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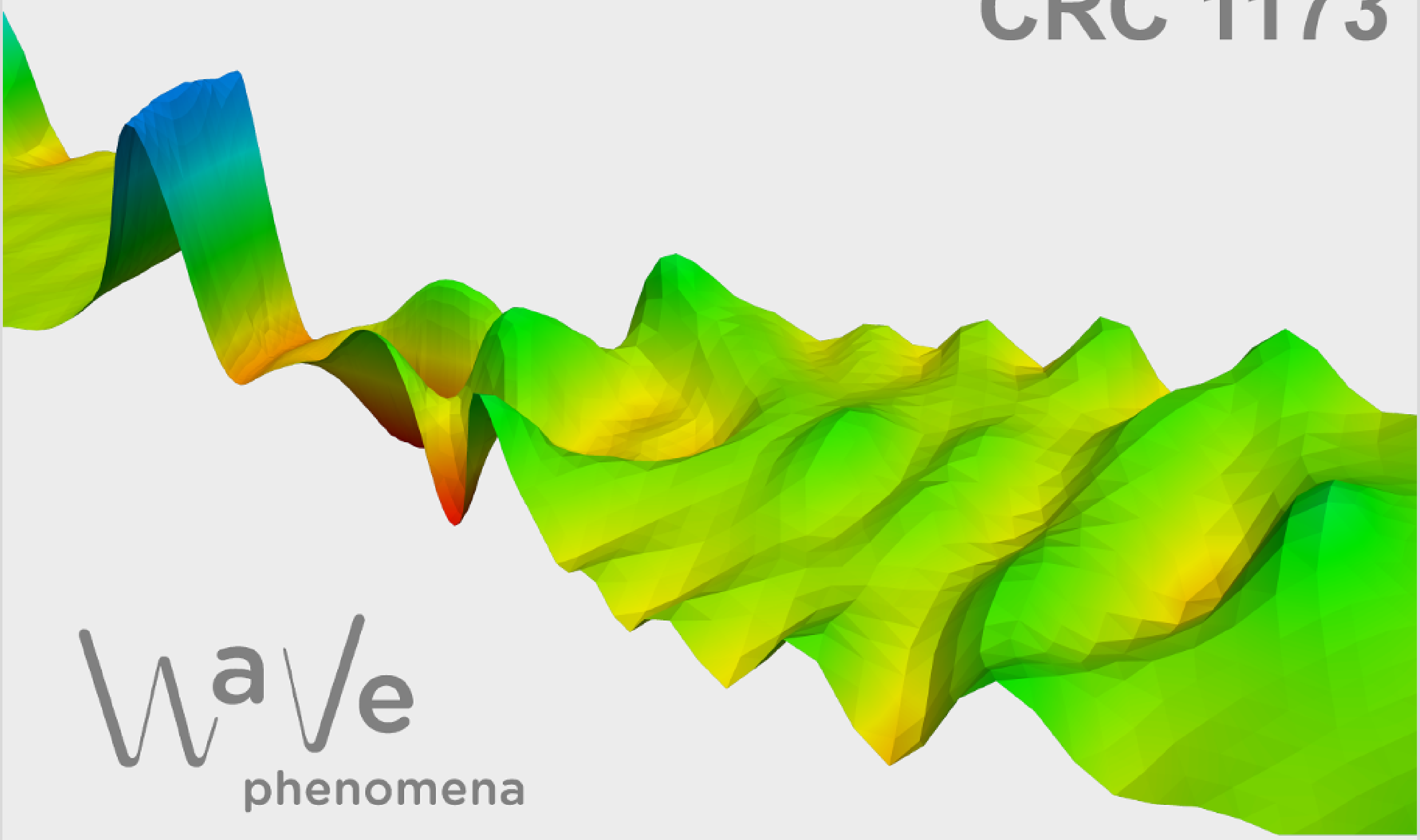

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Dispersion managed solitons in the presence of saturated nonlinearity

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Abstract

The averaged dispersion managed nonlinear Schrödinger equation with saturated nonlinearity is considered. It is shown that under rather general assumptions on the saturated nonlinearity, the ground state solution corresponding to the dispersion managed soliton can be found for both zero residual dispersion and positive residual dispersion. The same applies to diffraction management solitons, which are a discrete version describing certain waveguide arrays.

