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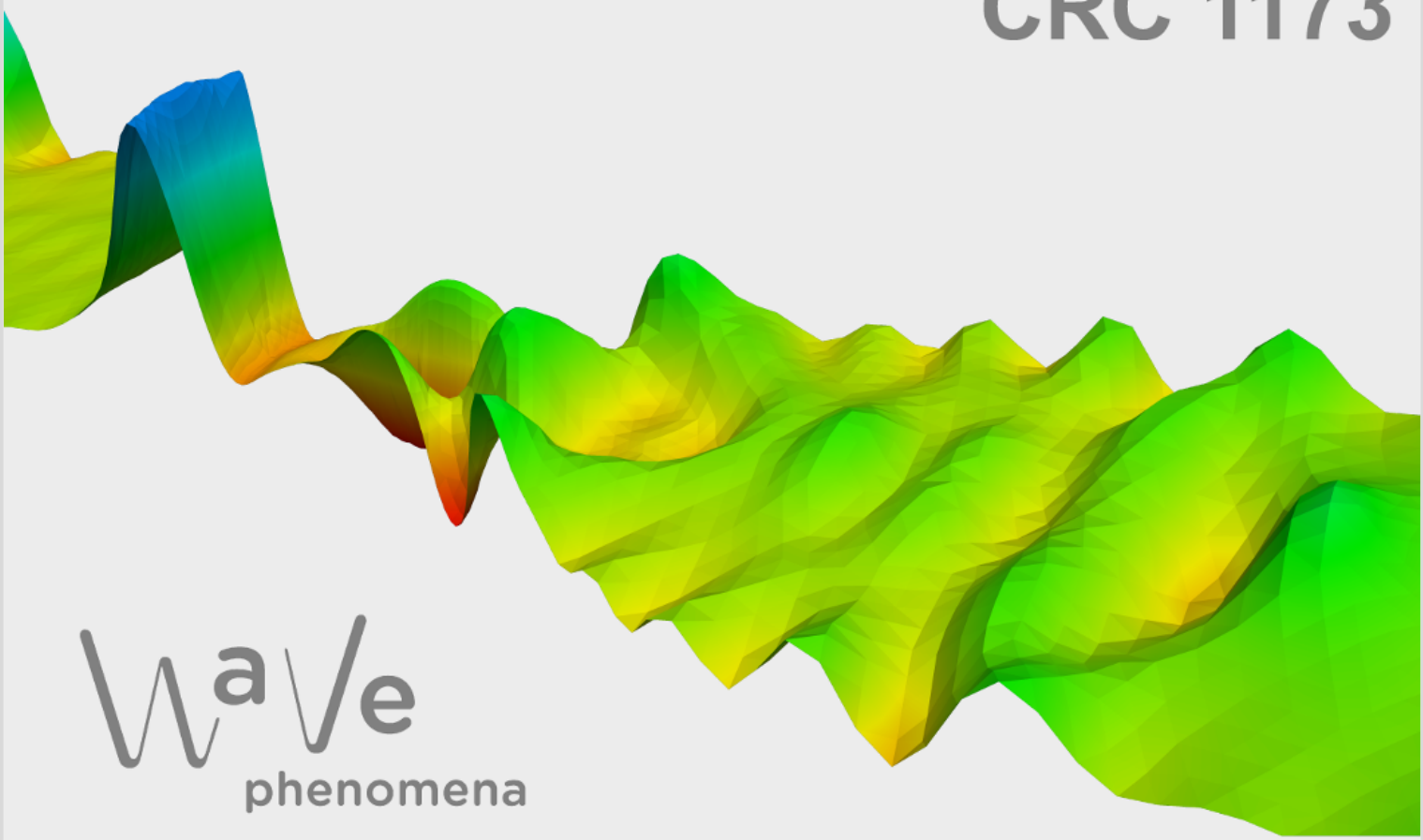

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Freezing Traveling and Rotating Waves in Second Order Evolution Equations

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Abstract. In this paper we investigate the implementation of the so-called *freezing method* for second order wave equations in one and several space dimensions. The method converts the given PDE into a partial differential algebraic equation which is then solved numerically. The reformulation aims at separating the motion of a solution into a co-moving frame and a profile which varies as little as possible. Numerical examples demonstrate the feasability of this approach for semilinear wave equations with sufficient damping. We treat the case of a traveling wave in one space dimension and of a rotating wave in two space dimensions. In addition, we investigate in arbitrary space dimensions the point spectrum and the essential spectrum of operators obtained by linearizing about the profile, and we indicate the consequences for the nonlinear stability of the wave.

Key words. Systems of damped wave equations, traveling waves, rotating waves, freezing method, second order evolution equations, point spectra, essential spectra.
AMS subject classification. 35K57, 35Pxx, 65Mxx (35Q56, 47N40, 65P40).

1. Introduction

The topic of this paper is the numerical computation and stability of waves occurring in second order evolution equations with damping terms. More specifically, we transfer the so called *freezing method* (see [7], [19], [4]) from first order to second order evolution equations, and we investigate its relation to the stability of the waves. Generally speaking, the method tries to separate the solution of a Cauchy problem into the motion of a co-moving frame and of a profile, where the latter is required to vary as little as possible or even become stationary. This is achieved by transforming the original PDE into a partial differential algebraic equation (PDAE). The PDAE involves extra unknowns specifying the frame, and extra constraints (so called *phase conditions*) enforcing the freezing principle for the profile. This methodology has been successfully applied to a wide range of PDEs which are of first order in time and of hyperbolic, parabolic or of mixed type, cf. [21], [23], [22], [6], [16], [17], [18], [4]. One aim of the theoretical underpinning is to prove that waves which are (asymptotically) stable with asymptotic phase for the PDE, become stable in the classical Lyapunov sense for the PDAE. While this has been rigorously proved for many systems in one space dimension and confirmed numerically in higher space dimensions, the corresponding theory for the multi-dimensional case is still in its early stages, see [1], [3], [2], [14].

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In this paper we develop the freezing formulation and perform the spectral calculations in an informal way, for the one-dimensional as well as the multi-dimensional case. Rigorous stability results for the one-dimensional damped wave equation may be found in [10], [9], [5].
Here we consider a nonlinear wave equation of the form

$$Mu_{tt} = Au_{xx} + f(u, u_x, u_t),\ x \in \mathbb{R},\ t \geqslant 0, \tag{1.1}$$

where $u(x,t) \in \mathbb{R}^m, A, M \in \mathbb{R}^{m,m}$ and $f : \mathbb{R}^{3m} \to \mathbb{R}^m$ is sufficiently smooth. In addition, we assume the matrix M to be nonsingular and $M^{-1}A$ to be positive diagonalizable, which will lead to local wellposedness of the Cauchy problem associated with (1.1). Our interest is in traveling waves

$$u_\star(x,t) = v_\star(x - \mu_\star t),\ x \in \mathbb{R},\ t \geqslant 0,$$

with constant limits at $\pm\infty$, i.e.

$$\lim_{\xi\to\pm\infty} v_\star(\xi) = v_\pm \in \mathbb{R}^m,\ \lim_{\xi\to\pm\infty} v_{\star,\xi}(\xi) = 0, \quad f(v_\pm, 0, 0) = 0. \tag{1.2}$$

Transforming (1.1) into a co-moving frame via $u(x,t) = v(\xi,t), \xi = x - \mu_\star t$ leads to the system

$$Mv_{tt} = (A - \mu_\star^2 M)v_{\xi\xi} + 2\mu_\star M v_{\xi t} + f(v, v_\xi, v_t - \mu_\star v_\xi),\ \xi \in \mathbb{R},\ t \geqslant 0. \tag{1.3}$$

This system has $v_\star$ as a steady state,

$$0 = (A - \mu_\star^2 M)v_{\star,\xi\xi} + f(v_\star, v_{\star,\xi}, -\mu_\star v_{\star,\xi}),\ \xi \in \mathbb{R}. \tag{1.4}$$

In Section 2 we work out the details of the freezing PDAE based on the ansatz $u(x,t) = v(x - \gamma(t), t)$, $x \in \mathbb{R}, t \geq 0$ with the additional unknown function $\gamma(t), t \geq 0$. Solving this PDAE numerically will then be demonstated for a special semilinear case, for which damping occurs and for which the nonlinearity is of quintic type with 5 zeros. We will also discuss in Section 2.2 the spectral properties of the linear operator obtained by linearizing the right-hand side of (1.3) about the profile $v_\star$. First, there is the eigenvalue zero due to shift equivariance, and then we analyze the dispersion curves which are part of the operator's essential spectrum. If there is sufficient damping in the system (depending on the derivative $D_3 f$), one can expect the whole nonzero spectrum to lie strictly to the left of the imaginary axis. We refer to [5] for a rigorous proof of nonlinear stability in such a situation, both stability of the wave with asymptotic phase for equation (1.3) and Lyapunov stability of the wave and its speed for the freezing equation.
The subsequent section is devoted to study corresponding problems for multi-dimensional wave equations

$$Mu_{tt} + Bu_t = A\Delta u + f(u),\ x \in \mathbb{R}^d,\ t \geqslant 0, \tag{1.5}$$

where the matrices A, M are as above, the damping matrix $B \in \mathbb{R}^{m,m}$ is given and $f : \mathbb{R}^m \to \mathbb{R}^m$ is again sufficiently smooth. We look for rotating waves of the form

$$u_\star(x,t) = v_\star(e^{-tS_\star}(x - x_\star)),\ x \in \mathbb{R}^d,\ t \geqslant 0,$$

where $x_\star \in \mathbb{R}^d$ denotes the center of rotation, $S_\star \in \mathbb{R}^{d,d}$ is a skew-symmetric matrix, and $v_\star : \mathbb{R}^d \to \mathbb{R}^m$ describes the profile. Transforming (1.5) into a co-rotating frame via $u(x,t) = v(e^{-tS_\star}(x - x_\star), t)$ now leads to the equation

$$Mv_{tt} + Bv_t = A\triangle v - Mv_{\xi\xi}(S_\star\xi)^2 + 2Mv_{\xi t}S_\star\xi - Mv_\xi S_\star^2\xi + Bv_\xi S_\star\xi + f(v),\ \xi \in \mathbb{R}^d,\ t \geqslant 0, \tag{1.6}$$

where our notation for derivatives uses multilinear calculus, e.g.

$$(v_{\xi\xi}h_1h_2)_i = \sum_{j=1}^d \sum_{k=1}^d v_{i,\xi_j\xi_k}(h_1)_j(h_2)_k, \quad (\triangle v)_i = \sum_{j=1}^d v_{i,\xi_j\xi_j} = \sum_{j=1}^d v_{i,\xi\xi}(e^j)^2.$$

The profile $v_\star$ of the wave is then a steady state solution of (1.6), i.e.

$$0 = A\triangle v_\star - Mv_{\star,\xi\xi}(S_\star\xi)^2 - Mv_{\star,\xi}S_\star^2\xi + Bv_{\star,\xi}S_\star\xi + f(v_\star),\ \xi \in \mathbb{R}^d. \tag{1.7}$$

As is known from first oder in time PDEs, there are several eigenvalues of the linearized operator on the imaginary axis caused by the Euclidean symmmetry, see e.g. [11], [12], [8], [1], [13]. The computations become more involved for the wave equation (1.6), but we will show that the eigenvalues on the imaginary axis are the same as in the parabolic case. Further, determining the dispersion relation, and thus curves in the essential spectrum, now amounts to solving a parameterized quadratic eigenvalue problem which in

general can only be solved numerically. Finally, we present a numerical example of a rotating wave for the cubic-quintic Ginzburg-Landau equation. The performance of the freezing method will be demonstrated, and we investigate the numerical eigenvalues approximating the point spectrum on (and close to) the imaginary axis as well as the essential spectrum in the left half-plane.

