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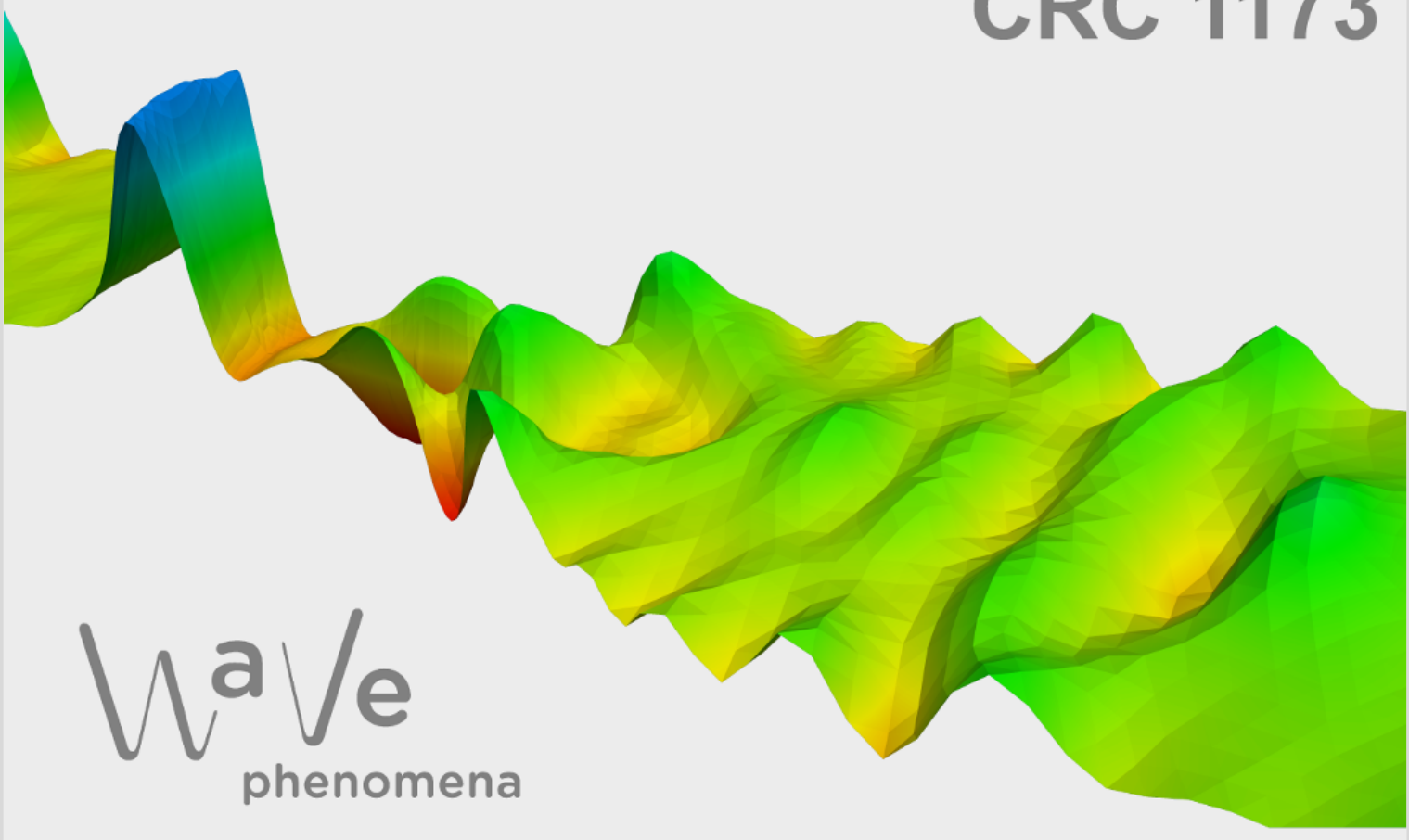

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AN EXPONENTIAL-TYPE INTEGRATOR FOR THE KDV EQUATION

by

Martina Hofmanová & Katharina Schratz

Abstract. — We introduce an exponential-type time-integrator for the KdV equation and prove its first-order convergence in H^1 for initial data in H^3. Furthermore, we outline the generalization of the presented technique to a second-order method.

We consider the Korteweg-de Vries (KdV) equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) = \frac{1}{2}\partial_x (u(t,x))^2, \quad u(0,x) = u_0(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{T} = [-\pi, \pi], \tag{1}$$

where for practical implementation issues we impose periodic boundary conditions. For local-wellposedness results of the periodic KdV equation in low regularity spaces we refer to [1, 5, 19].

In the context of the numerical time integration of (non)linear partial differential equations splitting methods as well as exponential integrators contribute attractive classes of integration methods. We refer to [7, 8, 9, 18] for an extensive overview, and in particular to [3, 4, 16] for the analysis of splitting methods for Schrödinger(-Poisson) equations. In recent years, splitting as well as exponential integration schemes (including Lawson type Runge-Kutta methods [15]) have also gained a lot of attention in the context of the numerical integration of the KdV equation, see for instance [10, 11, 12, 13, 14, 20] and the references therein. We also refer to [2] for a splitting approach for the Kadomtsev-Petviashvili equation.

In particular, a distinguished convergence result was obtained in [11, 10]. In the latter it was proven that the Strang splitting, where the right-hand side of the KdV equation is split into the linear and Burgers part, respectively, is second-order convergent in H^r for initial data in H^{r+5} for $r \geq 1$ assuming that the Burgers part is solved exactly.

Here we derive a first-order exponential-type time-integrator for the KdV equation (1) based on Duhamel's formula

$$u(t) = \mathrm{e}^{-\partial_x^3 t} u_0 + \frac{1}{2}\int_0^t \mathrm{e}^{-\partial_x^3 (t-s)} \partial_x (u(s))^2 \mathrm{d}s \tag{2}$$

Key words and phrases. — KdV equation – exponential-type time integrator – convergence.

looking at the "twisted variable" $v(t) = \mathrm{e}^{\partial_x^3 t} u(t)$. This idea of "twisting" the variable is widely used in the analysis of partial differential equations in low regularity spaces (see, for instance [1, 5, 19] for the periodic KdV equation) and also well known in the context of numerical analysis, see [15] for the introduction of Lawson type Runge-Kutta methods. However, instead of approximating the appearing integral with a Runge-Kutta method (see for instance [13]) we use the key relation

$$k_1^3 + k_2^3 - (k_1 + k_2)^3 = -3(k_1 + k_2)k_1 k_2 \tag{3}$$

which allows us to overcome the *loss of derivative* by integrating the *stiff parts* (i.e., the terms involving ∂_x^3) exactly. The derived exponential-type integrator is unconditionally stable and we will in particular show its first-order convergence in H^1 for initial data in H^3. A key tool in our convergence analysis is a variant of [10, Lemma 3.1].

The presented technique can be generalized to higher-order methods. We outline the construction of a second-order exponential-type integrator in Remark 1.4.

Notation: In the following we will denote the Fourier expansion of some function $f \in L^2(\mathbb{T})$ by $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{ikx}$. Furthermore, we will use the notation

$$(\partial_x^{-1})_k := \begin{cases} (ik)^{-1} & \text{if} \quad k \neq 0 \\ 0 & \text{if} \quad k = 0 \end{cases}, \qquad \text{i.e.,} \qquad \partial_x^{-1} f(x) = \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} (ik)^{-1} \hat{f}_k \mathrm{e}^{ikx}. \tag{4}$$

1. An exponential-type integrator

To illustrate the idea we first consider initial values with zero mean. In Remark 1.3 we point out the generalization to general initial values.

Assumption 1.1. — Assume that the zero-mode of the initial value is zero, i.e., $\hat{u}_0(0) = (2\pi)^{-1} \int_{\mathbb{T}} u(0,x)\mathrm{d}x = 0$. Note that the conservation of mass then implies that $\hat{u}_0(t) = 0$.

We will derive a scheme for the "twisted" variable $v(t) = \mathrm{e}^{\partial_x^3 t} u(t)$. With this transformation at hand the equation in v reads

$$v(t) = v_0 + \frac{1}{2} \int_0^t \mathrm{e}^{s\partial_x^3} \partial_x (\mathrm{e}^{-\partial_x^3 s} v(s))^2 \mathrm{d}s \tag{5}$$

such that

$$v(t_n + \tau) = v(t_n) + \frac{1}{2} \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \left(\mathrm{e}^{-\partial_x^3 (t_n+s)} v(t_n + s) \right)^2 \mathrm{d}s. \tag{6}$$

For a small time-step τ we iterate Duhamel's formula (5) and approximate the exact solution (6) as follows

$$v(t_n + \tau) \approx v(t_n) + \frac{1}{2} \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3} \partial_x \left(\mathrm{e}^{-\partial_x^3 (t_n+s)} v(t_n) \right)^2 \mathrm{d}s. \tag{7}$$

