma 5.1. *Let the family* $\left\{ f^\varepsilon \in L_2(\widetilde{\Omega}^\varepsilon) \right\}_\varepsilon$ *satisfy*

$$\|f^\varepsilon\|_{L_2(\widetilde{\Omega}^\varepsilon)} \leq C, \quad \widetilde{J}_1^\varepsilon f^\varepsilon \rightharpoonup f_1 \text{ in } L_2(0,1), \quad \widetilde{J}_2^\varepsilon f^\varepsilon \rightharpoonup f_2 \text{ in } L_2(0,1) \text{ as } \varepsilon \to 0. \tag{5.8}$$

We set $v_{f^\varepsilon}^\varepsilon = (\mathcal{H}^{\varphi^\varepsilon,\varepsilon} + \mu)^{-1} f^\varepsilon$, *where* $\mu > 0$. *Then*

$$\widetilde{J}_1^\varepsilon v_{f^\varepsilon}^\varepsilon \rightharpoonup v_1 \text{ in } H^1(0,1) \text{ as } \varepsilon \to 0,$$

where $v_1 \in H^2(0,1)$ *satisfies* $v_1(1) = \exp(i\varphi)v_1(0)$, $v_1'(1) = \exp(i\varphi)v_1'(0)$ *and solves the problem* (2.13) *on the interval* $(0,1)$. *Moreover*

$$\widetilde{J}_2^\varepsilon u^\varepsilon \rightharpoonup v_2 = \frac{q|B|^{1/2}}{q+\mu} v_1 + \frac{1}{q+\mu} f_2 \text{ in } L_2(0,1).$$

Proof. The proof is similar to the proof of Theorem 2.1. The only essential difference is that the test-function $w = w^\varepsilon(\mathbf{x})$ defined by (3.5) have to be modified in order to meet φ^ε-periodic boundary conditions. Namely, let $w_1 \in C^\infty(0,1)$ satisfy $w_1(1) = \exp(i\varphi)w_1(0)$. We introduce $w_1^\varepsilon \in C^\infty(0,1)$ by

$$w_1^\varepsilon(x) = w_1(x)\left((\exp(i\varphi^\varepsilon - i\varphi) - 1)x + 1\right). \tag{5.9}$$

Clearly $w_1^\varepsilon(x)$ satisfies $w_1^\varepsilon(1) = \exp(i\varphi^\varepsilon)w_1^\varepsilon(0)$ and

$$w_1^\varepsilon \to w_1 \text{ in } C^1(0,1) \text{ as } \varepsilon \to 0. \tag{5.10}$$

Finally, we define the function w by formula (3.5) with $w_1^\varepsilon(x)$ instead of $w_1(x)$. In view of (5.9) $w^\varepsilon \in \text{dom}(\mathfrak{h}^{\varphi^\varepsilon,\varepsilon})$. Then we plug the function w^ε into the equality

$$\int_{\widetilde{\Omega}^\varepsilon} \left(\nabla v_{f^\varepsilon}^\varepsilon \cdot \nabla w^\varepsilon + \mu v_{f^\varepsilon}^\varepsilon w^\varepsilon \right) d\mathbf{x} = \int_{\widetilde{\Omega}^\varepsilon} f^\varepsilon w^\varepsilon \, d\mathbf{x}$$

and pass to the limit as $\varepsilon \to 0$. Using the same arguments as in the proof of Theorem 2.1 (with account of (5.10)) we arrive at the statement of the lemma. $\square$

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We choose $f^\varepsilon = (\lambda + \mu)u^\varepsilon$. It is clear that in this case $v^\varepsilon_{f^\varepsilon} = u^\varepsilon$. Due to (5.4)-(5.5) conditions (5.8) hold true. Then by Lemma 5.1 u_1 belongs to $H^2(0,1)$ and satisfies (additionally to (5.7))

$$u_1'(1) = \exp(i\varphi)u_1'(0) \tag{5.11}$$

$$-u_1'' + \mu\left(1 + \frac{q|B|}{q+\mu}\right)u_1 = (\lambda+\mu)u_1 + \frac{q|B|^{1/2}}{q+\mu}(\lambda+\mu)u_2, \quad u_2 = \frac{q|B|^{1/2}}{q+\mu}u_1 + \frac{1}{q+\mu}(\lambda+\mu)u_2, \tag{5.12}$$

From (5.12), via simple calculations, we obtain the following equation for u_1:

$$-u_1'' = \rho(\lambda)u_1, \text{ where } \rho(\lambda) = \lambda\left(1 + \frac{q|B|}{q-\lambda}\right). \tag{5.13}$$

Since $u_1 \neq 0$ (5.7), (5.11), (5.13) imply that $\rho(\lambda) \in \sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}|_{L_2(\mathbb{R})}) = [0,\infty)$ or, equivalently, $\lambda \in [0,\infty) \setminus (q, q+q|B|)$. Then due to (2.18) $\lambda \in \sigma(\mathcal{H})$.