2. Traveling waves in one space dimension

2.1. Freezing traveling waves. Consider the Cauchy problem associated with (1.1)

$$Mu_{tt} = Au_{xx} + f(u, u_x, u_t), \qquad x \in \mathbb{R},\ t \geqslant 0, \tag{2.1a}$$

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = v_0, \qquad x \in \mathbb{R},\ t = 0, \tag{2.1b}$$

for some initial data $u_0, v_0 : \mathbb{R} \to \mathbb{R}^m$ and some nonlinearity $f \in C^3(\mathbb{R}^{3m}, \mathbb{R})$. Introducing new unknowns $\gamma(t) \in \mathbb{R}$ and $v(\xi, t) \in \mathbb{R}^m$ via the freezing ansatz for traveling waves

$$u(x, t) = v(\xi, t), \quad \xi := x - \gamma(t),\ x \in \mathbb{R},\ t \geqslant 0, \tag{2.2}$$

and inserting (2.2) into (2.1a) by taking

$$u_t = -\gamma_t v_\xi + v_t, \quad u_{tt} = -\gamma_{tt} v_\xi + \gamma_t^2 v_{\xi\xi} - 2\gamma_t v_{\xi t} + v_{tt} \tag{2.3}$$

into account, we obtain the equation

$$Mv_{tt} = (A - \gamma_t^2 M)v_{\xi\xi} + 2\gamma_t M v_{\xi t} + \gamma_{tt} M v_\xi + f(v, v_\xi, v_t - \gamma_t v_\xi),\ \xi \in \mathbb{R},\ t \geqslant 0. \tag{2.4}$$

Now it is convenient to introduce time-dependent functions $\mu_1(t) \in \mathbb{R}$ and $\mu_2(t) \in \mathbb{R}$ via

$$\mu_1(t) := \gamma_t(t), \quad \mu_2(t) := \mu_{1,t}(t) = \gamma_{tt}(t)$$

which allows us to transfer (2.4) into a coupled PDE/ODE-system

$$Mv_{tt} = (A - \mu_1^2 M)v_{\xi\xi} + 2\mu_1 M v_{\xi t} + \mu_2 M v_\xi + f(v, v_\xi, v_t - \mu_1 v_\xi), \qquad \xi \in \mathbb{R},\ t \geqslant 0, \tag{2.5a}$$

$$\mu_{1,t} = \mu_2, \qquad t \geqslant 0, \tag{2.5b}$$

$$\gamma_t = \mu_1, \qquad t \geqslant 0. \tag{2.5c}$$

The quantity $\gamma(t)$ denotes the position, $\mu_1(t)$ the velocity and $\mu_2(t)$ the acceleration of the profile $v(\xi, t)$ at time t. We next specify initial data for the system (2.5) as follows,

$$v(\cdot, 0) = u_0, \quad v_t(\cdot, 0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \gamma(0) = 0 \tag{2.6}$$

Note that if we require $\gamma(0) = 0$ and $\mu_1(0) = \mu_1^0$, then the first equation in (2.6) follows from (2.2) and (2.1b), while the second equation in (2.6) follows from (2.3), (2.1b) and (2.5c). Suitable values for μ_1^0 depend on the choice of phase condition to be discussed next.
We compensate the extra variable μ_2 in the system (2.5) by imposing an additional scalar algebraic constraint, also known as a phase condition, of the general form

$$\psi(v, v_t, \mu_1, \mu_2) = 0,\ t \geqslant 0. \tag{2.7}$$

Two possible choices are the fixed phase condition ψ_{fix} and the orthogonal phase condition ψ_{orth} given by

$$\begin{aligned} \psi_{\text{fix}}(v) &= \langle v - \hat{v}, \hat{v}_\xi \rangle_{L^2},\ t \geqslant 0, \\ \psi_{\text{orth}}(v_t) &= \langle v_t, v_\xi \rangle_{L^2},\ t \geqslant 0. \end{aligned} \tag{2.8}$$

These two types and their derivation are discussed in [5]. The function $\hat{v} : \mathbb{R} \to \mathbb{R}^m$ denotes a time-independent and sufficiently smooth template (or reference) function, e.g. $\hat{v} = u_0$. Suitable values for $\mu_1(0) = \mu_1^0$ can be derived from requiring consistent initial values for the PDAE. For example, consider (2.8) and take the time derivative at $t = 0$. Together with (2.6) this leads to $0 = \langle v_t(\cdot, 0), \hat{v}_\xi \rangle_{L^2} = \langle v_0, \hat{v}_\xi \rangle_{L^2} + \mu_1^0 \langle u_{0,\xi}, \hat{v}_\xi \rangle_{L^2}$. If $\langle u_{0,\xi}, \hat{v}_\xi \rangle_{L^2} \neq 0$ this determines a unique value for μ_1^0.
Let us summarize the set of equations obtained by the freezing method of the original Cauchy problem (2.1). Combining the differential equations (2.5), the initial data (2.6) and the phase condition (2.7),

we arrive at the following partial differential algebraic evolution equation (short: PDAE) to be solved numerically:

$$\begin{aligned} &Mv_{tt} = (A-\mu_1^2 M)v_{\xi\xi} + 2\mu_1 M v_{\xi,t} + \mu_2 M v_\xi + f(v, v_\xi, v_t - \mu_1 v_\xi), \\ &\mu_{1,t} = \mu_2, \quad \gamma_t = \mu_1, \end{aligned} \qquad t \geqslant 0, \tag{2.9a}$$

$$0 = \psi(v, v_t, \mu_1, \mu_2), \qquad t \geqslant 0, \tag{2.9b}$$

$$v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \gamma(0) = 0. \tag{2.9c}$$

The system (2.9) depends on the choice of phase condition ψ and is to be solved for $(v, \mu_1, \mu_2, \gamma)$ with given initial data (u_0, v_0, μ_1^0). It consists of a PDE for v that is coupled to two ODEs for μ_1 and γ (2.9a) and an algebraic constraint (2.9b) which closes the system. A consistent initial value μ_1^0 for μ_1 is computed from the phase condition and the initial data. Further initialization of the algebraic variable μ_2 is usually not needed for a PDAE-solver but can be provided if necessary (see [5]).
The ODE for γ is called the reconstruction equation in [19]. It decouples from the other equations in (2.9) and can be solved in a postprocessing step. The ODE for μ_1 is the new feature of the PDAE for second order systems when compared to the first order parabolic and hyperbolic equations in [7, 15, 4].
Finally, note that $(v, \mu_1, \mu_2) = (v_\star, \mu_\star, 0)$ satisfies

$$\begin{aligned} 0 &= (A-\mu_\star^2 M)v_{\star,\xi\xi} + \mu_\star M v_{\star,\xi} + f(v_\star, v_{\star,\xi}, -\mu_\star v_{\star,\xi}), \ \xi \in \mathbb{R}, \\ 0 &= \mu_2, \\ 0 &= \psi(v_\star, 0, \mu_\star, 0), \end{aligned}$$

and hence is a stationary solution of (2.9a),(2.9b). Here we assume that $v_\star, \mu_\star$ have been selected to satisfy the phase condition. Obviously, in this case we have $\gamma(t) = \mu_\star t$. For a stable traveling wave we expect that solutions $(v, \mu_1, \mu_2, \gamma)$ of (2.9) show the limiting behavior

$$v(t) \to v_\star, \quad \mu_1(t) \to \mu_\star, \quad \mu_2(t) \to 0 \quad \text{as} \quad t \to \infty,$$

provided the initial data are close to their limiting values.

Example 2.1 (Freezing quintic Nagumo wave equation)**.** Consider the quintic Nagumo wave equation,

$$\varepsilon u_{tt} = A u_{xx} + f(u, u_x, u_t), \ x \in \mathbb{R}, \ t \geqslant 0, \tag{2.10}$$

with $u = u(x,t) \in \mathbb{R}$, $\varepsilon > 0$, $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$, and the nonlinear term

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(u, u_x, u_t) = -u_t + u(1-u)\prod_{j=1}^{3}(u - \alpha_j). \tag{2.11}$$

For the parameter values

$$M = \varepsilon = \frac{1}{2}, \quad A = 1, \quad \alpha_1 = \frac{2}{5}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = \frac{17}{20}, \tag{2.12}$$

equation (2.10) admits a traveling front solution connecting the asymptotic states $v_- = 0$ and $v_+ = 1$. Figure 2.1 shows a numerical simulation of the solution u of (2.10) on the spatial domain $(-50, 50)$ with homogeneous Neumann boundary conditions, with initial data

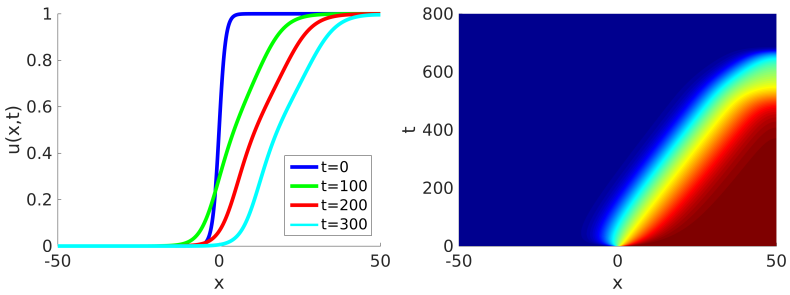
$$u_0(x) = \tfrac{1}{2}\left(1 + \tanh\left(\tfrac{x}{2}\right)\right), \quad v_0(x) = 0 \tag{2.13}$$

and parameters taken from (2.12). For the space discretization we use continuous piecewise linear finite elements with spatial stepsize $\triangle x = 0.1$. For the time discretization we use the BDF method of order 2 with absolute tolerance atol $= 10^{-3}$, relative tolerance rtol $= 10^{-2}$, temporal stepsize $\triangle t = 0.2$ and final time $T = 800$. Computations are performed with the help of the software COMSOL 5.2.
Let us now consider the frozen quintic Nagumo wave equation resulting from (2.9)

$$\begin{aligned} &\varepsilon v_{tt} + v_t = (1-\mu_1^2\varepsilon)v_{\xi\xi} + 2\mu_1\varepsilon v_{\xi,t} + (\mu_2\varepsilon + \mu_1)v_\xi + \tilde{f}(v), \\ &\mu_{1,t} = \mu_2, \quad \gamma_t = \mu_1, \end{aligned} \qquad t \geqslant 0, \tag{2.14a}$$

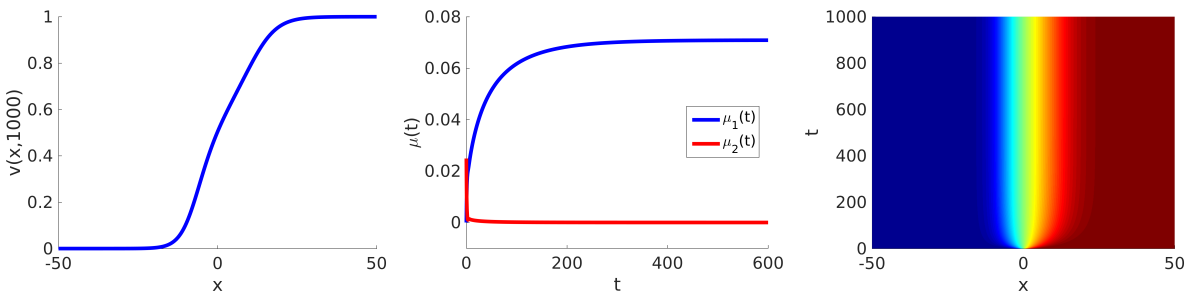
$$0 = \left\langle v_t(\cdot,t), \hat{v}_\xi \right\rangle_{L^2(\mathbb{R},\mathbb{R})}, \qquad t \geqslant 0, \tag{2.14b}$$

$$v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \gamma(0) = 0. \tag{2.14c}$$



(a) (b)

FIGURE 2.1. Traveling front of quintic Nagumo wave equation (2.10) at different time instances (a) and its time evolution (b) for parameters from (2.12).



(a) (b) (c)

FIGURE 2.2. Solution of the frozen quintic Nagumo wave equation (2.14): approximation of profile $v(x, 1000)$ (a) and time evolutions of velocity μ_1 and acceleration μ_2 (b) and of the profile v (c) for parameters from (2.12).

Figure 2.2 shows the solution $(v, \mu_1, \mu_2, \gamma)$ of (2.14) on the spatial domain $(-50, 50)$ with homogeneous Neumann boundary conditions, initial data u_0, v_0 from (2.13), and reference function $\hat{v} = u_0$. For the computation we used the fixed phase condition $\psi_{\text{fix}}(v)$ from (2.8) with consistent intial data $\mu_1^0 = 0$, see above. The spatial discretization data are taken as in the nonfrozen case. For the time discretization we used the BDF method of order 2 with absolute tolerance atol $= 10^{-3}$, relative tolerance rtol $= 10^{-2}$, temporal stepsize $\triangle t = 0.6$ and final time $T = 3000$. The diagrams show that after a very short transition phase the profile becomes stationary, the acceleration μ_2 converges to zero, and the speed μ_1 approaches an asymptotic value $\mu_\star^{\text{num}} \approx 0.0709$ which is close to the exact (not explicitly known) value $\mu_\star$.