The key relation (3) now allows us the following integration technique (cf. [1, 5, 19]): We have

$$
\begin{aligned}
&\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x(\mathrm{e}^{-\partial_x^3(t_n+s)}v(t_n))^2\mathrm{d}s\\
&=\sum_{k_1,k_2}\int_0^\tau \mathrm{e}^{-i(t_n+s)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}i(k_1+k_2)\hat{v}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\mathrm{d}s\\
&=\sum_{k_1,k_2}\frac{\mathrm{e}^{-i(t_n+\tau)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}-\mathrm{e}^{-it_n\left((k_1+k_2)^3-k_1^3-k_2^3\right)}}{-i\left((k_1+k_2)^3-k_1^3-k_2^3\right)}i(k_1+k_2)\hat{v}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x} \qquad (8)\\
&=\sum_{k_1,k_2}\left(\mathrm{e}^{-i(t_n+\tau)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}-\mathrm{e}^{-it_n\left((k_1+k_2)^3-k_1^3-k_2^3\right)}\right)\frac{1}{-3k_1k_2}\hat{v}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\\
&=\frac{1}{3}\mathrm{e}^{\partial_x^3(t_n+\tau)}\left(\mathrm{e}^{-\partial_x^3(t_n+\tau)}\partial_x^{-1}v(t_n)\right)^2-\frac{1}{3}\mathrm{e}^{\partial_x^3t_n}\left(\mathrm{e}^{-\partial_x^3t_n}\partial_x^{-1}v(t_n)\right)^2.
\end{aligned}
$$

Together with the approximation in (7) this yields that

$$
v^{n+1}=v^n+\frac{1}{6}\mathrm{e}^{\partial_x^3(t_n+\tau)}\left(\mathrm{e}^{-\partial_x^3(t_n+\tau)}\partial_x^{-1}v^n\right)^2-\frac{1}{6}\mathrm{e}^{\partial_x^3t_n}\left(\mathrm{e}^{-\partial_x^3t_n}\partial_x^{-1}v^n\right)^2, \qquad (9)
$$

where ∂_x^{-1} is defined in (4) and by construction $\hat{v}_0^{n+1}=0$, see Remark 1.2 below.

Remark 1.2. — The zero-mode is preserved by the scheme (9) as the key relation (3) implies that

$$
\hat{v}_0^{n+1}=\hat{v}_0^n-\frac{1}{6}\sum_{k_1+k_2=0}\left(\mathrm{e}^{-i(t_n+\tau)3(k_1+k_2)k_1k_2}-\mathrm{e}^{-it_n3(k_1+k_2)k_1k_2}\right)\frac{1}{k_1k_2}\hat{v}_{k_1}^n\hat{v}_{k_2}^n=\hat{v}_0^n.
$$

In order to obtain an approximation to the original solution $u(t_n)$ of the KdV equation (1) at time $t_n=n\tau$ we then "twist" the variable back again by setting $u^n=\mathrm{e}^{-\partial_x^3t_n}v^n$. This yields the following exponential-type integrator for the KdV equation (1)

$$
u^{n+1}=\mathrm{e}^{-\tau\partial_x^3}u^n+\frac{1}{6}\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}u^n\right)^2-\frac{1}{6}\mathrm{e}^{-\tau\partial_x^3}\left(\partial_x^{-1}u^n\right)^2, \qquad (10)
$$

where ∂_x^{-1} is defined in (4) and $\hat{u}_0^{n+1}=0$ thanks to Remark 1.2. For sufficiently smooth solutions the semi-discrete scheme (10) is first-order convergent, see Corollary 2.8 below for the precise convergence result.

Remark 1.3. — If $\hat{u}_0(0)=\alpha\neq 0$ we set $\tilde{u}:=u-\alpha$ and look at the modified KdV equation in $\tilde{u}$, i.e.,

$$
\partial_t\tilde{u}+\partial_x^3\tilde{u}=\alpha\partial_x\tilde{u}+\frac{1}{2}\partial_x(\tilde{u})^2. \qquad (11)
$$

Note that the solution $\tilde{u}(t)$ of the modified KdV equation (11) satisfies $\hat{\tilde{u}}_0(t)=0$ for all t as by the conservation of mass we have that $\hat{u}_0(t)\equiv\hat{u}_0(0)=\alpha$. Thus, we can proceed as above: We

look at the twisted variable $\tilde{v}(t) = \mathrm{e}^{(\partial_x^3 - \alpha\partial_x)t}\tilde{u}(t)$ and carry out an approximation as above, i.e.,

$$\tilde{v}(t_n+\tau) \approx \tilde{v}(t_n) + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)(\partial_x^3-\alpha\partial_x)}\partial_x\left(\mathrm{e}^{-(t_n+s)(\partial_x^3-\alpha\partial_x)}\tilde{v}(t_n)\right)^2 \mathrm{d}s.$$

The relation

$$-(k_1+k_2)^3 + \alpha(k_1+k_2) + k_1^3 + k_2^3 - \alpha k_1 - \alpha k_2 = -(k_1+k_2)^3 + k_1^3 + k_2^3$$

then allows us to derive similarly to above an exponential-type integration scheme

(12)
$$\tilde{v}^{n+1} = \tilde{v}^n + \frac{1}{6}\mathrm{e}^{(\partial_x^3-\alpha\partial_x)(t_n+\tau)}\left(\mathrm{e}^{-(\partial_x^3-\alpha\partial_x)(t_n+\tau)}\partial_x^{-1}\tilde{v}^n\right)^2 - \frac{1}{6}\mathrm{e}^{(\partial_x^3-\alpha\partial_x)t_n}\left(\mathrm{e}^{-(\partial_x^3-\alpha\partial_x)t_n}\partial_x^{-1}\tilde{v}^n\right)^2,$$

where ∂_x^{-1} is defined in (4) and $\hat{\tilde{v}}_0^{n+1} = 0$ cf. Remark 1.2. Finally, by setting $u^n = \mathrm{e}^{-(\partial_x^3-\alpha\partial_x)t_n}\tilde{v}^n + \alpha$ we then obtain an approximation to the exact solution $u(t_n)$ of the KdV equation (1) (with non-zero zero-mode) at time $t_n = n\tau$.

Note that higher-order approximations to the solution of the KdV equation (1) can be obtained by truncating the expansion in (6) later. In Remark 1.4 below we explain the construction of a second-order scheme in more detail.

***Remark 1.4* (A second-order exponential-type integrator).** — In order to derive a second-order approximation in the "twisted" variable $v(t)$ we need to include the second-order term in the Taylor-series expansion of $v(t_n+s)$ in (6). More precisely, plugging the formal expansion

$$v(t_n+s) = v(t_n) + sv'(t_n) + \mathcal{O}\left(s^2 v''\right)$$

into Duhamel's formula (6) yields that

$$\begin{aligned} v(t_n+\tau) &= v(t_n) + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-\partial_x^3(t_n+s)}\big(v(t_n)+sv'(t_n)\big)\right)^2 \mathrm{d}s + \mathcal{R}_1(\tau,t_n,v) \\ &= v(t_n) + \frac{1}{2}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\Big[\left(\mathrm{e}^{-\partial_x^3(t_n+s)}v(t_n)\right)^2 \\ &\qquad\qquad + 2s\left(\mathrm{e}^{-\partial_x^3(t_n+s)}v(t_n)\right)\left(\mathrm{e}^{-\partial_x^3(t_n+s)}v'(t_n)\right)\Big]\mathrm{d}s \\ &\quad + \mathcal{R}_1(\tau,t_n,v) + \mathcal{R}_2(\tau,t_n,v), \end{aligned} \tag{13}$$

where the remainders satisfy for $r > 1/2$ and some constant $c > 0$ that

$$\|\mathcal{R}_1(\tau,t_n,v)\|_r + \|\mathcal{R}_2(\tau,t_n,v)\|_r \le c\tau^3 \sup_{t_n\le t\le t_{n+1}}\Big(\|\partial_x\big(vv''\big)(t)\|_r + \|\partial_x(v')^2(t)\|_r\Big). \tag{14}$$

In order to construct a numerical scheme based on the expansion (13) we need to solve the appearing integral. The first term involving the product v^2 can be easily determined thanks

to (8). Note that $\hat{v}_0(t) = \hat{v}_0'(t) = 0$. Thus, similarly, we obtain for the $vv'-$ term with the aid of the key-relation (3) and integration by parts that

