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CRC Preprint 2016/25 (revised), June 2017

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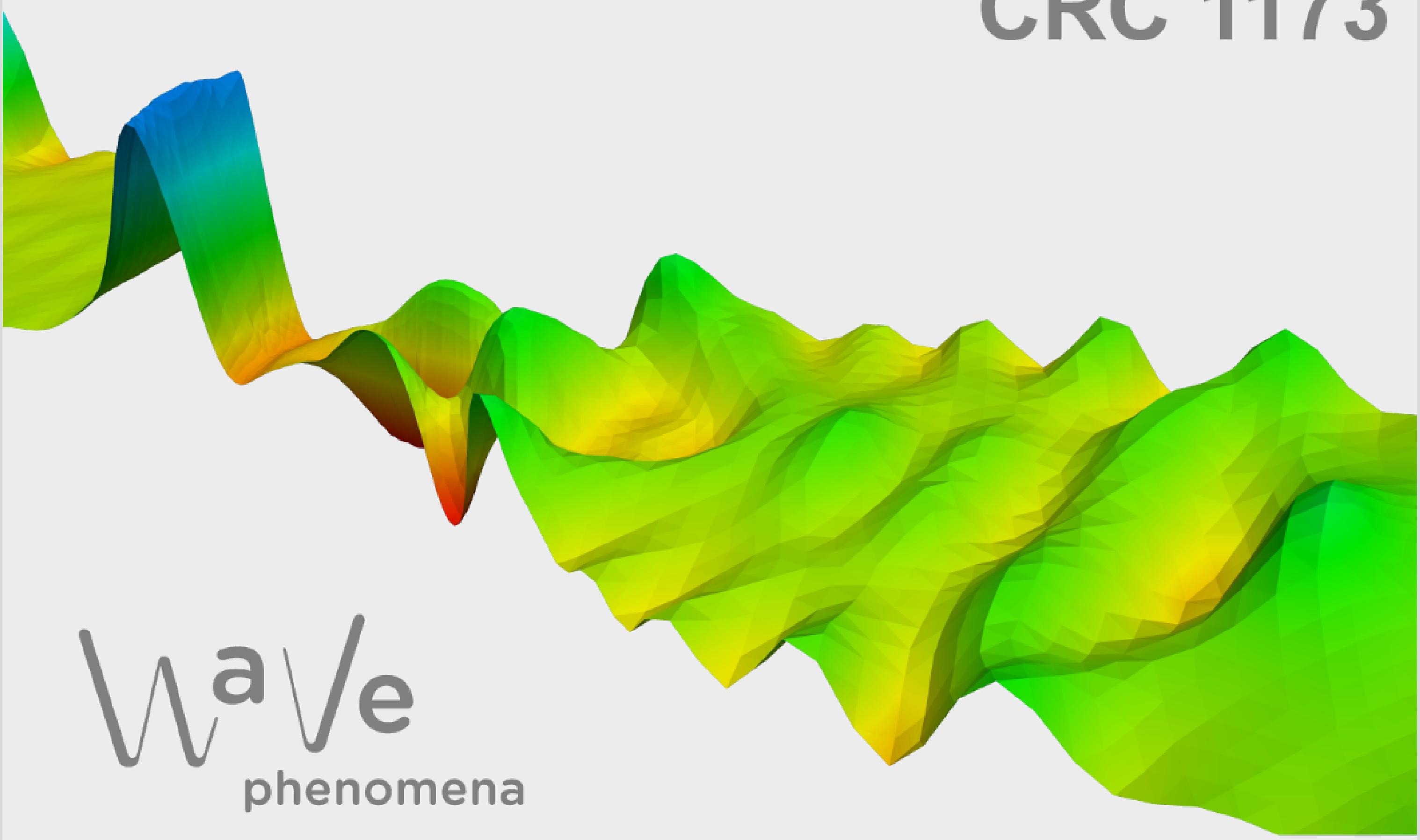
CRC 1173



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ISSN 2365-662X

Finite Element Heterogeneous Multiscale Method for Time-Dependent Maxwell's Equations

Marlis Hochbruck and Christian Stohrer

Abstract We propose a Finite Element Heterogeneous Multiscale Method (FE-HMM) for time dependent Maxwell's equations in second-order formulation in locally periodic materials. This method can approximate the effective behavior of an electromagnetic wave traveling through a highly oscillatory material without the need to resolve the microscopic details of the material. To prove an a-priori error bound for the semi-discrete FE-HMM scheme, we need a new generalization of a Strang-type lemma for second-order hyperbolic equations. Finally, we present a numerical example that is in accordance with the theoretical results.

Key words: time dependent Maxwell's equations, finite element heterogeneous multiscale method, Strang-type lemma for hyperbolic PDEs

1 Introduction

We want to simulate electromagnetic wave propagation in a highly oscillatory material. FE-HMMs have proven to be efficient and reliable methods for many multiscale problems, see e.g. [1, 2]. Their most important advantage is that the influence of the microscopic details of the material are taken into account, whilst only a macroscopic discretization of the whole computational domain is needed. These methods were first proposed for elliptic and parabolic equations. In [3] it was proven, that the same ideas can be applied to the acoustic wave equation. This equation can be seen as an easily manageable special case of Maxwell's equations. Therefore, it is reasonable

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that FE-HMM can also be generalized to second-order time-dependent Maxwell's equation.

Recently, FE-HMMs for time-harmonic Maxwell's equations in rapidly oscillatory materials were presented, see [12] and [7]. There, two types of micro problems were used to approximate the effective (or upscaled or homogenized) solution. These micro problems are solved on small sampling domains such that the overall computational cost does not become infeasibly large. Here, we apply the FE-HMM scheme from [7] to second-order time-dependent Maxwell's equation. To the best of our knowledge, this is the first FE-HMM scheme for this equation, while other multiscale schemes have already been proposed, see e.g. the recent article [6] and the references therein.