2.2. Spectra of traveling waves. Consider the linearized equation

$$Mv_{tt} - (A - \mu_\star^2 M)v_{\xi\xi} - 2\mu_\star M v_{\xi t} - (D_2 f_\star - \mu_\star D_3 f_\star)v_\xi - D_3 f_\star v_t - D_1 f_\star v = 0 \tag{2.15}$$

which is obtained from the co-moving frame (1.3) linearized at the profile $v_\star$. In (2.15) we use the short form $D_j f_\star = D_j f(v_\star, v_{\star,\xi}, -\mu_\star v_{\star,\xi})$. Looking for solutions of the form $v(\xi, t) = e^{\lambda t} w(\xi)$ to (2.15) yields the quadratic eigenvalue problem

$$\mathcal{P}(\lambda)w = \left(\lambda^2 P_2 + \lambda P_1 + P_0\right) w = 0,\ \xi \in \mathbb{R} \tag{2.16}$$

with differential operators P_j defined by

$$P_2 = M, \quad P_1 = -2\mu_\star M\partial_\xi - D_3 f_\star, \quad P_0 = -(A - \mu_\star^2 M)\partial_\xi^2 - (D_2 f_\star - \mu_\star D_3 f_\star)\partial_\xi - D_1 f_\star.$$

We are interested in solutions (λ, w) of (2.16) which are candidates for eigenvalues $\lambda \in \mathbb{C}$ and eigenfunctions $w : \mathbb{R} \to \mathbb{C}^m$ in suitable function spaces. In fact, it is usually imposssible to determine the spectrum $\sigma(\mathcal{P})$ analytically, but one is able to analyze certain subsets. Let us first calculate the symmetry set $\sigma_{\text{sym}}(\mathcal{P})$,

which belongs to the point spectrum $\sigma_{\rm pt}(\mathcal{P})$ and is affected by the underlying group symmetries. Then, we calculate the dispersion set $\sigma_{\rm disp}(\mathcal{P})$, which belongs to the essential spectrum $\sigma_{\rm ess}(\mathcal{P})$ and is affected by the far-field behavior of the wave. Let us first derive the symmetry set of $\mathcal{P}$. This is a simple task for traveling waves but becomes more involved when analyzing the symmetry set for rotating waves (see Section 3.2.1).

2.2.1. Point Spectrum and symmetry set. Applying ∂_ξ to the traveling wave equation (1.4) yields $P_0 v_{\star,\xi} = 0$ which proves the following result.

Proposition 2.2 (Point spectrum of traveling waves)**.** Let $f \in C^1(\mathbb{R}^{3m}, \mathbb{R}^m)$ and let $v_\star \in C^3(\mathbb{R}, \mathbb{R}^m)$ be a nontrivial classical solution of (1.4) for some $\mu_\star \in \mathbb{R}$. Then, $w = v_{\star,\xi}$ and $\lambda = 0$ is a classical solution of the eigenvalue problem (2.16). In particular, the symmetry set

$$\sigma_{\rm sym}(\mathcal{P}) = \{0\}$$

belongs to the point spectrum $\sigma_{\rm pt}(\mathcal{P})$ of $\mathcal{P}$.

Of course, a rigorous statement of this kind requires to specify the function spaces involved, e.g. $L^2(\mathbb{R}, \mathbb{R}^m)$ or $H^1(\mathbb{R}, \mathbb{R}^m)$, see [10], [9], [5].

2.2.2. Essential Spectrum and dispersion set.

1. **The far-field operator.** It is a well known fact that the essential spectrum is affected by the limiting equation obtained from (2.16) as $\xi \to \pm\infty$. Therefore, we let formally $\xi \to \pm\infty$ in (2.16) and obtain

$$\left(\lambda^2 P_2 + \lambda P_1^\pm + P_0^\pm\right) w = 0, \ \xi \in \mathbb{R}. \tag{2.17}$$

with the constant coefficient operators

$$P_2 = M, \quad P_1^\pm = -2\mu_\star M \partial_\xi - D_3 f_\pm, \quad P_0^\pm = -(A - \mu_\star^2 M)\partial_\xi^2 - (D_2 f_\pm - \mu_\star D_3 f_\pm)\partial_\xi - D_1 f_\pm,$$

where $v_\pm$ are from (1.2) and $D_j f_\pm = D_j f(v_\pm, 0, 0)$. We may then write equation (2.16) as

$$\left(\lambda^2 P_2 + \lambda(P_1^\pm + Q_1^\pm(\xi)) + (P_0^\pm + Q_2^\pm(\xi)\partial_\xi + Q_3^\pm(\xi))\right) w = 0, \ \xi \in \mathbb{R}$$

with the perturbation operators defined by

$$Q_1^\pm(\xi) = D_3 f_\pm - D_3 f_\star, \ \ Q_2^\pm(\xi) = D_2 f_\pm - D_2 f_\star + \mu_\star(D_3 f_\star - D_3 f_\pm), \ \ Q_3^\pm(\xi) = D_1 f_\pm - D_1 f_\star,$$

Note that $v_\star(\xi) \to v_\pm$ implies $Q_j^\pm(\xi) \to 0$ as $\xi \to \pm\infty$ for $j = 1, 2, 3$.

2. **Spatial Fourier transform.** For $\omega \in \mathbb{R}$, $z \in \mathbb{C}^m$, $|z| = 1$ we apply the spatial Fourier transform $w(\xi) = e^{i\omega\xi} z$ to equation (2.17) which leads to the m-dimensional quadratic eigenvalue problem

$$\left(\lambda^2 A_2 + \lambda A_1^\pm(\omega) + A_0^\pm(\omega)\right) z = 0 \tag{2.18}$$

with matrices $A_2 \in \mathbb{R}^{m,m}$ and $A_1^\pm, A_0^\pm \in \mathbb{C}^{m,m}$ given by

$$A_2 = M, \ A_1^\pm(\omega) = -2i\omega\mu_\star M - D_3 f_\pm, \ A_0^\pm(\omega) = \omega^2(A - \mu_\star^2 M) - i\omega(D_2 f_\pm - \mu_\star D_3 f_\pm) - D_1 f_\pm. \tag{2.19}$$

3. **Dispersion relation and dispersion set.** The dispersion relation for traveling waves of second order evolution equations states the following: Every $\lambda \in \mathbb{C}$ satisfying

$$\det\left(\lambda^2 A_2 + \lambda A_1^\pm(\omega) + A_0^\pm(\omega)\right) = 0 \tag{2.20}$$

for some $\omega \in \mathbb{R}$ belongs to the essential spectrum of $\mathcal{P}$, i.e. $\lambda \in \sigma_{\rm ess}(\mathcal{P})$. Solving (2.20) is equivalent to finding all zeros of a polynomial of degree $2m$. Note that the limiting case $M = 0$ in (2.20) leads to the dispersion relation for traveling waves of first order evolution equations, which is well-known in the literature, see [20].

Proposition 2.3 (Essential spectrum of traveling waves)**.** Let $f \in C^1(\mathbb{R}^{3m}, \mathbb{R}^m)$ with $f(v_\pm, 0, 0) = 0$ for some $v_\pm \in \mathbb{R}^m$. Let $v_\star \in C^2(\mathbb{R}, \mathbb{R}^m)$, $\mu_\star \in \mathbb{R}$ be a nontrivial classical solution of (1.4) satisfying $v_\star(\xi) \to v_\pm$ as $\xi \to \pm\infty$. Then, the dispersion set

$$\sigma_{\rm disp}(\mathcal{P}) = \{\lambda \in \mathbb{C} : \lambda \text{ satisfies (2.20) for some } \omega \in \mathbb{R}, \text{ and } + \text{ or } -\}$$

belongs to the essential spectrum $\sigma_{\rm ess}(\mathcal{P})$ of $\mathcal{P}$.

Example 2.4 (Spectrum of quintic Nagumo wave equation)**.** As shown in Example 2.1 the quintic Nagumo wave equation (2.10) with coefficients and parameters (2.12) has a traveling front solution $u_\star(x,t) = v_\star(x-\mu_\star t)$ with velocity $\mu_\star \approx 0.0709$, whose profile $v_\star$ connects the asymptotic states $v_- = 0$ and $v_+ = 1$ according to (1.2).
We solve numerically the eigenvalue problem for the quintic Nagumo wave equation

$$\left(\lambda^2\varepsilon + \lambda\left(-2\mu_\star\varepsilon\partial_\xi - D_3 f_\star\right) + \left(-(1-\mu_\star^2\varepsilon)\partial_\xi^2 - (D_2 f_\star - \mu_\star D_3 f_\star)\partial_\xi - D_1 f_\star\right)\right) w = 0. \tag{2.21}$$

Both approximations of the profile $v_\star$ and the velocity $\mu_\star$ in (2.21) are chosen from the solution of (2.14) at time $t = 3000$ in Example 2.1. Due to Proposition 2.2 we expect $\lambda = 0$ to be an isolated eigenvalue belonging to the point spectrum. Let us next discuss the dispersion set from Proposition 2.3. The quintic Nagumo nonlinearity (2.11) satisfies

$$f_\pm = 0, \quad D_3 f_\pm = -1, \quad D_2 f_\pm = 0, \quad D_1 f_- = -\alpha_1\alpha_2\alpha_3, \quad D_1 f_+ = -\prod_{j=1}^{3}(1-\alpha_j).$$

The matrices A_2, $A_1^\pm(\omega)$, $A_0^\pm(\omega)$ from (2.19) of the quadratic problem (2.18) are given by

$$A_2 = \varepsilon, \quad A_1^\pm(\omega) = -2i\omega\mu_\star\varepsilon + 1, \quad A_0^\pm(\omega) = \omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - D_1 f_\pm.$$

The dispersion relation (2.20) for the quintic Nagumo front states that every $\lambda \in \mathbb{C}$ satisfying

$$\lambda^2\varepsilon + \lambda(-2i\omega\mu_\star\varepsilon + 1) + (\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - D_1 f_\pm) = 0 \tag{2.22}$$

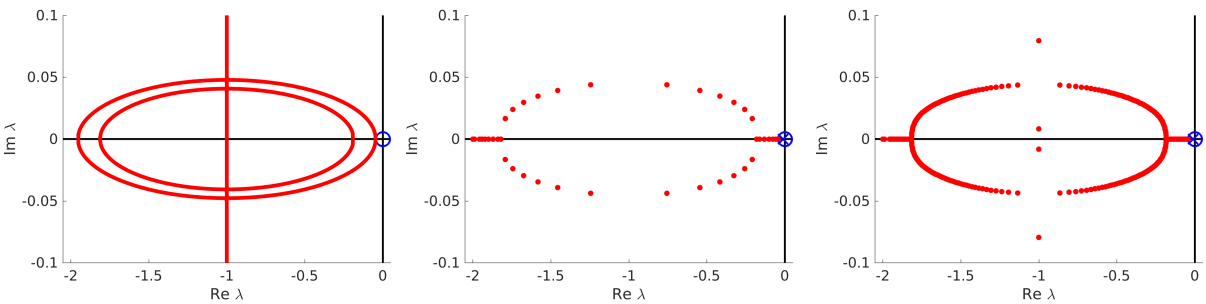
for some $\omega \in \mathbb{R}$, and for $+$ or $-$, belongs to $\sigma_{\text{ess}}(\mathcal{P})$. We introduce a new unknown $\tilde\lambda \in \mathbb{C}$ via $\lambda = \tilde\lambda + i\omega\mu_\star$ and solve the transformed equation

$$\tilde\lambda^2 + \frac{1}{\varepsilon}\tilde\lambda + \frac{1}{\varepsilon}(\omega^2 - D_1 f_\pm) = 0.$$

obtained from (2.22). Thus, the quadratic eigenvalue problem (2.22) has the solutions

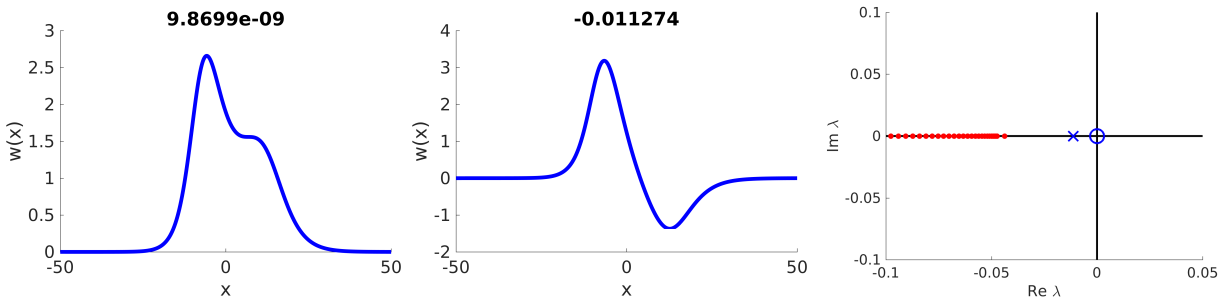
$$\lambda = -\frac{1}{2\varepsilon} + i\omega\mu_\star \pm \frac{1}{2\varepsilon}\sqrt{1 - 4\varepsilon(\omega^2 - D_1 f_\pm)}, \ \omega \in \mathbb{R}.$$

These solutions lie on the line $\text{Re} = -\frac{1}{2\varepsilon}$ and on two ellipses if $-4D_1 f_\pm \varepsilon < 1$ (cf. Figure 2.3(a)).