$$
\begin{aligned}
&\int_0^\tau s\mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-\partial_x^3(t_n+s)}v(t_n)\right)\left(\mathrm{e}^{-\partial_x^3(t_n+s)}v'(t_n)\right)\mathrm{d}s\\
&=\sum_{k_1,k_2}\int_0^\tau s\cdot\mathrm{e}^{-i(t_n+s)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}i(k_1+k_2)\hat{v}_{k_1}(t_n)\hat{v}'_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\mathrm{d}s\\
&=\sum_{\substack{k_1\neq0,k_2\neq0\\k_1+k_2\neq0}}\int_0^\tau s\cdot\mathrm{e}^{-i(t_n+s)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}i(k_1+k_2)\hat{v}_{k_1}(t_n)\hat{v}'_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\mathrm{d}s\\
&=\tau\sum_{\substack{k_1\neq0,k_2\neq0\\k_1+k_2\neq0}}\mathrm{e}^{-i(t_n+\tau)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}\frac{1}{-3k_1k_2}\hat{v}_{k_1}(t_n)\hat{v}'_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\\
&-\sum_{\substack{k_1\neq0,k_2\neq0\\k_1+k_2\neq0}}\int_0^\tau\mathrm{e}^{-i(t_n+s)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}\frac{1}{-3k_1k_2}\hat{v}_{k_1}(t_n)\hat{v}'_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\mathrm{d}s\\
&=\tau\sum_{\substack{k_1\neq0,k_2\neq0\\k_1+k_2\neq0}}\mathrm{e}^{-i(t_n+\tau)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}\frac{1}{-3k_1k_2}\hat{v}_{k_1}(t_n)\hat{v}'_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\\
&-\sum_{\substack{k_1\neq0,k_2\neq0\\k_1+k_2\neq0}}\frac{\mathrm{e}^{-i(t_n+\tau)\left((k_1+k_2)^3-k_1^3-k_2^3\right)}-\mathrm{e}^{-it_n\left((k_1+k_2)^3-k_1^3-k_2^3\right)}}{9ik_1^2k_2^2(k_1+k_2)}\hat{v}_{k_1}(t_n)\hat{v}'_{k_2}(t_n)\mathrm{e}^{i(k_1+k_2)x}\mathrm{d}s\\
&=\frac{\tau}{3}\mathrm{e}^{(t_n+\tau)\partial_x^3}\left(\mathrm{e}^{-\partial_x^3(t_n+\tau)}\partial_x^{-1}v(t_n)\right)\left(\mathrm{e}^{-\partial_x^3(t_n+\tau)}\partial_x^{-1}v'(t_n)\right)\\
&-\frac{1}{9}\mathrm{e}^{(t_n+\tau)\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-(t_n+\tau)\partial_x^3}(\partial_x^{-1})^2v(t_n)\right)\left(\mathrm{e}^{-(t_n+\tau)\partial_x^3}(\partial_x^{-1})^2v'(t_n)\right)\\
&+\frac{1}{9}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}(\partial_x^{-1})^2v(t_n)\right)\left(\mathrm{e}^{-t_n\partial_x^3}(\partial_x^{-1})^2v'(t_n)\right)
\end{aligned}
\tag{15}
$$

with ∂_x^{-1} defined in (4). Plugging the relations given in (8) and (15) together with the definition

$$
v'^n := \frac{1}{2}\mathrm{e}^{t_n\partial_x^3}\partial_x\left(\mathrm{e}^{-t_n\partial_x^3}v^n\right)^2 \tag{16}
$$

(see (5)) into the expansion (13) builds the basis of our numerical scheme: As a second-order approximation to the solution $v(t_n+\tau)$ of (5) we take the exponential-type integration scheme

$$\begin{aligned} v^{n+1} &= v^n + \frac{1}{6}\mathrm{e}^{\partial_x^3(t_n+\tau)}\left(\mathrm{e}^{-\partial_x^3(t_n+\tau)}\partial_x^{-1}v^n\right)^2 - \frac{1}{6}\mathrm{e}^{\partial_x^3 t_n}\left(\mathrm{e}^{-\partial_x^3 t_n}\partial_x^{-1}v^n\right)^2 \\ &+ \frac{\tau}{3}\mathrm{e}^{(t_n+\tau)\partial_x^3}\left(\mathrm{e}^{-\partial_x^3(t_n+\tau)}\partial_x^{-1}v^n\right)\left(\mathrm{e}^{-\partial_x^3(t_n+\tau)}\partial_x^{-1}v'^n\right) \\ &- \frac{1}{9}\mathrm{e}^{(t_n+\tau)\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-(t_n+\tau)\partial_x^3}(\partial_x^{-1})^2v^n\right)\left(\mathrm{e}^{-(t_n+\tau)\partial_x^3}(\partial_x^{-1})^2v'^n\right) \\ &+ \frac{1}{9}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}(\partial_x^{-1})^2v^n\right)\left(\mathrm{e}^{-t_n\partial_x^3}(\partial_x^{-1})^2v'^n\right) \end{aligned}$$

with v'^n given in (16), ∂_x^{-1} defined in (4) and by construction $\hat v_0^{n+1}=0$ (cf. Remark 1.2).

The approximation to the original solution $u(t_n)$ of the KdV equation (1) at time $t_n=n\tau$ is then obtained by "twisting" the variable back again, i.e., by setting $u^n=\mathrm{e}^{-\partial_x^3 t_n}v^n$. This yields that

(17)

$$\begin{aligned} u^{n+1} &= \mathrm{e}^{-\partial_x^3\tau}u^n + \frac{1}{6}\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}u^n\right)^2 - \frac{1}{6}\mathrm{e}^{-\partial_x^3\tau}\left(\partial_x^{-1}u^n\right)^2 + \frac{\tau}{3}\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}u^n\right)\left(\mathrm{e}^{-\partial_x^3\tau}\tfrac{1}{2}(u^n)^2\right) \\ &- \frac{1}{9}\partial_x^{-1}\left(\mathrm{e}^{-\partial_x^3\tau}(\partial_x^{-1})^2u^n\right)\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\tfrac{1}{2}(u^n)^2\right) + \frac{1}{9}\partial_x^{-1}\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\left((\partial_x^{-1})^2u^n\right)\left(\partial_x^{-1}\tfrac{1}{2}(u^n)^2\right), \end{aligned}$$

where ∂_x^{-1} is defined in (4) and by construction $\hat u_0^{n+1}=0$ (cf. Remark 1.2). The semi-discret exponential-type integration scheme (17) *allows formally second-order convergence* in H^r for sufficiently smooth solutions $u(t)\in H^{r+5}$ with $r>1/2$ thanks to the local error bound (14) together with the observation that

$$\|v''(t)\|_r \le c\left(\|\partial_x^2 v(t)\|_r^3 + \|\partial_x^4 v(t)\|_r^2\right) = c\left(\|\partial_x^2 u(t)\|_r^3 + \|\partial_x^4 u(t)\|_r^2\right).$$

We do not pursuit this here and only underline the second-order convergence rate numerically in Section 3.

In the following section we give a detailed convergence analysis of the first-order exponential-type integration scheme (10).

2. Error analysis

For simplicity we carry out the error analysis for initial values satisfying Assumption 1.1. Furthermore, in the following we denote by $\langle\cdot,\cdot\rangle$ the L^2 scalar product, i.e., $\langle f,g\rangle=\int_{\mathbb{T}} fg\mathrm{d}x$ and by $\|\cdot\|_{L^2}$ the corresponding L^2 norm.

In order to obtain a convergence result in H^1 we follow the strategy presented in [**16, 10**]: We first prove convergence order of one half of the numerical scheme (9) in H^2 for solutions in H^3, see Section 2.1 Theorem 2.6. This yields essential a priori bounds on the numerical solution in H^2 and allows us to prove first-order convergence globally in H^1, see Theorem 2.7 in Section 2.2. The latter in particular implies first-order convergence of the exponential-type

integration scheme (10) towards the KdV solution (1), see Corollary 2.8 below for the precise convergence result.

2.1. Error analysis in H^2. — We commence with the error analysis of the numerical scheme (9) in H^2. In Section 2.1.1 we carry out the stability analysis in H^2. In Section 2.1.2 we show that the method is consistent of order one half in H^2 for solutions in H^3.

2.1.1. Stability analysis. — Set

$$\Phi_t^\tau(v) := v + \frac{1}{6}\mathrm{e}^{\partial_x^3(t+\tau)}\left(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}v\right)^2 - \frac{1}{6}\mathrm{e}^{\partial_x^3 t}\left(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}v\right)^2 \tag{18}$$

such that for all k we have $v^{k+1} = \Phi_{t_k}^\tau(v^k)$. The following stability result holds for the numerical flow Φ_t^τ:

Lemma 2.1. — *Let $f \in H^2$ and $g \in H^3$. Then, for all $t \in \mathbb{R}$ we have*

$$\left\|\partial_x^2(\Phi_t^\tau(f) - \Phi_t^\tau(g))\right\|_{L^2} \leq \exp(\tau L)\|\partial_x^2(f-g)\|_{L^2},$$

where L depends on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^3 g\|_{L^2}$.

Proof. — Note that

$$\begin{aligned}
&\|\partial_x^2(\Phi_t^\tau(f) - \Phi_t^\tau(g))\|_{L^2}^2 = \|\partial_x^2(f-g)\|_{L^2}^2 \\
&+ \frac{1}{3}\langle \partial_x^2 \mathrm{e}^{\partial_x^3(t+\tau)}\Big[\left(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}f\right)^2 - \left(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}g\right)^2\Big], \partial_x^2(f-g)\rangle \\
&- \frac{1}{3}\langle \partial_x^2 \mathrm{e}^{\partial_x^3 t}\Big[\left(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}f\right)^2 - \left(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}g\right)^2\Big], \partial_x^2(f-g)\rangle \\
&+ \frac{1}{6^2}\|\partial_x^2 \mathrm{e}^{\partial_x^3(t+\tau)}\Big[\left(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}f\right)^2 - \left(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}g\right)^2\Big] \\
&- \partial_x^2 \mathrm{e}^{\partial_x^3 t}\Big[\left(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}f\right)^2 - \left(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}g\right)^2\Big]\|_{L^2}^2 \\
&=: \|\partial_x^2(f-g)\|_{L^2}^2 + \frac{1}{3}I_1 + \frac{1}{6^2}I_2.
\end{aligned}$$

Lemma 2.3 and Lemma 2.4 below allow us the following bounds on I_1 and I_2: We have

$$|I_1 + I_2| \leq \tau L\|\partial_x^2(f-g)\|_{L^2}^2, \tag{19}$$

where L depends on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^3 g\|_{L^2}$. Hence,

$$\|\partial_x^2\big(\Phi_t^\tau(f) - \Phi_t^\tau(g)\big)\|_{L^2}^2 \leq (1+\tau L)\|\partial_x^2(f-g)\|_{L^2}^2$$

which yields the assertion. □

In the rest of Section 2.1.1 we will show the essential bound (19). We start with a useful Lemma.