We consider a multiscale material with permittivity ε^η and permeability μ^η, where η denotes the characteristic microscopic length of the material. We assume that η is much smaller than the diameter of the computational domain Ω. In this article we restrict ourselves to locally periodic materials, see Definition 1 below, for simplicity. We are convinced that the Finite Element Heterogeneous Multiscale Methods (FE-HMM) presented here can be adapted to more general situations, but a rigorous justification thereof is ongoing research and beyond the scope of the current article. For a locally periodic material, η denotes the length of the microscopic oscillations in it.

The multiscale second order time-dependent Maxwell's equation is given by

$$\partial_{tt}\varepsilon^\eta(x)\mathbf{E}^\eta(t;x) + \nabla\times\big(\nu^\eta(x)(\nabla\times\mathbf{E}^\eta(t;x)\big) = \boldsymbol{f}(t;x) \quad \text{in } (0,T)\times\Omega, \qquad (1)$$

where $\mathbf{E}^\eta$ is the unknown multiscale electric field and

$$\nu^\eta = (\mu^\eta)^{-1}$$

is the inverse of the magnetic permeability. To derive this equation from the standard first-order Maxwell's equations we assumed that the electric field is generated by a density free current and that the conductivity is zero (lossless material). The precise functional analytic setting, the initial and boundary conditions are given in Section 2, where we also recall a homogenization result derived from [18, Theorem 3.2]. In a nutshell, it states that $\mathbf{E}^\eta$ converges to the solution $\mathbf{E}^{\text{eff}}$ of an effective Maxwell's equation as the characteristic length η tends to zero. In Section 3 we describe how the idea of [7] can be used to build a FE-HMM for (1) to approximate $\mathbf{E}^{\text{eff}}$. All the advantages of FE-HMM schemes mentioned above carry over to the time-dependent case. We give an a-priori estimate of the difference between the FE-HMM and the effective solution in Section 4. This estimate is based on a improved version of the Strang-type Lemma given in [3]. To conclude this article we give a numerical example that corroborates our theoretical findings.

Notation. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, with $d = 2,3$. We denote by $H^\ell(\Omega)$ the standard Sobolev spaces and set $L^2(\Omega) = H^0(\Omega)$ as usual. Vector valued function spaces are denoted in bold face, e.g. we set $\boldsymbol{H}^\ell(\Omega) := H^\ell(\Omega)^d$. We denote the corresponding scalar product and norm by $(\cdot,\cdot)_{\ell,\Omega}$, and $\|\cdot\|_{\ell,\Omega}$ respectively. The space $\boldsymbol{H}(\mathbf{curl};\Omega)$ consists of all $\boldsymbol{L}^2(\Omega)$ functions with a bounded curl. This space is a Hilbert space with respect to the scalar product

$$(\mathbf{v},\mathbf{w})_{\mathbf{curl},\Omega} = (\mathbf{v},\mathbf{w})_{0,\Omega} + (\mathbf{curl}\,\mathbf{v},\mathbf{curl}\,\mathbf{w})_{0,\Omega}.$$

We denote by $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ the closure of $\boldsymbol{C}_0^\infty(\Omega)$ in $\boldsymbol{H}(\mathbf{curl};\Omega)$. This is the subspace of $\boldsymbol{H}(\mathbf{curl};\Omega)$ of functions with vanishing tangential components on the boundary $\partial\Omega$. Details about these spaces can e.g. be found in [17]. We denote likewise periodic boundary condition. For example for the centered unit cube $Y = (-1/2, 1/2)^d$, we denote by $\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};Y)$ the closure of $\boldsymbol{C}_{\mathrm{per}}^\infty(Y)$.

2 Analytic setting

As already mentioned in the introduction, we assume that the permittivity ε^η and the inverse permeability ν^η are locally periodic.

Definition 1. A tensor $\xi^\eta : \Omega \to \mathbb{R}^{d\times d}$ is *locally periodic* if there is a tensor $\xi : \Omega\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$, which is Y-periodic ($Y = (-1/2, 1/2)^d$) in its second argument, such that $\xi^\eta(x) = \xi(x, x/\eta)$ for almost every $x \in \Omega$. We call such a function ξ *blueprint* of ξ^η.

In addition to the local periodicity we make from now on the following regularity assumptions on the tensors ε^η and ν^η:

The blueprints of ε^η and ν^η are symmetric and in $\left(C(\Omega;L^\infty_{\mathrm{per}}(Y))\right)^{d\times d}$. (A$_1$)

The tensors ε^η and ν^η are uniformly bounded and positive definite. (A$_2$)

Assumption (A$_2$) means that there are $0 < \alpha \leq \beta$ such that for $\xi \in \{\varepsilon^\eta, \nu^\eta\}$ and almost every $x \in \Omega$

$$\alpha|z|^2 \leq \xi(x)z\cdot z \quad \text{and} \quad \xi(x)z\cdot\tilde{z} \leq \beta|z||\tilde{z}| \qquad \text{for all } z,\tilde{z}\in\mathbb{R}^d. \tag{A$_2'$}$$

We consider the variational formulation of (1).

$$\begin{cases} \text{Find } \mathbf{E}^\eta : (0,T) \to \boldsymbol{H}_0(\mathbf{curl};\Omega), \text{ such that for all } \mathbf{v}\in\boldsymbol{H}_0(\mathbf{curl};\Omega) \\ \left(\partial_{tt}\varepsilon^\eta\mathbf{E}^\eta(t),\mathbf{v}\right)_{0,\Omega} + \left(\nu^\eta\,\mathbf{curl}\,\mathbf{E}^\eta(t),\mathbf{curl}\,\mathbf{v}\right)_{0,\Omega} = \left(\mathbf{f}(t),\mathbf{v}\right)_{0,\Omega}, \\ \quad \mathbf{E}^\eta(0) = \mathbf{E}_0, \qquad \text{and} \qquad \partial_t\mathbf{E}^\eta(0) = \mathbf{E}_0'. \end{cases} \tag{2}$$

This problem has a unique solution if, see e.g. [14, Chap. 3, Thm. 8.1],

$$\mathbf{E}_0 \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \quad \mathbf{E}_0' \in \boldsymbol{L}^2(\Omega), \quad \text{and} \quad \mathbf{f}\in\boldsymbol{L}^2(0,T;L^2(\Omega)).$$

Note that by the choice of the space $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ we use boundary conditions of a perfect electric conductor. This means that the tangential component of $\mathbf{E}^\eta$ vanishes at the boundary.