(a) (b) (c)

FIGURE 2.3. Spectrum of the quintic Nagumo wave equation for parameters (2.12) (a) and the numerical spectrum on the spatial domain $[-R, R]$ for $R = 50$ (b) and $R = 400$ (c) both for spatial stepsize $\triangle x = 0.1$.



(a) (b) (c)

FIGURE 2.4. Eigenfunctions of the quintic Nagumo wave equation for parameters (2.12) belonging to the isolated eigenvalues $\lambda_1 \approx 0$ (a), $\lambda_2 \approx -0.011274$ (b), and a zoom into the spectrum from Fig.2.3(c) in (c).

Figure 2.3(a) shows the part of the spectrum of the quintic Nagumo wave which is guaranteed by Proposition 2.2 and 2.3. It is subdivided into the symmetry set $\sigma_{\text{sym}}(\mathcal{P})$ (blue circle), which is determined by Proposition 2.2 and belongs to the point spectrum $\sigma_{\text{pt}}(\mathcal{P})$, and the dispersion set $\sigma_{\text{disp}}(\mathcal{P})$ (red lines), which is determined by Proposition 2.3 and belongs to the essential spectrum $\sigma_{\text{ess}}(\mathcal{P})$. In general, there may be further essential spectrum in $\sigma_{\text{ess}}(\mathcal{P})\backslash\sigma_{\text{disp}}(\mathcal{P})$ and further isolated eigenvalues in $\sigma_{\text{pt}}(\mathcal{P})\backslash\sigma_{\text{sym}}(\mathcal{P})$. In fact, for the quintic Nagumo wave equation we find an extra eigenvalue with negative real part, cf. Figure 2.4(c). The numerical spectrum of the quintic Nagumo wave equation on the spatial domain $[-R, R]$ equipped with periodic boundary conditions is shown in Figure 2.3(b) for $R = 50$ and in Figure 2.3(c) for $R = 400$. Each of them consists of the approximations of the point spectrum subdivided into the symmetry set (blue circle) and an additional isolated eigenvalue (blue plus sign), and of the essential spectrum (red dots). The missing line inside the ellipse in Figure 2.3(b) gradually appears numerically when enlarging the spatial domain, see Figure 2.3(c). The second ellipse only develops on even larger domains.

3. Rotating waves in several space dimensions

3.1. Freezing rotating waves. Consider the Cauchy problem associated with (1.5)

$$Mu_{tt} + Bu_t = A\triangle u + f(u) \qquad , x \in \mathbb{R}^d,\ t > 0, \tag{3.1a}$$

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = v_0 \qquad , x \in \mathbb{R}^d,\ t = 0, \tag{3.1b}$$

for some initial data $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}^m$, where u_0 denotes the initial displacement and v_0 the initial velocity. The damped wave equation (3.1) has a more special nonlinearity than in the one-dimensional case, see (1.5). This will simplify some of the computations below.

In the following, let $\text{SE}(d) = \text{SO}(d) \ltimes \mathbb{R}^d$ denote the special Euclidean group and $\text{SO}(d)$ the special orthogonal group. Let us introduce new unknowns $(Q(t), \tau(t)) \in \text{SE}(d)$ and $v(\xi, t) \in \mathbb{R}^m$ via the rotating wave ansatz

$$u(x, t) = v(\xi, t), \quad \xi := Q(t)^\top (x - \tau(t)),\ x \in \mathbb{R}^d,\ t \geqslant 0. \tag{3.2}$$

Inserting (3.2) into (3.1a) and suppressing arguments of u and v leads to

$$\begin{aligned} \triangle_x u =& \triangle_\xi v, \quad f(u) = f(v), \quad u_t = v_\xi \left(Q_t^\top (x - \tau) - Q^\top \tau_t\right) + v_t, \\ u_{tt} =& v_{\xi\xi} \left(Q_t^\top (x - \tau) - Q^\top \tau_t\right)^2 + v_\xi \left(Q_{tt}^\top (x - \tau) - 2Q_t^\top \tau_t - Q^\top \tau_{tt}\right) \\ &+ 2v_{\xi t} \left(Q_t^\top (x - \tau) - Q^\top \tau_t\right) + v_{tt}. \end{aligned} \tag{3.3}$$

Hence equation (3.1a) turns into

$$\begin{aligned} Mv_{tt} + Bv_t = A\triangle v &- Mv_{\xi\xi} \left(Q_t^\top Q\xi - Q^\top \tau_t\right)^2 - 2Mv_{\xi t} \left(Q_t^\top Q\xi - Q^\top \tau_t\right) \\ &- Mv_\xi \left(Q_{tt}^\top Q\xi - 2Q_t^\top \tau_t - Q^\top \tau_{tt}\right) - Bv_\xi \left(Q_t^\top Q\xi - Q^\top \tau_t\right) + f(v). \end{aligned} \tag{3.4}$$

It is convenient to introduce time-dependent functions $S_1(t), S_2(t) \in \mathbb{R}^{d,d}$, $\mu_1(t), \mu_2(t) \in \mathbb{R}^d$ via

$$S_1 := Q^\top Q_t, \quad S_2 := S_{1,t}, \quad \mu_1 := Q^\top \tau_t, \quad \mu_2 := \mu_{1,t}.$$

Obviously, S_1 and S_2 satisfy $S_1^\top = -S_1$ and $S_2^\top = -S_2$, which follows from $Q^\top Q = I_d$ by differentiation. Moreover, we obtain

$$\begin{aligned} &Q_t^\top Q = -S_1, \quad Q^\top \tau_t = \mu_1, \quad Q_t^\top \tau_t + Q^\top \tau_{tt} = \mu_2, \\ &Q_{tt}^\top Q = -S_{1,t} - S_1^\top S_1 = -S_2 + S_1^2, \quad -Q_t^\top \tau_t = -Q_t^\top Q Q^\top \tau_t = S_1 \mu_1, \end{aligned}$$

which transforms (3.4) into the system

$$\begin{aligned} Mv_{tt} + Bv_t = A\triangle v - Mv_{\xi\xi}\left(S_1\xi + \mu_1\right)^2 + 2Mv_{\xi t}\left(S_1\xi + \mu_1\right) \\ + Mv_\xi\left((S_2 - S_1^2)\xi - S_1\mu_1 + \mu_2\right) + Bv_\xi\left(S_1\xi + \mu_1\right) + f(v), \end{aligned} \tag{3.5a}$$

$$\begin{pmatrix} S_1 \\ \mu_1 \end{pmatrix}_t = \begin{pmatrix} S_2 \\ \mu_2 \end{pmatrix}, \tag{3.5b}$$

$$\begin{pmatrix} Q \\ \tau \end{pmatrix}_t = \begin{pmatrix} QS_1 \\ Q\mu_1 \end{pmatrix}. \tag{3.5c}$$

The quantity $(Q(t), \tau(t))$ describes the position by its spatial shift $\tau(t)$ and the rotation $Q(t)$. Moreover, $S_1(t)$ denotes the rotational velocities, $\mu_1(t)$ the translational velocities, $S_2(t)$ the angular acceleration and $\mu_2(t)$ the translational acceleration of the rotating wave v at time t. Note that in contrast to the traveling waves the leading part $A\triangle - M\partial_\xi^2(S_1\xi + \mu_1)^2$ not only depends on the velocities S_1 and μ_1, but also on the spatial variable ξ, which means that the leading part has unbounded (linearly growing) coefficients. We next specify initial data for the system (3.5) as follows,

$$\begin{aligned} v(\cdot, 0) = u_0, \quad v_t(\cdot, 0) = v_0 + u_{0,\xi}(S_1^0\xi + \mu_1^0), \\ S_1(0) = S_1^0, \quad \mu_1(0) = \mu_1^0, \quad Q(0) = I_d, \quad \tau(0) = 0. \end{aligned} \tag{3.6}$$

Note that, requiring $Q(0) = I_d$, $\tau(0) = 0$, $S_1(0) = S_1^0$ and $\mu_1(0) = \mu_1^0$ for some $S_1^0 \in \mathbb{R}^{d,d}$ with $(S_1^0)^\top = -S_1^0$ and $\mu_1^0 \in \mathbb{R}^d$, the first equation in (3.6) follows from (3.2) and (3.1b), while the second condition in (3.6) can be deduced from (3.3), (3.1b), (3.5c) and the first condition in (3.6).

The system (3.5) comprises evolution equations for the unknowns v, S and μ_1. In order to specify the remaining variables S_2 and μ_2 we impose $\dim \mathrm{SE}(d) = \frac{d(d+1)}{2}$ additional scalar algebraic constraints, also known as phase conditions

$$\psi(v, v_t, (S_1, \mu_1), (S_2, \mu_2)) = 0 \in \mathbb{R}^{\frac{d(d+1)}{2}}, \quad t \geqslant 0. \tag{3.7}$$

Two possible choices of such a phase condition are

$$\psi_{\mathrm{fix}}(v) := \begin{pmatrix} \langle v - \hat{v}, D_l \hat{v} \rangle_{L^2} \\ \langle v - \hat{v}, D^{(i,j)} \hat{v} \rangle_{L^2} \end{pmatrix} = 0, \ t \geqslant 0, \tag{3.8}$$

$$\psi_{\mathrm{orth}}(v_t) := \begin{pmatrix} \langle v_t, D_l v \rangle_{L^2} \\ \langle v_t, D^{(i,j)} v \rangle_{L^2} \end{pmatrix} = 0, \ t \geqslant 0, \tag{3.9}$$

for $l = 1, \ldots, d$, $i = 1, \ldots, d-1$ and $j = i+1, \ldots, d$ with $D_l := \partial_{\xi_l}$ and $D^{(i,j)} := \xi_j \partial_{\xi_i} - \xi_i \partial_{\xi_j}$. Condition (3.8) is obtained from the requirement that the distance

$$\rho(Q, \tau) := \left\| v(\cdot, t) - \hat{v}(Q^\top(\cdot - \tau)) \right\|_{L^2}^2$$

attains a local minimum at $(Q, \tau) = (I_d, 0)$. Since $D_l, D^{(i,j)}$ are the generators of the Euclidean group action, condition (3.9) requires the time derivative of v to be orthogonal to the group orbit of v at any time instance.

Combining the differential equations (3.5), the initial data (3.6) and the phase condition (3.7), we obtain the following partial differential algebraic evolution equation (PDAE)

$$\begin{aligned} Mv_{tt} + Bv_t = A\triangle v - Mv_{\xi\xi}\left(S_1\xi + \mu_1\right)^2 + 2Mv_{\xi t}\left(S_1\xi + \mu_1\right) \\ + Mv_\xi\left((S_2 - S_1^2)\xi - S_1\mu_1 + \mu_2\right) + Bv_\xi\left(S_1\xi + \mu_1\right) + f(v), \qquad \xi \in \mathbb{R}^d, \, t > 0, \end{aligned} \tag{3.10a}$$

$$v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + u_{0,\xi}(S_1^0\xi + \mu_1^0), \qquad \xi \in \mathbb{R}^d,\ t = 0, \tag{3.10b}$$

$$0 = \psi(v, v_t, (S_1,\mu_1), (S_2,\mu_2)), \qquad t \geqslant 0, \tag{3.10c}$$

$$\begin{pmatrix} S_1 \\ \mu_1 \end{pmatrix}_t = \begin{pmatrix} S_2 \\ \mu_2 \end{pmatrix}, \quad \begin{pmatrix} S_1(0) \\ \mu_1(0) \end{pmatrix} = \begin{pmatrix} S_1^0 \\ \mu_1^0 \end{pmatrix}, \qquad t \geqslant 0, \tag{3.10d}$$

$$\begin{pmatrix} Q \\ \tau \end{pmatrix}_t = \begin{pmatrix} QS_1 \\ Q\mu_1 \end{pmatrix}, \quad \begin{pmatrix} Q(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix}, \qquad t \geqslant 0. \tag{3.10e}$$

The system (3.10) depends on the choice of phase condition and must be solved for $(v, S_1, \mu_1, S_2, \mu_2, Q, \tau)$ for given $(u_0, v_0, S_1^0, \mu_1^0)$. It consists of a PDE for v in (3.10a)–(3.10b), two systems of ODEs for (S_1, μ_1) in (3.10d) and for (Q, τ) in (3.10e) and $\frac{d(d+1)}{2}$ algebraic constraints for (S_2, μ_2) in (3.10c). The ODE (3.10e) for (Q, τ) is the reconstruction equation (see [19]), it decouples from the other equations in (3.10) and can be solved in a postprocessing step. Note that in the frozen equation for first order evolution equations, the ODE for (S_1, μ_1) does not appear, see [13, (10.26)]. The additional ODE is a new component of the PDAE and is caused by the second order time derivative.
Finally, note that $(v, S_1, \mu_1, S_2, \mu_2) = (v_\star, S_\star, \mu_\star, 0, 0)$ satisfies

$$\begin{aligned} 0 &= A\triangle v - Mv_{\star,\xi\xi}\left(S_\star\xi + \mu_\star\right)^2 - Mv_{\star,\xi}S_\star\left(S_\star\xi + \mu_\star\right) + Bv_{\star,\xi}\left(S_\star\xi + \mu_\star\right) + f(v_\star),\ \xi \in \mathbb{R}^d, \\ 0 &= \begin{pmatrix} S_2 \\ \mu_2 \end{pmatrix}. \end{aligned}$$

If, in addition, it has been arranged that $v_\star, S_\star, \mu_\star$ satisfy the phase condition $\psi(v_\star, 0, S_\star, \mu_\star, 0, 0) = 0$ then $(v_\star, S_\star, \mu_\star, 0, 0)$ is a stationary solution of the system (3.10a),(3.10c),(3.10d). For a stable rotating wave we expect that solutions $(v, S_1, \mu_1, S_2, \mu_2)$ of (3.10a)–(3.10d) satisfy

$$v(t) \to v_\star, \quad (S_1(t), \mu_1(t)) \to (S_\star, \mu_\star), \quad (S_2(t), \mu_2(t)) \to (0,0), \quad \text{as} \quad t \to \infty,$$

provided the initial data are close to their limiting values.