Now, we inspect the case
$$u_1 = 0.$$
We show that in this case $\lambda = q$ and hence (see (2.18)) $\lambda \in \sigma(\mathcal{H})$.

Recall that $\lambda^\varepsilon = \lambda^{\varphi^\varepsilon,\varepsilon}_{k^\varepsilon}$, $u^\varepsilon = u^{\varphi^\varepsilon,\varepsilon}_{k^\varepsilon}$. We express the eigenfunction u^ε in the form

$$u^\varepsilon = v^\varepsilon - w^\varepsilon + \delta^\varepsilon,$$

where

$$v^\varepsilon(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \widetilde{\Pi}^\varepsilon, \\ \langle u^\varepsilon \rangle_{B^\varepsilon_j}(h^\varepsilon)^{-1}x_2, & \mathbf{x} \in T^\varepsilon_j, \\ \langle u^\varepsilon \rangle_{B^\varepsilon_j}, & \mathbf{x} \in B^\varepsilon_j, \end{cases} \qquad w^\varepsilon(\mathbf{x}) = \sum_{k=1}^{k^\varepsilon-1}(v^\varepsilon, u^{\varphi^\varepsilon,\varepsilon}_k)_{L_2(\widetilde{\Omega}^\varepsilon)}u^{\varphi^\varepsilon,\varepsilon}_k(\mathbf{x})$$

and δ^ε is a remainder term. It is clear that

$$v^\varepsilon, w^\varepsilon \in \mathrm{dom}(\mathfrak{h}^{\varphi^\varepsilon,\varepsilon}) \quad \text{and} \quad v^\varepsilon - w^\varepsilon \in \left(\mathrm{span}\left\{u^{\varphi^\varepsilon,\varepsilon}_1, \ldots, u^{\varphi^\varepsilon,\varepsilon}_{k^\varepsilon-1}\right\}\right)^\perp. \tag{5.14}$$

Lemma 5.2. *One has for each $u \in H^1(T^\varepsilon_j \cup Y^\varepsilon_j)$:*

$$\|u\|^2_{L_2(T^\varepsilon_j)} \leq C\left((h^\varepsilon)^2\|u\|^2_{L_2(Y^\varepsilon_j)} + d^\varepsilon|\ln d^\varepsilon|h^\varepsilon\|\nabla u\|^2_{L_2(Y^\varepsilon_j)} + (h^\varepsilon)^2\|\nabla u\|^2_{L_2(T^\varepsilon_j)}\right). \tag{5.15}$$

Proof. By the density arguments it is enough to prove the lemma only for smooth functions. Let u be an arbitrary function from $C^1(\overline{T^\varepsilon_j \cup Y^\varepsilon_j})$. Let $\mathbf{x} = (x_1, x_2) \in T^\varepsilon_j$, $\mathbf{y} = (x_1, 0) \in S^\varepsilon_j$. One has

$$u(\mathbf{x}) = u(\mathbf{y}) + \int\limits_0^{x_2} \frac{\partial u(\xi(\tau))}{\partial\tau}\,\mathrm{d}\tau, \text{ where } \xi(\tau) = (x_1, \tau),$$

whence, using (3.17), we obtain:

$$\|u^\varepsilon\|^2_{L_2(T^\varepsilon_j)} = \int\limits_0^{h^\varepsilon}\int\limits_{x^\varepsilon_j-d^\varepsilon/2}^{x^\varepsilon_j+d^\varepsilon/2}|u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \leq 2h^\varepsilon\int\limits_{x^\varepsilon_j-d^\varepsilon/2}^{x^\varepsilon_j+d^\varepsilon/2}|u(x_1,0)|^2\,\mathrm{d}x_1 + 2(h^\varepsilon)^2\int\limits_0^{h^\varepsilon}\int\limits_{x^\varepsilon_j-d^\varepsilon/2}^{x^\varepsilon_j+d^\varepsilon/2}|\partial_{x_2}u(x_1,x_2)|^2\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$\leq Ch^\varepsilon\left(d^\varepsilon\varepsilon^{-2}\|u\|^2_{L_2(Y^\varepsilon_j)} + d^\varepsilon|\ln d^\varepsilon|\|\nabla u\|^2_{L_2(Y^\varepsilon_j)}\right) + 2(h^\varepsilon)^2\|\nabla u^\varepsilon\|^2_{L_2(T^\varepsilon_j)}. \tag{5.16}$$

From (5.16), taking into account that $d^\varepsilon\varepsilon^{-2} = O(h^\varepsilon)$ (see (2.6) for $q < \infty$), we arrive at (5.15). $\qquad\square$

Estimate (5.15) yields

$$\|u^\varepsilon\|_{L_2\left(\cup_{j\in\mathcal{N}^\varepsilon}T_j^\varepsilon\right)} \to 0 \text{ as } \varepsilon \to 0. \tag{5.17}$$

Also, one has the following Poincaré inequality:

$$\sum_{j\in\mathcal{N}^\varepsilon} \|u^\varepsilon - \langle u^\varepsilon\rangle_{B_j^\varepsilon}\|^2_{L_2(B_j^\varepsilon)} \leq C\varepsilon^2\|\nabla u^\varepsilon\|^2_{L_2(B_j^\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{5.18}$$

Since $u_1 = 0$, then, using (2.10) (evidently, it holds with $\widetilde{\Pi}^\varepsilon$ and $(0,1)$ instead of Π^ε and $\mathbb{R}$), we get

$$\|u^\varepsilon\|^2_{L_2(\widetilde{\Pi}^\varepsilon)} \leq \|J_1^\varepsilon u^\varepsilon\|^2_{L_2(0,1)} + C\varepsilon^2\|\nabla u^\varepsilon\|^2_{L_2(\widetilde{\Pi}^\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{5.19}$$

Since $\|u^\varepsilon\|_{L_2(\widetilde{\Omega}^\varepsilon)} = 1$ then

$$1 = \|u^\varepsilon\|^2_{L_2(\widetilde{\Pi}^\varepsilon)} + \sum_{j\in\mathcal{N}^\varepsilon} \|u^\varepsilon\|^2_{L_2(T_j^\varepsilon)} + \sum_{j\in\mathcal{N}^\varepsilon} |B_j^\varepsilon|\left|\langle u^\varepsilon\rangle_{B_j^\varepsilon}\right|^2 + \sum_{j\in\mathcal{N}^\varepsilon} \|u^\varepsilon - \langle u^\varepsilon\rangle_{B_j^\varepsilon}\|^2_{L_2(B_j^\varepsilon)},$$

and hence, in view of (5.17)-(5.19), we obtain

$$\sum_{j\in\mathcal{N}^\varepsilon} |B_j^\varepsilon|\left|\langle u^\varepsilon\rangle_{B_j^\varepsilon}\right|^2 = 1 + o(1) \text{ as } \varepsilon \to 0. \tag{5.20}$$

From (5.20), taking into account (2.2), (2.6), we obtain the asymptotics for v^ε:

$$\|\nabla v^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} = \sum_{j\in\mathcal{N}^\varepsilon} d^\varepsilon(h^\varepsilon)^{-1}\left|\langle u^\varepsilon\rangle_{B_j^\varepsilon}\right|^2 = q + o(1) \text{ as } \varepsilon \to 0, \tag{5.21}$$

$$\sum_{j\in\mathcal{N}^\varepsilon} \|v^\varepsilon\|^2_{L_2(B_j^\varepsilon)} = \sum_{j\in\mathcal{N}^\varepsilon} |B_j^\varepsilon|\left|\langle u^\varepsilon\rangle_{B_j^\varepsilon}\right|^2 = 1 + o(1) \text{ as } \varepsilon \to 0, \tag{5.22}$$

$$\sum_{j\in\mathcal{N}^\varepsilon} \|v^\varepsilon\|^2_{L_2(T_j^\varepsilon)} = \frac{1}{3|B|}(h^\varepsilon)^2\frac{d^\varepsilon}{h^\varepsilon\varepsilon^2} \sum_{j\in\mathcal{N}^\varepsilon} |B_j^\varepsilon|\left|\langle u^\varepsilon\rangle_{B_j^\varepsilon}\right|^2 = o(1) \text{ as } \varepsilon \to 0. \tag{5.23}$$

