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Asymptotic behavior of the ground state energy of a Fermionic Fröhlich multipolaron in the strong coupling limit

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In this article, we investigate the asymptotic behavior of the ground state energy of the Fröhlich Hamiltonian for a Fermionic multipolaron in the so-called strong coupling limit. We prove that it is given to leading order by the ground state energy of the Pekar-Tomasevich functional with Fermionic statistics, which is a much simpler model. Our main theorem is new because none of the previous results on the strong coupling limit have taken into account the Fermionic statistics and the spin of the electrons. A binding result for Fröhlich multipolarons is a corollary of our main theorem combined with the binding result for multipolarons in the Pekar-Tomasevich model by [AG14]. Our analysis strongly relies on [Wel15] which in turn used and generalized methods developed in [LT97], [FLST11] and [GW13]. In order to take the Fermionic statistics into account, we employ a localization method given in [LL05].

1. Introduction

Consider a conducting electron traveling through a polar crystal such as NaF. We assume that the ions in the crystal are not rigidly fixed. Note that we use the expression "ion" in a more general meaning, where we not only include the case of metals but also the case of covalent semi-conductors. As the electron moves along the crystal, it causes distortions of the crystal lattice in a neighborhood, moving together with the electron. The ensemble consisting of the traveling electron and the induced distortions of the lattice is called a polaron. If we instead consider two or more electrons in a polar crystal, we analogously call the resulting ensemble of electrons and lattice distortions a bipolaron or a multipolaron, respectively. The (quantized) lattice distortions are described by phonons. Energetically it is more favorable if the electrons deform the lattice in a small region, hence they tend to stay close together. Therefore, an attractive force mediated by the lattice distortions operates between the electrons which is counteracted by their Coulomb repulsion. As a consequence, even though electrons always

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repel each other, polarons may attract each other. This phenomenon attracted the attention of many physicists because it was believed for a long time that this could cause the formation of Cooper pairs, which in turn causes superconductivity.

In order to understand the question of whether polarons attract each other or not, one needs an insight into the behavior of the ground state energies of polarons and multipolarons. A well known and established model describing these is the so-called Fröhlich Hamiltonian derived by Herbert Fröhlich in 1954 ([Frö54]). The Fröhlich Hamiltonian is informally given by

$$H^{(N,\alpha)} = \sum_{j=1}^N (D_{A,x_j}^\dagger D_{A,x_j} + V(x_j) + \sqrt{\alpha}\phi(x_j)) + H_{ph} + UV_C(x_1, x_2, \dots, x_N),$$

where $N \in \mathbb{N}$ and $\alpha > 0$. In here, $D_{A,x_j} := -i\nabla_{x_j} + A(x_j)$ denotes the canonical momentum operator for a given external vector potential A . V represents an external electric potential, U indicates the rescaled Coulomb coupling strength between electrons and

$$V_C(x_1, x_2, \dots, x_N) := \sum_{i < j} \frac{1}{|x_i - x_j|}.$$

Moreover, $\sqrt{\alpha}$ denotes the coupling constant between an electron and the phonon field. The interaction term is given by $\sqrt{\alpha} \sum_{j=1}^N \phi(x_j)$ with

$$\phi(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^3} (a(k)e^{ikx} + a^\dagger(k)e^{-ikx}) \frac{dk}{|k|}$$

and the energy of the phonon field is given by

$$H_{ph} := \int_{\mathbb{R}^3} a^\dagger(k)a(k)dk.$$

$a(k)$ denotes the annihilation operator for a phonon with momentum k and we consider $a^\dagger(k)$ (informally) as its adjoint, both seen as acting as quadratic forms on an appropriate dense subspace of the physical space

$$\mathcal{H}_N := \mathcal{E}_N \otimes \mathcal{F}. \tag{1.1}$$

The physical space of the phonons is the bosonic Fock space \mathcal{F} over $L^2(\mathbb{R}^3)$ and the physical space \mathcal{E}_N of the electrons is

$$\mathcal{E}_N = \bigwedge_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2). \tag{1.2}$$

The last definition reflects the fact that electrons are spin-1/2 Fermions. However, we neglect the coupling of the spin to the magnetic field $B = \nabla \times A$. We endow $\mathcal{H}_N \subseteq L^2(\mathbb{R}^{3N}; \mathbb{C}^{2^N}) \otimes \mathcal{F} = \bigoplus_{j=1}^{2^N} L^2(\mathbb{R}^{3N}; \mathcal{F})$ with the following inner product:

$$\langle f, g \rangle_{\mathcal{H}_N} = \sum_{j=1}^{2^N} \int_{\mathbb{R}^{3N}} \langle f_j(x), g_j(x) \rangle_{\mathcal{F}} dx.$$

An appropriate choice of a form domain of $H^{(N,\alpha)}$ for which there exists an associated closable

quadratic form, see Theorem 2.7, is

$$\mathcal{Q}_N := \mathcal{E}_{N,0} \otimes \mathcal{F}_0, \quad (1.3)$$

where

$$\mathcal{F}_0 := \{(\eta_n)_{n \in \mathbb{N}} \in \mathcal{F} \mid \forall n \in \mathbb{N} : \eta_n \in \mathcal{C}_c^\infty(\mathbb{R}^{3n}) \\ \exists n_0 \in \mathbb{N} : \eta_n = 0 \ \forall n \geq n_0\} \quad (1.4)$$

and

$$\mathcal{E}_{N,0} := \bigwedge_{i=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2). \quad (1.5)$$

Note that up to now we understand $H^{(N,\alpha)}$ only as an informal notation for a (self-adjoint) Hamiltonian for which we will show its existence. So far, the notation for the Fröhlich Hamiltonian does not define an operator—on a larger space than the null-space—because $a^\dagger(f_{x_1, \dots, x_N}^{(N)})\eta \notin \mathcal{F}$ for $f_{x_1, \dots, x_N}^{(N)}(k) := \sum_{j=1}^N \frac{e^{-ik \cdot x_j}}{\sqrt{2\pi|k|}}$ unless $\eta = 0$, due to the fact that $f_{x_1, \dots, x_N}^{(N)} \notin L^2$. If $\Phi \in \mathcal{Q}_N$ and $x = (x_1, \dots, x_N)$, then we define

$$(a(f^{(N)})(\Phi))(x) := \int_{\mathbb{R}^3} \overline{f_x^{(N)}(k)} \left(\sqrt{n} \Phi^{(n)}(x, \vec{k}_{n-1}, k) \Big|_{\vec{k}_{n-1} \in \mathbb{R}^{3(n-1)}} \right)_{n \in \mathbb{N}} dk \quad (1.6)$$

in the sense of a Bochner integral in \mathcal{F} .

We prove the existence of the Fröhlich Hamiltonian with quadratic form methods. The proof gives a form core with nice properties. This is important in several steps of the proof of the main theorem. More precisely, we will prove that the quadratic form $q^{(N,\alpha)} : \mathcal{Q}_N^2 \rightarrow \mathbb{C}$, defined in Equation (2.2) below, is indeed associated with a self-adjoint operator on \mathcal{H}_N which is called the Fröhlich Hamiltonian. As it turns out, $q^{(N,\alpha)}$ is bounded from below, and therefore so is $H^{(N,\alpha)}$. This means that the ground state energy of the actual Fröhlich Hamiltonian exists.

An advantage of this approach is that, by Remark 2.3, Proposition 2.5 and Theorem 2.7 below, minimizing $\Psi \mapsto \langle \Psi, H^{(N,\alpha)} \Psi \rangle$ on $\mathcal{D}(H^{(N,\alpha)}) \cap \mathbb{S}_{\mathcal{H}_N}$ will be equivalent to minimizing $\Psi \mapsto q^{(N,\alpha)}(\Psi, \Psi)$ on $\mathcal{Q}_N \cap \mathbb{S}_{\mathcal{H}_N}$, where \mathbb{S}_X denotes the unit sphere in a normed space $(X, \|\cdot\|)$. More precisely, the ground state energy $E^{(N,\alpha)}(A, V, U)$ of $H^{(N,\alpha)}$ satisfies for appropriate choices of A and V ¹

$$E^{(N,\alpha)}(A, V, U) \stackrel{\text{def}}{=} \inf_{\substack{\Phi \in \mathcal{D}(H^{(N,\alpha)}), \\ \|\Phi\|=1}} \langle \Phi, H^{(N,\alpha)} \Phi \rangle = \inf_{\substack{\Phi \in \mathcal{Q}_N, \\ \|\Phi\|=1}} q^{(N,\alpha)}(\Phi, \Phi). \quad (1.7)$$

The Pekar-Tomasevich functional $\mathcal{P}^{(N,\alpha)}(A, V, U, \cdot)$ is given for $\psi \in \mathcal{E}_{N,0}$ by

$$\mathcal{P}^{(N,\alpha)}(A, V, U, \psi) \stackrel{\text{def}}{=} \inf_{\substack{\eta \in \mathcal{F}_0, \\ \|\eta\|=1}} q^{(N,\alpha)}(\psi \otimes \eta, \psi \otimes \eta). \quad (1.8)$$

We then minimize with respect to $\psi \in \mathcal{E}_{N,0} \cap \mathbb{S}_{\mathcal{E}_N}$ and denote the infimum by

$$C^{(N,\alpha)}(A, V, U) := \inf_{\substack{\psi \in \mathcal{E}_{N,0}, \\ \|\psi\|=1}} \mathcal{P}^{(N,\alpha)}(A, V, U, \psi). \quad (1.9)$$

¹Unless it should remain unclear to the reader, we will neither specify inner products nor norms.

We obviously have

$$E^{(N,\alpha)} \leq C^{(N,\alpha)}. \quad (1.10)$$

A now standard calculation, originally presented in [Pek54], gives an explicit expression for the Pekar-Tomasevich functional. For convenience of the reader we outline the calculation. We have for $\|\eta\|_{\mathcal{F}} = \|\psi\|_{\bigwedge_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2)} = 1$

$$\begin{aligned} q^{(N,\alpha)}(\psi \otimes \eta, \psi \otimes \eta) &= \sum_{j=1}^N \langle D_{A,x_j} \psi, D_{A,x_j} \psi \rangle + \sum_{j=1}^N \langle \psi, V(x_j) \psi \rangle + U \langle \psi, V_C \psi \rangle \\ &+ \int \langle \eta, \left(a^\dagger(k) a(k) + \frac{\sqrt{\alpha}}{\sqrt{2\pi}|k|} \left(a(k) \widehat{\rho_\psi}(k) + a^\dagger(k) \widehat{\rho_\psi}(k) \right) \right) \eta \rangle dk. \end{aligned}$$

Here $\widehat{\rho_\psi}$ denotes the Fourier transform of the electron density defined by

$$\rho_\psi(x) = \sum_{j=1}^N \int_{\mathbb{R}^{3(N-1)}} |\psi(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_N)|^2 dx_1 \dots \widehat{dx}_j \dots dx_N, \quad (1.12)$$

where $|\cdot|$ always denotes the Euclidean norm in the respective space. For example, $|\psi(x_1, \dots, x_N)|$ denotes the Euclidean norm of $\psi(x_1, \dots, x_N)$ in \mathbb{C}^{2^N} . By completing the square, we arrive at

$$\begin{aligned} q^{(N,\alpha)}(\psi \otimes \eta, \psi \otimes \eta) &= \sum_{j=1}^N \langle D_{A,x_j} \psi, D_{A,x_j} \psi \rangle + \sum_{j=1}^N \langle \psi, V(x_j) \psi \rangle + U \langle \psi, V_C \psi \rangle \\ &+ \int \left\| \left(a(k) + \frac{\sqrt{\alpha} \widehat{\rho_\psi}(k)}{\sqrt{2\pi}|k|} \right) \eta \right\|^2 dk - \frac{\alpha}{2\pi^2} \int \frac{1}{|k|^2} |\widehat{\rho_\psi}|^2 dk. \end{aligned}$$