Lemma 2.2. — *The following estimates hold for $u, v, w \in H^2$*

$$
\begin{aligned}
&|\langle \partial_x^2 u, vw - \mathrm{e}^{\partial_x^3 \tau}\Big[\Big(\mathrm{e}^{-\partial_x^3\tau} v\Big)\Big(\mathrm{e}^{-\partial_x^3\tau} w\Big)\Big]\rangle| \leq c\tau \|\partial_x^2 u\|_{L^2}\|\partial_x^2 v\|_{L^2}\|\partial_x^2 w\|_{L^2} \\
&|\langle u, (\partial_x v)^2 - \mathrm{e}^{\partial_x^3\tau}\Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x v\Big)^2\rangle| \leq c\tau \|\partial_x^2 u\|_{L^2}\|\partial_x^2 v\|_{L^2}^2
\end{aligned} \tag{20}
$$

for some constant $c > 0$.

Proof. — The key relation (3) together with the Cauchy-Schwarz inequality allows us the following bound

$$
\begin{aligned}
&|\langle \partial_x^2 u, vw - \mathrm{e}^{\partial_x^3 \tau}\Big[\Big(\mathrm{e}^{-\partial_x^3\tau} v\Big)\Big(\mathrm{e}^{-\partial_x^3\tau} w\Big)\Big]\rangle| \\
&= |\sum_{k_1,k_2} (k_1+k_2)^2 \hat{u}_{-(k_1+k_2)}\Big(1 - \mathrm{e}^{-i\tau\left((k_1+k_2)^3 - k_1^3 - k_2^3\right)}\Big)\hat{v}_{k_1}\hat{w}_{k_2}| \\
&= |\sum_{k_1,k_2} (k_1+k_2)^2 \hat{u}_{-(k_1+k_2)}\Big(1 - \mathrm{e}^{-i\tau 3k_1k_2(k_1+k_2)}\Big)\hat{v}_{k_1}\hat{w}_{k_2}| \\
&\leq 3\tau \sum_{k_1,k_2} |(k_1+k_2)^2 \hat{u}_{-(k_1+k_2)}||(k_1+k_2)k_1k_2\hat{v}_{k_1}\hat{w}_{k_2}| \\
&= 3\tau \sum_{l,k} l^2 |\hat{u}_{-l}||lk(l-k)||\hat{v}_k\hat{w}_{l-k}| \\
&\leq 3\tau \sum_{l,k} l^2 |\hat{u}_{-l}|\left(|k(l-k)^2||\hat{v}_k\hat{w}_{l-k}| + |k|^2|l-k||\hat{v}_k\hat{w}_{l-k}|\right) \\
&\leq 3\tau \Big(\sum_l l^4|\hat{u}_l|^2\Big)^{1/2}\Big(\sum_l\Big(\sum_k |k||\hat{v}_k||l-k|^2|\hat{w}_{l-k}|\Big)^2\Big)^{1/2} \\
&+ 3\tau \Big(\sum_l l^4|\hat{u}_l|^2\Big)^{1/2}\Big(\sum_l\Big(\sum_k |k|^2|\hat{v}_k||l-k||\hat{w}_{l-k}|\Big)^2\Big)^{1/2} \\
&\leq 3\tau \|\partial_x^2 u\|_{L^2}\big(\|v^{(1)} * w^{(2)}\|_{l^2} + \|v^{(2)} * w^{(1)}\|_{l^2}\big),
\end{aligned} \tag{21}
$$

where $v^{(j)}(k) := |k|^j|\hat{v}_k|$ and $w^{(j)}(k) := |k|^j|\hat{w}_k|$. By the Young and Cauchy-Schwarz inequality we furthermore obtain that

$$
\begin{aligned}
\|v^{(1)} * w^{(2)}\|_{l^2} + \|v^{(2)} * w^{(1)}\|_{l^2} &\leq \|v^{(1)}\|_{l^1}\|w^{(2)}\|_{l^2} + \|w^{(1)}\|_{l^1}\|v^{(2)}\|_{l^2} \leq c\|v^{(2)}\|_{l^2}\|w^{(2)}\|_{l^2} \\
&\leq c\|\partial_x^2 v\|_{L^2}\|\partial_x^2 w\|_{L^2}
\end{aligned} \tag{22}
$$

for some constant $c > 0$. Plugging (22) into (21) yields the first assertion.

Similarly we have that

$$
\begin{aligned}
&|\langle u, (\partial_x v)^2 - \mathrm{e}^{\partial_x^3\tau}\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x v\right)^2\rangle| \leq 3\tau \sum_{k_1,k_2} |(k_1+k_2)\hat{u}_{-(k_1+k_2)}||k_1k_2|^2|\hat{v}_{k_1}\hat{v}_{k_2}| \\
&= 3\tau \sum_{k,l} |l||\hat{u}_{-l}|k^2(l-k)^2|\hat{v}_k\hat{v}_{k-l}| \leq 3\tau\Big(\sum_k k^4|\hat{v}_k|^2\Big)^{1/2}\|u^{(1)} * v^{(2)}\|_{l^2} \qquad (23)\\
&\leq c\tau\|\partial_x^2 v\|_{L^2}\|u^{(1)}\|_{l^1}\|v^{(2)}\|_{l^2} \leq c\tau\|\partial_x^2 v\|_{L^2}^2\|\partial_x^2 u\|_{L^2}
\end{aligned}
$$

which yields the second assertion. □

***Lemma 2.3* (Bound on I_1).** — *We have*

$$|I_1| \leq c\tau\Big(\|\partial_x^2(f-g)\|_{L^2} + \|\partial_x^3 g\|_{L^2}\Big)\|\partial_x^2(f-g)\|_{L^2}^2$$

for some constant $c > 0$.

Proof. — Note that for all $t \in \mathbb{R}$ the following relation holds

$$\langle \mathrm{e}^{t\partial_x^3} f, g\rangle = \langle f, \mathrm{e}^{-t\partial_x^3} g\rangle.$$

Thus, by setting $\tilde{f} = \mathrm{e}^{-t\partial_x^3} f$ and $\tilde{g} = \mathrm{e}^{-t\partial_x^3} g$ we obtain that

$$
\begin{aligned}
I_1 = &\langle \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2, \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde{f}-\tilde{g})\rangle \\
&- \langle \partial_x^2\left(\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\partial_x^{-1}\tilde{g}\right)^2, \partial_x^2(\tilde{f}-\tilde{g})\rangle.
\end{aligned}
$$

Using the relation $f^2 - g^2 = (f-g)^2 + 2(f-g)g$ as well as the chain rule yields that

$$
\begin{aligned}
I_1 = &\langle \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}(\tilde{f}-\tilde{g})\right)^2 + 2\partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}(\tilde{f}-\tilde{g})\right)\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right), \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde{f}-\tilde{g})\rangle \\
&- \langle \partial_x^2\left(\partial_x^{-1}(\tilde{f}-\tilde{g})\right)^2 + 2\partial_x^2\left(\partial_x^{-1}(\tilde{f}-\tilde{g})\right)\left(\partial_x^{-1}\tilde{g}\right), \partial_x^2(\tilde{f}-\tilde{g})\rangle \\
&= 2\langle\left(\mathrm{e}^{-\partial_x^3\tau}(\tilde{f}-\tilde{g})\right)^2 + \left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}(\tilde{f}-\tilde{g})\right)\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x(\tilde{f}-\tilde{g})\right), \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde{f}-\tilde{g})\rangle \\
&+ 2\langle\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x(\tilde{f}-\tilde{g})\right)\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right) + 2\left(\mathrm{e}^{-\partial_x^3\tau}(\tilde{f}-\tilde{g})\right)\left(\mathrm{e}^{-\partial_x^3\tau}\tilde{g}\right), \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde{f}-\tilde{g})\rangle \\
&+ 2\langle\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}(\tilde{f}-\tilde{g})\right)\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x\tilde{g}\right), \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde{f}-\tilde{g})\rangle \\
&- 2\langle\left(\partial_x(\tilde{f}-\tilde{g})\right)\left(\partial_x^{-1}(\tilde{f}-\tilde{g})\right) + (\tilde{f}-\tilde{g})^2, \partial_x^2(\tilde{f}-\tilde{g})\rangle \\
&- 2\langle\left(\partial_x^{-1}(\tilde{f}-\tilde{g})\right)\partial_x\tilde{g} + 2(\tilde{f}-\tilde{g})\tilde{g} + \left(\partial_x(\tilde{f}-\tilde{g})\right)\left(\partial_x^{-1}\tilde{g}\right), \partial_x^2(\tilde{f}-\tilde{g})\rangle.
\end{aligned}
$$

Next we use another key fact namely that

$$\langle vu, \partial_x u\rangle = \frac{1}{2}\langle v, \partial_x(u)^2\rangle = -\frac{1}{2}\langle \partial_x v, u^2\rangle$$

as well as that

$$\langle u-v, (\partial_x(u-v))^2\rangle = -\frac{1}{2}\langle (u-v)^2, \partial_x^2(u-v)\rangle.$$