Homogenization theory. In [18] homogenization results for time-dependent first order Maxwell's equations have been proven, that answer the question how $\mathbf{E}^\eta$ behaves as $\eta \to 0$. In the case of lossless materials with no charge density, it is easy to rewrite this result in a second-order formulation. Similar results can be found in [5], [13], and [15]. Let us first introduce the involved micro problems.

Definition 2. Let $Y_\eta(x) = x + \eta Y$ be the scaled and shifted unit cell. The *first micro problem at $x \in \Omega$ constrained with a given* $\mathbf{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$ is defined as follows.

$$\begin{cases} \text{Find } \varphi^{\mathbf{v}}(x,\cdot) \in \varphi^{\mathbf{v}}_{\text{lin}}(x,\cdot) + H^1_{\text{per}}(Y_\eta(x)), \text{ such that } \displaystyle\int_{Y_\eta(x)} \varphi^{\mathbf{v}}(x,y)\,dy = 0 \text{ and} \\ \left(\varepsilon\left(x,\frac{\cdot}{\eta}\right)\nabla_y \varphi^{\mathbf{v}}(x,\cdot), \nabla\zeta\right)_{0,Y_\eta(x)} = 0, \qquad \text{for all } \zeta \in H^1_{\text{per}}(Y_\eta(x)), \end{cases} \tag{3}$$

where $\varphi^{\mathbf{v}}_{\text{lin}}(x,y) = \mathbf{v}(x)\cdot(y-x)$.

Definition 3. The *second micro problem at $x \in \Omega$ constrained with a given* $\mathbf{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$ is defined as follows.

$$\begin{cases} \text{Find } \big(\mathbf{u}^{\mathbf{v}}(x,\cdot),p\big) \in \big(\mathbf{u}^{\mathbf{v}}_{\text{lin}} + \boldsymbol{H}_{\text{per}}(\mathbf{curl};Y_\eta(x))\big) \times H^1_{\text{per}}(Y_\eta(x)), \\ \text{such that} \displaystyle\int_{Y_\eta(x)} \mathbf{u}^{\mathbf{v}}(x,y)\,dy = \mathbf{0}, \int_{Y_\eta(x)} p(y)\,d\mathbf{y} = 0, \text{ and} \\ \left(\nu\left(x,\frac{\cdot}{\eta}\right)\mathbf{curl}_y\,\mathbf{u}^{\mathbf{v}}(x,\cdot), \mathbf{curl}\,\mathbf{z}\right)_{0,Y_\eta(x)} + \big(\mathbf{u}^{\mathbf{v}}(x,\cdot),\nabla q\big)_{0,Y_\eta(x)} + \big(\mathbf{z},\nabla p\big)_{0,Y_\eta(x)} = 0, \\ \text{for all } (\mathbf{z},q) \in \boldsymbol{H}_{\text{per}}(\mathbf{curl};Y_\eta(x)) \times H^1_{\text{per}}(Y_\eta(x)), \end{cases} \tag{4}$$

where $\mathbf{u}^{\mathbf{v}}_{\text{lin}}(x,y) = \mathbf{v}(x) + \frac{1}{2}\,\mathbf{curl}\,\mathbf{v}(x) \times (y-x)$.

Note that the first micro problem is the well-known elliptic cell problem of classical homogenization theory posed over the shifted sampling domain $Y_\eta(x)$ instead of the unit square Y if one chooses $\mathbf{v}$ to be a (constant) unit vector of $\mathbb{R}^d$. The second micro problem is used less frequently and related to the first one through "dual formulas", see [5, Ch. 1, Rem. 5.9]. We recall the following homogenization result.

Theorem 1 (cf. [18, Thm. 3.2]). *Let ε^η and ν^η be locally periodic with blueprints ε, respectively ν, which fulfill the assumptions* (A_1) *and* (A_2)*. For $\eta > 0$ let $\mathbf{E}^\eta$ be the solution of the multiscale Maxwell's equation* (2)*. Then, as $\eta \to 0$, $\mathbf{E}^\eta$ converges weakly-$*$ in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ to $\mathbf{E}^{\text{eff}}$, where $\mathbf{E}^{\text{eff}}$ is the solution of the following effective Maxwell's equation.*

$$\begin{cases} \textit{Find } \mathbf{E}^{\text{eff}} : (0,T) \to \boldsymbol{H}_0(\mathbf{curl};\Omega), \textit{ such that for all } \mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \\ S^{\text{eff}}(\partial_{tt}\mathbf{E}^{\text{eff}}(t),\mathbf{v}) + B^{\text{eff}}(\mathbf{E}^{\text{eff}}(t),\mathbf{v}) = (\mathbf{f}(t),\mathbf{v})_{0,\Omega}, \\ \mathbf{E}^{\text{eff}}(0) = \mathbf{E}_0, \qquad \textit{and} \qquad \partial_t\mathbf{E}^{\text{eff}}(0) = \mathbf{E}_0'. \end{cases} \tag{5}$$