Asymptotics (5.22)-(5.23) together with $v^\varepsilon|_{\widetilde{\Pi}^\varepsilon} = 0$ yield

$$\|v^\varepsilon\|_{L_2(\widetilde{\Omega}^\varepsilon)} = 1 + o(1) \text{ as } \varepsilon \to 0. \tag{5.24}$$

By virtue of (5.17)-(5.19) and (5.23)

$$\|u^\varepsilon - v^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} = \sum_{j\in\mathcal{N}^\varepsilon}\left(\|u^\varepsilon - \langle u^\varepsilon\rangle_{B_j^\varepsilon}\|^2_{L_2(B_j^\varepsilon)} + \|u^\varepsilon - v^\varepsilon\|^2_{L_2(T_j^\varepsilon)}\right) + \|u^\varepsilon\|^2_{L_2(\widetilde{\Pi}^\varepsilon)} \xrightarrow[\varepsilon\to0]{} 0. \tag{5.25}$$

Since $(u^\varepsilon, u_k^{\varphi^\varepsilon,\varepsilon})_{L_2(\widetilde{\Omega}^\varepsilon)} = 0$ for $k = 1, \ldots, k^\varepsilon - 1$, we get, using the Bessel inequality:

$$\|w^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} = \sum_{k=1}^{k^\varepsilon-1}\left|(v^\varepsilon, u_k^{\varphi^\varepsilon,\varepsilon})_{L_2(\widetilde{\Omega}^\varepsilon)}\right|^2 = \sum_{k=1}^{k^\varepsilon-1}\left|(v^\varepsilon - u^\varepsilon, u_k^{\varphi^\varepsilon,\varepsilon})_{L_2(\widetilde{\Omega}^\varepsilon)}\right|^2 \leq \|v^\varepsilon - u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)},$$

$$\|\nabla w^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} = \sum_{k=1}^{k^\varepsilon-1}\lambda_k^{\varphi^\varepsilon,\varepsilon}\left|(v^\varepsilon, u_k^{\varphi^\varepsilon,\varepsilon})_{L_2(\widetilde{\Omega}^\varepsilon)}\right|^2 = \sum_{k=1}^{k^\varepsilon-1}\lambda_k^{\varphi^\varepsilon,\varepsilon}\left|(v^\varepsilon - u^\varepsilon, u_k^{\varphi^\varepsilon,\varepsilon})_{L_2(\widetilde{\Omega}^\varepsilon)}\right|^2 \leq \lambda^\varepsilon\|v^\varepsilon - u^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)},$$

whence, in view of (5.25),

$$\|w^\varepsilon\|^2_{H^1(\widetilde{\Omega}^\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{5.26}$$

Now, we are in position to estimate the remainder δ^ε. One has the following variational character-ization for λ^ε (see, e.g., [30]):

$$\lambda^\varepsilon = \inf \left\{ \frac{\|\nabla u\|^2_{L_2(\widetilde{\Omega}^\varepsilon)}}{\|u\|^2_{L_2(\widetilde{\Omega}^\varepsilon)}}, \ 0 \neq u \in \text{dom}(\mathfrak{h}^{\varphi^\varepsilon,\varepsilon}) \cap \left(\text{span} \left\{ u_1^{\varphi^\varepsilon,\varepsilon}, \ldots, u_{k^\varepsilon-1}^{\varphi^\varepsilon,\varepsilon} \right\} \right)^\perp \right\}. \tag{5.27}$$

From (5.27) we get, taking into account (5.14):

$$\lambda^\varepsilon = \|\nabla u^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} \leq \frac{\|\nabla \tilde{v}^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)}}{\|\tilde{v}^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)}}, \ \text{where } \tilde{v}^\varepsilon = v^\varepsilon - w^\varepsilon. \tag{5.28}$$

Inequality (5.28) is equivalent to

$$\|\nabla \delta^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} \leq \|\nabla \tilde{v}^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} \left(\|\tilde{v}^\varepsilon\|^{-2}_{L_2(\widetilde{\Omega}^\varepsilon)} - 1 \right) - 2(\nabla \tilde{v}^\varepsilon, \nabla \delta^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)}. \tag{5.29}$$