In order to minimize this expression, we choose a sequence in \mathcal{F}_0 approximating a coherent state such that the second to last term vanishes. By applying Plancherel and the fact that the inverse Fourier transform of $\frac{1}{|k|^2}$ satisfies

$$\widehat{|k|^{-2}}(x) = (2\pi)^{-3} \int \frac{1}{|k|^2} e^{ikx} dk = 2\pi^2 \frac{1}{|x|}, \quad (1.14)$$

we then obtain

$$\begin{aligned} \mathcal{P}^{(N,\alpha)}(A, V, U, \psi) &= \sum_{j=1}^N \langle D_{A,x_j} \psi, D_{A,x_j} \psi \rangle + \sum_{j=1}^N \langle \psi, V(x_j) \psi \rangle + U \langle \psi, V_C \psi \rangle \\ &- \alpha \int \frac{\rho_\psi(x) \rho_\psi(y)}{|x-y|} d(x, y). \end{aligned}$$

The goal is to show that the Pekar-Tomasevich functional, which is a simpler model than the full Fröhlich Hamiltonian, yields the behavior of the ground state energy of the Fröhlich Hamiltonian up to a relative error which vanishes in the strong coupling limit $\alpha \rightarrow \infty$. In view of inequality (1.10) an appropriate lower bound for the ground state energy of the Fröhlich Hamiltonian is sufficient. Such a lower bound is provided in Theorem 1.1 below.

Assumptions. As in [Wel15] we assume the following.

- (i) $A_k \in L^2_{loc}(\mathbb{R}^3; \mathbb{R})$ and $V \in L^1_{loc}$.

(II) $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is *infinitesimally $-\Delta$ -form-bounded*, i.e., for any $\varepsilon > 0$ there is $C_\varepsilon \in \mathbb{R}$ such that for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)$ we have

$$|\langle \phi, V\phi \rangle| \leq \varepsilon \|\nabla \phi\|^2 + C_\varepsilon \|\phi\|^2. \quad (1.16)$$

(III) A, V are such that

$$C^{(n,1)}(A, V, \nu) + C^{(m,1)}(A, V, \nu) \geq C^{(m+n,1)}(A, V, \nu), \quad \forall \nu \geq 0, \quad \forall m, n \in \mathbb{N}. \quad (1.17)$$

Note that the space of all infinitesimally $-\Delta$ -form-bounded potentials $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a real vector space. Now we will give an example of external fields A and V for which assumption (III) is satisfied.

Example. *In addition to assumptions (I) and (II), assume that we have a periodic electric potential and a periodic magnetic field. More precisely, suppose that there exist $f \in H^2(\mathbb{R}^3)$ and $w \in \mathbb{R}^3/\{0\}$ such that $f(x+w) = f(x)$, $A(x+w) = A(x) + \nabla f(x)$ and $V(x+w) = V(x)$ for all $x \in \mathbb{R}^3$. Then assumption (III) is fulfilled.*

Proof. It is known that $C^{(n,1)}(A, V, \nu)$ exists for all $\nu \geq 0, n \in \mathbb{N}$ when A, V fulfill assumptions (I) and (II). We reprove that at the end of Section 2 but we assume it for now and we proceed with the proof of (1.17). The argument is also standard but we repeat it for convenience of the reader. We choose $\phi_l \in \bigwedge_{j=1}^l \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)$ to be approximate minimizers of $\mathcal{P}^{(l,1)}$ with error $\varepsilon/3$ for $l \in \{m; n\}$. Define for any $k \in \mathbb{N}$

$$\phi_m^{(k)}(x) := \phi_m(x_1 + kw, \dots, x_m + kw) e^{ik \sum_{j=1}^m f(x_j)} \quad \forall x = (x_1, \dots, x_m)^T \in \mathbb{R}^{3m}.$$

Note that

$$\mathcal{P}^{(m,1)}(A, V, U, \phi_m^{(k)}) = \mathcal{P}^{(m,1)}(A, V, U, \phi_m). \quad (1.18)$$

We choose k so large that ρ_{ϕ_n} and $\rho_{\phi_m^{(k)}}$ have disjoint supports. Then we have

$$\rho_{\frac{\phi_m^{(k)} \wedge \phi_n}{\|\phi_m^{(k)} \wedge \phi_n\|}} = \rho_{\phi_m^{(k)}} + \rho_{\phi_n}$$

which in turn yields

$$\mathcal{P}^{(m+n,1)}(A, V, U, \frac{\phi_m^{(k)} \wedge \phi_n}{\|\phi_m^{(k)} \wedge \phi_n\|}) = \tilde{\mathcal{P}}^{(m+n,1)}(A, V, U, \phi_m^{(k)} \otimes \phi_n). \quad (1.19)$$

Here $\tilde{\mathcal{P}}^{(m+n,1)}$ denotes the Pekar-Tomasevich functional extended to $\bigotimes_{j=1}^{m+n} \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)$. By enlarging $k \in \mathbb{N}$ such that

$$\sum_{i=1}^m \sum_{j=m+1}^{m+n} \langle \phi_m^{(k)} \otimes \phi_n, \frac{U}{|x_i - x_j|} \phi_m^{(k)} \otimes \phi_n \rangle \leq \frac{\varepsilon}{3}$$

and employing Equations (1.18) and (1.19), we arrive at

$$\mathcal{P}^{(m+n,1)}(A, V, U, \frac{\phi_m^{(k)} \wedge \phi_n}{\|\phi_m^{(k)} \wedge \phi_n\|}) \leq C^{(m,1)}(A, V, U) + C^{(n,1)}(A, V, U) + \varepsilon.$$

□

We are now ready to formulate the main result of the article.

1.1 Theorem. *Suppose the assumptions (I), (II) and (III) hold. Let $A_\alpha(x) := \alpha A(\alpha x)$, and $V_\alpha(x) := \alpha^2 V(\alpha x)$. Then*

(i) *There exists $c(A, V) \in \mathbb{R}$ such that for any $N \in \mathbb{N}$, $\nu \geq 2$, and $\alpha > 0$*

$$E^{(N, \alpha)}(A_\alpha, V_\alpha, \alpha\nu) \geq \alpha^2 C^{(N, 1)}(A, V, \nu) - c(A, V) \alpha^{42/23} N^4.$$

(ii) *If, in addition, $A_k \in L^3_{loc}(\mathbb{R}^3)$ and $V \in L^{3/2}_{loc}(\mathbb{R}^3)$, we find for $N \in \mathbb{N}$ and $\nu \geq 2$*

$$\lim_{\alpha \rightarrow \infty} \alpha^{-2} E^{(N, \alpha)}(A, V, \alpha\nu) = C^{(N, 1)}(0, 0, \nu).$$

1.2 Remark. *The importance of Theorem 1.1 lies in the fact that the Pekar-Tomasevich functional is a much simpler model which is defined only on the electronic space $\bigwedge_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2)$. Therefore, conclusions for the simpler Pekar-Tomasevich functional can be used for conclusions for the more fundamental Fröhlich Hamiltonian. As an example for the last claim, we have the following application in Section 4: Combining a previous binding result in the Fermionic Pekar-Tomasevich model, i.e., Theorem 1.1 in [AG14] with our Theorem 1.1, a binding result for Fermionic Fröhlich polarons follows.*

1.3 Remark. *Note that a simple rescaling argument shows that*

$$C^{(N, \alpha)}(A_\alpha, V_\alpha, \alpha\nu) = \alpha^2 C^{(N, 1)}(A, V, \nu).$$

Therefore the leading term in part (i) of Theorem 1.1 is the ground state energy of the Pekar-Tomasevich functional with the same fields and coupling constant as the Fröhlich Hamiltonian.

Several previous results exist for the strong coupling limit. In 1980, Adamowski, Gerlach and Leschke sketched a proof of the convergence² $\alpha^{-2} E^{(1, \alpha)}(0, 0) \rightarrow C^{(1, 1)}(0, 0)$ in [AGL80] with the help of large deviation theory of Brownian motion. In 1983, Donsker and Varadhan provided a complete and rigorous proof. However, information on the rate of convergence was not provided. In 1997, Lieb and Thomas found in [LT97] a simpler proof which also gives rates of convergence. In 2013 Wellig and Griesemer ([GW13]) extended this result by including external electric and magnetic fields. In the same year the multipolaron case with no external fields was studied by the first author and Landon in [AL13]. They combined the methods of Lieb and Thomas and also applied Feynman-Kac formulas, which were applied in [FLST11] to show that bipolarons do not form a bound state above a threshold of ν . A few months later Wellig extended in [Wel15] this result by including external fields. Wellig followed [LT97] as well, and he generalized a localization method of the phonons, originally developed in [FLST11] for bipolarons. He further used arguments in [GW13], which were applied for the case of a polaron. Neither in [AL13] nor in [Wel15] was the Fermionic statistics taken into account. Ghanta ([Gha15]) recently investigated the one-dimensional polaron in the strong coupling limit. He proved, for $N = 1$, $A = 0$ and some assumptions on V , convergence of approximate minimizers of the Fröhlich Hamiltonian to the (unique) minimizer of the Pekar-Tomasevich functional in a certain sense. In this work, we follow [Wel15] quite closely but we take care of the Fermionic statistics. To this end, we employ a localization method developed in [LL05].

²Note that when $N = 1$ we trivially have no Coulomb repulsion term and therefore ν is irrelevant. For that reason the third input of $E^{(1, \alpha)}$ and $C^{(1, 1)}$ is omitted.

The paper is organized as follows: In Section 2 we show that the Fröhlich Hamiltonian may be derived as a closable quadratic form defined on \mathcal{Q}_N . In Section 3 we prove Theorem 1.1. In Section 4 we present a corollary of Theorem 1.1 and Theorem 1.1 in [AG14] which shows that Fermionic Fröhlich polarons can bind.

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2. Form core of the Fröhlich Hamiltonian

2.1. Preparations

As it turns out, it is very difficult to describe the domain of self-adjointness even for the bare one-particle Fröhlich Hamiltonian, i.e., without external fields. This has recently been done in [GW15]. In order to avoid this problem and since it is sufficient for our purposes to only have a form core for the above mentioned reasons, we will derive the Fröhlich Hamiltonian from a closable quadratic form defined on \mathcal{Q}_N , see Theorem 2.7 below.

The main idea behind this construction of the Hamiltonian is the duality between a positive self-adjoint operator A and a positive closed quadratic form (both defined on a same Hilbert space). For convenience of the reader we repeat some knowledge on quadratic forms.

2.1 Definition. Let $q : Q^2 \rightarrow \mathbb{C}$ be a densely defined and semi-bounded quadratic form on a Hilbert space with $q(\phi, \phi) \geq -b\|\phi\|^2$ for any $\phi \in Q$ and some $b \geq 0$. We define a scalar product on Q by

$$\langle \cdot, \cdot \rangle_q := (1 + b)\langle \cdot, \cdot \rangle + q(\cdot, \cdot), \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of the ambient Hilbert space.

(i) We call q closed iff Q endowed with $\langle \cdot, \cdot \rangle_q$ is complete.

(ii) We call q closable iff it has a closed extension on the ambient Hilbert space. We then denote the closure by \bar{q} and the domain of the closure by $Q(q)$.