The effective scalar product S^{eff} is given by

$$S^{\text{eff}}(\mathbf{v},\mathbf{w}) = \int_\Omega \frac{1}{|Y_\eta(x)|}\left(\varepsilon\left(x,\frac{\cdot}{\eta}\right)\boldsymbol{\nabla}_y\,\varphi^{\mathbf{v}}(x,\cdot),\boldsymbol{\nabla}_y\,\varphi^{\mathbf{w}}(x,\cdot)\right)_{0,Y_\eta(x)} dx,$$

for all $\mathbf{v},\mathbf{w}\in \boldsymbol{H}(\mathbf{curl};\Omega)$*, where* $\varphi^{\mathbf{v}}$ *and* $\varphi^{\mathbf{w}}$ *are the solutions of the first micro problem at x constrained with* $\mathbf{v}$*, respectively* $\mathbf{w}$*, see Definition 2. The effective bilinear form* B^{eff} *is given by*

$$B^{\text{eff}}(\mathbf{v},\mathbf{w}) = \int_\Omega \frac{1}{|Y_\eta(x)|}\left(\nu\left(x,\frac{\cdot}{\eta}\right)\mathbf{curl}_y\,\mathbf{u}^{\mathbf{v}}(x,\cdot),\mathbf{curl}_y\,\mathbf{u}^{\mathbf{w}}(x,\cdot)\right)_{0,Y_\eta(x)} dx$$

for all $\mathbf{v},\mathbf{w}\in \boldsymbol{H}(\mathbf{curl};\Omega)$*, where* $\mathbf{u}^{\mathbf{v}}$ *and* $\mathbf{u}^{\mathbf{w}}$ *are the solutions of the second micro problem at x constrained with* $\mathbf{v}$*, respectively* $\mathbf{w}$*, see Definition 3.*

We choose to give the effective scalar product and the effective bilinear form in a non-standard version, since it reveals well the connection with our multiscale scheme defined below.

Nevertheless, we would like to mention that S^{eff} and B^{eff} could also be given with the help of an effective permittivity ε^{eff} and an effective inverse permeability ν^{eff} as

$$S^{\text{eff}}(\mathbf{v},\mathbf{w}) = (\varepsilon^{\text{eff}}\mathbf{v},\mathbf{w})_{0,\Omega} \qquad \text{and} \qquad B^{\text{eff}}(\mathbf{v},\mathbf{w}) = (\nu^{\text{eff}}\,\mathbf{curl}\,\mathbf{v},\mathbf{curl}\,\mathbf{w})_{0,\Omega}. \quad (6)$$

Explicit formulas for the effective tensors ε^{eff} and ν^{eff} in terms of the solutions of the micro problems can e.g. be found in [5, Rem. 5.8]. This rewriting process has been shown in [7] for discretized versions of S^{eff} and B^{eff}, but one can follow the lines of the given proof also in the continuous case. We mention here the involved ideas. With the help of the "dual formulas" one can rewrite the effective equation as effective first order Maxwell's equations with effective electric permittivity and effective magnetic permeability. These effective equations are simplified versions of the ones given in [18]. The simplification originates by considering only lossless materials. In [18] the notion of two-scale convergence [4] was applied to Maxwell's equation to derive the convergence result.

Note, that it is well known that ε^{eff} and ν^{eff} only vary on a macroscopic length scale and that they are again uniformly bounded and positive definite. More precisely, we have that (A_2') holds for $\xi\in\{\varepsilon^{\text{eff}},\nu^{\text{eff}}\}$ with the same constants α and β. For the bilinear forms S^{eff} and B^{eff} this means, that there are $0<\lambda_S\le\Lambda_S$ and $0<\lambda_B\le\Lambda_B$, such that

$$\begin{aligned}\lambda_S\|\mathbf{v}\|_{0,\Omega}^2 &\le S^{\text{eff}}(\mathbf{v},\mathbf{v}), \quad S^{\text{eff}}(\mathbf{v},\mathbf{w})\le\Lambda_S\|\mathbf{v}\|_{0,\Omega}\|\mathbf{w}\|_{0,\Omega},\\ \lambda_B\|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2 &\le B^{\text{eff}}(\mathbf{v},\mathbf{v}), \quad B^{\text{eff}}(\mathbf{v},\mathbf{w})\le\Lambda_B\|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}\|\mathbf{curl}\,\mathbf{w}\|_{0,\Omega}.\end{aligned} \quad (7)$$

3 Multiscale Algorithm

As usual for FE-HMM schemes our algorithm consists of a macro and a micro solver. For the macro solver we discretize the effective equation (5) with edge elements from Nédélec's first family. Let $\mathscr{T}_H$ be a shape regular triangulation of the computational domain Ω into simplicial elements K. We let H be the largest diameter of all elements K in $\mathscr{T}_H$. Note that H can be much larger than the characteristic length η of the material. By $\boldsymbol{V}_H \subset \boldsymbol{H}_0(\mathbf{curl};\Omega)$ we denote the corresponding finite element space, for instance consisting of edge elements. The finite element discretization of (5) reads as follows.

$$\begin{cases} \text{Find } \mathbf{E}_H^{\text{eff}} : (0,T) \to \boldsymbol{V}_H, \text{ such that for all } \mathbf{v}_H \in \boldsymbol{V}_H \\ S^{\text{eff}}(\partial_{tt}\mathbf{E}_H^{\text{eff}}(t), \mathbf{v}_H) + B^{\text{eff}}(\mathbf{E}_H^{\text{eff}}(t), \mathbf{v}_H) = (\mathbf{f}(t), \mathbf{v}_H), \\ \mathbf{E}_H^{\text{eff}}(0) = \Pi_H \mathbf{E}_0, \qquad \text{and} \qquad \partial_t \mathbf{E}_H^{\text{eff}}(0) = \Pi_H \mathbf{E}_0', \end{cases} \tag{8}$$

where Π_H is a suitable $\boldsymbol{L}^2$-projection onto $\boldsymbol{V}_H$. Yet, this formulation can not be used directly, since the evaluation of S^{eff} and B^{eff} would require the exact solution of micro problems at every point $x \in \Omega$, i.e. of infinitely many micro problems.