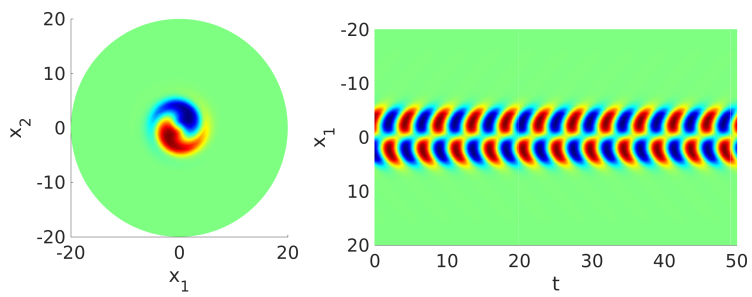
Example 3.1 (Cubic-quintic complex Ginzburg-Landau wave equation)**.** Consider the cubic-quintic complex Ginzburg-Landau wave equation

$$\varepsilon u_{tt} + \rho u_t = \alpha\triangle u + u(\delta + \beta|u|^2 + \gamma|u|^4),\ x \in \mathbb{R}^d,\ t \geqslant 0 \tag{3.11}$$

with $u = u(x,t) \in \mathbb{C}$, $d \in \{2,3\}$, $\varepsilon, \rho, \alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\operatorname{Re}\alpha > 0$. For the parameter values

$$\varepsilon = 10^{-4}, \quad \rho = 1, \quad \alpha = \frac{3}{5}, \quad \gamma = -1 - \frac{1}{10}i, \quad \beta = \frac{5}{2} + i, \quad \delta = -0.73. \tag{3.12}$$

equation (3.11) admits a spinning soliton solution.



(a) (b)

FIGURE 3.1. Solution of cubic-quintic complex Ginzburg-Landau wave equation (3.11): Spinning soliton $u(x,t)$ at time $t = 50$ (a) and its time evolution along $x_2 = 0$ (b) for parameters from (3.12).

Figure 3.1 shows a numerical simulation of the solution u of (3.11) on the ball $B_R(0)$ with radius $R = 20$, homogeneous Neumann boundary conditions parameters from (3.12). The initial data u_0 and v_0 come

from a simulation. To generate the initial data we consider the case $\varepsilon = 0$ and $\rho = 1$, solve the frozen QCGL for parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \gamma = -1 - \frac{1}{10}i, \quad \beta = \frac{5}{2} + i, \quad \delta = -\frac{1}{2}.$$

as in [13]. Since this value of α leads to an ill-posed wave equation, we gradually change the value of δ and α to arrive at the parameter setting (3.12). For the space discretization we use continuous piecewise linear finite elements with spatial stepsize $\triangle x = 0.8$. For the time discretization we use the BDF method of order 2 with absolute tolerance atol $= 10^{-4}$, relative tolerance rtol $= 10^{-3}$, temporal stepsize $\triangle t = 0.1$ and final time $T = 50$. Computations are performed with the help of the software COMSOL 5.2.
Let us now consider the frozen cubic-quintic complex Ginzburg-Landau wave equation resulting from (3.10)

$$\begin{aligned}
&(3.13a) \quad \varepsilon v_{tt} + \rho v_t = \alpha \triangle v - \varepsilon v_{\xi\xi} \left(S_1 \xi + \mu_1\right)^2 + 2\varepsilon v_{\xi t} \left(S_1 \xi + \mu_1\right) \\
&\qquad\qquad + \varepsilon v_\xi \left((S_2 - S_1^2)\xi - S_1 \mu_1 + \mu_2\right) + \rho v_\xi \left(S_1 \xi + \mu_1\right) + f(v), \qquad \xi \in \mathbb{R}^d,\ t > 0, \\
&(3.13b) \quad v(\cdot, 0) = u_0, \quad v_t(\cdot, 0) = v_0 + u_{0,\xi}(S_1^0 \xi + \mu_1^0), \qquad \xi \in \mathbb{R}^d,\ t = 0, \\
&(3.13c) \quad 0 = \psi_{\text{fix}}(v) := \begin{pmatrix} \langle v - \hat{v}, D_l \hat{v} \rangle_{L^2} \\ \langle v - \hat{v}, D^{(i,j)} \hat{v} \rangle_{L^2} \end{pmatrix}, \qquad t \geqslant 0, \\
&(3.13d) \quad \begin{pmatrix} S_1 \\ \mu_1 \end{pmatrix}_t = \begin{pmatrix} S_2 \\ \mu_2 \end{pmatrix}, \quad \begin{pmatrix} S_1(0) \\ \mu_1(0) \end{pmatrix} = \begin{pmatrix} S_1^0 \\ \mu_1^0 \end{pmatrix}, \qquad t \geqslant 0, \\
&(3.13e) \quad \begin{pmatrix} Q \\ \tau \end{pmatrix}_t = \begin{pmatrix} QS_1 \\ Q\mu_1 \end{pmatrix}, \quad \begin{pmatrix} Q(0) \\ \tau(0) \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix}, \qquad t \geqslant 0.
\end{aligned}$$

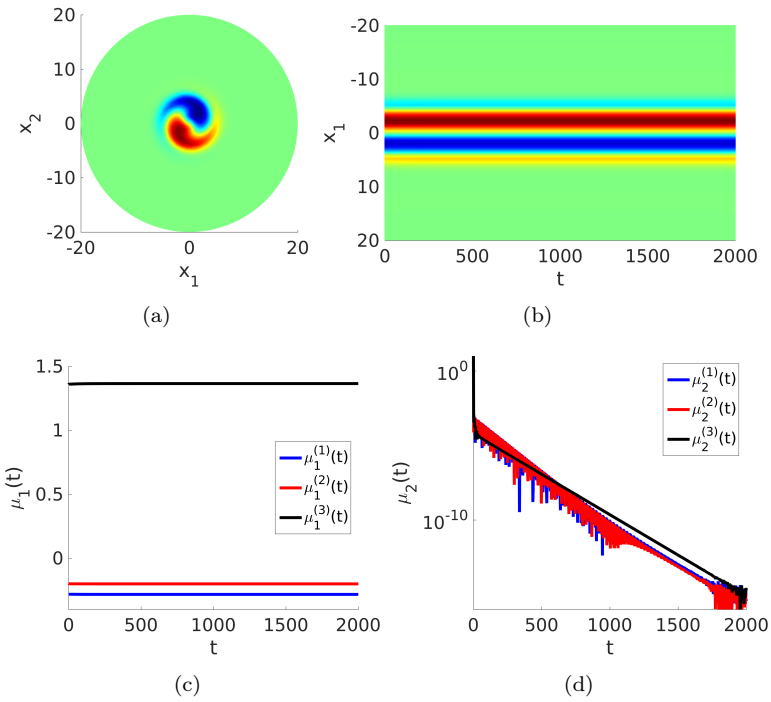


FIGURE 3.2. Solution of the frozen cubic-quintic complex Ginzburg-Landau wave equation (3.13): profile $v(x,t)$ at time $t = 2000$ (a), its time evolution along $x_2 = 0$ (b), velocities $\mu_1(t)$ (c), and accelerations $\mu_2(t)$ (d) for parameters from (3.12).

Figure 3.2 shows the solution $(v, S_1, \mu_1, S_2, \mu_2, Q, \tau)$ of (3.13) on the ball $B_R(0)$ with radius $R = 20$, homogeneous Neumann boundary conditions, initial data u_0, v_0 as in the nonfrozen case, and reference function $\hat{v} = u_0$. For the computation we used the fixed phase condition $\psi_{\text{fix}}(v)$ from (3.8). The spatial discretization data are taken as in the nonfrozen case. For the time discretization we used the BDF method of order 2 with absolute tolerance atol $= 10^{-3}$, relative tolerance rtol $= 10^{-2}$, maximal temporal stepsize $\triangle t = 0.5$, initial step 10^{-4}, and final time $T = 2000$. Due to the choice of initial data, the profile becomes immediately stationary, the acceleration μ_2 converges to zero, while the speed μ_1 and the nontrivial entry S_{12} of S approach asymptotic values

$$\mu_1^{(1)} = -0.2819, \quad \mu_1^{(2)} = -0.1999, \quad \mathcal{S}_{12} = 1.3658.$$

Note that we have a clockwise rotation if $S_{12} > 0$, and a counter clockwise rotation, if $S_{12} < 0$. Thus, the spinning soliton rotates clockwise. The center of rotation $x_\star$ and the temporal period $T^{2\text{D}}$, that the spinning soliton in $\mathbb{R}^2$ needs for exactly one rotation, are given by, see [13, Exa.10.8],

$$x_\star = \frac{1}{S_{12}} \begin{pmatrix} \mu_1^{(2)} \\ -\mu_1^{(1)} \end{pmatrix} = \begin{pmatrix} -0.1464 \\ 0.2064 \end{pmatrix}, \qquad T^{2\text{D}} = \frac{2\pi}{|S_{12}|} = 4.6004.$$

3.2. Spectra of rotating waves. Consider the linearized equation

$$M v_{tt} + B v_t - A\triangle v + M v_{\xi\xi}(S_\star \xi)^2 - 2M v_{\xi t} S_\star \xi + M v_\xi S_\star^2 \xi - B v_\xi S_\star \xi - Df(v_\star) v = 0 \tag{3.14}$$

where we set $S = S_\star$. Equation (3.14) is obtained from the co-rotating frame equation (1.6) when linearizing at the profile $v_\star$. Moreover, we assume $\mu_\star = 0$, that is the wave that rotates about the origin. Shifting the center of rotation does not influence the stability properties, see the discussion in [3]. Looking for solutions of the form $v(\xi, t) = e^{\lambda t} w(\xi)$ to (3.14) yields the quadratic eigenvalue problem

$$\mathcal{P}(\lambda) w := \left(\lambda^2 P_2 + \lambda P_1 + P_0\right) w = 0, \; \xi \in \mathbb{R}^d \tag{3.15}$$

with differential operators P_j defined by

$$\begin{aligned} P_2 =& M, \quad P_1 = B - 2M\left(\partial_\xi \cdot\right) S_\star \xi = B - 2M \sum_{j=1}^d (S_\star \xi)_j \partial_{\xi_j}, \\ P_0 =& -A\triangle \cdot + M\left(\partial_\xi^2 \cdot\right)(S_\star \xi)^2 + M\left(\partial_\xi \cdot\right) S_\star^2 \xi - B\left(\partial_\xi \cdot\right) S_\star \xi - Df(v_\star) \cdot \\ =& -A \sum_{j=1}^d \partial_{\xi_j}^2 + M \sum_{j=1}^d \sum_{\nu=1}^d (S_\star \xi)_j (S_\star \xi)_\nu \partial_{\xi_j} \partial_{\xi_\nu} + M \sum_{j=1}^d (S_\star^2 \xi)_j \partial_{\xi_j} - B \sum_{j=1}^d (S_\star \xi)_j \partial_{\xi_j} - Df(v_\star). \end{aligned} \tag{3.16}$$