(iii) Let q be closed and $D \subseteq Q$. We say that D is a form core of q iff $\overline{q|_D} = q$.

2.2 Definition. Let $q : Q^2 \rightarrow \mathbb{C}$ be a closed quadratic form and A be a self-adjoint operator defined on the same Hilbert space. We say that q is associated with A iff

$$\begin{aligned} \mathcal{D}(A) &\subseteq Q, \\ \langle \phi, A\psi \rangle &= q(\phi, \psi) \quad \forall \phi \in Q, \psi \in \mathcal{D}(A). \end{aligned}$$

More generally, we say that a closable quadratic form is associated with a self-adjoint operator A iff its closure is associated with A . Conversely, we say that A is associated with a closable quadratic form q if q is associated with A .

2.3 Remark. Employing a simple argument given in [BS87], it follows that if a closable quadratic form q is associated with A , then A is semi-bounded by some constant iff q is semi-bounded by the same constant.

The following result associates certain quadratic forms with self-adjoint operators.

2.4 Proposition. Let q be a closed semi-bounded symmetric quadratic form on a Hilbert space. Then there is a unique self-adjoint operator A associated with q . In particular, A has the same lower bound as q .

This is a corollary to Theorem 2 in [BS87], Chapter 10. For the opposite direction of the duality, we state Theorem 1 in [BS87], Chapter 10.

2.5 Proposition. Given a positive definite self-adjoint operator A there exists a unique closed quadratic form q associated with A , and q is defined by

$$q(\phi, \psi) := \langle A^{\frac{1}{2}}\phi, A^{\frac{1}{2}}\psi \rangle \quad \forall \phi, \psi \in Q(q) = \mathcal{D}(A^{\frac{1}{2}}).$$

2.6 Remark. We say that a semi-bounded self-adjoint operator A has form core D iff the closed quadratic form q associated with A has form core D .

The next theorem is the main theorem of this section.

2.7 Theorem. Under the assumptions **(I)**, **(II)** of Theorem 1.1, the quadratic form $q^{(N,\alpha)} : \mathcal{Q}_N^2 \rightarrow \mathbb{C}$ defined by

$$\begin{aligned} q^{(N,\alpha)}(\Phi, \Psi) := & \sum_{j=1}^N \langle D_{A,x_j} \Phi, D_{A,x_j} \Psi \rangle + \sum_{j=1}^N \langle \Phi, V(x_j) \Psi \rangle \\ & + \langle \Phi, (UV_C + H_{ph}) \Psi \rangle + 2\sqrt{\alpha} \Re \langle \Phi, (a(f^{(N)})) \Psi \rangle \end{aligned} \quad (2.2)$$

is closable and semi-bounded from below. In particular, the closure of $q^{(N,\alpha)}$ is associated with a unique self-adjoint operator $H^{(N,\alpha)}$. Moreover, $E^{(N,\alpha)}(A, V, U) > -\infty$.

To prove this theorem we use the following two auxiliary lemmas.

2.8 Lemma (KLMN Theorem for quadratic forms). Let $q : D^2 \rightarrow \mathbb{C}$ be a closable positive quadratic form on a Hilbert space and $\beta : D^2 \rightarrow \mathbb{C}$ be a symmetric quadratic form. Assume there exist $a < 1$ and $b \in \mathbb{R}$ with

$$|\beta(\phi, \phi)| \leq aq(\phi, \phi) + b\|\phi\|^2 \quad (2.3)$$

for any $\phi \in D$. Then

$$\gamma : D^2 \rightarrow \mathbb{C}, \quad (\phi, \psi) \mapsto q(\phi, \psi) + \beta(\phi, \psi)$$

is closable and semi-bounded from below by $-b$ and $Q(q) = Q(\gamma)$.

The proof is analogous to the proof of the actual KLMN Theorem given in [RS75], Theorem X.17.

Whereas the KLMN Theorem is a powerful tool to "add" quadratic forms acting on the same space, we will use the following Lemma to add quadratic forms acting on different spaces.

2.9 Lemma. Let $q_j : D_j^2 \rightarrow \mathbb{C}$, $j \in \{1; 2\}$, be two positive closable quadratic forms on Hilbert spaces h_j . Then the quadratic form $q_1 + q_2 : (D_1 \otimes D_2)^2 \rightarrow \mathbb{C}$ determined by

$$(q_1 + q_2)(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2) := q_1(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2) + q_2(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2)$$

is closable and positive, where we identify q_j with its obvious extension to $D_1 \otimes D_2$.

For a proof we refer to appendix A.

2.2. Structure of the proof of Theorem 2.7

We prove that there is a closable quadratic form $\tilde{q}^{(N,\alpha)}$ on $L^2(\mathbb{R}^{3N}; \mathbb{C}^{2^N}) \otimes \mathcal{F}$ including all above indicated interactions before we include the Fermionic statistics. Recall $A_j \in L_{loc}^2(\mathbb{R}^3)$ and let $A := (A_1, A_2, A_3)^T$. We will start by proving that

$$\tilde{q}_{e,0}^{(N)} : \left(\bigotimes_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2) \right)^2 \rightarrow \mathbb{C}, (\varphi, \psi) \mapsto \sum_{j=1}^N \langle D_{A,x_j} \varphi, D_{A,x_j} \psi \rangle \quad (2.4)$$

is closable and that the domain of the closure is the magnetic Sobolev space $H_{\mathbf{A}}^1(\mathbb{R}^{3N}; \mathbb{C}^{2^N})$ which will be specified below. Next we will employ the KLMN Theorem to add an infinitesimally $\sum_{j=1}^N (-\Delta_{x_j})$ -form-bounded external electric potential $\sum_{j=1}^N V(x_j)$, see inequality (1.16) respectively Remark 2.12 below, and the Coulomb repulsion UV_C between the electrons. Due to Hardy's inequality

$$\frac{1}{|x|^2} \leq -4\Delta \quad (2.5)$$

in three dimensions, UV_C is infinitesimally $\sum_{j=1}^N (-\Delta_{x_j})$ -form-bounded. We then obtain a closable form $\tilde{q}_e^{(N)} : (\bigotimes_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2))^2 \rightarrow \mathbb{C}$ that includes all electronic interactions.

Next we will use that \mathcal{F}_0 , defined in Equation (1.4), is a form-core of the phonon occupation number operator H_{ph} in order to define a closable form $\tilde{q}^{(N,0)} := \tilde{q}_e^{(N)} + q_{ph} : \tilde{\mathcal{Q}}_N^2 \rightarrow \mathbb{C}$ as in Lemma 2.9. In here, q_{ph} is the quadratic form associated with H_{ph} , defined in Equation (2.7) below, and $\tilde{\mathcal{Q}}_N$ is analogously defined to \mathcal{Q}_N replacing $\bigwedge_{j=1}^N$ in Equation (1.5) respectively (1.3) by $\bigotimes_{j=1}^N$. In general, the notation with tilde ($\tilde{}$) stresses the fact that we are not taking care of the Fermionic statistics, yet. The main advantage of using the language of quadratic forms relies on the fact that the KLMN Theorem only preserves form cores and not the domain of essential self-adjointness. Again, this is sufficient for our purposes.

Finally, Lemma 2.15 tells us that including the electron-phonon interaction $\sqrt{\alpha} \sum_{j=1}^N \phi(x_j) : \tilde{\mathcal{Q}}_N^2 \rightarrow \mathbb{C}$ still yields a closable quadratic form $\tilde{q}^{(N,\alpha)}$ due to the KLMN Theorem. We are then ready to prove Theorem 2.7 by showing that the restriction of $\tilde{q}^{(N,\alpha)}$ to \mathcal{Q}_N , which is equal to $q^{(N,\alpha)}$, is still a closable quadratic form.

2.3. Actual proof of Theorem 2.7

From now on, we will always assume that assumptions **(I)** and **(II)** of Theorem 1.1 hold. For $d, k \in \mathbb{N}$ and $a_j \in L^2_{loc}(\mathbb{R}^d)$, $1 \leq j \leq k$, we define the space

$$H_a^1(\mathbb{R}^d; \mathbb{C}^k) := \{ \Phi \in L^2(\mathbb{R}^d; \mathbb{C}^k) \mid ((-i\partial_j + a_j(\cdot))\Phi) \in L^2(\mathbb{R}^d; \mathbb{C}^k) \\ \forall 1 \leq j \leq d \}.$$

We are especially interested in the case $d = 3N$, $k = 2^N$ and $a = \mathbf{A}$ with

$$\mathbf{A}(x_1, \dots, x_N) := (A_1(x_1), A_2(x_1), A_3(x_1), \dots, A_1(x_N), A_2(x_N), A_3(x_N))^T.$$

Instead of Lemma 2.9, we will use another approach to establish a closable quadratic form inducing the multi-particle magnetic Laplacian. By this we mean the self-adjoint operator assigned to $\sum_{j=1}^N D_{A, x_j}^\dagger D_{A, x_j}$ in the informal expression of the Fröhlich Hamiltonian. The advantage of this approach is that it gives us an explicit description of the domain of the closure of $\tilde{q}_{e,0}^{(N)}$.

2.10 Proposition. $\tilde{q}_{e,0}^{(N)}$ defined in (2.4) is closable and positive and the domain of its closure is $H_{\mathbf{A}}^1(\mathbb{R}^{3N}; \mathbb{C}^{2^N})$.

Proof. First of all, we refer to Theorem 7.22 in [LL96] to obtain that $\mathcal{C}_c^\infty(\mathbb{R}^{3N})$ is dense in $H_{\mathbf{A}}^1(\mathbb{R}^{3N})$. As mentioned in the definition 7.20 in [LL96], one can show that $H_{\mathbf{A}}^1(\mathbb{R}^{3N})$ is complete with an argument similar to the case of $H^1(\mathbb{R}^d)$. Therefore, $H_{\mathbf{A}}^1(\mathbb{R}^{3N}; \mathbb{C}^{2^N}) = \bigoplus_{j=1}^{2^N} H_{\mathbf{A}}^1(\mathbb{R}^{3N})$ is also complete. In addition, $\mathcal{C}_c^\infty(\mathbb{R}^{3N}; \mathbb{C}^{2^N}) = \bigoplus_{j=1}^{2^N} \mathcal{C}_c^\infty(\mathbb{R}^{3N})$ is dense in $H_{\mathbf{A}}^1(\mathbb{R}^{3N}; \mathbb{C}^{2^N})$. Thus it remains to show that $\bigotimes_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)$ is dense in $\mathcal{C}_c^\infty(\mathbb{R}^{3N}; \mathbb{C}^{2^N})$ with respect to the topology of the ambient space $H_{\mathbf{A}}^1(\mathbb{R}^{3N}; \mathbb{C}^{2^N})$. This follows from Lemma B.1 in Appendix B. \square

Next we will add an external electrical field and the Coulomb interaction between the electrons. For that purpose we will use Lemma 2.11.

2.11 Lemma (Diamagnetic inequality). *Let $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $a_j \in L^2_{loc}(\mathbb{R}^d)$ for any $j \in \{1, \dots, d\}$. Let $f = (f_1, \dots, f_k) \in H_a^1(\mathbb{R}^d; \mathbb{C}^k)$ and $|f|_V := (|f_1|, \dots, |f_k|)$. Then we have $|f|_V \in H^1(\mathbb{R}^d; \mathbb{C}^k)$ and*

$$\|(-i\nabla)|f|_V(x)\|_{d \times k} \leq \|((-i\nabla + a(\cdot))f)(x)\|_{d \times k} \quad \text{for almost all } x \in \mathbb{R}^d,$$

where $\|\cdot\|_{d \times k}$ denotes the ℓ^2 -norm on the space $\mathbb{C}^{d \times k}$ of complex matrices.