This yields that

$$\begin{aligned}
I_1 &= 2\langle \Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f-\tilde g)\Big)^2, \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde f-\tilde g)\rangle \\
&+ \frac{1}{2}\langle \Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f-\tilde g)\Big)^2, \partial_x^2\mathrm{e}^{-\partial_x^3\tau}(\tilde f-\tilde g)\rangle \\
&- \langle \mathrm{e}^{-\partial_x^3\tau}\tilde g, \Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x(\tilde f-\tilde g)\Big)^2\rangle \\
&+ 4\langle \Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f-\tilde g)\Big)\Big(\mathrm{e}^{-\partial_x^3\tau}\tilde g\Big), \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde f-\tilde g)\rangle \\
&+ 2\langle \Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}(\tilde f-\tilde g)\Big)\Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x\tilde g\Big), \mathrm{e}^{-\partial_x^3\tau}\partial_x^2(\tilde f-\tilde g)\rangle \\
&- \frac{1}{2}\langle (\tilde f-\tilde g)^2, \partial_x^2(\tilde f-\tilde g)\rangle \\
&- 2\langle (\tilde f-\tilde g)^2, \partial_x^2(\tilde f-\tilde g)\rangle \\
&- 2\langle \Big(\partial_x^{-1}(\tilde f-\tilde g)\Big)\partial_x\tilde g, \partial_x^2(\tilde f-\tilde g)\rangle \\
&- 4\langle (\tilde f-\tilde g)\tilde g, \partial_x^2(\tilde f-\tilde g)\rangle \\
&+ \langle \tilde g, \Big(\partial_x(\tilde f-\tilde g)\Big)^2\rangle.
\end{aligned}$$

Thus, rearranging the terms leads to

$$\begin{aligned}
I_1 &= \frac{5}{2}\langle \mathrm{e}^{\partial_x^3\tau}\Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f-\tilde g)\Big)^2 - (\tilde f-\tilde g)^2, \partial_x^2(\tilde f-\tilde g)\rangle \\
&- \langle \mathrm{e}^{\partial_x^3\tau}\Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x(\tilde f-\tilde g)\Big)^2 - \Big(\partial_x(\tilde f-\tilde g)\Big)^2, \tilde g\rangle \\
&+ 4\langle \mathrm{e}^{\partial_x^3\tau}\Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f-\tilde g)\Big)\Big(\mathrm{e}^{-\partial_x^3\tau}\tilde g\Big) - (\tilde f-\tilde g)\,(\tilde g)\,, \partial_x^2(\tilde f-\tilde g)\rangle \\
&+ 2\langle \mathrm{e}^{\partial_x^3\tau}\Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}(\tilde f-\tilde g)\Big)\Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x\tilde g\Big) - \Big(\partial_x^{-1}(\tilde f-\tilde g)\Big)\,(\partial_x\tilde g)\,, \partial_x^2(\tilde f-\tilde g)\rangle.
\end{aligned}$$

With the aid of Lemma 2.2 we thus obtain that

$$|I_1| \le \tau c\Big(\|\partial_x^2(f-g)\|_{L^2} + \|\partial_x^3 g\|_{L^2}\Big)\|\partial_x^2(f-g)\|_{L^2}^2$$

for some constant $c>0$. □

***Lemma 2.4* (Bound on I_2).** — *We have*

$$|I_2| \le \tau M\|\partial_x^2(f-g)\|_{L^2}^2,$$

where M depends on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^2 g\|_{L^2}$.

Proof. — In the following let M denote a constant depending on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^2 g\|_{L^2}$. Setting $\tilde{f} = \mathrm{e}^{-t\partial_x^3} f$ and $\tilde{g} = \mathrm{e}^{-t\partial_x^3} g$ yields that

$$
\begin{aligned}
I_2 &= \langle \partial_x^2 \left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2, \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2\rangle \\
&- 2\langle \partial_x^2 \mathrm{e}^{\partial_x^3\tau}\Big[\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2\Big], \partial_x^2\left(\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\partial_x^{-1}\tilde{g}\right)^2\rangle \\
&+ \langle \partial_x^2\left(\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\partial_x^{-1}\tilde{g}\right)^2, \partial_x^2\left(\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\partial_x^{-1}\tilde{g}\right)^2\rangle \\
&= I_2^a + I_2^b
\end{aligned} \tag{24}
$$

with

$$
\begin{aligned}
I_2^a =& \langle \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2 - \partial_x^2\mathrm{e}^{-\partial_x^3\tau}\Big[\left(\partial_x^{-1}\tilde{f}\right)^2 - \left(\partial_x^{-1}\tilde{g}\right)^2\Big], \\
& \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2\rangle \\
I_2^b =& -\langle \partial_x^2\mathrm{e}^{\partial_x^3\tau}\Big[\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2\Big] - \partial_x^2\left(\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\partial_x^{-1}\tilde{g}\right)^2, \\
& \partial_x^2\left(\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\partial_x^{-1}\tilde{g}\right)^2\rangle.
\end{aligned}
$$

Similarly to Lemma 2.2 we obtain with $F := \partial_x^2\left(\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\partial_x^{-1}\tilde{g}\right)^2$ by the key relation (3) using the Cauchy-Schwarz and Young inequality that

$$
\begin{aligned}
|I_2^b| &= |\sum_{k_1,k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)^2}{k_1k_2}\left(1-\mathrm{e}^{-i\tau\left((k_1+k_2)^3-k_1^3-k_2^3\right)}\right)\left(\hat{\tilde{f}}_{k_1}\hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1}\hat{\tilde{g}}_{k_2}\right)| \\
&= |\sum_{k_1,k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)^2}{k_1k_2}\left(1-\mathrm{e}^{-i\tau 3k_1k_2(k_1+k_2)}\right)\left(\hat{\tilde{f}}_{k_1}\hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1}\hat{\tilde{g}}_{k_2}\right)| \\
&\leq 3\tau \sum_{k_1,k_2} |\hat{F}_{-(k_1+k_2)}(k_1+k_2)||(k_1+k_2)^2\left(\hat{\tilde{f}}_{k_1}\hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1}\hat{\tilde{g}}_{k_2}\right)| \\
&\leq 3\tau \sum_{k,l} |l\hat{F}_{-l}|\big((l-k)^2 + 2|(l-k)k| + k^2\big)\big|(\tilde{f}_k - \tilde{g}_k)\tilde{f}_{l-k} + \tilde{g}_k(\tilde{f}_{l-k} - \tilde{g}_{l-k})\big| \\
&\leq 6\tau\|\partial_x F\|_{L^2}\Big(\sum_{j=0,1}\|(\tilde{f}-\tilde{g})^{(2j)} * \tilde{g}^{(2-2j)}\|_{l^2} + \|(\tilde{f}-\tilde{g})^{(2j)} * \tilde{f}^{(2-2j)}\|_{l^2}\Big) \\
&+ 6\tau\|\partial_x F\|_{L^2}\big(\|(\tilde{f}-\tilde{g})^{(1)} * \tilde{f}^{(1)}\|_{l^2} + \|(\tilde{f}-\tilde{g})^{(1)} * \tilde{g}^{(1)}\|_{l^2}\big) \\
&\leq c\tau\|\partial_x F\|_{L^2}\|\partial_x^2(f-g)\|_{L^2}(\|\partial_x^2 f\|_{L^2} + \|\partial_x^2 g\|_{L^2}) \\
&\leq \tau M\|\partial_x^2(f-g)\|_{L^2}^2,
\end{aligned} \tag{25}
$$

where again we used the notation $\Phi^{(j)}(k) := |k|^j|\hat{\Phi}_k|$. Similarly, we obtain for I_2^a with $F := \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{f}\right)^2 - \partial_x^2\left(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}\tilde{g}\right)^2$ that

$$
\begin{aligned}
|I_2^a| &= |\sum_{k_1,k_2} \hat{F}_{-(k_1+k_2)}\frac{(k_1+k_2)^2}{k_1k_2}\left(\mathrm{e}^{i\tau(k_1^3+k_2^3)} - \mathrm{e}^{i\tau(k_1+k_2)^3}\right)\left(\hat{\tilde{f}}_{k_1}\hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1}\hat{\tilde{g}}_{k_2}\right)| \\
&\le \sum_{k_1,k_2} |\hat{F}_{-(k_1+k_2)}\frac{(k_1+k_2)^2}{k_1k_2}|\left|1-\mathrm{e}^{-i\tau 3k_1k_2(k_1+k_2)}\right||\left(\hat{\tilde{f}}_{k_1}\hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1}\hat{\tilde{g}}_{k_2}\right)| \\
&\le \tau M\|\partial_x^2(f-g)\|_{L^2}^2.
\end{aligned} \tag{26}
$$

Plugging the bounds (25) and (26) into (24) yields the assertion. □

2.1.2. Local error analysis. — Let ϕ^t denote the exact flow associated to the reformulated KdV equation (5), i.e., $v(t) = \phi^t(v(0))$. The following local error bound holds for the exponential-type integrator Φ^τ defined in (18) with $v^{k+1} = \Phi_{t_k}^\tau(v^k)$.

Lemma 2.5. — *Let $v(t_k+t) = \phi^t(v(t_k)) \in H^3$ for $0 \le t \le \tau$. Then*

$$\|\partial_x^2\big(\phi^\tau(v(t_k)) - \Phi_{t_k}^\tau(v(t_k))\big)\|_{L^2} \le c\tau^{3/2},$$

where c depends on $\sup_{0\le t\le\tau}\|\phi^t(v(t_k))\|_{H^3}$.