As in the one-dimensional case we cannot solve equation (3.15) in general. Rather, our aim is to determine the dispersion set $\sigma_{\text{disp}}(\mathcal{P})$ as a subset of the essential spectrum $\sigma_{\text{ess}}(\mathcal{P})$, and the symmetry set $\sigma_{\text{sym}}(\mathcal{P})$ as a subset of the point spectrum $\sigma_{\text{pt}}(\mathcal{P})$. The essential spectrum depends on the far-field behavior of the wave while the point spectrum is affected by the underlying group symmetries.
In a first step let us transform the skew-symmetric matrix $S_\star$ into quasi-diagonal real form. Let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ be the nonzero eigenvalues of $S_\star$ so that 0 is a semisimple eigenvalue of multiplicity $d - 2k$. There is an orthogonal matrix $P \in \mathbb{R}^{d,d}$ such that

$$S_\star = P \Lambda P^\top, \quad \Lambda = \operatorname{diag}\left(\Lambda_1, \ldots, \Lambda_k, \mathbf{0}\right), \quad \Lambda_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad \mathbf{0} \in \mathbb{R}^{d-2k, d-2k}.$$

The transformation $\tilde{w}(y) = w(Py), \tilde{v}_\star(y) = v_\star(Py)$ transfers (3.15),(3.16) into the form

$$(\lambda^2 \tilde{P}_2 + \lambda \tilde{P}_1 + \tilde{P}_0)\tilde{w} = 0. \tag{3.17}$$

With the abbreviations

$$D_j = \partial_{y_j}, \quad D^{(i,j)} = y_j D_i - y_i D_j, \quad K = \sum_{l=1}^k \sigma_l D^{(2l-1, 2l)} \tag{3.18}$$

the operators $\tilde{P}_j$ are given by

$$
\begin{aligned}
\tilde{P}_2 =& M, \quad \tilde{P}_1 = B - 2M\sum_{j=1}^{d}(\Lambda y)_j D_j = B - 2MK, \\
\tilde{P}_0 =& -A\triangle + M\sum_{j=1}^{d}\sum_{\nu=1}^{d}(\Lambda y)_j(\Lambda y)_\nu D_j D_\nu + M\sum_{j=1}^{d}(\Lambda^2 y)_j D_j - B\sum_{j=1}^{d}(\Lambda y)_j D_j - Df(\tilde{v}_\star) \\
=& -A\triangle + MK^2 - BK - Df(\tilde{v}_\star).
\end{aligned} \tag{3.19}
$$

In the following we present the recipe for computing the subsets $\sigma_{\mathrm{disp}}(\mathcal{P}) \subseteq \sigma_{\mathrm{ess}}(\mathcal{P})$ and $\sigma_{\mathrm{sym}}(\mathcal{P}) \subseteq \sigma_{\mathrm{pt}}(\mathcal{P})$.

3.2.1. Essential spectrum and dispersion set.

1. **The far-field operator.** Assume that $v_\star$ has an asymptotic state $v_\infty \in \mathbb{R}^m$, i.e. $f(v_\infty) = 0$ and $v_\star(\xi) \to v_\infty \in \mathbb{R}^m$ as $|\xi| \to \infty$. In the limit $|y| \to \infty$ the eigenvalue problem (3.17) turns into the far-field problem

$$
\left(\lambda^2\tilde{P}_2 + \lambda\tilde{P}_1 + \tilde{P}_\infty\right)\tilde{w} = 0,\ y \in \mathbb{R}^d, \quad \tilde{P}_\infty = -A\triangle + MK^2 - BK - Df(v_\infty). \tag{3.20}
$$

2. **Transformation into several planar polar coordinates:** Since we have k angular derivatives in k different planes it is advisable to transform into several planar polar coordinates via

$$
\begin{pmatrix} y_{2l-1} \\ y_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l\cos\phi_l \\ r_l\sin\phi_l \end{pmatrix},\ \phi_l \in [-\pi, \pi),\ r_l \in (0, \infty),\ l = 1, \dots, k.
$$

All further coordinates, i.e. $y_{2k+1}, \dots, y_d$, remain fixed. The transformation $\hat{w}(\psi) := \tilde{w}(T_2(\psi))$ with $T_2(\psi) = (T(r_1, \phi_1), \dots, T(r_k, \phi_k), y_{2k+1}, \dots, y_d)$ for $\psi = (r_1, \phi_1, \dots, r_k, \phi_k, y_{2k+1}, \dots, y_d)$ in the domain $\Omega = ((0, \infty) \times [-\pi, \pi))^k \times \mathbb{R}^{d-2k}$ transfers (3.20) into

$$
\left(\lambda^2\hat{P}_2 + \lambda\hat{P}_1 + \hat{P}_\infty\right)\hat{w} = 0,\ \psi \in \Omega \tag{3.21}
$$

with

$$
\begin{aligned}
&\hat{P}_2 = M, \quad \hat{P}_1 = B + 2M\sum_{l=1}^{k}\sigma_l\partial_{\phi_l}, \\
&\hat{P}_\infty = -A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2 + \frac{1}{r_l}\partial_{\phi_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right] + M\sum_{l,n=1}^{k}\sigma_l\sigma_n\partial_{\phi_l}\partial_{\phi_n} + B\sum_{l=1}^{k}\sigma_l\partial_{\phi_l} - Df(v_\infty).
\end{aligned}
$$

3. **Simplified far-field operator:** The far-field operator (3.21) can be further simplified by letting $r_l \to \infty$ for any $1 \leqslant l \leqslant k$ which turns (3.21) into

$$
\left(\lambda^2\hat{P}_2 + \lambda\hat{P}_1 + P_\infty^{\mathrm{sim}}\right)\hat{w} = 0,\ \psi \in \Omega \tag{3.22}
$$

with

$$
P_\infty^{\mathrm{sim}} = -A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{y_l}^2\right] + M\sum_{l,n=1}^{k}\sigma_l\sigma_n\partial_{\phi_l}\partial_{\phi_n} + B\sum_{l=1}^{k}\sigma_l\partial_{\phi_l} - Df(v_\infty). \tag{3.23}
$$

4. **Angular Fourier transform:** Finally, we solve for eigenvalues and eigenfunctions of (3.23) by separation of variables and an angular resp. radial Fourier ansatz with $\omega \in \mathbb{R}^k$, $\rho, y \in \mathbb{R}^{d-2k}$, $n \in \mathbb{Z}^k$, $z \in \mathbb{C}^m$, $|z| = 1$, $r \in (0, \infty)^k$, $\phi \in (-\pi, \pi]^k$:

$$
\hat{w}(\psi) = \exp\left(i\sum_{l=1}^{k}\omega_l r_l\right)\exp\left(i\sum_{l=1}^{k}n_l\phi_l\right)\exp\left(i\sum_{l=2k+1}^{d}\rho_l y_l\right)z = \exp\left(i\langle\omega, r\rangle + i\langle n, \phi\rangle + i\langle\rho, y\rangle\right)z.
$$

Inserting this in (3.22) leads to the m-dimensional quadratic eigenvalue problem

$$
\left(\lambda^2 A_2 + \lambda A_1(n) + A_\infty(\omega, n, \rho)\right)z = 0 \tag{3.24}
$$

with matrices $A_2 \in \mathbb{R}^{m,m}$ and $A_1(n), A_\infty(\omega, n, \rho) \in \mathbb{C}^{m,m}$ given by

$$\begin{aligned} A_2 =& M, \quad A_1(n) = B + 2i\langle \sigma, n\rangle M, \\ A_\infty(\omega, n, \rho) =& \left(|\omega|^2 + |\rho|^2\right) A - \langle \sigma, n\rangle^2 M + i\langle \sigma, n\rangle B - Df(v_\infty). \end{aligned} \tag{3.25}$$

The Fourier ansatz is a well-known tool for investigating essential spectra, see e.g. [8].

5. Dispersion relation and dispersion set: As in Section 2.2.2 we consider the dispersion set consisting of all values $\lambda \in \mathbb{C}$ satisfying the dispersion relation

$$\det\left(\lambda^2 A_2 + \lambda A_1(n) + A_\infty(\omega, n, \rho)\right) = 0 \tag{3.26}$$

for some $\omega \in \mathbb{R}^k$, $\rho \in \mathbb{R}^{d-2k}$ and $n \in \mathbb{Z}^k$. Of course, one can replace $|\omega|^2 + |\rho|^2$ by any nonnegative real number. Solving (3.26) is equivalent to finding all zeros of a parameterized polynomial of degree $2m$. Note that the limiting case $M = 0$ and $B = I_m$ in (3.26) leads to the dispersion relation for rotating waves of first order evolution equations, see [1] for $d = 2$, and [13, Sec. 7.4 and 9.4], [2] for general $d \geqslant 2$.

Using standard cut-off arguments as in [1],[13],[2], the following result can be shown for suitable function spaces (e.g. $L^2(\mathbb{R}^d, \mathbb{R}^m)$):

Proposition 3.2 (Essential spectrum of rotating waves)**.** Let $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ with $f(v_\infty) = 0$ for some $v_\infty \in \mathbb{R}^m$. Let $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ with skew-symmetric $S_\star \in \mathbb{R}^{m,m}$ be a classical solution of (1.7) satisfying $v_\star(\xi) \to v_\infty$ as $|\xi| \to \infty$. Then, the dispersion set

$$\sigma_{\text{disp}}(\mathcal{P}) = \{\lambda \in \mathbb{C} \mid \lambda \text{ satisfies (3.26) for some } \omega \in \mathbb{R}^k,\ \rho \in \mathbb{R}^{d-2k},\ n \in \mathbb{Z}^k\}$$

belongs to the essential spectrum $\sigma_{\text{ess}}(\mathcal{P})$ of the operator polynomial $\mathcal{P}$ from (3.15).

3.2.2. Point Spectrum and symmetry set. Recall the SE(d)-group action

$$[a(R, \tau)u](x) = u(R^{-1}(x - \tau)),\ x \in \mathbb{R}^d,\ (R, \tau) \in \text{SE}(d).$$

whose generators are $D_l, l = 1, \dots, d$ and $D^{(a,b)}, a = 1, \dots, d-1,\ b = a+1, \dots, d$ from (3.18). As with the eigenvalue problem (3.15) we transform the steady state equation (1.7) into y-coordinates via $\tilde{v}_\star(y) = v_\star(Py)$ to obtain

$$0 = L\tilde{v}_\star + f(\tilde{v}_\star), \quad L = A\triangle - MK^2 + BK. \tag{3.27}$$

We apply the generators (3.18) to this equation and find

$$\begin{aligned} 0 =& D_l L\tilde{v}_\star + Df(\tilde{v}_\star)D_l\tilde{v}_\star, \quad l = 1, \dots, d, \\ 0 =& D^{(a,b)}L\tilde{v}_\star + Df(\tilde{v}_\star)D^{(a,b)}\tilde{v}_\star, \quad a = 1, \dots, d-1, b = a+1, \dots, d. \end{aligned} \tag{3.28}$$

Moreover, we can write the eigenvalue problem (3.17), (3.19) as follows

$$(L + Df(\tilde{v}_\star))\tilde{w} = (\lambda^2 M + \lambda(B - 2MK))\tilde{w}. \tag{3.29}$$

1. Linear combination of generators: In view of (3.28),(3.29) it is natural to seek eigenfunctions as a linear combination of generators applied to the profile

$$\tilde{w} = \sum_{a=1}^{d-1}\sum_{b=a+1}^{d} \alpha_{ab}^{\text{rot}} D^{(a,b)}\tilde{v}_\star + \sum_{c=1}^{d} \alpha_c^{\text{tra}} D_c\tilde{v}_\star, \quad \alpha_{ab}^{\text{rot}}, \alpha_c^{\text{tra}} \in \mathbb{C} \tag{3.30}$$

This is to be plugged into (3.29) and reduced to an eigenvalue problem for $\alpha^{\text{rot}}, \alpha^{\text{tra}}$ by using the equations (3.28).