For a proof see Theorem 7.21 in [LL96]. Before continuing, we recall the following notion.

2.12 Remark. *Let $q_1, q_2 : D^2 \rightarrow \mathbb{C}$ be two closable and symmetric quadratic forms on a Hilbert space. We say that q_1 is relatively q_2 -form-bounded with bound $a > 0$ iff there is some $C \in \mathbb{R}$ such that*

$$|q_1(\phi, \phi)| \leq a q_2(\phi, \phi) + C \|\phi\|^2$$

for any $\phi \in D$. We call q_1 infinitesimally q_2 -form-bounded iff for any $a > 0$ q_1 is relatively q_2 -form-bounded with bound a . Due to the duality between positive closed quadratic forms and positive self-adjoint operators, we adopt this manner of speaking to positive self-adjoint operators. Note that this is consistent with the formulation of assumption **(II)**.

Recall that $\tilde{q}_{e,0}^{(N)}$ is defined in Equation (2.4).

2.13 Corollary. $UV_C(x_1, \dots, x_N) + \sum_{j=1}^N V(x_j)$ is infinitesimally $\tilde{q}_{e,0}^{(N)}$ -form-bounded. In particular,

$$\tilde{q}_e^{(N)} := \tilde{q}_{e,0}^{(N)} + UV_C(x_1, \dots, x_N) + \sum_{j=1}^N V(x_j) \quad (2.6)$$

is closable and semi-bounded from below and the domain of its closure is $H_{\mathbf{A}}^1(\mathbb{R}^{3N}; \mathbb{C}^{2^N})$.

Proof. Define

$$V_0(x_1, \dots, x_N) := UV_C(x_1, \dots, x_N) + \sum_{j=1}^N V(x_j).$$

We already saw in the last subsection that V_C is infinitesimally $\sum_{j=1}^N (-\Delta_{x_j})$ -form-bounded, and therefore by assumption **(II)** so is V_0 . Let $\Phi \in \bigotimes_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)$ and $\varepsilon > 0$. For a suitable choice of $C_\varepsilon \in \mathbb{R}$ we find

$$\begin{aligned} |\langle \Phi, V_0 \Phi \rangle| &= |\langle |\Phi|_V, V_0 |\Phi|_V \rangle| \\ &\leq \varepsilon \langle -i\nabla |\Phi|_V, -i\nabla |\Phi|_V \rangle + C_\varepsilon \|\Phi\|^2 \\ &\leq \varepsilon \|(-i\nabla + \mathbf{A})\Phi\|^2 + C_\varepsilon \|\Phi\|^2 \\ &= \varepsilon \tilde{q}_{e,0}^{(N)}(\Phi, \Phi) + C_\varepsilon \|\Phi\|^2, \end{aligned}$$

where $|\cdot|_V$ is defined as in Lemma 2.11. The second inequality follows from Lemma 2.11. Subsequently, V_0 is infinitesimally $\tilde{q}_{e,0}^{(N)}$ -form-bounded and by Lemma 2.8 Corollary 2.13 follows. \square

So far, we have only taken into account all the electronic interactions. Next, we will add the phononic part. One can easily show that $\sqrt{H_{ph}}$ is essentially self-adjoint on \mathcal{F}_0 . Then Theorem 1 in Chapter 10 in [BS87] yields that \mathcal{F}_0 is a form core of H_{ph} . We have that q_{ph} with

$$q_{ph}(\phi, \psi) := \langle \sqrt{H_{ph}}\phi, \sqrt{H_{ph}}\psi \rangle \quad \forall \phi, \psi \in \mathcal{F}_0 \quad (2.7)$$

is associated with H_{ph} . Before continuing, recall that $\tilde{\mathcal{Q}}_N$ is defined by

$$\tilde{\mathcal{Q}}_N := \left(\bigotimes_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2) \right) \otimes \mathcal{F}_0. \quad (2.8)$$

The next proposition is an easy corollary of Corollary 2.13 if we employ Lemma 2.9.

2.14 Proposition. The quadratic form $\tilde{q}^{(N,0)} : \tilde{\mathcal{Q}}_N^2 \rightarrow \mathbb{C}$ given by

$$\tilde{q}^{(N,0)} := \tilde{q}_e^{(N)} + q_{ph} \quad (2.9)$$

is closable and semi-bounded from below.

The upper index $(N, 0)$ indicates that we deal with N electrons and no coupling to the phonons. This notation is consistent with the notation $\tilde{q}^{(N,\alpha)}$ below for the case of positive coupling $\alpha > 0$. We have been using \tilde{q} for a quadratic form with form domain $\bigotimes_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)$ respectively $\tilde{\mathcal{Q}}_N$, where we have not imposed anti-symmetry on the electronic part of the wave functions.

Finally, we will include the electron-phonon interaction

$$\tilde{q}_{e,p}^{(N)} : \tilde{\mathcal{Q}}_N^2 \rightarrow \mathbb{C}, (\Phi, \Psi) \mapsto \langle \Phi, a(f^{(N)})\Psi \rangle + \langle a(f^{(N)})\Phi, \Psi \rangle.$$

Recall that we defined $f_{x_1, \dots, x_N}^{(N)}(k) := \sum_{j=1}^N \frac{e^{-ik \cdot x_j}}{\sqrt{2\pi|k|}}$ for $k \in \mathbb{R}^3$. We suppress the arguments of $f^{(N)}$ in the notation for $\tilde{q}_{e,p}^{(N)}$ to avoid imprecision.³

2.15 Lemma. $\tilde{q}_{e,p}^{(N)}$ is infinitesimally $\tilde{q}^{(N,0)}$ -form-bounded

Proof. Let $\varepsilon > 0$ be arbitrary and $\Lambda > 0$ be some fixed real number and define

$$\begin{aligned} f_{x,\Lambda}(k) &:= f_x^{(N)}(k) \cdot \mathbb{1}_{|\cdot| \leq \Lambda}(k) \\ f_{x,\Lambda,\infty}(k) &:= f_x^{(N)}(k) - f_{x,\Lambda}(k) \end{aligned}$$

for $k \in \mathbb{R}^3$ and $x = (x_1, \dots, x_N) \in \mathbb{R}^{3N}$. Throughout this article, $\mathbb{1}_A$ will denote the characteristic function of a measurable set A . As mentioned above for $f^{(N)}$, we will suppress the electron positions in the notation for $a(f_\Lambda)$ and $a(f_{\Lambda,\infty})$. Now let $\Phi \in \tilde{\mathcal{Q}}_N$. We have

$$|\langle \Phi, a(f^{(N)})\Phi \rangle| \leq |\langle \Phi, a(f_\Lambda)\Phi \rangle| + |\langle \Phi, a(f_{\Lambda,\infty})\Phi \rangle|. \quad (2.10)$$

One readily sees

$$|\langle \Phi, a(f_\Lambda)\Phi \rangle| \leq \|\Phi\| \cdot \|a(f_\Lambda)\Phi\|. \quad (2.11)$$

Then we compute

$$\begin{aligned} & |(a(f_\Lambda)\Phi)^{(n-1)}(x, k_1, \dots, k_{n-1})| \\ & \leq \sqrt{n} \int_{|k| \leq \Lambda} \left| \sum_{j=1}^N \frac{e^{-ik \cdot x_j}}{\sqrt{2\pi|k|}} \right| \cdot |\Phi^{(n)}(x, k_1, \dots, k_{n-1}, k)| dk \\ & \leq \sqrt{n} \cdot \|\Phi^{(n)}(x, k_1, \dots, k_{n-1}, \cdot)\|_2 \cdot M_\Lambda \\ & = \|(\sqrt{H_{ph}}\Phi)^{(n)}(x, k_1, \dots, k_{n-1}, \cdot)\|_2 \cdot M_\Lambda, \end{aligned} \quad (2.12)$$

where

$$M_\Lambda := \left(\int_{|k| \leq \Lambda} \frac{N^2}{2\pi^2|k|^2} dk \right)^{\frac{1}{2}}$$

is some positive number depending on Λ .

Let $\Gamma \in \mathbb{R}$ such that $\tilde{q}_e^{(N)} + \Gamma \geq 0$. Then also $\tilde{q}^{(N,0)} + \Gamma = \tilde{q}_e^{(N)} + \Gamma + q_{ph} \geq 0$ and we find using Estimates (2.11) and (2.12) and Equation (2.7)

$$\begin{aligned} |\langle \Phi, a(f_\Lambda)\Phi \rangle| &\leq \frac{\varepsilon}{2} q_{ph}(\Phi, \Phi) + \frac{M_\Lambda^2}{2\varepsilon} \|\Phi\|^2 \\ &\leq \frac{\varepsilon}{2} (\tilde{q}^{(N,0)} + \Gamma)(\Phi, \Phi) + \frac{M_\Lambda^2}{2\varepsilon} \|\Phi\|^2. \end{aligned} \quad (2.13)$$

³The scalar product is taken in \mathcal{H}_N but the lower indices of $f^{(N)}$ only refer to the electron positions (and not to the phonon momenta).

Bounding the second term in Equation (2.10) is more involved. Define

$$g_{x,\Lambda}(k) := \sum_{j=1}^N \frac{e^{-ik \cdot x_j} k^{(j)}}{\sqrt{2\pi} |k|^3} \cdot \mathbb{1}_{|\cdot| > \Lambda}(k),$$

where

$$k^{(j)} := (0_{\mathbb{R}^{3(j-1)}}, k_1, k_2, k_3, 0_{\mathbb{R}^{3(N-j)}})^T$$

and $0_{\mathbb{R}^d}$ denotes the zero vector in \mathbb{R}^d . For the subsequent estimates we define by abuse of notation

$$a(g_\Lambda) := (a((g_\Lambda)_i))_{1 \leq i \leq 3N} \quad \text{and} \quad a^\dagger(g_\Lambda) := (a^\dagger((g_\Lambda)_i))_{1 \leq i \leq 3N}.$$

Then an easy calculation gives

$$\begin{aligned} \langle \Phi, a(f_{\Lambda,\infty}) \Phi \rangle &= \langle (-i\nabla + \mathbf{A}) \Phi, a(g_\Lambda) \Phi \rangle \\ &\quad - \langle a^\dagger(g_\Lambda) \Phi, (-i\nabla + \mathbf{A}) \Phi \rangle, \end{aligned} \quad (2.14)$$

where the scalar product is taken in $L^2(\mathbb{R}^{3N}; \mathbb{C}^{2N \times 3N}) \otimes \mathcal{F} = \bigoplus_{j=1}^{2N \cdot 3N} L^2(\mathbb{R}^{3N}; \mathcal{F})$ and analogously to the above

$$\begin{aligned} \|a(g_\Lambda) \Phi\| &\leq K_\Lambda \sqrt{q_{ph}(\Phi, \Phi)}, \\ \|a^\dagger(g_\Lambda) \Phi\| &\leq K_\Lambda \sqrt{q_{ph}(\Phi, \Phi) + \|\Phi\|^2}, \end{aligned}$$

where

$$K_\Lambda := \left(\int_{|k| > \Lambda} \frac{N}{2\pi^2 |k|^4} dk \right)^{\frac{1}{2}}. \quad (2.15)$$