Due to (5.21), (5.24), (5.26)

$$\|\nabla \tilde{v}^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} \left(\|\tilde{v}^\varepsilon\|^{-2}_{L_2(\widetilde{\Omega}^\varepsilon)} - 1 \right) \to 0 \text{ as } \varepsilon \to 0. \tag{5.30}$$

Now, let us estimate the last term in the right-hand-side of (5.29). One has

$$(\nabla \tilde{v}^\varepsilon, \nabla \delta^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} = (\nabla v^\varepsilon, \nabla \delta^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} - (\nabla w^\varepsilon, \nabla \delta^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)}$$
$$= (\nabla v^\varepsilon, \nabla u^\varepsilon - \nabla v^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} + (\nabla v^\varepsilon, \nabla w^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} - (\nabla w^\varepsilon, \nabla \delta^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)}. \tag{5.31}$$

Integrating by parts and taking into account that $\Delta v_j^\varepsilon = 0$ in T_j^ε we get

$$(\nabla v^\varepsilon, \nabla u^\varepsilon - \nabla v^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} = \sum_{j \in \mathcal{N}^\varepsilon} \int_{T_j^\varepsilon} \nabla v^\varepsilon \cdot \nabla (u^\varepsilon - v^\varepsilon) \mathrm{d}\mathbf{x} = \frac{d^\varepsilon}{h^\varepsilon} \sum_{j \in \mathcal{N}^\varepsilon} \langle u^\varepsilon \rangle_{B_j^\varepsilon} \left(-\langle u^\varepsilon \rangle_{S_j^\varepsilon} + \langle u^\varepsilon \rangle_{C_j^\varepsilon} - \langle u^\varepsilon \rangle_{B_j^\varepsilon} \right).$$

Using (2.3), (2.6), (3.13), (3.14), (5.19), (5.20) we obtain

$$\left| (\nabla v^\varepsilon, \nabla u^\varepsilon - \nabla v^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} \right|^2 \leq \left(\frac{d^\varepsilon}{h^\varepsilon} \right)^2 |B_j^\varepsilon|^{-1} \left\{ \sum_{j \in \mathcal{N}^\varepsilon} |B_j^\varepsilon| \left| \langle u^\varepsilon \rangle_{B_j^\varepsilon} \right|^2 \right\} \left\{ \sum_{j \in \mathcal{N}^\varepsilon} \left| \langle u^\varepsilon \rangle_{S_j^\varepsilon} + \langle u^\varepsilon \rangle_{C_j^\varepsilon} - \langle u^\varepsilon \rangle_{B_j^\varepsilon} \right|^2 \right\}$$

$$\leq C_1 \left(\frac{d^\varepsilon}{h^\varepsilon \varepsilon^2} \right)^2 \varepsilon^2 |B_j^\varepsilon|^{-1} \sum_{j \in \mathcal{N}^\varepsilon} \left(\varepsilon^2 \left| \langle u^\varepsilon \rangle_{Y_j^\varepsilon} \right|^2 + \varepsilon^2 \left| \langle u^\varepsilon \rangle_{S_j^\varepsilon} - \langle u^\varepsilon \rangle_{Y_j^\varepsilon} \right|^2 + \varepsilon^2 \left| \langle u^\varepsilon \rangle_{C_j^\varepsilon} - \langle u^\varepsilon \rangle_{B_j^\varepsilon} \right|^2 \right)$$

$$\leq C_2 \left(\|u^\varepsilon\|^2_{L_2(\widetilde{\Pi}^\varepsilon)} + \varepsilon^2 |\ln d^\varepsilon| \|\nabla u^\varepsilon\|^2_{L_2(\bigcup_{j \in \mathcal{N}^\varepsilon} Y_j^\varepsilon)} + \varepsilon^2 |\ln d^\varepsilon| \|\nabla u^\varepsilon\|^2_{L_2(\bigcup_{j \in \mathcal{N}^\varepsilon} B_j^\varepsilon)} \right) \to 0 \text{ as } \varepsilon \to 0. \tag{5.32}$$