2. Commutator relations and eigenvalues of $S_\star$: From (3.18) it is straightforward to establish the commutator relations

$$\begin{aligned} &D_c D_j = D_j D_c, \quad D_c D^{(i,j)} = D^{(i,j)} D_c + \delta_{cj} D_i - \delta_{ci} D_j \\ &D^{(a,b)} D^{(i,j)} = D^{(i,j)} D^{(a,b)} + \delta_{aj} D^{(i,b)} + \delta_{ai} D^{(b,j)} + \delta_{bi} D^{(j,a)} + \delta_{bj} D^{(a,i)}. \end{aligned} \tag{3.31}$$

From this we find the commutators with K and $\triangle$ to be

$$\begin{aligned} &D_cK = KD_c\ (c=2k+1,\dots,d),\ D^{(2\nu-1,2\nu)}\triangle = \triangle D^{(2\nu-1,2\nu)},\ D^{(2\nu-1,2\nu)}K = KD^{(2\nu-1,2\nu)},\\ &D_{2\nu-1}K = KD_{2\nu-1} - \sigma_\nu D_{2\nu},\quad D_{2\nu}K = KD_{2\nu} + \sigma_\nu D_{2\nu-1}\ (\nu=1,\dots,k). \end{aligned} \tag{3.32}$$

First, all operators $D_c, c=2k+1,\dots,d$ commute with L. Hence, by (3.28) the operator $L+Df(\tilde v_\star)$ has the eigenvalue 0 with geometric multiplicity at least $d-2k$ with eigenfunctions given by $D_c\tilde v_\star, c=2k+1,\dots d$. Furthermore, we obtain from (3.32) for $\nu=1,\dots,k$

$$\begin{aligned} &D_{2\nu-1}K^2 = K^2D_{2\nu-1} - 2\sigma_\nu KD_{2\nu} - \sigma_\nu^2 D_{2\nu-1},\quad D_{2\nu}K^2 = K^2D_{2\nu} + 2\sigma_\nu KD_{2\nu-1} - \sigma_\nu^2 D_{2\nu},\\ &D_{2\nu-1}L = LD_{2\nu-1} - \sigma_\nu(B-2MK)D_{2\nu} + \sigma_\nu^2 MD_{2\nu-1},\\ &D_{2\nu}L = LD_{2\nu} + \sigma_\nu(B-2MK)D_{2\nu-1} + \sigma_\nu^2 MD_{2\nu}. \end{aligned} \tag{3.33}$$

Combining the last two rows with (3.28) finally yields

$$(L+Df(\tilde v_\star))(D_{2\nu}+iD_{2\nu-1})\tilde v_\star = \left[(i\sigma_\nu)(B-2MK) + (i\sigma_\nu)^2M\right](iD_{2\nu-1}+D_{2\nu})\tilde v_\star, \tag{3.34}$$

and similarly for the complex conjugate. Therefore, the quadratic problem (3.29) has eigenvalues $\pm i\sigma_\nu$ with eigenfunctions $\pm iD_{2\nu-1}\tilde v_\star + D_{2\nu}\tilde v_\star, \nu=1,\dots k$.

3. Commutator relations and sums of eigenvalues of $S_\star$: Now we use the relation (3.28) for indices $a=2\mu-1,2\mu$ and $b=2\nu-1,2\nu$ with $1\le\mu<\nu\le k$. The following commutator relations follow from (3.31)

$$\mathcal{D}^{[\mu,\nu]}K = (KI_4+\Sigma)\mathcal{D}^{[\mu,\nu]},\quad \mathcal{D}^{[\mu,\nu]} = \begin{pmatrix} D^{(2\mu-1,2\nu-1)}\\ D^{(2\mu-1,2\nu)}\\ D^{(2\mu,2\nu-1)}\\ D^{(2\mu,2\nu)}\end{pmatrix},\quad \Sigma = \begin{pmatrix} 0 & -\sigma_\nu & -\sigma_\mu & 0\\ \sigma_\nu & 0 & 0 & -\sigma_\mu\\ \sigma_\mu & 0 & 0 & -\sigma_\nu\\ 0 & \sigma_\mu & \sigma_\nu & 0\end{pmatrix}. \tag{3.35}$$

Note that Σ is skew-symmetric with eigenvalues $\pm i(\sigma_\nu\pm\sigma_\mu)$. We look for eigenfunctions of the special form

$$\tilde w = L_\alpha\tilde v_\star,\quad L_\alpha = \sum_{a=2\mu-1,2\mu}\sum_{b=2\nu-1,2\nu}\alpha_{a,b}D^{(a,b)} = \alpha^\top\mathcal{D}^{[\mu,\nu]}. \tag{3.36}$$

Let $\lambda\in\mathbb{C}$ be an eigenvalue of Σ with eigenvector $\alpha\in\mathbb{C}^4$, so that $\alpha^\top\Sigma = -\lambda\alpha^\top$ by skew-symmetry. Then we obtain from (3.35)

$$L_\alpha K = \alpha^\top\mathcal{D}^{[\mu,\nu]}K = \alpha^\top(KI_4+\Sigma)\mathcal{D}^{[\mu,\nu]} = (K-\lambda)L_\alpha,\quad L_\alpha K^2 = (K^2-2\lambda K+\lambda^2)L_\alpha. \tag{3.37}$$

Using (3.27), (3.32) and the fact that scalar differential operators commute with matrices, yields

$$\begin{aligned} L_\alpha L =& L_\alpha(\triangle A - K^2M + KB) = \triangle L_\alpha A - (K^2-2\lambda K+\lambda^2)L_\alpha M + (K-\lambda)L_\alpha B\\ =& LL_\alpha - (\lambda^2M + \lambda(B-2MK))L_\alpha. \end{aligned}$$

Finally, we note that $L_\alpha L\tilde v_\star = -Df(\tilde v_\star)L_\alpha\tilde v_\star$ follows from (3.28), which then gives

$$(L+Df(\tilde v_\star))L_\alpha\tilde v_\star = (LL_\alpha - L_\alpha L)\tilde v_\star = (\lambda^2M+\lambda(B-2MK))L_\alpha\tilde v_\star.$$

Hence $\tilde w = L_\alpha\tilde v_\star$ solves the quadratic eigenvalue problem (3.29). All 4 eigenfunctions can now be read off from the columns of the unitary matrix

$$V = \frac14\begin{pmatrix} 1 & 1 & 1 & 1\\ -i & -i & i & i\\ -i & i & -i & i\\ -1 & 1 & 1 & -1\end{pmatrix},\quad \Sigma V = V\operatorname{diag}(i(\sigma_\nu+\sigma_\mu), i(\sigma_\nu-\sigma_\mu), i(\sigma_\mu-\sigma_\nu), -i(\sigma_\nu+\sigma_\mu)).$$

The computations for the remaining eigenfunctions are similar. Take $\mu=1,\dots,k$, $c=2k+1,\dots,d$ and replace (3.35) by the relation

$$\mathcal{D}^{[\mu,c]}K = (KI_4+\Sigma)\mathcal{D}^{[\mu,c]},\quad \mathcal{D}^{[\mu,\nu]} = \begin{pmatrix} D^{(2\mu-1,2\mu)}\\ D^{(2\mu-1,c)}\\ D^{(2\mu,c)}\end{pmatrix},\quad \Sigma = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -\sigma_\mu\\ 0 & \sigma_\mu & 0\end{pmatrix}.$$

For the zero row and column in Σ compare (3.32). Then by the same computations as above one finds that eigenvalues λ of Σ are also eigenvalues of (3.29). The corresponding eigenfunctions are $\tilde{w} = \alpha^\top \mathcal{D}^{[\mu,c]}\rangle \tilde{v}_\star$ where $\Sigma\alpha = \lambda\alpha$. Therefore, we have further eigenfunctions $D^{(2\mu-1,2\mu)}\tilde{v}_\star$ for the eigenvalue 0 and $D^{(2\mu-1,c)}\tilde{v}_\star \pm iD^{(2\mu,c)}\tilde{v}_\star$ for the eigenvalue $\pm i\sigma_\mu$, $\mu = 1,\dots,k$. Finally, note that for any two indices $2k < b < c \le d$ the operator $D^{(b,c)}$ commutes with K and $\triangle$, and thus produces another eigenfunction $\tilde{w} = D^{(b,c)}\tilde{v}_\star$ which belongs to the eigenvalue 0.
Collecting all computations we have found a total of $d+4\frac{k(k-1)}{2}+2k(d-2k)+k+\frac{(d-2k)(d-2k-1)}{2} = \frac{1}{2}d(d+1)$ eigenfunctions corresponding to eigenvalues of $S_\star$ and to sums of different eigenvalues of $S_\star$. In a generic sense we expect these eigenfunctions to be linearly independent, but, of course, one cannot prove this in general. Let us summarize the result in a proposition.

Proposition 3.3 (Point spectrum of rotating waves)**.** Let $f \in C^2(\mathbb{R}^m,\mathbb{R}^m)$ and let $v_\star \in C^3(\mathbb{R}^d,\mathbb{R}^m)$ be a classical solution of (1.7) with skew-symmetric $S_\star \in \mathbb{R}^{m,m}$. Further, let $\lambda_j, j = 1,\dots,d$ be the eigenvalues of $S_\star$ repeated according to their multiplicity. Then the symmetry set

$$\sigma_{\mathrm{sym}}(\mathcal{P}) = \{\lambda \in \mathbb{C} : \lambda = \lambda_j \text{ or } \lambda = \lambda_i + \lambda_j \text{ for some } 1 \le i < j \le d\}$$

belongs to the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{P})$ of the quadratic operator polynomial $\mathcal{P}(\lambda)$ in (3.15), (3.16) obtained by linearizing at the wave. The eigenvalues and eigenfunctions of the transformed problem (3.17) are displayed in the following table (setting $r = d - 2k$):

eigenvalue	eigenfunction	index	number
0	$D_c\tilde{v}_\star$	$2k < c \le d$	r
$\pm i\sigma_\nu$	$(D_{2\nu} \pm iD_{2\nu-1})\tilde{v}_\star$	$1 \le \nu \le k$	$2k$
$\pm i(\sigma_\nu + \sigma_\mu)$	$(D^{(2\mu-1,2\nu-1)} \mp iD^{(2\mu-1,2\nu)} \mp iD^{(2\mu,2\nu-1)} - D^{(2\mu,2\nu)})\tilde{v}_\star$	$1 \le \mu < \nu \le k$	$k(k-1)$
$\pm i(\sigma_\nu - \sigma_\mu)$	$(D^{(2\mu-1,2\nu-1)} \pm iD^{(2\mu-1,2\nu)} \mp iD^{(2\mu,2\nu-1)} + D^{(2\mu,2\nu)})\tilde{v}_\star$	$1 \le \mu < \nu \le k$	$k(k-1)$
$\pm i\sigma_\nu$	$(D^{(2\nu-1,c)} \pm iD^{(2\nu,c)})\tilde{v}_\star$	$1 \le \nu \le k, 2k < c \le d$	$2kr$
0	$D^{(2\nu-1,\nu)}\tilde{v}_\star$	$1 \le \nu \le k$	k
0	$D^{(b,c)}\tilde{v}_\star$	$2k < b < c \le d$	$\frac{r}{2}(r-1)$

Table 1: Eigenfunctions and eigenvalues (with multiplicities) on the imaginary axis for the linearized quadratic eigenvalue problem (3.17).

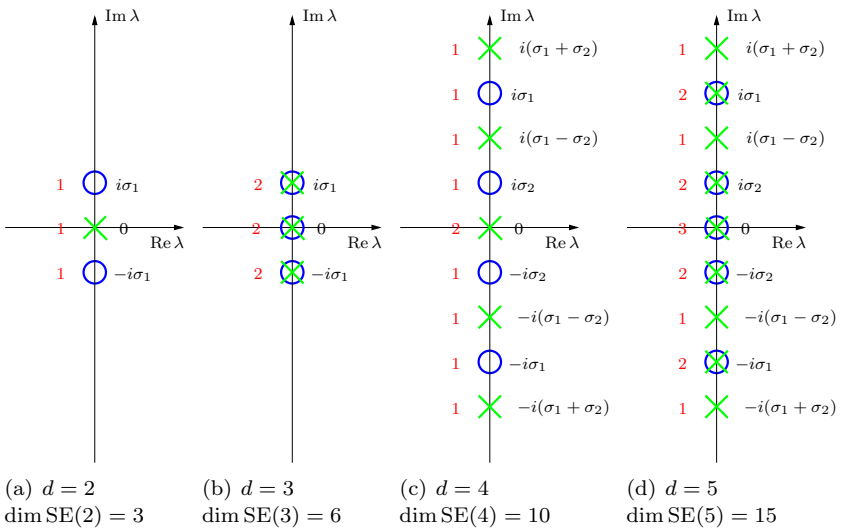
Note that we did not assume any limit behavior of $v_\star(\xi)$ for $\xi \to \infty$ as in Proposition 3.2. Therefore, Proposition 3.3 also applies to rotating waves that are not localized, e.g. spiral waves. This has been confirmed in numerical experiments.
Figure 3.3 shows the eigenvalues $\lambda \in \sigma_{\mathrm{sym}}(\mathcal{P})$ from Proposition 3.3 and their corresponding multiplicities for different space dimensions $d = 2,3,4,5$. The eigenvalues $\lambda \in \sigma(S_\star)$ are indicated by blue circles, the eigenvalues $\lambda \in \{\lambda_i + \lambda_j \mid \lambda_i, \lambda_j \in \sigma(S_\star),\ 1 \leqslant i < j \leqslant d\}$ by green crosses. The imaginary values to the right of the symbols denote eigenvalues and the numbers to the left their corresponding multiplicities. As expected, there are $\frac{d(d+1)}{2}$ eigenvalues on the imaginary axis in case of space dimension d. Lower bounds for the geometric multiplicities can be derived from our table as follows

$$\mathrm{mult}(0) \ge k + \frac{1}{2}(d-2k)(d-2k+1), \quad \mathrm{mult}(\pm i\sigma_\nu) \ge d - 2k + 1, \quad \nu = 1,\dots,k.$$

It is a remarkable feature that the eigenvalues coincide with those for first order evolution equations, see [2], [13].