Then the Cauchy-Schwarz inequality yields

$$\begin{aligned} |\langle \Phi, a(f_{\Lambda,\infty}) \Phi \rangle| &\leq \|(-i\nabla + \mathbf{A}) \Phi\| \cdot \|a(g_\Lambda) \Phi\| \\ &\quad + \|a^\dagger(g_\Lambda) \Phi\| \cdot \|(-i\nabla + \mathbf{A}) \Phi\| \\ &\leq 2K_\Lambda \sqrt{\tilde{q}_{e,0}^{(N)}(\Phi, \Phi)} \left(q_{ph}(\Phi, \Phi) + \frac{\|\Phi\|^2}{2} \right). \end{aligned} \quad (2.16)$$

Let $V_0(x_1, x_2, \dots, x_N) := UV_C(x_1, x_2, \dots, x_N) + \sum_{j=1}^N V(x_j)$. Then Corollary 2.13 implies that for some $0 < \eta < 1$ there is $C_\eta \in \mathbb{R}$ such that

$$V_0 \geq -\eta \tilde{q}_{e,0}^{(N)} - C_\eta,$$

i.e.,

$$\tilde{q}_e^{(N)} = \tilde{q}_{e,0}^{(N)} + V_0 \geq (1 - \eta) \tilde{q}_{e,0}^{(N)} - C_\eta.$$

So we have

$$\tilde{q}_{e,0}^{(N)} \leq \frac{1}{1 - \eta} (\tilde{q}_e^{(N)} + \Gamma) + \tilde{C}_\eta \leq r(1 + \tilde{q}_e^{(N)} + \Gamma)$$

for obvious choices of $\tilde{C}_\eta \in \mathbb{R}$ and $r > 1$. The last inequality together with (2.4) and (2.16)

imply

$$\begin{aligned} |\langle \Phi, a(f_{\Lambda, \infty}) \Phi \rangle| &\leq K_{\Lambda}(\tilde{q}_{e,0}^{(N)}(\Phi, \Phi) + q_{ph}(\Phi, \Phi)) + K_{\Lambda} \|\Phi\|^2 \\ &\leq rK_{\Lambda}(\tilde{q}^{(N,0)} + \Gamma)(\Phi, \Phi) + K_{\Lambda}(r+1) \|\Phi\|^2. \end{aligned}$$

Since $K_{\Lambda} \rightarrow 0$ as $\Lambda \rightarrow \infty$, we may fix $\Lambda = \Lambda_0 > 0$ so large that $rK_{\Lambda_0} < \varepsilon/2$. This together with the Estimates (2.10) and (2.13) finishes the proof. \square

Define

$$\tilde{q}^{(N,\alpha)} := \tilde{q}^{(N,0)} + \sqrt{\alpha} \tilde{q}_{e,p}^{(N)}. \quad (2.17)$$

Proposition 2.14 and Lemma 2.15 together with Lemma 2.8 imply that $\tilde{q}^{(N,\alpha)}$ is closable and semi-bounded from below. Note that we have

$$q^{(N,\alpha)} = \tilde{q}^{(N,\alpha)}|_{\mathcal{Q}_N^2}.$$

The notation without tilde indicates that we finally take care of the Fermionic statistics for the electronic part of the wave function. We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7. $q^{(N,\alpha)}$ is semi-bounded from below since it is the restriction of a quadratic form which is semi-bounded from below. One can easily verify that $q^{(N,\alpha)}$ is densely defined and as the restriction of a closable quadratic form, it is closable itself. It remains to show that the domain of the closure may be embedded into \mathcal{H}_N . Note that $\tilde{q}^{(N,\alpha)}$ being closable means that the domain of its closure may be embedded into $L^2(\mathbb{R}^{3N}; \mathbb{C}^{2^N}) \otimes \mathcal{F} \supseteq \mathcal{H}_N$ and thus we only have $Q(q^{(N,\alpha)}) \subseteq L^2(\mathbb{R}^{3N}; \mathbb{C}^{2^N}) \otimes \mathcal{F}$.

Let $q := \tilde{q}^{(N,\alpha)}$ and $(\bar{q}, Q(q))$ be its closure. Let further

$$\pi^{(N)} : \bigotimes_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F} \rightarrow \bigwedge_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F} = \mathcal{H}_N$$

be the canonical projection. We have that $\bar{q}|_{\pi^{(N)}(Q(q))}$ is densely defined and closed due to the fact that $\pi^{(N)}|_{Q(q)}$ is also an orthogonal projection with respect to $\langle \cdot, \cdot \rangle_{\bar{q}}^4$. Since $\mathcal{Q}_N = \pi^{(N)} \tilde{\mathcal{Q}}_N \subseteq \mathcal{H}_N$, \mathcal{Q}_N is dense in $\pi^{(N)}(Q(q))$ with respect to $\|\cdot\|_{\bar{q}}$ and thus

$$\overline{q|_{\mathcal{Q}_N}} = \bar{q}|_{\pi^{(N)}(Q(q))}.$$

This means that the domain of the closure of $q|_{\tilde{\mathcal{Q}}_N}$ may be embedded into \mathcal{H}_N .

We may finally apply Prop. 2.4 to associate $q^{(N,\alpha)}$ with the Fröhlich Hamiltonian $H^{(N,\alpha)}$. \square

At this point, we will briefly argue why $C^{(N,\alpha)}(A, V, \nu)$ even exists when Assumptions **(I)**, **(II)** are fulfilled. This directly follows from $C^{(N,\alpha)} \geq E^{(N,\alpha)} > -\infty$ due to Theorem 2.7 and inequality (1.10) but we may obtain the existence apart from the above discussion. It is a well-known argument but for convenience of the reader, we will repeat it. Let $\psi \in \bigwedge_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2) = \mathcal{E}_{N,0}$

⁴N.b.: We have $\bar{q}(T_\sigma \Phi, T_\sigma \Psi) = \bar{q}(\Phi, \Psi)$ for any $\sigma \in \mathcal{S}^{(N)}$, where T_σ permutes the electron coordinates with respect to σ .

be normalized. Recall

$$\begin{aligned} \mathcal{P}^{(N,\alpha)}(A, V, U, \psi) &= \sum_{j=1}^N \langle (D_{A,x_j} \psi, D_{A,x_j} \psi) \rangle + \sum_{j=1}^N \langle \psi, V(x_j) \psi \rangle + U \langle \psi, V_C \psi \rangle \\ &\quad - \alpha \int_{\mathbb{R}^6} \frac{\rho_\psi(x) \rho_\psi(y)}{|x-y|} d(x, y). \end{aligned}$$

We have that $\sum_{j=1}^N V(x_j) + UV_C(x_1, x_2, \dots, x_N)$ is relatively form-bounded with respect to $\sum_{j=1}^N D_{A,x_j}^\dagger D_{A,x_j}$, see Corollary 2.13, so it suffices to bound the last term in the last equation. Using Equation (1.12) we obtain

$$\begin{aligned} \int_{\mathbb{R}^6} \frac{\rho_\psi(x) \rho_\psi(y)}{|x-y|} d(x, y) &= \sum_{j=1}^N \int_{\mathbb{R}^{3(N+1)}} \frac{|\psi(x_1, \dots, x_N)|^2 \rho_\psi(y)}{|x_j - y|} d(x_1, \dots, x_N, y) \\ &\leq \frac{\varepsilon}{4N} \sum_{j=1}^N \int_{\mathbb{R}^{3(N+1)}} \frac{|\psi(x_1, \dots, x_N)|^2 \rho_\psi(y)}{|x_j - y|^2} d(x_1, \dots, x_N, y) + \frac{N^3}{\varepsilon} \\ &\leq \varepsilon \sum_{j=1}^N \langle D_{A,x_j} \psi, D_{A,x_j} \psi \rangle + \frac{N^3}{\varepsilon}. \end{aligned}$$

In the last step, we applied the Hardy inequality together with the Diamagnetic inequality, see Lemma 2.11, and we used the fact that $\int_{\mathbb{R}^3} \rho_\psi(y) dy = N$.

3. Proof of Theorem 1.1

For the purpose of proving Theorem 1.1, we closely follow the analysis in [Wel15]. We need in addition to take care of the Fermionic statistics. To this end we use a localization method given by [LL05]. We recall that we always assume that assumptions **(I)** and **(II)** of Theorem 1.1 hold.

3.1. Localization of a multipolaron

Let $p \in \mathbb{N}$, and let M_1, M_2, \dots, M_p be subsets of \mathbb{R}^3 . Then we define

$$\Omega(M_1 \times M_2 \times \dots \times M_p) := \bigcup_{\sigma \in \mathcal{S}_p} M_{\sigma(1)} \times M_{\sigma(2)} \times \dots \times M_{\sigma(p)},$$

where \mathcal{S}_p denotes the p^{th} symmetric group.

3.1 Lemma. *Let $R > 0$ and $\Phi \in \mathcal{Q}_N$ be normalized. Then there exist $m, n_1, n_2, \dots, n_m \in \mathbb{N}$ with $\sum_{i=1}^m n_i = N$, and balls B_1, B_2, \dots, B_m , where B_i has radius R_i , with the following properties.*

(i) $\text{dist}(B_i, B_j) \geq R$ for $i \neq j$.

(ii) $R_i = \frac{1}{2}(3n_i - 1)R \quad \forall 1 \leq i \leq m$.

(iii) *There is a normalized $\Phi_0 \in \mathcal{Q}_N$ with*

$$\text{supp}_e(\Phi_0) \subseteq \Omega\left(\bigtimes_{i=1}^m B_i^{n_i}\right), \tag{3.1}$$

where supp_e refers to the support with respect to the electronic coordinates and $B_i^{n_i} := \times_{j=1}^{n_i} B_j$, such that

$$\langle \Phi, H^{(N,\alpha)} \Phi \rangle \geq \langle \Phi_0, H^{(N,\alpha)} \Phi_0 \rangle - 2\pi^2 N^2 R^{-2}. \quad (3.2)$$

Remark. In contrast to [Wel15], the growth of the error in the lower bound now is of order N^2 and not N .

Proof. As in [Wel15] we will subdivide the proof into two steps: In the first step, we will construct an appropriate localized normalized function Φ_0 which satisfies Estimate (3.2) and having (electronic) support within $\Omega(\times_{i=1}^N B_R(y_i))$ for some $y_1, y_2, \dots, y_N \in \mathbb{R}^3$. In the second step, we will inscribe $\times_{i=1}^N B_R(y_i)$ into the cartesian product $\times_{i=1}^m B_i^{n_i}$ of possibly bigger balls and use the fact that then

$$\Omega(\times_{i=1}^N B_R(y_i)) \subseteq \Omega(\times_{i=1}^m B_i^{n_i}).$$

Indeed, for any $k \in \mathbb{N}$ and any subsets $A_1 \subseteq \tilde{A}_1, A_2 \subseteq \tilde{A}_2 \dots A_k \subseteq \tilde{A}_k$ of \mathbb{R}^3 we have

$$\Omega(A_1 \times A_2 \times \dots \times A_k) \subseteq \Omega(\tilde{A}_1 \times \tilde{A}_2 \times \dots \times \tilde{A}_k).$$

This argument combined with the antisymmetry of Φ_0 gives (3.1) and concludes the proof.