To overcome these issues we replace S^{eff} and B^{eff} by there discretized counterparts. In this process, two discretization steps are involved. Firstly, the outer integral over the computational domain Ω is replaced by a quadrature formula: In every element $K \in \mathscr{T}_H$ we choose J quadrature nodes x_j^K and corresponding quadrature weights ω_j^K, $j = 1,\dots,J$. Then we approximate

$$\int_\Omega g(x)\,dx \approx \sum_{K\in\mathscr{T}_H} \sum_{j=1}^{J} \omega_j^K g(x_j^K) =: \sum_{K,j} \omega_j^K g(x_j^K).$$

Secondly, the micro problems are not solved analytically, but the solutions are approximated using finite elements. Therefore, we consider microscopic triangulations $\mathscr{T}_h(x)$ of the sampling domains $Y_\eta(x)$ into simplicial elements with maximal diamater h. Let $\varphi_h^{\mathbf{v}}$ be the FE solution of the first micro problem (3). This means, that $\varphi_h^{\mathbf{v}}$ is the solution of (3), where the space $H^1_{\text{per}}(Y_\eta(x))$ has been replaced with the space $W_{h,\text{per}}$ of Lagrange finite elements with periodic boundary conditions defined over $\mathscr{T}_h(x)$ of a given order. Similarly, let $\mathbf{u}_h^{\mathbf{v}}$ be the FE solution of the second micro problem (4). Here we replace additionally the space $\boldsymbol{H}_{\text{per}}(\mathbf{curl};Y_\eta(x))$ with an edge element space $\boldsymbol{V}_{h,\text{per}}$ with periodic boundary conditions defined again over $\mathscr{T}_h(x)$. With these notations, we can define the HMM scalar product and bilinear form by

$$S_H^{\text{HMM}}(\mathbf{v}_H, \mathbf{w}_H) = \sum_{K,j} \frac{\omega_j^K}{|Y_\eta|} \left(\varepsilon\left(x_j^K, \frac{\cdot}{\eta}\right) \boldsymbol{\nabla}_y \varphi_h^{\mathbf{v}_H}(x_j^K,\cdot), \boldsymbol{\nabla}_y \varphi_h^{\mathbf{w}_H}(x_j^K,\cdot) \right)_{0,Y_\eta(x_j^K)},$$

$$B_H^{\text{HMM}}(\mathbf{v}_H, \mathbf{w}_H) = \sum_{K,j} \frac{\omega_j^K}{|Y_\eta|} \left(\nu\left(x_j^K, \frac{\cdot}{\eta}\right) \mathbf{curl}_y\, \mathbf{u}_h^{\mathbf{v}_H}(x_j^K,\cdot), \mathbf{curl}_y\, \mathbf{u}_h^{\mathbf{w}_H}(x_j^K,\cdot) \right)_{0,Y_\eta(x_j^K)}.$$

Remark 1. From the definition, it is obvious, that S^{HMM} and B^{HMM} are symmetric. Furthermore, it can be shown, that (7) holds as well for S^{HMM} and B^{HMM}, if ε^η, ν^η are sufficiently smooth and if the quadrature formula is accurate enough, with respect to the chosen macroscopic FE space $\boldsymbol{V}_H$. This is well known for FE-HMM, see [1, 2] and the references therein. For the specific case of Maxwell's equation a detailed discussion on the regularity assumptions can be found in [7]. Regarding the quadrature formula, we also refer to [9, Chapter 4].

Finally the FE-HMM scheme for second-order time-dependent Maxwell's equation can be written as follows.

$$\begin{cases} \text{Find } \mathbf{E}_H^{\mathrm{HMM}} : (0,T) \to \boldsymbol{V}_H, \text{ such that for all } \mathbf{v}_H \in \boldsymbol{V}_H \\ S_H^{\mathrm{HMM}}(\partial_{tt}\mathbf{E}_H^{\mathrm{HMM}}(t), \mathbf{v}_H) + B_H^{\mathrm{HMM}}(\mathbf{E}_H^{\mathrm{HMM}}(t), \mathbf{v}_H) = (\mathbf{f}(t), \mathbf{v}_H), \\ \quad \mathbf{E}_H^{\mathrm{HMM}}(0) = \Pi_H \mathbf{E}_0, \qquad \text{and} \qquad \partial_t \mathbf{E}_H^{\mathrm{HMM}}(0) = \Pi_H \mathbf{E}_0'. \end{cases} \tag{9}$$

Note that this FE-HMM scheme leads to a system of second-order ordinary differential equations.

For the full discretization the scheme, an appropriate time integration method has to be applied, e.g. the leap-frog or the Crank-Nicolson scheme. We refer to [8] for an error analysis for second-order Maxwell's equation for these two methods.

4 Error analysis

FE-HMM schemes can be seen as non-conforming FE methods, since the true effective and the HMM bilinear form differ from each other. In [7] the FE-HMM for time harmonic Maxwell's equation was analyzed using the notion of T-coercivity. Since we now consider a hyperbolic time-dependent PDE we can no longer use this theory. However, the present situation is closely related to the one in [3], where a FE-HMM scheme for the scalar valued acoustic wave equation was introduced. There, a Strang-type lemma for wave equations was proven, where only the bilinear forms, but not the involved scalar products may differ from each other. Here we generalize it, such that it is applicable to our FE-HMM scheme.