1. Introduction

1.1. Background

The dispersion managed nonlinear Schrödinger equation (DM NLS) by now is a well established model in nonlinear science. There is a good review article on the subject by Turitsyn-Brandon-Fedoruk [17]. Initially, the main motivation to study this equation came from fiber optics applications, after the introduction of the dispersion compensation technique (which itself appeared due to the invention of fibers with anomalous dispersion). Nowadays, DM NLS became a paradigm of a nonlinear dispersive equation with periodically varying coefficients that in some regime, e.g. strong dispersion management, leads to a dispersion averaged nonlinearity. This nonlocal equation and its solutions can easily have properties which are qualitatively different from what one is used from the local NLS. For example, it can have ground states which have strongly oscillating tails, see [16]. One should also note an interesting related development in pure mathematics where several works have appeared on best constants in space-time inequalities, such as the celebrated Strichartz inequality [3, 4, 14, 9, 15, 10], which are related to dispersion managed solitons.

The evolutionary equation for the propagation of the wave envelope of an optical pulse in a single mode fiber is given by [11, 1]

$$\mathrm{i}u_t + \left(\alpha + \frac{1}{\epsilon} d\left(\frac{t}{\epsilon}\right)\right) u_{xx} + \gamma\, f(|u|)u = 0,$$

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where t is the distance along the fiber, x is the retarded time, $d(\tau)$ is the mean-zero periodically varying dispersion profile (with period L), α is the constant residual dispersion, and f represents a nonlinearity that is usually given by $f(|u|) = |u|^2$ in fiber optics and $\epsilon > 0$ and $-\infty < \gamma < +\infty$ are parameters. We ignored attenuation and amplification effects which can be transformed out by an appropriate change of variables.

In the limit of small ϵ, after rescaling time $t = \tau\epsilon$, changing the variable $u = T(\tau)v$, and averaging over the fast dispersion action, one obtains an averaged DM NLS [11, 1]

$$\mathrm{i}v_\tau + \epsilon\alpha v_{xx} + \frac{\epsilon\gamma}{L}\int_0^L T^{-1}(s)\Big(f(|T(s)v|)T(s)v\Big)\mathrm{d}s = 0,$$

where $T(s)$ is the fundamental solution of $\mathrm{i}w_\tau + d(\tau)w_{xx} = 0$. The operator $T(s)$ is periodic because $d(\tau)$ has zero mean. This averaged equation was first obtained by Gabitov and Turitsyn [11] and afterwards systematically derived and solved numerically by Ablowitz and Biondini [1]. The averaged equation retains the Hamiltonian structure and the corresponding averaged Hamiltonian functional is given by

$$H(v) = \epsilon\alpha\int_{-\infty}^{+\infty} |v_x|^2\mathrm{d}x - \frac{\epsilon\gamma}{L}\int_0^L\int_{-\infty}^{+\infty} F(|T(t)v|)\mathrm{d}x\mathrm{d}t.$$

With our definitions of the nonlinearity f and the "nonlinearity potential" F, the relation between the two is given by $F'(s) = f(s)s$ for $s \geq 0$.

Next, one can use the variational approach and look for ground state solutions, that is, solutions of the averaged DM NLS of the forms $v(\tau, x) = e^{i\epsilon\omega\tau}u(x)$, which minimize the Hamiltonian, subject to the energy constraint

$$P_E = \inf\left\{H(u) : \int_{-\infty}^{+\infty} |u|^2\mathrm{d}x = E\right\}.$$

While the Hamiltonian functional is related to the standard NLS functional, the fact that the nonlinearity is averaged over the dispersion action produces several nice properties, one of which is that ground states can exist even in the absence of the gradient term ($\alpha = 0$). For a large class of nonlinearities it is now well understood [5] when the ground state exists, including the case of $\alpha = 0$. One of the key properties to assure the presence of the ground state is that the nonlinearity potential F is "sufficiently" nonlinear. Formally, one needs to verify the so-called sub-additivity condition

$$P_{E_1+E_2} < P_{E_1} + P_{E_2},$$

which heuristically speaking means that if a hypothetical ground state is split into two parts, preserving the total energy, and these two parts are moved infinitely far away from each other, then the value of the Hamiltonian will increase. Clearly, that is a necessary condition for the tightness of the ground state, as an example with $F(|u|) = |u|^2$ illustrates. Note that in the case of the quantum harmonic oscillator the tightness of the ground state comes from the confining quadratic potential.