Step 1: Let $\chi \in L^2(\mathbb{R}^3; \mathbb{R})$ be a non-negative, normalized and C^∞ function with $\text{supp}(\chi) \subseteq B_R(0)$. Further let $Y^{(N)} := (y_1, y_2, \dots, y_N)^T \in \mathbb{R}^{3N}$ be arbitrary. We will later fix a value for $Y^{(N)}$. We employ the proof of Lemma 4.1 in [LL05]. Define

$$G(\cdot, Y^{(N)}) : \mathbb{R}^{3N} \rightarrow \mathbb{R}, X^{(N)} = (x_1, x_2, \dots, x_N)^T \mapsto \sum_{\pi \in \mathcal{S}_N} \prod_{i=1}^N \chi(x_i - y_{\pi(i)}),$$

where again \mathcal{S}_N denotes the N^{th} symmetry group and

$$P : \mathbb{R}^{3N} \rightarrow \mathbb{R}, X^{(N)} = (x_1, x_2, \dots, x_N)^T \mapsto \int_{\mathbb{R}^{3N}} G(X^{(N)}, Z^{(N)})^2 dZ^{(N)}.$$

As stated in [LL05], we have that for any $X^{(N)} = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^{3N}$

$$P(X^{(N)}) = N! \cdot \text{Per}(M), \quad (3.3)$$

where $M_{ij} := M(X^{(N)})_{ij} := \int_{\mathbb{R}^3} \chi(x_i - y) \chi(x_j - y) dy \geq 0$ for any $1 \leq i, j \leq N$ and $\text{Per}(M)$ denotes the permanent of the matrix M . M is a hermitian matrix with only positive elements. Hence we obtain

$$\text{Per}(M) \geq \prod_{i=1}^N M_{ii} = 1.$$

Together with (3.3) this yields for any $X^{(N)} \in \mathbb{R}^{3N}$

$$P(X^{(N)}) \geq N!. \quad (3.4)$$

Thus we may define

$$W(X^{(N)}, Y^{(N)}) := G(X^{(N)}, Y^{(N)}) P(X^{(N)})^{-\frac{1}{2}}$$

and

$$\Psi_{Y^{(N)}}(\cdot) := W(\cdot, Y^{(N)})\Phi(\cdot),$$

where we suppress the phonon indices in the notation. Note that we have $\Psi_{Y^{(N)}} \in \mathcal{Q}_N$ due to the symmetry of $W(X^{(N)}, Y^{(N)})$ with respect to relabeling indices in $X^{(N)} = (x_1, \dots, x_N)^T$ and due to the regularity of $W(\cdot, Y^{(N)})$. We abbreviate

$$\Delta E(Y^{(N)}) := \langle \Psi_{Y^{(N)}}, H^{(N,\alpha)} \Psi_{Y^{(N)}} \rangle - \left(\langle \Phi, H^{(N,\alpha)} \Phi \rangle + b \frac{N^2}{R^2} \right) \langle \Psi_{Y^{(N)}}, \Psi_{Y^{(N)}} \rangle$$

with $b := 2\pi^2$. Note that b is twice the lowest eigenvalue of the Dirichlet Laplacian on $B_1(0)$. We will show $\int \Delta E(Y) dY < 0$ for an appropriate choice of χ . In particular, this yields the existence of some $Y_0^{(N)} \in \mathbb{R}^{3N}$ such that $\Psi_{Y_0^{(N)}} \neq 0$ and

$$\langle \Psi_{Y_0^{(N)}}, H^{(N,\alpha)} \Psi_{Y_0^{(N)}} \rangle \leq \left(\langle \Phi, H^{(N,\alpha)} \Phi \rangle + b \frac{N^2}{R^2} \right) \langle \Psi_{Y_0^{(N)}}, \Psi_{Y_0^{(N)}} \rangle \quad (3.5)$$

which then finishes the proof.

First of all, we notice $\int W(X^{(N)}, Y^{(N)})^2 dY^{(N)} = 1$ due to the choice of W and thus by Fubini's Theorem

$$\int \langle \Psi_{Y^{(N)}}, \Psi_{Y^{(N)}} \rangle dY^{(N)} = \langle \Phi, \Phi \rangle = 1.$$

The multiplication with $W(\cdot, Y^{(N)})$ obviously commutes with all but the kinetic part of the Hamiltonian. In particular, if we define

$$q^{(I)} := q^{(N,\alpha)} - \tilde{q}_{e,0}^{(N)}|_{\mathcal{Q}_N^2}, \quad (3.6)$$

see Equations (2.2) and (2.4), we find

$$\int q^{(I)}(\Psi_{Y^{(N)}}, \Psi_{Y^{(N)}}) dY^{(N)} = q^{(I)}(\Phi, \Phi). \quad (3.7)$$

In order to calculate the commutator with the kinetic part $\tilde{q}_{e,0}^{(N)} = \sum_{j=1}^N D_{A,x_j}^\dagger D_{A,x_j}$, we compute

$$\begin{aligned} \langle (D_{A,x_j})_k \Psi_{Y^{(N)}}, (D_{A,x_j})_k \Psi_{Y^{(N)}} \rangle &= \langle [(\nabla_{x_j})_k W]^2 \Phi, \Phi \rangle \\ &\quad + \Re \langle [(-i\nabla_{x_j})_k W^2] \Phi, (D_{A,x_j})_k \Phi \rangle \\ &\quad + \langle W^2 (D_{A,x_j})_k \Phi, (D_{A,x_j})_k \Phi \rangle. \end{aligned}$$

We have that $\int W(X^{(N)}, Y^{(N)})^2 dY^{(N)} = 1$ implies $\int (\nabla_{x_j})_k [W(\cdot, Y^{(N)})]^2(X^{(N)}) dY^{(N)} = 0$ and so the second term of the r.h.s. of the last equation yields zero when we integrate with respect to $dY^{(N)}$. Again by Fubini's Theorem, we find that the last term gives $\langle (D_{A,x_j})_k \Phi, (D_{A,x_j})_k \Phi \rangle$

when we integrate with respect to $dY^{(N)}$. We arrive at

$$\begin{aligned}
\int \tilde{q}_{e,0}^{(N)}(\Psi_{Y^{(N)}}, \Psi_{Y^{(N)}}) dY^{(N)} &= \sum_{j=1}^N \int \langle D_{A,x_j} \Psi_{Y^{(N)}}, D_{A,x_j} \Psi_{Y^{(N)}} \rangle dY^{(N)} \\
&= \sum_{j=1}^N (\langle D_{A,x_j} \Phi, D_{A,x_j} \Phi \rangle + \langle \Phi, F_j(X^{(N)}) \Phi \rangle) \\
&= \tilde{q}_{e,0}^{(N)}(\Phi, \Phi) + \sum_{j=1}^N \langle \Phi, F_j(X^{(N)}) \Phi \rangle,
\end{aligned} \tag{3.8}$$

where $F_j(X^{(N)})$ is given by

$$F_j(X^{(N)}) = \int |\nabla_{x_j} [G(\cdot, Y^{(N)}) P(\cdot)^{-1/2}](X^{(N)})|^2 dY^{(N)}.$$

As in the proof of Lemma 4.1 in [LL05] we argue in Appendix C that we have

$$F_j(X^{(N)}) \leq N \int |\nabla \chi(x)|^2 dx. \tag{3.9}$$

We continue as in [LL05] and let $\chi \in \mathcal{C}_c^\infty$ approximate the lowest Dirichlet eigenfunction of $-\Delta$ in $B_R(0)$ such that

$$N^2 \int |\nabla \chi(x)|^2 dx \leq N^2 \pi^2 R^{-2} + \varepsilon,$$

with $\varepsilon := N^2 \pi^2 R^{-2}/2$. Note that $\pi^2 R^{-2}$ is the lowest eigenvalue of the Dirichlet Laplacian on $B_R(0)$. This then gives us

$$F_j(X^{(N)}) \leq 3N\pi^2/(2R^2). \tag{3.10}$$

Equations (3.6), (3.7), (3.8), and inequality (3.10) imply inequality (3.5). Thus defining $\Phi_0 := \Psi_{Y_0^{(N)}}/||\Psi_{Y_0^{(N)}}||$ we finish the proof of Step 1.

Step 2: Since the second step does not involve the shape of the given Hamiltonian but only the geometry of the support of the given function, we can apply the proof of Lemma 3.1 given in [Wel15]. For the sake of completeness we will repeat the argument in here. We will show that there are $m \leq N$, $n_1, \dots, n_m \in \mathbb{N}$ with $\sum_i n_i = N$, balls B_1, B_2, \dots, B_m and a permutation $\sigma \in \mathcal{S}_N$ such that $\times_{j=1}^N B_R(y_{\sigma(j)}) \subseteq \times_{i=1}^m B_i^{n_i}$ and (i) and (ii) in the assertion hold by running induction on N . Due to the antisymmetry of Φ_0 we may assume without loss of generality that σ is the identity and thus Φ_0 satisfies the assertion.

$N = 1$ is trivial. Assume the induction hypothesis holds true for N electrons. So choose $\tilde{m} \leq N$, $\tilde{n}_1, \dots, \tilde{n}_{\tilde{m}} \in \mathbb{N}$ with $\sum_i \tilde{n}_i = N$, balls $\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_{\tilde{m}}$ and a permutation $\tilde{\sigma} \in \mathcal{S}_N$ such that $\times_{j=1}^N B_R(y_{\tilde{\sigma}(j)}) \subseteq \times_{i=1}^{\tilde{m}} \tilde{B}_i^{\tilde{n}_i}$ and (i) and (ii) hold. Consider one further ball $B_R(y_{N+1})$. Two cases can arise:

Case 1: For any $1 \leq i \leq \tilde{m}$ we have $\text{dist}(\tilde{B}_i, B_R(y_{N+1})) \geq R$. Then we define $\sigma \in \mathcal{S}_{N+1}$ by $\sigma|_{\{1, \dots, N\}} := \tilde{\sigma}$ and $\sigma(N+1) := N+1$, $m := \tilde{m} + 1$, $n_m := 1$, $n_i := \tilde{n}_i$, $B_i := \tilde{B}_i$, $1 \leq i \leq m-1$, and $B_m := B_R(y_{N+1})$. Hence, $\times_{j=1}^{N+1} B_R(y_{\sigma(j)}) \subseteq \times_{i=1}^m B_i^{n_i}$ and (i) and (ii) hold.