Let $V \subset H \sim H' \subset V'$ be a Gelfand triple of Hilbert spaces and $W \subset V$ be a closed subset. We consider the following problem.

$$\begin{cases} \text{Find } u : (0,T) \to W, \text{ such that for all } w \in W \\ \quad S\big(\partial_{tt} u(t), w\big) + B\big(u(t), w\big) = \langle f(t), w \rangle, \\ \quad u(0) = u_0, \qquad \text{and} \qquad \partial_t u(0) = u_0', \end{cases} \tag{10}$$

where $S, B : W \times W \to \mathbb{R}$ are symmetric bilinear forms. S and B are assumed to be H-coercive and V-coercive, respectively, i.e., there are constants $0 < \lambda \leq \Lambda$ with

$$S(v,v) \geq \lambda \|v\|_H^2, \qquad S(v,w) \leq \Lambda \|v\|_H \|w\|_H, \tag{11a}$$

$$B(v,v) \geq \lambda \|v\|_V^2, \qquad B(v,w) \leq \Lambda \|v\|_V \|w\|_V, \tag{11b}$$

for all $v, w \in W$. We denote the norms of bilinear forms by

$$\|B\|_V := \sup_{v,w\in W\setminus\{0\}} \frac{|B(v,w)|}{\|v\|_V\|w\|_V}, \qquad \|S\|_H := \sup_{v,w\in W\setminus\{0\}} \frac{|S(v,w)|}{\|v\|_H\|w\|_H}.$$

In the following, we will drop the explicit indication of the time dependence whenever possible, for better readability. Additionally, for the energy norm we use the abbreviation

$$\|v\|_{E(H,V)} = \|\partial_t v\|_{L^\infty(0,T;H)} + \|v\|_{L^\infty(0,T;V)} \qquad \text{for } v \in V.$$

Theorem 2 (Strang-type lemma for second-order hyperbolic equations). *Let $S,\tilde{S},B,\tilde{B} : W\times W \to \mathbb{R}$ be symmetric bilinear forms satisfying* (11a) *and* (11b)*, respectively. For given $f : [0,T] \to V'$ and $u_0,u_0' \in W$, let u be the solution of* (10)*. Furthermore, let $\tilde{u}$ be the solution of* (10) *with S and B being replaced by $\tilde{S}$ and $\tilde{B}$, respectively. If $\partial_t^r u, \partial_t^r \tilde{u} \in C(0,T;V)$ for $r \in \{0,1,2\}$, then there is a constant C (depending on T and $\partial_t^r u$ for $r \in \{0,1,2\}$) such that*

$$\|u-\tilde{u}\|_{E(H,V)} \le C\big(\|S-\tilde{S}\|_H + \|B-\tilde{B}\|_V\big).$$

Proof. The proof consists of three steps. The key idea is to consider the projection $\hat{u}(t) \in W$ of $u(t)$ given by

$$\tilde{B}\big(\hat{u}(t),w\big) = B\big(u(t),w\big) \qquad \text{for all } w \in W \tag{12}$$

and splitting the error into

$$e := u-\tilde{u} = \hat{e}+\tilde{e}, \qquad \text{where} \qquad \hat{e} := u-\hat{u} \qquad \text{and} \qquad \tilde{e} := \hat{u}-\tilde{u}. \tag{13}$$

(a) Due to the continuous embedding of $H^1(0,T;V)$ into the Bochner space $C([0,T];V)$, see e.g. [10, Sec. 5.9.2], we have for $v \in H^1(0,T;V)$

$$\|v\|_{L^\infty(0,T;V)} \le C\big(\|v\|_{L^2(0,T;V)} + \|\partial_t v\|_{L^2(0,T;V)}\big). \tag{14}$$

Using (14) for $v = \hat{e}$ and $v = \partial_t \hat{e}$, respectively, we obtain

$$\|e\|_{E(H,V)} \le C\big(\|\hat{e}\|_{L^2(0,T;V)} + \|\partial_t \hat{e}\|_{L^2(0,T;V)} + \|\partial_t^2 \hat{e}\|_{L^2(0,T;V)}\big) + \|\tilde{e}\|_{E(H,V)}.$$

It remains to bound $\hat{e}$ and $\tilde{e}$ defined in (13).

(b) To bound $\hat{e}$ one can follow the lines of the first paragraph of the proof of [3, Lemma 4.4]

$$\|\partial_t^r \hat{e}\|_{L^2(0,T;V)} \le C\|B-\tilde{B}\|_V \|\partial_t^r u\|_{L^2(0,T;V)}, \qquad r = 0,1,2.$$

(c) Bounding $\tilde{e}$ is motivated by the second part of the proof of [3, Lemma 4.4]. However, here we have to deal with the different scalar products S and $\tilde{S}$. From the definitions of the projection $\hat{u}$ in (12) and $\tilde{e}$ in (13) we obtain

$$\tilde{S}(\partial_t^2 \tilde{e},w) + \tilde{B}(\tilde{e},w) = \tilde{S}(\partial_t^2 \hat{u},w) - S(\partial_t^2 u,w) \qquad \text{for all } w \in W.$$

Setting $w = \partial_t \tilde{e}$ yields

$$\frac{1}{2}\frac{d}{dt}\Big(\tilde{S}(\partial_t \tilde{e}, \partial_t \tilde{e}) + \tilde{B}(\tilde{e}, \tilde{e})\Big) = (\tilde{S} - S)(\partial_t^2 u, \partial_t \tilde{e}) - \tilde{S}(\partial_t^2 \hat{e}, \partial_t \tilde{e}).$$

By (11), we conclude

$$\frac{\lambda}{2}\frac{d}{dt}\big(\|\partial_t \tilde{e}\|_H^2 + \|\tilde{e}\|_V^2\big) \le \big(\|S - \tilde{S}\|_H \|\partial_t^2 u\|_H + \Lambda \|\partial_t^2 \hat{e}\|_H\big)\|\partial_t \tilde{e}\|_H.$$

Using the abbreviations

$$\rho = \|\partial_t \tilde{e}\|_H^2 + \|\tilde{e}\|_V^2 \quad \text{and} \quad \sigma = \|S - \tilde{S}\|_H \|\partial_t^2 u\|_H + \Lambda \|\partial_t^2 \hat{e}\|_H,$$

we find by applying Young's inequality

$$\frac{\lambda}{2}\frac{d}{dt}\rho(t) \le \sigma(t)\|\partial_t \tilde{e}(t)\|_H \le \frac{1}{2}\big(\sigma^2(t) + \rho(t)\big).$$