One important case when the sub-additivity condition does not hold uniformly is the so-called saturated nonlinearity, such as

$$f(|u|) = \frac{|u|^2}{1+\sigma|u|^2}.$$

This function approaches a constant for large values of $|u|$ and, as a result, the nonlinear term degenerates into a linear one.

While saturated nonlinearities are well-studied for the regular NLS [12], the DM NLS with saturated nonlinearities has not received much attention. This is perhaps due to the small values of optical power in fiber optics applications, which suggests that saturation effects are negligible. Nevertheless, theoretically speaking, it leaves an open question whether one can still construct ground sates. The task is especially delicate in the case of zero average dispersion, as we explain below. The reader will also see that our argument points to some possible limitations when ground states may fail to exist.

In the remaining part of this paper, we establish the existence of ground state solutions in a general class of averaged DM NLS with saturable nonlinearities. We first address the case of mean-zero dispersion which is more difficult and relevant. Next we extend these results to the DM NLS with positive residual dispersion. One can apply the same ideas and methods from dispersion management to the case of waveguide arrays, which are modelled by a discrete nonlinear Schrödinger equation. This was proposed in [7, 8] and the effective equation, the diffraction managed discrete nonlinear Schrödinger equation (DM DNLS), governing the regime of strong diffraction management was derived in [2]. We address the existence of diffraction managed solitons and the necessary changes of our argument to cover the discrete case in the last section.

1.2. Variational formulation

We consider the following averaged variational principle

$$P_E = \inf\left\{H(u) : \int_{-\infty}^{+\infty} |u|^2 \mathrm{d}x = E\right\} \tag{1}$$

where

$$H(u) = \alpha\int_{-\infty}^{+\infty} |u_x|^2 \mathrm{d}x - \int_0^1\int_{-\infty}^{+\infty} F(|T(t)u|)\mathrm{d}x\mathrm{d}t.$$

For the clarity of presentation, we put $L = \gamma = 1$ and we let $T(t) = e^{it\partial_x^2}$ (corresponding to the square wave dispersion profile), as one can easily extend our results to other values of those parameters and more general dispersion profiles. A particular choice of F that leads to the saturable nonlinearity discussed above is given by

$$F(s) = \frac{s^2}{2\sigma} - \frac{1}{2\sigma^2}\log(1+\sigma s^2).$$

As another natural example one can also consider the saturable version of a quartic potential

$$F(s) = \frac{s^4}{1+s^2},$$

which leads to a saturable type nonlinearity in the NLS equation:

$$f(s) = \frac{2s^2(2+s^2)}{(1+s^2)^2}.$$

One should keep in mind that saturation may destroy ground states in the case of nonlinearities for which there are ground states in the unsaturated case, as a simple example illustrates. Consider the local NLS without dispersion management and let $F(s) = \frac{s^2+s^4}{1+s^2}$. This is really $F(s) = s^2$ in disguise and obviously does not lead to a ground state. While $s^2 \ll s^4$ for large s and one would have solitons without saturation, the saturating denominator is just enough to destroy sub-additivity.

The general conditions on F, which generalize those typical nonlinearities to a much broader class, are stated in the next section. These conditions extend the notion that the saturated nonlinearity has superquadratic growth for small values but approaches a quadratic function for large values.

A natural strategy to establish the existence of a ground state is to show that there is a minimizer of the constrained variational principle by constructing a converging subsequence. One difficulty with the saturable nonlinearity is that the above sub-additivity property does not hold in the limit of large values. However, if the minimizing sequence has bounded amplitude, which we show, then sub-additivity holds in the relevant region.

We rely here on a recent result [5] by Choi-Hundertmark-Lee, where the existence of ground states for a large class of nonlinearities in the averaged DM NLS was established. We modify our saturated nonlinearity in such a way that their approach applies and then we establish a bound on the maximum of the ground state in the modified problem. Next we show that the ground state in the modified problem is also a ground state in the original problem.