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Also, by virtue of (5.21), (5.26),

$$(\nabla v^\varepsilon, \nabla w^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} \to 0 \text{ as } \varepsilon \to 0, \tag{5.33}$$

$$\left|(\nabla w^\varepsilon, \nabla \delta^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)}\right| \le \left|(\nabla w^\varepsilon, \nabla u^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)}\right| + \left|(\nabla w^\varepsilon, \nabla v^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)}\right| + \|\nabla w^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{5.34}$$

From (5.31)-(5.34) we get

$$\lim_{\varepsilon \to 0}(\nabla \tilde{v}^\varepsilon, \nabla \delta^\varepsilon)_{L_2(\widetilde{\Omega}^\varepsilon)} = 0. \tag{5.35}$$

Finally, combining (5.29), (5.30) and (5.35) we conclude that

$$\lim_{\varepsilon \to 0} \|\nabla \delta^\varepsilon\|_{L_2(\widetilde{\Omega}^\varepsilon)} = 0, \tag{5.36}$$

whence, taking into account (5.21), (5.26),

$$\lambda^\varepsilon = \|\nabla u^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} \sim \|\nabla v^\varepsilon\|^2_{L_2(\widetilde{\Omega}^\varepsilon)} \sim q \text{ as } \varepsilon \to 0. \text{ Q.E.D.}$$

Property (i) is completely proved.

5.2. *Proof of the property (ii)*

Let $\lambda \in \sigma(\mathcal{H})$; we have to show that there exists a family $\{\lambda^\varepsilon \in \sigma(\mathcal{H}^\varepsilon)\}_\varepsilon$ such that $\lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda$.

Let us assume the opposite. Then there exist a subsequence ε_k, $\varepsilon_k \searrow 0$ and $\delta > 0$ such that

$$(\lambda - \delta, \lambda + \delta) \cap \sigma(\mathcal{H}^\varepsilon) = \varnothing \text{ as } \varepsilon = \varepsilon_k. \tag{5.37}$$

Since $\lambda \in \sigma(\mathcal{H})$ there exists $F = (f_1, f_2) \in [L_2(\mathbb{R})]^2$, such that

$$F \notin \text{range}(\mathcal{H} - \lambda I). \tag{5.38}$$

Due to (5.37) λ is not in the spectrum of $\mathcal{H}^\varepsilon$ as $\varepsilon = \varepsilon_k$ and therefore for an arbitrary $f^\varepsilon \in L_2(\Omega^\varepsilon)$ there exists the unique solution u^ε of the problem

$$\mathcal{H}^\varepsilon u^\varepsilon - \lambda u^\varepsilon = f^\varepsilon, \ \varepsilon = \varepsilon_k. \tag{5.39}$$

Moreover the following estimates hold true:

$$\|u^\varepsilon\|_{L_2(\Omega^\varepsilon)} \le \frac{1}{\delta}\|f^\varepsilon\|_{L_2(\Omega^\varepsilon)}, \quad \|\nabla u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} = \lambda\|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + (f^\varepsilon, u^\varepsilon)_{L_2(\Omega^\varepsilon)} \le \left(\frac{\lambda}{\delta^2} + \frac{1}{\delta}\right)\|f^\varepsilon\|^2_{L_2(\Omega^\varepsilon)}. \tag{5.40}$$

Now, we choose f^ε in (5.39) by (2.14). The family $\{f^\varepsilon\}_\varepsilon$ satisfies (2.12), whence, taking into account (5.40), we conclude that there exist a subsequence (still indexed by ε_k) and $u_1 \in H^1(\mathbb{R})$, $u_2 \in L_2(\mathbb{R})$ such that (3.3) hold (as $\varepsilon = \varepsilon_k \to 0$).

Repeating word-by-word the proof of Theorem 2.1 we conclude that $U = (u_1, u_2)$ solves (5.38). We obtain a contradiction. Property (ii) is proved and this finishes the proof of Theorem 2.3.