Gronwall's lemma yields for $0 \le t \le T$

$$\rho(t) \le \mathrm{e}^{T/\lambda}\Big(\rho(0) + \int_0^t \sigma^2(s)\,ds\Big). \tag{15}$$

The initial condition of (10) imply $\tilde{e}(0) = -\hat{e}(0)$ and $\partial_t \tilde{e}(0) = -\partial_t \hat{e}(0)$. Using again that $H^1(0,T;V)$ is continuously embedded in $C([0,T];V)$ we have

$$\rho(0) \le C\|\partial_t \hat{e}\|_{L^\infty(0,T;V)}^2 + \|\hat{e}\|_{L^\infty(0,T;V)}^2.$$

Inserting the definition of ρ, taking square roots of the inequality (15), considering the supremum over $t \in [0,T]$, and using the bound (14) for $v = \hat{e}$ and $v = \partial_t \hat{e}$, proves the desired bound. $\square$

Our next goal is to apply Theorem 2 to FE-HMM. To get more insight in the following a-priori error bound, we will split it into macro and HMM error. To this end we approximate the effective scalar product and the effective bilinear form, c.f. (6), using numerical integration. For $\mathbf{v}_H, \mathbf{w}_H \in \boldsymbol{V}_H$ we set

$$S_H^{\mathrm{eff}}(\mathbf{v}_H, \mathbf{w}_H) = \sum_{K,j} \omega_j^K \varepsilon^{\mathrm{eff}}(x_j^K)\mathbf{v}_H(x_j^K) \cdot \mathbf{w}_H(x_j^K),$$
$$B_H^{\mathrm{eff}}(\mathbf{v}_H, \mathbf{w}_H) = \sum_{K,j} \omega_j^K \nu^{\mathrm{eff}}(x_j^K)\,\mathbf{curl}\,\mathbf{v}_H(x_j^K) \cdot \mathbf{curl}\,\mathbf{w}_H(x_j^K),$$

and define

$$\Delta S_{\mathrm{mac}} = \|S^{\mathrm{eff}} - S_H^{\mathrm{eff}}\|_{\boldsymbol{L}^2(\Omega)}, \qquad \Delta B_{\mathrm{mac}} = \|B^{\mathrm{eff}} - B_H^{\mathrm{eff}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},$$
$$\Delta S_{\mathrm{HMM}} = \|S_H^{\mathrm{eff}} - S_H^{\mathrm{HMM}}\|_{\boldsymbol{L}^2(\Omega)}, \qquad \Delta B_{\mathrm{HMM}} = \|B_H^{\mathrm{eff}} - B_H^{\mathrm{HMM}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

Corollary 1. *As above, let* $\mathbf{E}^{\text{eff}}$, $\mathbf{E}_H^{\text{eff}}$*, and* $\mathbf{E}_H^{\text{HMM}}$ *be the solution of* (5), (8)*, and* (9)*, respectively. Suppose that* $\partial_t^r \mathbf{E}_H^{\text{eff}}, \partial_t^r \mathbf{E}_H^{\text{HMM}} \in C(0,T;\boldsymbol{H}_0(\mathbf{curl};\Omega))$ *for* $r \in \{0,1,2\}$*. If* ε^η, ν^η *are sufficiently smooth and if the quadrature formulas are accurate enough, then*

$$\begin{aligned}\|\mathbf{E}^{\text{eff}} - \mathbf{E}_H^{\text{HMM}}\|_{E(\boldsymbol{L}^2(\Omega),\boldsymbol{H}(\mathbf{curl};\Omega))} \leq{}& \|\mathbf{E}^{\text{eff}} - \mathbf{E}_H^{\text{eff}}\|_{E(\boldsymbol{L}^2(\Omega),\boldsymbol{H}(\mathbf{curl};\Omega))} \\ &+ C(\Delta S_{\text{mac}} + \Delta B_{\text{mac}} + \Delta S_{\text{HMM}} + \Delta B_{\text{HMM}}).\end{aligned} \tag{16}$$

Proof. We only have to bound $\|\mathbf{E}_H^{\text{eff}} - \mathbf{E}_H^{\text{HMM}}\|_{E(\boldsymbol{L}^2(\Omega),\boldsymbol{H}(\mathbf{curl};\Omega))}$ due to the triangle inequality. For this we can apply Theorem 2 with $H = \boldsymbol{L}^2(\Omega)$, $V = \boldsymbol{H}_0(\mathbf{curl};\Omega)$, and $W = \boldsymbol{V}_H$. Since the bilinear forms B^{eff} and B_H^{HMM} are not W-elliptic, we consider the following modified bilinear forms

$$\begin{aligned} B(\cdot,\cdot) &= B^{\text{eff}}(\cdot,\cdot) + \tfrac{\lambda_S}{2}(\cdot,\cdot)_{0,\Omega}, & \tilde{B}(\cdot,\cdot) &= B_H^{\text{HMM}}(\cdot,\cdot) + \tfrac{\lambda_S}{2}(\cdot,\cdot)_{0,\Omega}, \\ S(\cdot,\cdot) &= S^{\text{eff}}(\cdot,\cdot) - \tfrac{\lambda_S}{2}(\cdot,\cdot)_{0,\Omega}, & \tilde{S}(\cdot,\cdot) &= S_H^{\text{HMM}}(\cdot,\cdot) - \tfrac{\lambda_S}{2}(\cdot,\cdot)_{0,\Omega}. \end{aligned}$$