2. Zero residual dispersion

Consider now the minimization problem (1) with zero residual dispersion $\alpha = 0$

$$H(u) = -\int_0^1 \int_{-\infty}^{+\infty} F(|T(t)u|)\mathrm{d}x\mathrm{d}t.$$

2.1. Assumptions on the nonlinearity potential F

First we state our assumptions on the nonlinearity potential:

1. **(positivity)** $F(s) > 0$ for any $s > 0$.

2. **(polynomial bound)** F is continuously differentiable for $s \in (0, \infty)$ and satisfies the inequality

$$F'(s) \leq C(s^{\gamma_1} + s^{\gamma_2}),$$

where $2 \leq \gamma_1 \leq \gamma_2 < 4$ and $C > 0$ is a constant.

3. **(superquadratic growth)** For any $A > 0$, there exists γ_0 that can depend on A such that

$$W(s) := \frac{F'(s)s}{F(s)} \geq \gamma_0(A) > 2, \ \ s \in (0, A).$$

4. **(saturation condition)** $W(s)$ is a monotonically decreasing function with limit

$$\lim_{s \to \infty} W(s) = 2.$$

Remark 2.1. *The growth rate function $W(s)$ is an important measure of polynomial growth and $W(s) > 2$ implies faster than quadratic growth. The two saturated nonlinearities that we mentioned in the introduction satisfy these four conditions. One can also construct many more examples.*

Remark 2.2. *For positive residual dispersion (DM NLS) and arbitrary non-negative residual diffraction (DM DNLS), we can extend the range of the parameters in assumption 2 to $1 < \gamma_1 \leq \gamma_2 < \infty$.*

Remark 2.3. *If together with the first two assumptions one imposes a strong superquadratic growth condition, i.e. no saturation, $W(s) > \gamma_0 > 2$ for all $s \in (0, \infty)$ then the existence theorem in [5] (DM NLS) and in [6] (DM DNLS) by Choi-Hundertmark-Lee applies and one obtains a ground state.*

2.2. The modified functional

Consider the growth rate function which will be modified first

$$W(s) = \frac{F'(s)s}{F(s)} = (\log F(s))'s > 2, \text{ for } s > 0.$$

For any $\mu > 0$, modify $W(s)$ for $s > \mu$ as follows: let $\delta = W(\mu) - 2$ and restrict $\delta \in (0, 1)$. This is possible, since by assumption 4 we always have $\delta > 0$ and it can be taken arbitrarily small by simply increasing μ.

Let W_m be a smooth modification consisting of a parabola and a horizontal line:

$$W_m(s) = W(s), \text{ if } s \in [0, \mu]$$

$$W_m(s) = a(s - \sigma)^2 + 2 + \delta/2 \text{ for } s \in [\mu, \sigma]$$

and

$$W_m(s) = 2 + \delta/2, \ \text{ if } s > \sigma.$$

By assumption $W'(\mu) \leq 0$ and we compute

$$a = \frac{W'(\mu)^2}{2(W(\mu)-2)}, \quad \sigma = \mu - \frac{W(\mu)-2}{W'(\mu)}.$$

In a way, the modified W_m is not worse than the original one. Also, the modified potential F_m, given by

$$F_m(s) = F(s), \quad 0 \leq s \leq \mu,$$

and for $s > \mu$ as the solution of the ODE $W_m(s) = (\log F_m(s))' s$ with initial condition $F_m(\mu) = F(\mu)$. By construction, the modified potential $F_m(s)$ satisfies a superquadratic growth condition

$$F_m'(s)s \geq \left(2 + \frac{\delta}{2}\right) F_m(s)$$

with $\delta = W(\mu) - 2 > 0$, but now *for all* $s > 0$, which, together with the following Lemma shows that F_m satisfies the appropriate conditions from [5], see Remark 2.3, that guarantee the existence of ground states.

Lemma 2.1. *The modified potential satisfies the bounds*

$$\begin{aligned} F_m(s) &\leq \frac{C}{3}(s^{\gamma_1+1} + s^{\gamma_2+1}) \\ F_m'(s) &\leq C(s^{\gamma_1} + s^{\gamma_2}). \end{aligned}$$

for $\mu > 0$, *where* C *is the constant from assumption* 2 *on* F.