Proof. — As $\mathrm{e}^{t\partial_x^3}$ is a linear isometry in H^r for all $t\in\mathbb{R}$ the iteration of Duhamel's formula (5) yields that

$$
\begin{aligned}
\|\phi^\tau(v(t_k)) - \Phi_{t_k}^\tau(v(t_k))\|_{H^2} &\le \int_0^\tau \|\left(\mathrm{e}^{-\partial_x^3(t_k+s)}\phi^s(v(t_k))\right)^2 - \left(\mathrm{e}^{-\partial_x^3(t_k+s)}v(t_k)\right)^2\|_{H^3}\mathrm{d}s \\
&\le \tau c_1 \sup_{0\le t\le\tau}\|\phi^t(v(t_k)) - v(t_k)\|_{H^3},
\end{aligned} \tag{27}
$$

where c_1 depends on $\sup_{0\le t\le\tau}\|\phi^t(v(t_k))\|_{H^3}$. Duhamel's formula (5) and integration by parts furthermore yields that

$$
\begin{aligned}
&\|\phi^t(v(t_j)) - v(t_j)\|_{H^3} \le \|\int_0^t \mathrm{e}^{(t_j+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-\partial_x^3(t_j+s)}v(t_j+s)\right)^2\mathrm{d}s\|_{H^3} \\
&\le \|\sum_{k_1,k_2}\frac{1}{k_1k_2}\mathrm{e}^{-3it_jk_1k_2(k_1+k_2)}\left(\mathrm{e}^{-3itk_1k_2(k_1+k_2)}-1\right)\hat{v}_{k_1}(t_j+t)\hat{v}_{k_2}(t_j+t)\mathrm{e}^{i(k_1+k_2)x}\|_{H^3} \\
&+ \|\sum_{k_1,k_2}\frac{1}{k_1k_2}\mathrm{e}^{-3it_jk_1k_2(k_1+k_2)}\big(\hat{v}_{k_1}(t_j+t)\hat{v}_{k_2}(t_j+t) - \hat{v}_{k_1}(t_j)\hat{v}_{k_2}(t_j)\big)\mathrm{e}^{i(k_1+k_2)x}\|_{H^3} \\
&+ \|\int_0^t\sum_{k_1,k_2}\mathrm{e}^{-3i(t_j+s)k_1k_2(k_1+k_2)}\frac{1}{k_1k_2}\frac{\mathrm{d}}{\mathrm{d}s}\big(\hat{v}_{k_1}(t_j+s)\hat{v}_{k_2}(t_k+s)\big)\mathrm{d}s\mathrm{e}^{i(k_1+k_2)x}\|_{H^3} \\
&\le ct^{1/2}\sup_{k_1,k_2\in\mathbb{Z}_{\neq 0}}\frac{|k_1+k_2|^{1/2}}{|k_1k_2|^{1/2}}\|v(t_j+t)\|_{H^3}^2 + ct\sup_{k_1,k_2\in\mathbb{Z}_{\neq 0}}\frac{|k_1|+|k_2|}{|k_1k_2|}\sup_{0\le s\le t}\|v(t_j+s)\|_{H^3}^3.
\end{aligned} \tag{28}
$$

Plugging (28) into (27) yields the assertion. □

2.1.3. Global error bound. — The stability analysis in Section 2.1.1 and local error analysis in Section 2.1.2 allows us the following global error bound in H^2.

Theorem 2.6. — *Let the solution of (5) satisfy $v(t) \in H^3$ for $t \leq T$. Then there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ we have*

$$\|v(t_n) - v^n\|_{H^2} \leq c\tau^{1/2},$$

where c depends on $\sup_{0\leq t\leq t_n}\|v(t)\|_{H^3}$ and t_n, but can be chosen independently of τ.

Proof. — The triangular inequality yields that

$$\begin{aligned}\|v(t_{k+1}) - v^{k+1}\|_{H^2} &= \|\phi^\tau(v(t_k)) - \Phi^\tau_{t_k}(v^k)\|_{H^2} \\ &\leq \|\phi^\tau(v(t_k)) - \Phi^\tau_{t_k}(v(t_k))\|_{H^2} + \|\Phi^\tau_{t_k}(v(t_k)) - \Phi^\tau_{t_k}(v^k)\|_{H^2}.\end{aligned} \tag{29}$$

Thus, iterating the estimate (29) we obtain with the aid of Lemma 2.1 (with $g = v(t_k) \in H^3$) and Lemma 2.5 that as long as $v^k \in H^2$ (for $0 \leq k \leq n$) we have that

$$\begin{aligned}\|v(t_{n+1}) - v^{n+1}\|_{H^2} &\leq c\tau^{3/2} + \mathrm{e}^{\tau L}\|v(t_n) - v^n\|_{H^2} \leq c\tau^{3/2} + \mathrm{e}^{\tau L}\left(c\tau^{3/2} + \mathrm{e}^{\tau L}\|v(t_{n-1}) - v^{n-1}\|_{H^2}\right) \\ &\leq c\tau^{3/2}\sum_{k=0}^{n}\mathrm{e}^{t_k L} \leq c\tau^{1/2}t_n\mathrm{e}^{t_n L},\end{aligned}$$

where c depends on $\sup_{0\leq t\leq t_{n+1}}\|v(t)\|_{H^3}$, L depends on $\sup_{0\leq k\leq n}\|v(t_k)\|_{H^3}$ as well as on $\sup_{0\leq k\leq n}\|v^k\|_{H^2}$ and we have used the fact that $\hat{v}_0(t_n) \equiv \hat{v}_0^n$. The assertion then follows by a bootstrap, respectively, "Lady Windermere's fan" argument, see, for example [**3**, **6**, **10**, **16**]. □

2.2. Error analysis in H^1. — The error analysis in H^2 of the numerical scheme (9) given in Section 2.1 yields a priori bounds on the numerical solution in H^2 for solutions in H^3. This allows us to derive the following first-order convergence bound in H^1.

Theorem 2.7. — *Let the solution of (5) satisfy $v(t) \in H^3$ for $t \leq T$. Then there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ we have*

$$\|v(t_n) - v^n\|_{H^1} \leq c\tau,$$

where c depends on $\sup_{0\leq t\leq t_n}\|v(t)\|_{H^3}$ and t_n, but can be chosen independently of τ.

Proof. — Note that Duhamel's formula (5) implies the first-order consistency bound

$$\begin{aligned}\|\partial_x\big(\phi^\tau(v(t_k)) - \Phi^\tau_{t_k}(v(t_k))\big)\|_{L^2} &\leq \int_0^\tau \|\partial_x^2\Big[\left(\mathrm{e}^{-\partial_x^3(t_k+s)}\phi^s(v(t_k))\right)^2 - \left(\mathrm{e}^{-\partial_x^2(t_k+s)}v(t_k)\right)^2\Big]\|_{L^2}\mathrm{d}s \\ &\leq \tau c_1 \sup_{0\leq t\leq\tau}\|\phi^t(v(t_k)) - v(t_k)\|_{H^2} \\ &\leq \tau^2 c_1 \sup_{0\leq t\leq\tau}\|\phi^t(v(t_k))\|_{H^3},\end{aligned} \tag{30}$$

where c_1 depends on $\sup_{0\leq t\leq\tau}\|\phi^t(v(t_k))\|_{H^2}$.

Furthermore, as $v(t) \in H^3$ for $t \leq T$ we have the boundedness of the numerical solution in H^2 a priori thanks to Theorem 2.6, i.e., there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ $v^n \in H^2$ as long as $t_n \leq T$. In particular a stability estimate of type

$$\|\partial_x\big(\Phi_t^\tau(f) - \Phi_t^\tau(g)\big)\|_{L^2} \leq \exp(\tau L)\|\partial_x(f-g)\|_{L^2}, \quad L = L(\|\partial_x^2 f\|_{L^2}, \|\partial_x^3 g\|_{L^2}) \tag{31}$$

is therefore sufficient for our bootstrapping argument in H^1 by choosing $f = v^n \in H^2$ and $g = v(t_n) \in H^3$. The stability bound (31) follows similarly to Lemma 2.1: Note that

$$\begin{aligned}
&\|\partial_x(\Phi_t^\tau(f) - \Phi_t^\tau(g))\|_{L^2}^2 = \|\partial_x(f-g)\|_{L^2}^2 \\
&+ \frac{1}{3}\langle \partial_x \mathrm{e}^{\partial_x^3(t+\tau)}\Big[\Big(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}f\Big)^2 - \Big(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}g\Big)^2\Big], \partial_x(f-g)\rangle \\
&- \frac{1}{3}\langle \partial_x \mathrm{e}^{\partial_x^3 t}\Big[\Big(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}f\Big)^2 - \Big(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}g\Big)^2\Big], \partial_x(f-g)\rangle \\
&+ \frac{1}{6^2}\|\partial_x \mathrm{e}^{\partial_x^3(t+\tau)}\Big[\Big(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}f\Big)^2 - \Big(\mathrm{e}^{-\partial_x^3(t+\tau)}\partial_x^{-1}g\Big)^2\Big] \\
&- \partial_x \mathrm{e}^{\partial_x^3 t}\Big[\Big(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}f\Big)^2 - \Big(\mathrm{e}^{-\partial_x^3 t}\partial_x^{-1}g\Big)^2\Big]\|_{L^2}^2 \\
&=: \|\partial_x(f-g)\|_{L^2}^2 + \frac{1}{3}I_1 + \frac{1}{6^2}I_2.
\end{aligned} \tag{32}$$