6. Proof of Lemma 2.1

In the proof we deal with domains $B_0^\varepsilon, T_0^\varepsilon, Y_0^\varepsilon$. For convenience hereinafter we omit the index "0". We denote

$$G^\varepsilon = Y^\varepsilon \cup T^\varepsilon, \quad D^\varepsilon = B^\varepsilon \cup G^\varepsilon.$$

The set D^ε is the smallest period cell for the operators $\mathcal{H}^\varepsilon$. Let $\varphi \in \mathbb{R} \backslash (2\pi\mathbb{Z})$. In $L_2(D^\varepsilon)$ we introduce the sesquilinear form $\mathfrak{a}^{\varphi,\varepsilon}$ by

$$\mathfrak{a}^{\varphi,\varepsilon}[u,v] = \int_{D^\varepsilon} \nabla u \cdot \overline{\nabla v} \, d\mathbf{x}, \; \operatorname{dom}(\mathfrak{a}^{\varphi,\varepsilon}) = \left\{ u \in H^1(D^\varepsilon) : u(\varepsilon, x_2) = \exp(i\varphi)u(0, x_2) \text{ for } x_2 \in (-\varepsilon, 0) \right\}. \quad (6.1)$$

We denote by $\mathcal{A}^{\varphi,\varepsilon}$ the operator associated with this form, by $\left\{ \mu_k^{\varphi,\varepsilon} \right\}_{k \in \mathbb{N}}$ we denote the sequence of its eigenvalues arranged in the ascending order and repeated according to their multiplicity.

Again using Floquet-Bloch theory we get the representation

$$\sigma(\mathcal{H}^\varepsilon) = \bigcup_{k \in \mathbb{N}} \left\{ \mu_k^{\varphi,\varepsilon} : \varphi \in \mathbb{R} \backslash (2\pi\mathbb{Z}) \right\}. \quad (6.2)$$

The sets $\left\{ \mu_k^{\varphi,\varepsilon} : \varphi \in \mathbb{R} \backslash (2\pi\mathbb{Z}) \right\}$ are compact intervals. Our goal is to prove that

$$\mu_2^{\varphi,\varepsilon} \to \infty \text{ as } \varepsilon \to 0 \text{ provided } \varphi \neq 0, \quad (6.3)$$

then Lemma 2.1 follows from directly from (6.2)-(6.3).

To prove (6.3) we consider the following operator in $L_2(D^\varepsilon) = L_2(B^\varepsilon) \oplus L_2(G^\varepsilon)$:

$$\widehat{\mathcal{A}}^{\varphi,\varepsilon} = (-\Delta_{B^\varepsilon}) \oplus \left(-\Delta_{G^\varepsilon}^\varphi \right).$$

Here $-\Delta_{B^\varepsilon}$ is the Neumann Laplacian on B^ε, and $-\Delta_{G^\varepsilon}^\varphi$ is the operator acting in $L_2(G^\varepsilon)$ being associated with the sesqulilinear form, which is defined by (6.1) with G^ε instead of D^ε. We denote by $\left\{ \widehat{\mu}_k^{\varphi,\varepsilon} \right\}_{k \in \mathbb{N}}$ the sequence of eigenvalues of $\widehat{\mathcal{A}}^{\varphi,\varepsilon}$. It is easy to see that $\operatorname{dom}(\widehat{\mathcal{A}}^{\varphi,\varepsilon}) \supset \operatorname{dom}(\mathcal{A}^{\varphi,\varepsilon})$ and $\widehat{\mathcal{A}}^{\varphi,\varepsilon} = \mathcal{A}^{\varphi,\varepsilon}$ on $\operatorname{dom}(\mathcal{A}^{\varphi,\varepsilon})$. Then, by the min-max principle,

$$\forall k \in \mathbb{N}: \quad \widehat{\mu}_k^{\varphi,\varepsilon} \leq \mu_k^{\varphi,\varepsilon}. \quad (6.4)$$

The first eigenvalue of $-\Delta_{B^\varepsilon}$ is equal to zero, therefore $\widehat{\mu}_1^{\varphi,\varepsilon} = 0$. Let us prove that the first eigenvalue of $-\Delta_{G^\varepsilon}^\varphi$ tends to infinity. For an arbitrary $u \in H^1(Y^\varepsilon)$ one has, using the Poincaré inequality:

$$\|u\|_{L_2(Y^\varepsilon)}^2 = \|u - \langle u \rangle_{Y^\varepsilon}\|_{L_2(Y^\varepsilon)}^2 + \varepsilon^2 |\langle u \rangle_{Y^\varepsilon}|^2 \leq C\varepsilon^2 \|\nabla u\|_{L_2(Y^\varepsilon)}^2 + \varepsilon^2 |\langle u \rangle_{Y^\varepsilon}|^2. \quad (6.5)$$