Proof: First, by integrating the bound from assumption 2 from zero to $s \leq \mu$, we have

$$F(s) \leq \frac{C}{\gamma_1+1} s^{\gamma_1+1} + \frac{C}{\gamma_2+1} s^{\gamma_2+1} \leq \frac{C}{3}\left(s^{\gamma_1+1} + s^{\gamma_2+1}\right)$$

for all $0 < s \leq \mu$. Solving the ODE $(\log F_m(s))' s = W_m(s)$ for F_m, we obtain for $s \geq \mu$

$$\begin{aligned} F_m(s) &= F(\mu) \exp\left(\int_\mu^s \frac{W_m(\tau)}{\tau} \mathrm{d}\tau\right) \leq F(\mu) \left(\frac{s}{\mu}\right)^{2+\delta} \leq \frac{C}{3}(\mu^{\gamma_1+1} + \mu^{\gamma_2+1}) \left(\frac{s}{\mu}\right)^{2+\delta} \\ &= \frac{C}{3}(\mu^{\gamma_1-\delta-1} + \mu^{\gamma_2-\delta-1}) s^{2+\delta} \leq \frac{C}{3}(s^{\gamma_1+1} + s^{\gamma_2+1}) \end{aligned}$$

and then using the defining differential equation for F_m, we also have

$$F_m'(s) = F_m(s) \cdot \frac{W_m(s)}{s} \leq \frac{C}{3}(s^{\gamma_1+1} + s^{\gamma_2+1}) \frac{2+\delta}{s} \leq C(s^{\gamma_1} + s^{\gamma_2}),$$

which implies the result.

QED.

The modified functional

$$H_m(u) = -\int_0^1 \int_{-\infty}^{+\infty} F_m(|T(t)u|)\mathrm{d}x\mathrm{d}t,$$

subject to the energy constraint, possesses a ground state u^* according to Choi-Hundertmark-Lee [5], see also Remark 2.3. This ground state depends on μ and E, but we suppress this dependence in the following, for simplicity of notation.

The ground state u^* must satisfy the corresponding Euler-Lagrange equation

$$\omega u^* = Q_m(u^*) := \int_0^1 T^{-1}(t)\left(F_m'(|T(t)u^*|)\frac{T(t)u^*}{|T(t)u^*|}\right)\mathrm{d}t.$$

Lemma 2.2. *The ground state u^* is uniformly bounded independently of μ, and, moreover, $|(T(r)u^*)(x)| \le K$ for all x and r for some constant $K < \infty$.*

Proof:
First we show that the Lagrange multiplier $\omega = \omega(\mu) \ge c > 0$, independently of the modification. By multiplying the Euler-Lagrange equation with $\bar{u}$ and integrating, we have

$$\omega \int_{-\infty}^{+\infty} |u|^2 \mathrm{d}x = \int_{-\infty}^{+\infty} Q_m(u)\bar{u}\,\mathrm{d}x = \int_0^1 \int_{-\infty}^{+\infty} F_m'(|T(t)u|)|T(t)u|\mathrm{d}x\mathrm{d}t.$$

To assure that the Lagrange multiplier ω is bounded away from zero, we look at the ratio

$$\omega = \frac{\int_{-\infty}^{+\infty} Q_m(u^*)\overline{u^*}\mathrm{d}x}{\int_{-\infty}^{+\infty} |u^*|^2 \mathrm{d}x}. \tag{2}$$

The lower bound of the ratio depends on the energy $\int |u^*|^2\mathrm{d}x$ but it is fixed in the minimization procedure. Consider the numerator

$$\int_{-\infty}^{+\infty} Q_m(u^*)\overline{u^*}\mathrm{d}x = \int_0^1 \int_{-\infty}^{+\infty} F_m'(|T(t)u^*|)|T(t)u^*|\mathrm{d}x\mathrm{d}t \ge 2\int_0^1 \int_{-\infty}^{+\infty} F_m(|T(t)u^*|)\mathrm{d}x\mathrm{d}t$$