The coercivity of B, S, $\tilde{B}$ and $\tilde{S}$ follows from (7) and Remark 1. Moreover, assumption (11) holds with $\lambda = \min\{\lambda_B, \lambda_S/2\}$ and $\Lambda = \lambda_S/2 + \max\{\Lambda_B, \Lambda_S\}$. With these choices we get from Theorem 2

$$\begin{aligned}\|\mathbf{E}_H^{\text{eff}} - \mathbf{E}_H^{\text{HMM}}\|_{E(\boldsymbol{L}^2(\Omega),\boldsymbol{H}(\mathbf{curl};\Omega))} &\leq C\big(\|S - \tilde{S}\|_{\boldsymbol{L}^2(\Omega)} + \|B - \tilde{B}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\big) \\ &= C\big(\|S^{\text{eff}} - S_H^{\text{eff}}\|_{\boldsymbol{L}^2(\Omega)} + \|B^{\text{eff}} - B_H^{\text{eff}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\big) \\ &\leq C\big(\Delta S_{\text{mac}} + \Delta B_{\text{mac}} + \Delta S_{\text{HMM}} + \Delta B_{\text{HMM}}\big). \qquad \square\end{aligned}$$

The first term on the right hand side of (16) can be bounded by standard FE theory. E.g. for $\boldsymbol{V}_H$ being chosen as lowest order $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming edge element from Nédélec's first family, we have under appropriate regularity conditions, see [16, Thm. 3.1],

$$\|\mathbf{E}^{\text{eff}} - \mathbf{E}_H^{\text{eff}}\|_{E(\boldsymbol{L}^2(\Omega),\boldsymbol{H}(\mathbf{curl};\Omega))} \leq C\big(\|\mathbf{E}_0' - \Pi_H \mathbf{E}_0')\|_{0,\Omega} + \|\mathbf{E}_0 - \Pi_H \mathbf{E}_0\|_{\mathbf{curl},\Omega} + H\big).$$

Convergence rates for the differences in the scalar products and bilinear forms in terms of H and h can be found in [7].

5 Numerical example

We present a first simple numerical example corroborating our analytical results. More involved examples will presented in a forthcoming publication. Let $\mathscr{T}_H$ be a triangulation of the computational domain $\Omega = [0,1]^2$ into uniform meshes of different mesh sizes H. Furthermore, define the function g^η by

$$g^\eta(x) = \sqrt{2} + \sin\left(2\pi \frac{x}{\eta}\right)$$

and let the electric permittivity and the inverse magnetic permeability be given by

$$\varepsilon^\eta(x_1,x_2) = \frac{g^\eta(x_1)g^\eta(x_2)}{\sqrt{2}}, \qquad \nu^\eta(x_1,x_2) = \frac{2}{g^\eta(x_1)g^\eta(x_2)},$$

with $\eta = 2^{-8} \approx 0.004$. For this particular case the effective parameters can be computed analytically and one finds $\varepsilon^{\text{eff}} = \nu^{\text{eff}} = 1$. We choose the source term

$$\boldsymbol{f}(t;x_1,x_2) = \begin{pmatrix} -\pi^2 \sin(-\pi t)\cos(\pi x_1)\sin(\pi x_2) \\ \pi^2 \sin(\pi t)\sin(\pi x_1)\cos(\pi x_2) \end{pmatrix},$$

such that the solution of the effective Maxwell's equation (5) is given by

$$\mathbf{E}^{\text{eff}}(t;x_1,x_2) = \begin{pmatrix} -\sin(\pi t)\cos(\pi x_1)\sin(\pi x_2) \\ \sin(-\pi t)\sin(\pi x_1)\cos(\pi x_2) \end{pmatrix}.$$

We discretize using lowest order $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming edge element from Nédélec's first family for the macro solver. For the micro solver we use Lagrange and edge elements of order one. For this particular choice it is shown in [7, Section 5] that we have

$$\Delta S_{\text{mac}} = \Delta B_{\text{mac}} = 0 \qquad \text{and} \qquad \Delta S_{\text{HMM}}, \Delta B_{\text{HMM}} \leq C\left(\frac{h}{\eta}\right)^2,$$

where C is independent of h and η.

In Figure 1 we show the maximal $\boldsymbol{H}(\mathbf{curl};\Omega)$-error between $\mathbf{E}^{\text{eff}}$ and $\mathbf{E}_H^{\text{HMM}}$ for various values of H. If $r = H_1/H_2$ denotes the refinement factor between two macro meshes $\mathscr{T}_{H_1}$ and $\mathscr{T}_{H_2}$, then we use $\sqrt{r}$ as the refinement factor between the corresponding micro meshes. This simultaneous refinement strategy accounts for the different convergence orders (1 for the macro and 2 for the micro solver). As expected from the theoretical consideration above, we see that the proposed FE-HMM scheme (9) converges linearly for the above choices of the finite element spaces.

Acknowledgement We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173 and the Klaus Tschira Stiftung. In addition we thank the anonymous referee for helpful suggestions.

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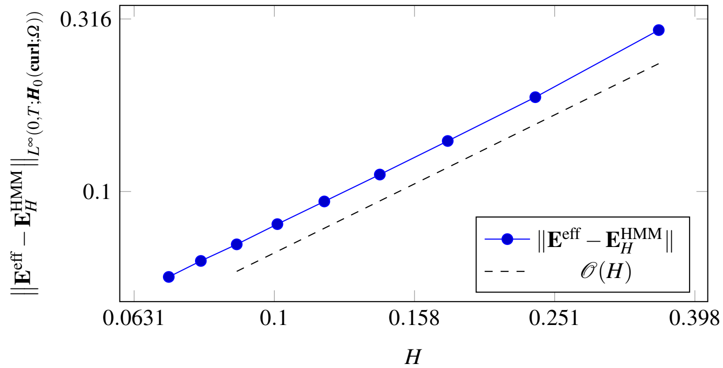

Fig. 1 Maximal difference between the effective and the FE-HMM solution, computed with first order elements. As expected we retrieve first order convergence. The experiment was conducted with `FreeFem++` [11].

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