We denote:

$$Z_0^\varepsilon = \{\mathbf{x} \in \mathbb{R}^2 : x_1 = 0, \ x_2 \in (-\varepsilon, 0)\}, \quad Z_1^\varepsilon = \{\mathbf{x} \in \mathbb{R}^2 : x_1 = \varepsilon, \ x_2 \in (-\varepsilon, 0)\}.$$

Employing the trace and the Poincaré inequalities one has

$$k = 0, 1: \quad \left| \langle u \rangle_{Z_k^\varepsilon} - \langle u \rangle_{Y^\varepsilon} \right|^2 = \left| \langle u - \langle u \rangle_{Y^\varepsilon} \rangle_{Z_k^\varepsilon} \right|^2 \leq \varepsilon^{-1} \|u - \langle u \rangle_{Y^\varepsilon}\|_{L_2(Z_k^\varepsilon)}^2$$

$$\leq C\left(\varepsilon^{-2} \|u - \langle u \rangle_{Y^\varepsilon}\|_{L_2(Y^\varepsilon)}^2 + \|\nabla u\|_{L_2(Y^\varepsilon)}^2 \right) \leq C_1 \|\nabla u\|_{L_2(Y^\varepsilon)}^2. \quad (6.6)$$

Now, suppose that u is not only in $H^1(G^\varepsilon)$, but also $u(\varepsilon, x_2) = \exp(i\varphi)u(x_2)$ for $x_2 \in (-\varepsilon, 0)$. Then it follows from (6.6) that (recall: $\varphi \neq 0$, whence $1 - \exp(i\varphi) \neq 0$)

$$
|\langle u \rangle_{Y^\varepsilon}|^2 = |1 - \exp(i\varphi)|^{-2} \left| \langle u \rangle_{Y^\varepsilon} - \langle u \rangle_{Z_1^\varepsilon} + \exp(i\varphi)\langle u \rangle_{Z_0^\varepsilon} - \exp(i\varphi)\langle u \rangle_{Y^\varepsilon} \right|^2
$$

$$
\leq 2|1 - \exp(i\varphi)|^{-2} \left(\left| \langle u \rangle_{Y^\varepsilon} - \langle u \rangle_{Z_1^\varepsilon} \right|^2 + \left| \langle u \rangle_{Z_0^\varepsilon} - \langle u \rangle_{Y^\varepsilon} \right|^2 \right) \leq C \|\nabla u\|_{L_2(Y^\varepsilon)}^2. \quad (6.7)
$$

Combining (6.5) and (6.7) we arrive at

$$
\|u\|_{L_2(Y^\varepsilon)}^2 \leq C\varepsilon^2 \|\nabla u\|_{L_2(Y^\varepsilon)}^2. \quad (6.8)
$$

Then, using Lemma 5.2 and inequality (6.8), we obtain the estimate

$$
\|u\|_{L_2(T^\varepsilon)}^2 \leq C \left((h^\varepsilon \varepsilon)^2 \|\nabla u\|_{L_2(Y^\varepsilon)}^2 + d^\varepsilon |\ln d^\varepsilon| h^\varepsilon \|\nabla u\|_{L_2(Y^\varepsilon)}^2 + (h^\varepsilon)^2 \|\nabla u\|_{L_2(T_j^\varepsilon)}^2 \right). \quad (6.9)
$$

It follows from (6.8)-(6.9) that for each $u \in \left\{ v \in H^1(G^\varepsilon) : v(\varepsilon, x_2) = \exp(i\varphi)v(x_2) \text{ for } x_2 \in (-\varepsilon, 0) \right\}$

$$
\|u\|_{L_2(G^\varepsilon)}^2 \leq C\eta^\varepsilon \|\nabla u\|_{L_2(G^\varepsilon)}^2, \text{ where } \eta^\varepsilon \to 0 \text{ as } \varepsilon \to 0. \quad (6.10)
$$

Inequality (6.10) implies that the first eigenvalue of the operator $-\Delta_{L_2(G^\varepsilon)}^\varphi$ tends to infinity as $\varepsilon \to 0$. Evidently, the second eigenvalue of $-\Delta_{B^\varepsilon}^\varphi$ also tends to infinity. Therefore

$$
\widehat{\mu}_2^{\varphi,\varepsilon} \to \infty \text{ as } \varepsilon \to 0 \text{ provided } \varphi \neq 0,
$$

whence, using (6.4), we infer (6.3). Lemma 2.1 is proved.

Acknowledgements

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