Similarly to the proof of Lemma 2.3 we can rewrite I_1 as

$$\begin{aligned}
I_1 &= \langle \tilde f - \tilde g, (\tilde f - \tilde g)^2 - \mathrm{e}^{\partial_x^3\tau}\Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f - \tilde g)\Big)^2\rangle \\
&+ \langle \tilde g, (\tilde f - \tilde g)^2 - \mathrm{e}^{\partial_x^3\tau}\Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f - \tilde g)\Big)^2\rangle \\
&- 2\langle \tilde f - \tilde g, \tilde g(\tilde f - \tilde g) - \mathrm{e}^{\partial_x^3\tau}\Big[\Big(\mathrm{e}^{-\partial_x^3\tau}\tilde g\Big)\Big(\mathrm{e}^{-\partial_x^3\tau}(\tilde f - \tilde g)\Big)\Big]\rangle \\
&- 2\langle \tilde f - \tilde g, (\partial_x \tilde g)\Big(\partial_x^{-1}(\tilde f - \tilde g)\Big) - \mathrm{e}^{\partial_x^3\tau}\Big[\Big(\mathrm{e}^{-\partial_x^3\tau}(\partial_x\tilde g)\Big)\Big(\mathrm{e}^{-\partial_x^3\tau}\partial_x^{-1}(\tilde f - \tilde g)\Big)\Big]\rangle.
\end{aligned} \tag{33}$$

As in Lemma 2.2 we obtain by the key relation (3) that

$$I(u,v,w) := |\langle u, vw - \mathrm{e}^{\partial_x^3\tau}\Big[\Big(\mathrm{e}^{-\partial_x^3\tau}v\Big)\Big(\mathrm{e}^{-\partial_x^3\tau}w\Big)\Big]\rangle| \leq 3\tau \sum_{k_1,k_2} |(k_1+k_2)\hat u_{-(k_1+k_2)}||k_1k_2||\hat v_{k_1}\hat w_{k_2}|.$$

The Cauchy-Schwarz and Young inequality furthermore yield that

$$\begin{aligned}
I(u,v,w) &\leq 3\tau \sum_{k,l} |l||\hat u_{-l}||(l-k)k||\hat v_k \hat w_{l-k}| \leq 3\tau\Big(\sum_l |l|^2|\hat u_l|^2\Big)^{1/2}\|v^{(1)} * w^{(1)}\|_{l^2} \\
&\leq c\tau\|\partial_x u\|_{L^2}\min\Big(\|v^{(1)}\|_{l^1}\|w^{(1)}\|_{l^2}, \|v^{(1)}\|_{l^2}\|w^{(1)}\|_{l^1}\Big) \\
&\leq c\tau\|\partial_x u\|_{L^2}\min\left(\|\partial_x^2 v\|_{L^2}\|\partial_x w\|_{L^2}, \|\partial_x^2 w\|_{L^2}\|\partial_x v\|_{L^2}\right).
\end{aligned} \tag{34}$$

The above bound allows us to control the first and last two terms in (33) as long as $f-g, g \in H^2$. Furthermore,

$$
\begin{aligned}
I(u,v,v) &\le 3\tau \sum_{k,l} |l||\hat{u}_{-l}||(l-k)k||\hat{v}_k\hat{v}_{l-k}| \le 3\tau \big(\sum_k |k|^2|\hat{v}_k|^2\big)^{1/2} \|u^{(1)} * v^{(1)}\|_{l^2} \\
&\le 3\tau \|\partial_x v\|_{L^2} \|u^{(1)}\|_{l^1} \|v^{(1)}\|_{l^2} \le c\tau \|\partial_x v\|_{L^2}^2 \|\partial_x^2 u\|_{L^2},
\end{aligned} \tag{35}
$$

which allows us to control the second term in (33) as long as $f-g \in H^1$ and $g \in H^2$.

Using the bounds (34) and (35) in (33) yields that

$$
|I_1| \le \tau L \|\partial_x (f-g)\|_{L^2}^2, \quad L = L(\|\partial_x^2 (f-g)\|_{L^2}, \|\partial_x^2 g\|_{L^2}). \tag{36}
$$

Next we write $I_2 = I_2^a + I_2^b$ with

$$
\begin{aligned}
I_2^a =& \langle \partial_x \left(\mathrm{e}^{-\partial_x^3\tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\mathrm{e}^{-\partial_x^3\tau} \partial_x^{-1} \tilde{g} \right)^2 - \partial_x \mathrm{e}^{-\partial_x^3\tau} \Big[\left(\partial_x^{-1} \tilde{f} \right)^2 - \left(\partial_x^{-1} \tilde{g} \right)^2 \Big], \\
& \partial_x \left(\mathrm{e}^{-\partial_x^3\tau} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\mathrm{e}^{-\partial_x^3\tau} \partial_x^{-1} \tilde{g} \right)^2 \rangle \\
I_2^b =& - \langle \partial_x \mathrm{e}^{\partial_x^3\tau} \Big[\left(\mathrm{e}^{-\partial_x^3\tau} \partial_x^{-1} \tilde{f} \right)^2 - \left(\mathrm{e}^{-\partial_x^3\tau} \partial_x^{-1} \tilde{g} \right)^2 \Big] - \partial_x \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\partial_x^{-1} \tilde{g} \right)^2, \\
& \partial_x \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\partial_x^{-1} \tilde{g} \right)^2 \rangle.
\end{aligned}
$$

Note that by the Cauchy-Schwarz and Young inequality we have with $F := \partial_x \left(\partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left(\partial_x^{-1} \tilde{g} \right)^2$ that

$$
\begin{aligned}
|I_2^b| &= |\sum_{k_1,k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)}{k_1 k_2} \left(1 - \mathrm{e}^{-i\tau \left((k_1+k_2)^3 - k_1^3 - k_2^3 \right)} \right) \left(\hat{\tilde{f}}_{k_1} \hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1} \hat{\tilde{g}}_{k_2} \right)| \\
&= |\sum_{k_1,k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1+k_2)}{k_1 k_2} \left(1 - \mathrm{e}^{-i\tau 3 k_1 k_2 (k_1+k_2)} \right) \left(\hat{\tilde{f}}_{k_1} \hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1} \hat{\tilde{g}}_{k_2} \right)| \\
&\le 3\tau \sum_{k_1,k_2} |\hat{F}_{-(k_1+k_2)} (k_1+k_2)| |(k_1+k_2)| \left(\hat{\tilde{f}}_{k_1} \hat{\tilde{f}}_{k_2} - \hat{\tilde{g}}_{k_1} \hat{\tilde{g}}_{k_2} \right)| \\
&\le 3\tau \Big(\sum_l l^2 |\hat{F}_l|^2 \Big)^{1/2} \Big(\|(\tilde{f}-\tilde{g})^{(1)} * \tilde{f}^{(0)}\|_{l^2} + \|(\tilde{f}-\tilde{g})^{(0)} * \tilde{f}^{(1)}\|_{l^2} \\
&+ 3\tau \Big(\sum_l l^2 |\hat{F}_l|^2 \Big)^{1/2} \Big(\|(\tilde{f}-\tilde{g})^{(1)} * \tilde{g}^{(0)}\|_{l^2} + \|(\tilde{f}-\tilde{g})^{(0)} * \tilde{g}^{(1)}\|_{l^2} \Big) \\
&\le c\tau \|\partial_x F\|_{L^2} \Big(\|(\tilde{f}-\tilde{g})^{(1)}\|_{l^2} \big(\|\tilde{f}^{(0)}\|_{l^1} + \|\tilde{g}^{(0)}\|_{l^1} \big) + \|(\tilde{f}-\tilde{g})^{(0)}\|_{l^1} \big(\|\tilde{f}^{(1)}\|_{l^2} + \|\tilde{g}^{(1)}\|_{l^2} \big) \Big) \\
&\le M\tau \|\partial_x (\tilde{f}-\tilde{g})\|_{L^2}^2,
\end{aligned} \tag{37}
$$

where M depends on $\|\partial_x f\|_{L^2}$ and $\|\partial_x g\|_{L^2}$. A similar bound holds for I_2^a which implies that

$$|I_2| \leq M\tau \|\partial_x (f-g)\|_{L^2}^2, \quad M = M(\|\partial_x f\|_{L^2}, \|\partial_x g\|_{L^2}). \tag{38}$$

Plugging the bounds (36) as well as (38) into (32) yields the stability estimate (31).

With the aid of the stability estimate (31) and the local error bound (30) the proof then follows the line of argumentation to the proof of Theorem 2.6. □

Corollary 2.8. — *Let the solution of the KdV equation (1) satisfy $u(t) \in H^3$ for $t \leq T$. Then there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ the exponential-type integration scheme (10) is first-order convergent in H^1, i.e.,*

$$\|u(t_n) - u^n\|_{H^1} \leq c\tau,$$

where c depends on $\sup_{0\leq t\leq t_n}\|u(t)\|_{H^3}$ and t_n, but can be chosen independently of τ.

Proof. — The assertion follows from Theorem 2.7 as $\mathrm{e}^{t\partial_x^3}$ is a linear isometry in H^1 for all $t \in \mathbb{R}$. □

3. Numerical experiments

In this section, we numerically underline the first- and second-order convergence rates of the exponential-type integration schemes (10) and (17), respectively, towards the exact solution of the KdV equation (1). For the space discretization we use a Fourier pseudo spectral method, see [17], where we choose the largest Fourier mode $K = 2^{12}$. Details on the fully discrete scheme are given in Remark 3.1.