where we used superquadratic property of F_m, that is, $F_m'(s)s \ge (2+\delta/2)F_m(s) \ge 2F_m(s)$. Now use any test function g, e.g., a Gaussian and note that since u^* is an energy minimizer, we have

$$\int_0^1 \int_{-\infty}^{+\infty} F_m(|T(t)u^*|)\mathrm{d}x\mathrm{d}t \ge \int_0^1 \int_{-\infty}^{+\infty} F_m(|T(t)g|)\mathrm{d}x\mathrm{d}t \ge \int_0^1 \int_{-\infty}^{+\infty} F(|T(t)g|)\mathrm{d}x\mathrm{d}t$$

where we also used that $F_m(s) \ge F(s)$ for all $\mu > 1$ and $s \ge 0$.

Now, taking a test function, e.g. a Gaussian, we simply observe that the last integral is strictly positive (by assumption 1) and independent of μ.

Next we show that $Q_m(u^*)$ is bounded by using an argument due to Kunze [15]. First recall the well-known bound on the solution of the linear Schrödinger equation in one dimension

$$|T(t)u| \leq \frac{1}{|t|^{1/2}} \int_{-\infty}^{+\infty} |u| \,\mathrm{d}x, \tag{3}$$

and consider

$$T(r)Q_m(u) = \int_0^1 T(r)T^{-1}(t) \left(F_m'(|T(t)u|) \frac{T(t)u}{|T(t)u|} \right) \mathrm{d}t.$$

Proposition 2.3. *Let $\nu \in [2, 4)$ and assume that $F'(s) \leq Cs^\nu$ for all $s \geq 0$ and some constant C. Then*

$$|T(r)Q_m(u)| \leq C(E)$$

for all r, where $C(E)$ is independent of the modification, but may depend on ν and the energy $E = \int_{-\infty}^{+\infty} |u(x)|^2 \mathrm{d}x$.

Proof:
Using the dispersive estimate (3),

$$|T(r)Q_m(u)| \leq \int_0^1 \frac{1}{|r-t|^{1/2}} \|F_m'(|T(t)u|)\|_{L_x^1} \mathrm{d}t,$$

where the L^p norm of a function is defined by $\|u\|_{L^p} = \left(\int_{-\infty}^{+\infty} |u|^p \mathrm{d}x \right)^{1/p}$. Continuing the above inequality

$$\leq \int_0^1 \frac{C}{|r-t|^{1/2}} \| |T(t)u|^\nu \|_{L^1} \mathrm{d}t = C \int_0^1 \frac{1}{|r-t|^{1/2}} \|T(t)u\|_{L_x^\nu}^\nu \mathrm{d}t$$

and using Hölder's inequality, we obtain

$$|T(r)Q_m(u)| \leq C \left(\int_0^1 \frac{\mathrm{d}t}{|r-t|^{p/2}} \right)^{1/p} \cdot \left(\int_0^1 \|T(t)u\|_{L_x^\nu}^{\nu p'} \mathrm{d}t \right)^{1/p'},$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

To bound the first integral uniformly in r we need $p < 2$, i.e. $p' > 2$. For the convergence of the second integral, we need a general case of the Strichartz inequality [13] that we recall here:

$$\left(\int_{-\infty}^{+\infty} \|T(t)u\|_{L_x^\sigma}^\rho \mathrm{d}t \right)^{1/\rho} \leq S_\sigma \|u\|_{L^2}$$

as long as

$$\frac{1}{\sigma} + \frac{2}{\rho} = \frac{1}{2},$$

where $2 \leq \sigma \leq \infty$ and $4 \leq \rho \leq \infty$ and S_σ is some constant. Therefore, we need to satisfy the relation

$$\frac{1}{\nu} + \frac{2}{\nu p'} = \frac{1}{2}, \quad \text{i.e.} \quad \nu = 2 + \frac{4}{p'}. \tag{4}$$

This holds for appropriate $p' > 2$ if $\nu \in [2, 4)$. This ends the proof of Proposition 2.3.

QED.