Example 3.4 (Cubic-quintic Ginzburg-Landau wave equation)**.** As shown in Example 3.1 the cubic-quintic Ginzburg-Landau wave equation (3.11) with coefficients and parameters (3.12) has a spinning soliton solution $u_\star(x,t) = v_\star(e^{-tS_\star}(x - x_\star))$ with rotational velocity $\mu_1^{(3)} = 1.3658$.
We next solve numerically the eigenvalue problem for the cubic-quintic Ginzburg-Landau wave equation. For this purpose we consider the real valued version of (3.11)

$$MU_{tt} + BU_t = A\triangle U + F(U),\ x \in \mathbb{R}^d,\ t \geqslant 0 \tag{3.38}$$



(a) $d = 2$ $\dim \mathrm{SE}(2) = 3$ (b) $d = 3$ $\dim \mathrm{SE}(3) = 6$ (c) $d = 4$ $\dim \mathrm{SE}(4) = 10$ (d) $d = 5$ $\dim \mathrm{SE}(5) = 15$

FIGURE 3.3. Point spectrum of the linearization $\mathcal{P}$ on the imaginary axis $i\mathbb{R}$ for space dimension $d = 2, 3, 4, 5$ given by Proposition 3.3.

with

$$
\begin{aligned}
&M = \begin{pmatrix} \varepsilon_1 & -\varepsilon_2 \\ \varepsilon_2 & \varepsilon_1 \end{pmatrix}, \quad B = \begin{pmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\
&F(U) = \begin{pmatrix} (U_1\delta_1 - U_2\delta_2) + (U_1\beta_1 - U_2\beta_2)(U_1^2 + U_2^2) + (U_1\gamma_1 - U_2\gamma_2)(U_1^2 + U_2^2)^2 \\ (U_1\delta_2 + U_2\delta_1) + (U_1\beta_2 + U_2\beta_1)(U_1^2 + U_2^2) + (U_1\gamma_2 + U_2\gamma_1)(U_1^2 + U_2^2)^2 \end{pmatrix},
\end{aligned}
\tag{3.39}
$$

where $u = u_1 + iu_2$, $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\rho = \rho_1 + i\rho_2$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, $\delta = \delta_1 + i\delta_2$ and $\varepsilon_j, \rho_j, \alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R}$.
Now, the eigenvalue problem for the cubic-quintic Ginzburg-Landau wave equation is, cf. (3.15), (3.16),

$$
\left(\lambda^2 M \cdot + \lambda \left[B \cdot - 2M(\partial_\xi \cdot)S\xi\right] + \left[-A\triangle \cdot + M(\partial_\xi^2 \cdot)(S\xi)^2 + M(\partial_\xi \cdot)S^2\xi - B(\partial_\xi \cdot)S\xi - DF(v_\star)\cdot\right]\right) w = 0. \tag{3.40}
$$

Both approximations of the profile $v_\star$ and the velocity matrix $S = S_\star$ in (3.40) are chosen from the solution of (3.13) at time $t = 2000$ in Example 3.1. By Proposition 3.3 the problem (3.40) has eigenvalues $\lambda = 0, \pm i\sigma$. These eigenvalues will be isolated and hence belong to the point spectrum, if the differential operator is Fredholm of index 0 in suitable function spaces. For the parabolic case ($M = 0$) this has been established in [2] and we expect it to hold in the general case as well. Let us next discuss the dispersion set from Proposition 3.2. The cubic-quintic Ginzburg-Landau nonlinearity $F : \mathbb{R}^2 \to \mathbb{R}^2$ from (3.39) satisfies

$$
DF(v_\infty) = \begin{pmatrix} \delta_1 & -\delta_2 \\ \delta_2 & \delta_1 \end{pmatrix} \quad \text{for} \quad v_\infty = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.41}
$$

The matrices A_2, $A_1(n)$, $A_\infty(\omega, n)$ from (3.25) of the quadratic problem (3.24) are given by

$$
A_2 = M, \quad A_1(n) = B + 2i\sigma n M, \quad A_\infty(\omega, n) = \omega^2 A - \sigma^2 n^2 M + i\sigma n B - DF(v_\infty)
$$

for M, B, A from (3.39), $DF(v_\infty)$ from (3.41), $\omega \in \mathbb{R}$, $n \in \mathbb{Z}$ and $\sigma = \mu_1^{(3)}$. The dispersion relation (3.26) for the spinning solitons of the Ginzburg-Landau wave equation in $\mathbb{R}^2$ states that every $\lambda \in \mathbb{C}$ satisfying

$$
\det\left(\lambda^2 M + \lambda(B + 2i\sigma n M) + (\omega^2 A - \sigma^2 n^2 M + i\sigma n B - DF(v_\infty))\right) = 0
$$

for some $\omega \in \mathbb{R}$ and $n \in \mathbb{Z}$, belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{P})$ of $\mathcal{P}$. We may rewrite this in complex notation and find the dispersion set

$$
\sigma_{\mathrm{disp}}(\mathcal{P}) = \{\lambda \in \mathbb{C} : \lambda^2\varepsilon + \lambda(\rho + 2i\sigma n\varepsilon) + (\omega^2\alpha - \sigma^2 n^2 \varepsilon + i\sigma n\rho - \delta) = 0 \text{ for some } \omega \in \mathbb{R}, n \in \mathbb{Z}\} \tag{3.42}
$$

The elements of the dispersion set are

$$\lambda_{1,2} = -\frac{\rho}{2\varepsilon} - i\sigma n \pm \frac{1}{2\varepsilon}\sqrt{\rho^2 - 4\varepsilon(\omega^2\alpha - \delta)}, \quad n \in \mathbb{Z}, \omega \in \mathbb{R}..$$

They lie on the vertical line $\mathrm{Re} = -\frac{\rho}{2\varepsilon}$ and on infinitely many horizontal lines given for $n \in \mathbb{Z}$ by $i\sigma n + \frac{1}{2\varepsilon}[-\rho - \sqrt{\rho^2 + 4\tau\delta}, \rho + \sqrt{\rho^2 + 4\tau\delta}]$, see Figure 3.4 (a),(b).

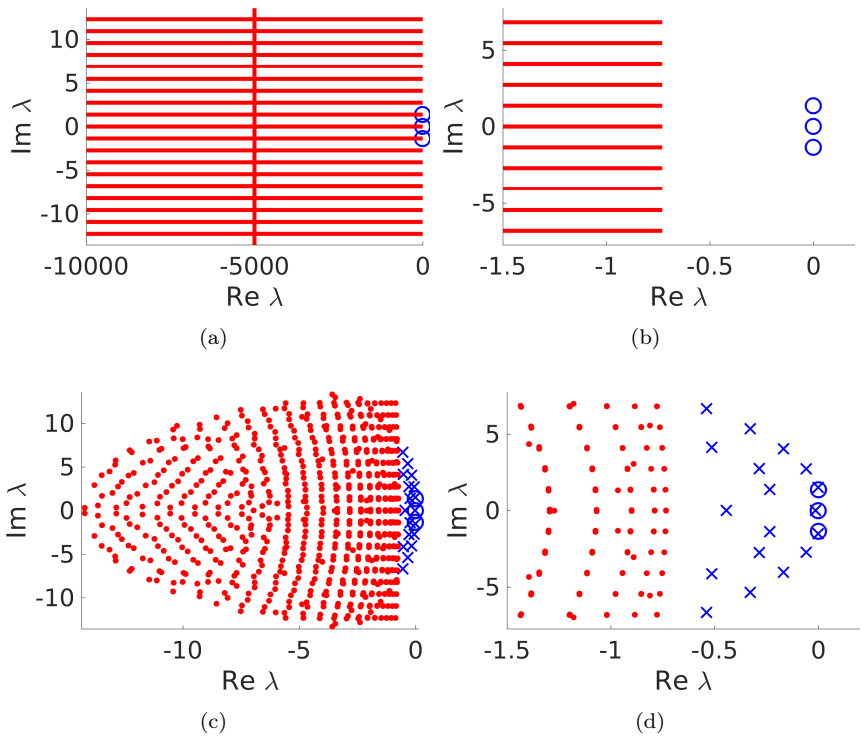


FIGURE 3.4. Subsets $\sigma_{\mathrm{disp}}(\mathcal{P})$ and $\sigma_{\mathrm{sym}}(\mathcal{P})$ of the spectrum for the cubic-quintic Ginzburg-Landau wave equation for $d = 2$ with parameters (3.12) (a),(b) and two different views of the numerical spectrum on a ball $B_R(0)$ with radius $R = 20$ (c),(d).

Figure 3.4(a) and (b) shows two different views for the part of the spectrum of the spinning solitons which is guaranteed by Proposition 3.2 and 3.3. It is subdivided into the symmetry set $\sigma_{\mathrm{sym}}(\mathcal{P})$ (blue circle), which is determined by Proposition 3.3 and belongs to the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{P})$, and the dispersion set $\sigma_{\mathrm{disp}}(\mathcal{P})$ (red lines), which is determined by Proposition 3.2 and belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{P})$. In general, there may be further essential spectrum in $\sigma_{\mathrm{ess}}(\mathcal{P}) \setminus \sigma_{\mathrm{disp}}(\mathcal{P})$ and further isolated eigenvalues in $\sigma_{\mathrm{pt}}(\mathcal{P}) \setminus \sigma_{\mathrm{sym}}(\mathcal{P})$. In fact, for the spinning solitons of the cubic-quintic Ginzburg-Landau wave equation we find 18 extra eigenvalues with negative real parts (8 complex conjugate pairs and 2 purely real eigenvalues), cf. Figure 3.4(c),(d). These Figures show two different views for the numerical spectrum of the cubic-quintic Ginzburg-Landau wave equation on the ball $B_R(0)$ with radius $R = 20$ equipped with homogeneous Neumann boundary conditions. They consist of the approximations of the point spectrum subdivided into the symmetry set (blue circle) and additional isolated eigenvalues (blue cross sign), and of the essential spectrum (red dots). Three of these isolated eigenvalues are very close to the imaginary axis, see Figure 3.5(c). Therefore, the spinning solitons seem to be only weakly stable. Finally, the approximated eigenfunctions belonging to the eigenvalues $\lambda \approx 0$ and $\lambda \approx +i\sigma$ are shown in Figure 3.5(a) and (b). In particular, Figure 3.5(a) is an approximation of the rotational term $\langle Sx, \nabla v_\star(x)\rangle$.

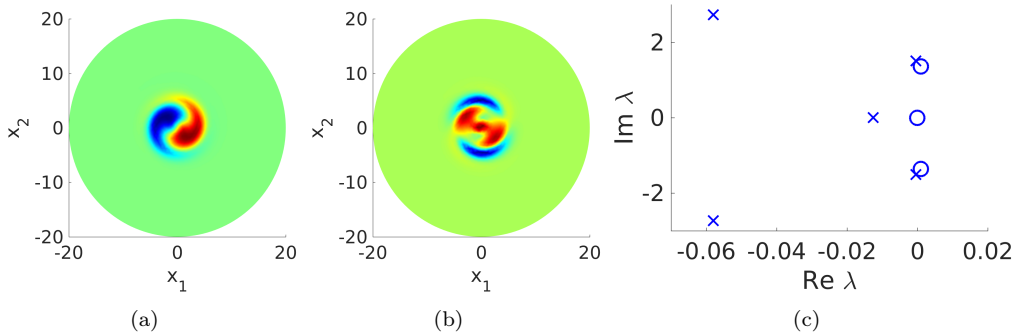


FIGURE 3.5. Eigenfunctions of the cubic-quintic Ginzburg-Landau wave equation for parameters (3.12) belonging to the isolated eigenvalues $\lambda_1 \approx 0$ (a) and $\lambda_2 \approx i\sigma$ (b) and a zoom into the spectrum from Fig.3.4(c) in (c).

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