Remark 3.1. — We employ the following fully discrete Fourier pseudo spectral version of (10): Set $B^K = \{-K/2, \dots, K/2-1\}$ and let $\mathcal{F}_K : B^K \to B^K$ denote the discrete Fourier transform and $\mathcal{F}_K^{-1}$ its inverse. Denote by $u^{K,0}$ the discretized initial value vector on the grid $x_a = \frac{2\pi}{K}a$, $a \in B^K$ and set

$$\xi^{K,0} = \mathcal{F}_K\left(u^{K,0}\right) = \left[(\mathcal{F}_K u^{K,0})_{-\frac{K}{2}}, (\mathcal{F}_K u^{K,0})_{-\frac{K}{2}+1}, \dots, (\mathcal{F}_K u^{K,0})_{\frac{K}{2}-1}\right].$$

With this notation at hand a fully discrete Fourier pseudo spectral version of (10) reads

$$\xi^{K,n+1} = \mathrm{e}^{-\tau\widehat{\partial}_{x,K}^3}\xi^{K,n} + \frac{1}{6}\mathcal{F}_K\left(\left[\mathcal{F}_K^{-1}\left(\mathrm{e}^{-\tau\widehat{\partial}_{x,K}^3}\widehat{\partial}_{x,K}^{-1}\xi^{K,n}\right)\right]^2\right) - \frac{1}{6}\mathrm{e}^{-\tau\widehat{\partial}_{x,K}^3}\mathcal{F}_K\left(\left[\mathcal{F}_K^{-1}\left(\widehat{\partial}_{x,K}^{-1}\xi^{K,n}\right)\right]^2\right)$$

with $u^{K,n+1} = \mathcal{F}_K^{-1}\left(\xi^{K,n+1}\right)$. Thereby, the multiplication of two vectors is taken point-wise, i.e.,

$$[x_1, \dots, x_K]\,[y_1, \dots, y_K] = [x_1y_1, \dots, x_Ky_K]$$

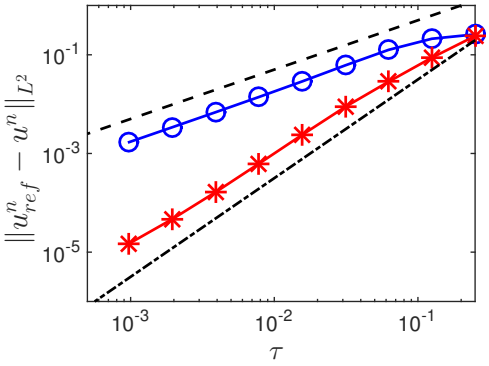
and the discrete differential operators acting in Fourier space are defined through

$$\widehat{\partial}_{x,K} := i\big[-K/2, \dots, K/2-1\big], \qquad \widehat{\partial}_{x,K}^{-1} := \tfrac{1}{i}\left[\tfrac{1}{-K/2}, \dots, -1, 0, 1, \dots \tfrac{1}{K/2-1}\right].$$

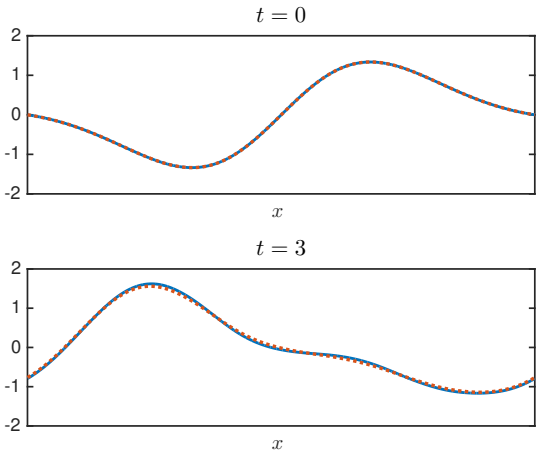
Example 3.2. — In the first numerical experiment we choose the initial value

$$u(0,x) = 2\,\mathrm{sech}^2\left(\tfrac{x}{2}\right)\sin(x) \qquad \text{with} \quad \mathrm{sech}(x) = \frac{1}{\cosh(x)} \tag{39}$$

and integrate the exponential-type integration schemes (10) and (17) up to $T = 2$. As the exact solution is unknown we take as a reference solution the second-order scheme itself with a very small time-step size $\tau = 10^{-7}$. The error between the numerical solutions and the reference solution at time $T = 2$ as well as a graph of the initial value, the reference solution and the first-order approximate solution is given in Figure 1.



(A) Orderplot (double logarithmic). Convergence rates of the first-order scheme (10) (blue, circle) and the second-order scheme (17) (red, star). The slopes of the dashed and dashed-dotted lines are one and two, respectively.



(B) Time evolution of the reference solution u^n_{ref} (orange, dotted) and the first-order approximate solution u^n with $\tau \approx 10^{-2}$ (blue, continuous).

FIGURE 1. (Initial value (39)) Numerical simulation of the first- and second-order exponential-type integration schemes (10) and (17).

Example 3.3 **(Solitary waves)**. — The KdV equation

$$\partial_t \phi + \partial_x^3 \phi + \frac{1}{2}\partial_x(\phi^2) = 0, \qquad x \in \mathbb{R}$$

allows solitary wave solutions of type

$$\phi(t,x) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - a)\right) \qquad \text{with} \quad a \in \mathbb{R},\ c > 0. \tag{40}$$

In order to test the resolution of solitary waves under the schemes (10) and (17) we choose a "large torus" $\mathbb{T}_L = [-\frac{\pi}{L}, \frac{\pi}{L}]$ with $L = 0.1$ such that boundary errors are negligible. Furthermore, we fix $c = 1$ and $a = 0$. The H^1-error between the first- and second-order exponential-type

integration schemes (10) and (17), respectively, and the exact solution (40) at time $T = 2$ is illustrated in Figure 2.

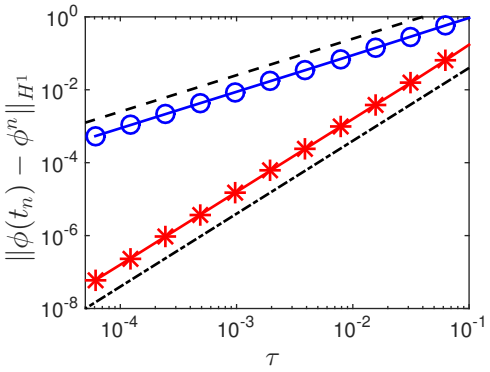
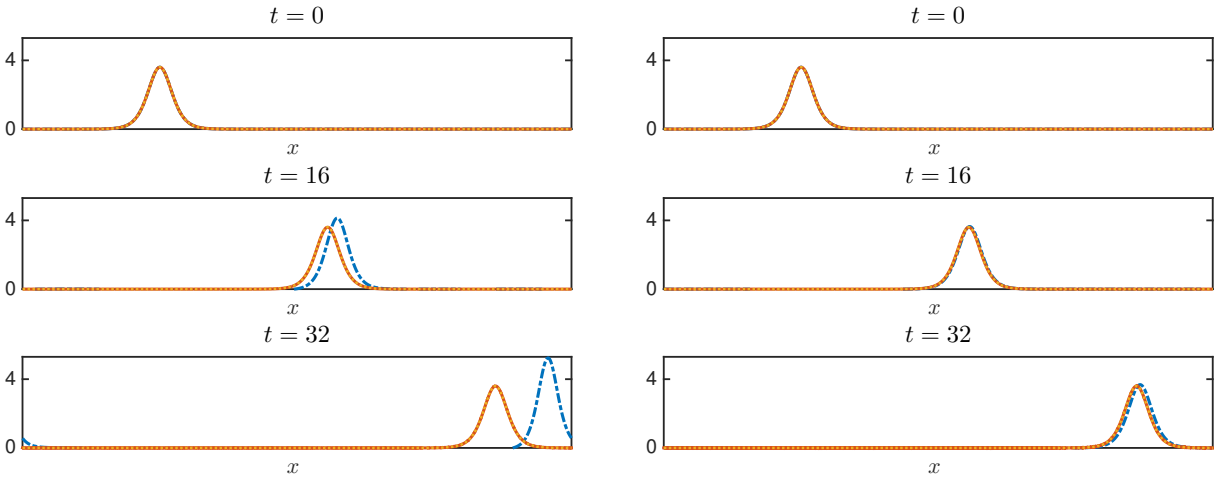


FIGURE 2. (Solitary wave) Orderplot (double logarithmic). Convergence rates of the first-order scheme (10) (blue, circle) and the second-order scheme (17) (red, star) measured in a discrete H^1 norm. The slopes of the dashed and dashed-dotted lines are one and two, respectively.

A graph of the time evolution of the solitary wave solution (40) (with $c = 1.2$, $a = -5\pi$) and the corresponding first- and second-order approximate solutions (12) and (17), respectively, for two different time step sizes is illustrated in Figure 3.



(A) $\tau = 10^{-2}$ (B) $\tau = 10^{-3}$

FIGURE 3. Time evolution of the solitary wave (40) (yellow, dotted), the first-order approximate solution (12) (blue, dashed-dotted) and second-order approximate solution (17) (red, continous) for two different time-step sizes τ.

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