Since u^* satisfies the Euler-Lagrange equation

$$u^* = \frac{1}{\omega} Q_m(u^*),$$

and F assumption 2, we can use Proposition 2.3 to obtain

$$|T(r)u^*(x)| = \frac{1}{\omega}|T(r)Q_m(u^*)(x)| \leq \frac{C_1(E) + C_2(E)}{c}$$

for all r and x, where we also used the uniform lower bound $\omega = \omega(\mu) \geq c > 0$, which we established in the beginning of the proof of Lemma 2.2. This ends the proof of Lemma 2.2.

QED.

Finally, if μ is chosen sufficiently large, so $|(T(r)u^*)(x)| < \mu$ for all x and $r \in [0, 1]$, which is possible by Lemma 2.2, then u^* is also a critical point of H. Further observing that $H_m(u) \leq H(u)$ and $H_m(u^*) = H(u^*)$, u^* has to be a ground state of H.

3. Positive residual dispersion

Now consider the functional

$$H(u) = \alpha \int_{-\infty}^{+\infty} |u_x|^2 \mathrm{d}x - \int_0^1 \int_{-\infty}^{+\infty} F(|T(t)u|) \mathrm{d}x \mathrm{d}t$$

for positive residual dispersion. Impose the same assumptions as in the previous section and modify the functional according to the same rule. By the argument in [5], there exists a minimizer $u^* \in H^1$. Using, for example, $F_m \geq F$ and some arguments from [5], it is easy to see that there is a uniform bound

$$\|T(r)u^*\|_{H^1} = ||u^*||_{H^1} \leq C(E),$$

which also implies the boundedness of $\sup_{x,r} |(T(r)u)(x)|$ by the Sobolev imbedding theorem. Taking again the modification parameter m large enough, we obtain that u^* is a critical point of the original functional. Note that for this argument to work, we do not need Lemma 2.2, so, in fact, we can allow for arbitrary $1 < \gamma_1 \leq \gamma_2 < +\infty$ in assumption 2 on F if $\alpha > 0$.

4. Diffraction management

Now we consider the equation for the ground states given by the stationary DM DNLS

$$\omega u = -\alpha \Delta_{\text{disc}} u - \int_0^1 T(t)^{-1}\big(F(|T(t)u|)\big)dt, \tag{5}$$

on the sequence space $l^2(Z)$, with Z the integers $0, \pm 1, \pm 2 \ldots$, with x indexing the position of the waveguide. Here Δ_{disc} is the discrete Laplacian, given by $\Delta_{\text{disc}} u(x) = u(x+1) - 2u(x) + u(x-1)$ and the solution operator $T(t)$ now corresponds to the discrete free Schrödinger equation and is simply given by the exponential $T(t) = e^{it\Delta_{\text{disc}}}$.

The ground state solutions can again be found as minimizers of the following averaged variational principle

$$P_E = \inf \left\{ H(u) : \sum_{x=-\infty}^{+\infty} |u(x)|^2 = E \right\} \tag{6}$$

where, with the forward difference $D_+u(x) = u(x+1) - u(x)$,

$$H(u) = \alpha \sum_{x=-\infty}^{+\infty} |D_+ u(x)|^2 - \int_0^1 \sum_{x=-\infty}^{+\infty} F(|T(t)u(x)|)\mathrm{d}t.$$

Again, for clarity of the presentation, we consider only the square wave diffraction profile. The discussion in [6] shows that one can, in fact, include any diffraction profile one can think of.

Since $\|u\|_{l^2} = \big(\sum_{x=-\infty}^{+\infty} |u(x)|\big)^{1/2}$, we have the very simple bound

$$\|u\|_{l^\infty} = \sup_{x \in Z} |u(x)| \le \|u\|_{l^2}\,.$$

Thus for any $u \in l^2(Z)$ with energy $\|u\|_{l^2}^2 = E$, we have the bound

$$\|T(t)u\|_{l^\infty} \le \|T(t)u\|_{l^2} = \|u\|_{l^2} = E^{1/2}\,.$$

for all t. So taking $\mu > E^{1/2}$ and modifying the functional according to the same rule as before, we get, according to [6], a solution of the modified functional, which, since $\mu > E^{1/2}$, is also a solution of the unmodified functional, as before. Note that this works for any residual diffraction $\alpha \ge 0$ and any $1 < \gamma_1 \le \gamma_2 < +\infty$ in assumption 2 on F.

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