Case 2: There is $i_1 \in \{1, \dots, \tilde{m}\}$ such that $\text{dist}(\tilde{B}_{i_1}, B_R(y_{N+1})) < R$. Then there is a ball $B^{(1)} \supseteq \tilde{B}_{i_1} \cup B_R(y_{N+1})$ with radius $\frac{1}{2}(3R + 2R_{i_1}) = \frac{1}{2}(3(\tilde{n}_{i_1} + 1) - 1)R$. If then for any $i_2 \in \{1, \dots, \hat{i}_1; \dots, \tilde{m}\}$ we have $\text{dist}(B^{(1)}, \tilde{B}_{i_2}) \geq R$, we define $B_{i_1} := B^{(1)}$ and $n_{i_1} := \tilde{n}_{i_1} + 1$, $m := (\tilde{m} + 1) - 1 = \tilde{m}$, $B_i := \tilde{B}_i$ and $n_i := \tilde{n}_i$ for any $i \neq i_1$. Otherwise, by iterating this

procedure, we finish step 2. □

In the last proof we have seen that we may ensure for Φ_0 that

$$\Phi_0 \in \{\pi^{(N)}(\Psi) | \text{supp}_e(\Psi) \subseteq \bigtimes_{i=1}^m B_i^{n_i}\} \subseteq \pi^{(N)}\tilde{\mathcal{Q}}_N = \mathcal{Q}_N,$$

where again $\pi^{(N)} : \bigotimes_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F} \rightarrow \bigwedge_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F} = \mathcal{H}_N$ is the canonical projection. For the definition of $\tilde{\mathcal{Q}}_N$ see Equation (2.8). Indeed, up to a constant prefactor, Φ_0 is given as

$$\begin{aligned} & P(X^{(N)})^{-1/2} \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^N \chi(x_j - y_{\sigma(j)}) \pi^{(N)}(\Psi)(X^{(N)}) \\ &= N! \cdot \pi^{(N)} \left(P(X^{(N)})^{-1/2} \prod_{j=1}^N \chi(x_j - y_j) \Psi(X^{(N)}) \right). \end{aligned}$$

Let $P^{(l)} : \bigotimes_{j=1}^l L^2(\mathbb{R}^3; \mathbb{C}^2) \rightarrow \bigwedge_{j=1}^l L^2(\mathbb{R}^3; \mathbb{C}^2)$ denote the canonical projection and by abuse of notation we extend $\bigotimes_{i=1}^m P^{(n_i)}$ to the whole of $\bigotimes_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}$. A simple calculation yields that for the above (disjoint) balls B_1, \dots, B_m and for $\text{supp}_e(\Psi) \subseteq \bigtimes_{i=1}^m B_i^{n_i}$ and $\|\pi^{(N)}(\Psi)\| = 1$ we have

$$\begin{aligned} \langle \pi^{(N)}(\Psi), H^{(N,\alpha)} \pi^{(N)}(\Psi) \rangle &= q^{(N,\alpha)}(\pi^{(N)}(\Psi), \pi^{(N)}(\Psi)) \\ &= \frac{\tilde{q}^{(N,\alpha)}(\bigotimes_{i=1}^m P^{(n_i)}(\Psi), \bigotimes_{i=1}^m P^{(n_i)}(\Psi))}{\|\bigotimes_{i=1}^m P^{(n_i)}(\Psi)\|^2}. \end{aligned}$$

For the definition of $\tilde{q}^{(N,\alpha)}$ see Equation (2.17) combined with (2.4), (2.6), (2.7), and (2.9). As a consequence of the last equation, we have that minimizing the energy of an N -polaron contained in m balls is equivalent to minimizing the energy of an ensemble of m multipolarons with total electron number equal to N . This means, that in the following we may restrict our considerations to the states in $\bigotimes_{i=1}^m P^{(n_i)}(\tilde{\mathcal{Q}}_N)$.

In our next step, we will bound the total many-particle ground state energy from below by the sum of the ground state energies for less particles supported on the balls described above and the interaction between the balls. For $n \in \mathbb{N}$ and a Borel measurable set $O \subseteq \mathbb{R}^3$, define

$$E_n^{(\alpha)}(O) := \inf_{\substack{\text{supp}_e(\Phi) \subseteq O^n, \\ \|\Phi\|=1}} \langle \Phi, H^{(n,\alpha)} \Phi \rangle.$$

In the physical sense, this denotes the ground state energy of an n -polaron whose electrons are contained in O . Note that $E_n^{(\alpha)}(O)$ clearly depends on the external fields A and V and on the coupling strength U .

The following proposition generalizes Lemma 3 in [FLST11]. As in [Wel15], the treatment of the bipolaron is transferred to the multipolaron case.

3.2 Proposition. *Let $\Psi \in \bigotimes_{i=1}^m P^{(n_i)}(\tilde{\mathcal{Q}}_N)$ be normalized with $\text{supp}_e(\Psi) \subseteq \bigtimes_{i=1}^m B_i^{n_i}$ such that*

$d_i := \min_{j \neq i} \text{dist}(B_i, B_j) > 0$ for any $1 \leq i \leq m$. Then Ψ satisfies

$$\tilde{q}^{(N, \alpha)}(\Psi, \Psi) \geq \sum_{i=1}^m E_{n_i}^{(\alpha)}(B_i) + (U - 2\alpha) \sum_{i < j} \sum_{\substack{k_i \in C_i \\ l_j \in C_j}} \langle \Psi, \frac{1}{|x_{k_i} - x_{l_j}|} \Psi \rangle - \frac{8\alpha N}{\pi^2} \sum_{i=1}^m \frac{n_i}{d_i},$$

where C_i denotes the set of electrons supported in B_i .

Proof. As in [Wel15], we start by subdividing \mathbb{R}^3 into the "area of influence" of the single multipolarons. More precisely, we define

$$S_i := \{x \in \mathbb{R}^3 \mid \text{dist}(B_i, x) < \text{dist}(B_j, x) \ \forall j \neq i\}$$

and find $S_i \cap S_j = \emptyset$ for any $i \neq j$. For $\text{dist}(B_i, B_j) > 0$ if $j \neq i$, we have $B_i \subseteq S_i$ as well as $\overline{\bigcup_{i=1}^m S_i} = \mathbb{R}^3$.

Next, we rewrite $\tilde{q}^{(N, \alpha)}$ with respect to this spatial partition. In the sense of a quadratic form on $\bigotimes_{i=1}^m P^{(n_i)}(\tilde{Q}_N)$, we write

$$\begin{aligned} \tilde{q}^{(N, \alpha)} &= \sum_{i=1}^m \left(\sum_{l \in C_i} (T_l + \sqrt{\alpha} \phi(x_l)) + U \sum_{r, s \in C_i} \frac{1}{|x_r - x_s|} \right) + H_{ph} \\ &\quad + U \sum_{i < j} \sum_{\substack{r_i \in C_i, \\ s_j \in C_j}} \frac{1}{|x_{r_i} - x_{s_j}|}, \end{aligned}$$

where $T_l = D_{A, x_l}^\dagger D_{A, x_l} + V(x_l)$. Then we define

$$\hat{a}(x) := \frac{1}{(2\pi)^{3/2}} \int e^{ikx} a(k) dk, \quad \hat{a}^\dagger(x) := \frac{1}{(2\pi)^{3/2}} \int e^{-ikx} a^\dagger(k) dk$$

in the sense of quadratic forms acting on \mathcal{F} . By Plancherel, we deduce

$$\phi(x) = \frac{1}{\pi^{3/2}} \int \frac{\hat{a}(y) + \hat{a}^\dagger(y)}{|x - y|^2} dy, \quad H_{ph} = \int \hat{a}^\dagger(y) \hat{a}(y) dy.$$

As in [Wel15], we introduce annihilation operators restricted to $S_i \supseteq B_i$ by $\hat{a}_i := (\hat{a} + g_i) \mathbf{1}_{S_i}$ with

$$g_i(y, x_1, \dots, x_N) := \frac{\sqrt{\alpha}}{\pi^{3/2}} \sum_{j \neq i} \sum_{l \in C_j} \frac{1}{|x_l - y|^2} \mathbf{1}_{S_i}(y).$$

We introduce some further notation to apply the result given in [Wel15]. First we define operators contributing to the inter-ball exchange energy

$$\begin{aligned} F_1(x_1, \dots, x_N) &:= \sum_{i=1}^m \|g_i(\cdot, x_1, \dots, x_N)\|^2 \\ F_2(x_1, \dots, x_N) &:= \frac{2\sqrt{\alpha}}{\pi^{3/2}} \sum_{i=1}^m \sum_{l \in C_i} \int_{S_i} \frac{g_i(y, x_1, \dots, x_N)}{|x_l - y|^2} dy. \end{aligned} \tag{3.11}$$

Second, in the sense of a quadratic form on $\bigwedge_{j=1}^{n_i} \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}(L^2(S_i))$, we define

$$K_i := \sum_{l \in C_i} \left(T_l + \frac{\sqrt{\alpha}}{\pi^{3/2}} \int_{S_i} \frac{\hat{a}_i(y) + \hat{a}_i^\dagger(y)}{|x_l - y|^2} dy \right) + \int_{S_i} \hat{a}_i^\dagger(y) \hat{a}_i(y) dy + U \sum_{\substack{r, s \in C_i \\ r < s}} \frac{1}{|x_r - x_s|}. \quad (3.12)$$

This term corresponds to the energy of the electrons supported within B_i and the phonons which are closer to this ball than to any other ball, including the electron-phonon interaction between these. Then by the proof of Proposition 3.2 in [Wel15] we obtain

$$\tilde{q}^{(N, \alpha)}(\Psi, \Psi) = \sum_{i=1}^m \langle \Psi, K_i \Psi \rangle + U \sum_{i < j} \sum_{\substack{r_i \in C_i \\ s_j \in C_j}} \langle \Psi, \frac{1}{|x_r - x_s|} \Psi \rangle - \langle \Psi, (F_1 + F_2) \Psi \rangle.$$

Recall $L^2(\mathbb{R}^3) = \bigoplus_{i=1}^m L^2(S_i)$ and thus $\mathcal{F}(L^2(\mathbb{R}^3)) = \bigotimes_{i=1}^m \mathcal{F}(L^2(S_i))$. In the last equation, we then identify the quadratic form indicated by K_i with its obvious extension to

$$\bigotimes_{l=1}^m \left(\left(\bigwedge_{j=1}^{n_l} L^2(\mathbb{R}^3; \mathbb{C}^2) \right) \otimes \mathcal{F}(L^2(S_i)) \right).$$

Using the last equation and Lemmata 3.3 and 3.4 below, the proposition follows. \square

3.3 Lemma. *Let $1 \leq i \leq m$ and K_i be defined as in Equation (3.12). Let $\psi \in (\bigwedge_{j=1}^{n_i} L^2(B_i; \mathbb{C}^2)) \otimes \mathcal{F}(L^2(S_i))$ be normalized. Then*

$$\langle \psi, K_i \psi \rangle \geq E_{n_i}^{(\alpha)}(B_i).$$

3.4 Lemma. *Let $\Psi \in \bigotimes_{i=1}^m P^{(n_i)}(\tilde{\mathcal{Q}}_N)$ fulfill the assumptions in Proposition 3.2 and F_1, F_2 be defined as in Equation (3.11). Then we have*

$$(i) \quad \langle \Psi, F_1 \Psi \rangle \leq N \frac{8\alpha}{\pi^2} \sum_{i=1}^m \frac{n_i}{d_i} \|\Psi\|^2 \text{ and}$$

$$(ii) \quad \langle \Psi, F_2 \Psi \rangle \leq 2\alpha \sum_{i < j} \sum_{\substack{r_i \in C_i \\ s_j \in C_j}} \langle \Psi, \frac{1}{|x_{r_i} - x_{s_j}|} \Psi \rangle.$$

These last two Lemmata directly follow from the proofs of Lemma 3.3 (for Lemma 3.3) and Lemma 3.4 (for Lemma 3.4) in [Wel15]. The proof of Lemma 3.3 in [Wel15] uses the Weyl operator

$$W(g_i) = e^{\hat{a}^\dagger(g_i) - \hat{a}(g_i)}.$$

It is important that g_i only depends on electron coordinates that are not contained in B_i . So $W(g_i)$ is not affected by permutations of electron coordinates within B_i .

3.2. Lower bound for the energy of a localized multipolaron

As in [Wel15], we will basically employ ideas from [LT97] (see also Erratum) to bound the energies of the single multipolarons found in the last part by the Pekar-Tomasevich functional given for $\psi \in \bigwedge_{j=1}^n \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)$ by

$$\mathcal{P}^{(n, \alpha)}(A, V, U, \psi) = \tilde{q}_e^{(n)}(\psi, \psi) - \alpha D(\rho_\psi)$$

(see Equation (2.6)), where $n > 0$ is an integer and

$$D(\rho) := \int_{\mathbb{R}^6} \frac{\rho(x)\rho(y)}{|x-y|} d(x,y)$$

indicates the self-interaction. Recall that

$$\rho_\psi(x) = \sum_{j=1}^n \int_{\mathbb{R}^{3(n-1)}} |\psi(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n)|^2 d(x_1, \dots, \hat{x}_j, \dots, x_n)$$

denotes the (electron) density. First we will use a ultra-violet cut-off for the phonon modes and then we will make use of the so-called *block modes* of which there are only finitely many contained in a finite ball.

Let $O \subseteq \mathbb{R}^3$ be a measurable set. Then we define

$$\mathbf{N}_O := \int_O a^\dagger(k)a(k)dk.$$

For a normalized $h \in L^2(\mathbb{R}^3)$ we will use the short-hand notation

$$a(h) := \int \overline{h(k)}a(k)dk.$$

If $\text{supp}(h) \subseteq O$, then $a^\dagger(h)a(h) \leq \mathbf{N}_O$.

At first, let $\Lambda > 0$ be some fixed positive real number. Let $B_\Lambda := B_\Lambda(0)$ and as in the proof of Lemma 2.15,

$$f_{x,\Lambda}^{(n)}(k) := \sum_{j=1}^n \frac{e^{-ikx_j}}{\sqrt{2\pi|k|}} \mathbf{1}_{B_\Lambda}(k), \quad x = (x_1, \dots, x_n),$$

and $\beta := 1 - \frac{2\alpha n}{\pi\Lambda}$. In order to abbreviate the notation henceforth, we introduce the total magnetic vector potential $\mathbf{A}_n := (A(x_1), \dots, A(x_n))^T$. For convenience, let $\mathbf{V}_n(x) := \sum_{j=1}^n V(x_j)$ and $D_{\mathbf{A}_n} := -i\nabla^{(n)} + \mathbf{A}_n$, where $\nabla^{(n)}$ denotes the derivative in \mathbb{R}^{3n} . Then we define the cut-off Fröhlich Hamiltonian

$$H_\Lambda^{(n,\alpha)} := \beta(D_{\mathbf{A}_n}^\dagger D_{\mathbf{A}_n} + UV_C(x)) + \mathbf{V}_n + \sqrt{\alpha}(a(f_\Lambda^{(n)}) + a^\dagger(f_\Lambda^{(n)})) + \mathbf{N}_{B_\Lambda}.$$

Arguing as in (2.12) we can show that $\sqrt{\alpha}(a(f_\Lambda^{(n)}) + a^\dagger(f_\Lambda^{(n)}))$ is \mathbf{N}_{B_Λ} -bounded with bound zero. From this it follows that $H_\Lambda^{(n,\alpha)}$ is a self-adjoint operator with the same domain as that of the free cut-off Hamiltonian $H_\Lambda^{(n,0)}$.

3.5 Lemma. *Let $\alpha > 0$ and $n \in \mathbb{N}$. Then for any choice of $\Lambda > 0$ we have, in the sense of quadratic forms on \mathcal{Q}_n ,*

$$H^{(n,\alpha)} \geq H_\Lambda^{(n,\alpha)} - \frac{1}{2}.$$

Proof. Let $\Phi \in \mathcal{Q}_n$. As in the proof of Lemma 2.15 we decompose

$$\langle \Phi, a(f_\Lambda^{(n)})\Phi \rangle = \langle \Phi, (a(f_\Lambda^{(n)}) + [D_{\mathbf{A}_n}, a(g_\Lambda^{(n)})])\Phi \rangle$$

in the sense of (2.14). If we use (2.16) and (2.15) with N replaced by n , we arrive at

$$\sqrt{\alpha}|\langle \Phi, [D_{\mathbf{A}_n}, a(g_\Lambda^{(n)})]\Phi \rangle| \leq \varepsilon \langle D_{\mathbf{A}_n} \Phi, D_{\mathbf{A}_n} \Phi \rangle + \frac{2\alpha n}{\varepsilon\pi\Lambda} \langle \Phi, \mathbf{N}_{B_\Lambda^c} \Phi \rangle + \frac{\alpha n}{\varepsilon\pi\Lambda} \|\Phi\|^2$$

for any arbitrary value of $\varepsilon > 0$. Now choose $\varepsilon = \frac{2\alpha n}{\pi\Lambda}$ and the result follows. \square

3.6 Lemma. *Let $P, \Lambda, r > 0$ and $y \in \mathbb{R}^3$ be arbitrary and $\Phi \in \mathcal{Q}_n$ be normalized such that $\text{supp}_\varepsilon(\Phi) \subseteq B_r(y)^n$. Then we have*

$$\langle \Phi, H^{(n,\alpha)} \Phi \rangle \geq \beta C^{(n,\beta^{-2}\alpha)}(A, \beta^{-1}V, U) - \frac{6n^2\alpha P^2 r^2 \Lambda}{(1-\beta)\pi} - \frac{1}{2} - \left(2\frac{\Lambda}{P} + 1\right)^3 \quad \forall U \geq 0.$$

Proof. Since $C^{(n,\alpha)}$ is constant with respect to shifting the fields $A(\cdot) \rightarrow A(\cdot - y)$ respectively $V(\cdot) \rightarrow V(\cdot - y)$, we may assume without loss of generality that $y = 0$.

We start by estimating $H^{(n,\alpha)}$ from below by the cut-off Hamiltonian $H_\Lambda^{(n,\alpha)}$ in the sense of quadratic forms on \mathcal{Q}_n . Then we compare the cut-off Hamiltonian with the Hamiltonian obtained by replacing the (continuous) phonon modes by (discrete) finitely many so-called block modes. Define

$$\begin{aligned} B(m) &:= \{k \in B_\Lambda \mid |k_i - m_i P| \leq P/2 \forall 1 \leq i \leq 3\} \quad \text{for any } m \in \mathbb{Z}^3, \\ \Xi &:= \{m \in \mathbb{Z}^3 \mid B(m) \neq \emptyset\} \end{aligned}$$

and for any $m \in \Xi$ choose some $k_m \in B(m)$. The *block modes* are defined as

$$a_m := \frac{1}{M_m} \int_{B(m)} \frac{1}{|k|} a(k) dk, \quad M_m := \left(\int_{B(m)} \frac{dk}{|k|^2} \right)^{1/2}.$$

We have that $h_m := M_m^{-1} \cdot |\cdot|^{-1} \mathbf{1}_{B(m)} \in L^2(\mathbb{R}^3)$ is normalized and $a_m = a(h_m)$. Let $\delta > 0$ be arbitrary. Then we define

$$\begin{aligned} H_{block}^{(n,\alpha)} &:= \beta(D_{\mathbf{A}_n}^\dagger D_{\mathbf{A}_n} + UV_C) + \mathbf{V}_n + (1-\delta)N_{block} \\ &\quad + \frac{\sqrt{\alpha}}{\sqrt{2\pi}} \sum_{j=1}^n \sum_{m \in \Xi} M_m (e^{ik_m x_j} a_m + e^{-ik_m x_j} a_m^\dagger), \end{aligned}$$

where $N_{block} = \sum_{m \in \Xi} a_m^\dagger a_m$. Since only the phononic part is of interest, we may apply the exact same steps as in the proof of Proposition 4.2 in [Wel15] to arrive at the assertion. \square

Henceforth, the arguments in [Wel15] do not involve the statistics, so we will state the results without proof. The next proposition follows as in [Wel15] from Lemmata 3.5 and 3.6 for suitable choices for Λ and P .

Recall that in Theorem 1.1 we defined A_α and V_α as $A_\alpha(x) := \alpha A(\alpha x)$ and $V_\alpha(x) := \alpha^2 V(\alpha x)$ where A and V satisfy the assumptions **(I)** and **(II)** of Theorem 1.1. We further denote by $B_{R,n}$ an arbitrary ball of radius $\frac{1}{2}(3n-1)R$ if $R > 0$ and $n \in \mathbb{N}$. Since we explicitly need the dependence of the energy on all of the parameters n, α, A, V and U we will emphasize it in the notation by writing $E_n^{(\alpha, A, V, U)}$.

3.7 Proposition. *In addition to assumptions **(I)**, **(II)**, let A, V satisfy assumption **(III)** of Theorem 1.1. Then there is $c(A, V)$ such that*

$$E_n^{(\alpha, A_\alpha, V_\alpha, \alpha\nu)}(B_{R,n}) \geq \alpha^2 C^{(n,1)}(A, V, \nu) - 3R^2 \alpha^{80/23} n^5 - c(A, V) \alpha^{42/23} n^3$$

for any $\alpha, \nu, R > 0$, and $n \in \mathbb{N}$. Furthermore, if we even have $A_k \in L_{loc}^3(\mathbb{R}^3)$ and $V \in L_{loc}^{3/2}(\mathbb{R}^3)$, then $c(A_{\alpha^{-1}}, V_{\alpha^{-1}})$ is uniformly bounded for α large enough, e.g., for $\alpha \geq 1$.

3.3. Conclusion of the proof of Theorem 1.1

We are finally ready to prove the main Theorem 1.1 analogously to the proof of Theorem 1.1 in [Wel15].

Proof of 1.1. Let $\Phi \in \mathcal{Q}_N$ be normalized. In order to emphasize the dependence of $H^{(N,\alpha)}$ on the parameters A , V and U , we will denote it within this proof by

$$H^{(N,\alpha,A,V,U)}.$$

Lemma 3.1, Proposition 3.2 and the assumption $\nu \geq 2$ of Theorem 1.1 imply

$$\langle \Phi, H^{(N,\alpha,A_\alpha,V_\alpha,\alpha\nu)} \Phi \rangle \geq \sum_{i=1}^m E_{n_i}^{(\alpha,A_\alpha,V_\alpha,\alpha\nu)}(B_i) - \frac{8\alpha N^2}{\pi^2 R} - 2\pi^2 N^2 R^{-2}$$

with the above notation. If we now choose $R = N^{-1}\alpha^{-19/23}$, we may apply Proposition 3.7 to obtain

$$E_{n_i}^{(\alpha,A_\alpha,V_\alpha,\alpha\nu)}(B_i) \geq \alpha^2 C^{(n_i,1)}(A, V, \nu) - \tilde{c}(A, V) \alpha^{42/23} n_i^3 \left(1 + \frac{n_i^2}{N^2}\right)$$

for some $\tilde{c}(A, V)$. In here, recall that for $\|\psi\| = 1$ and $\psi_\alpha(x) := \alpha^{\frac{3n}{2}} \psi(\alpha x)$ we have $\|\psi_\alpha\| = 1$ and

$$\mathcal{P}^{(n,\alpha)}(A_\alpha, V_\alpha, \alpha\nu, \psi_\alpha) = \alpha^2 \mathcal{P}^{(n,1)}(A, V, \nu, \psi).$$

Thus we obtain $C^{(n,\alpha)}(A_\alpha, V_\alpha, \alpha\nu) = \alpha^2 C^{(n,1)}(A, V, \nu)$. Now we use $\sum_{i=1}^m n_i^3 \leq N^3$ and the assumption (1.17) on the external fields to conclude

$$\langle \Phi, H^{(N,\alpha,A_\alpha,V_\alpha,\alpha\nu)} \Phi \rangle \geq \alpha^2 C^{(N,1)}(A, V, \nu) - \hat{c}(A, V) \alpha^{42/23} N^3 - \alpha^{38/23} 2\pi^2 N^4$$

for some $\hat{c}(A, V)$. Since \mathcal{Q}_N is a form core of $H^{(N,\alpha)}$ due to Theorem 2.7, we finally arrive at the initial assertion

$$E^{(N,\alpha)}(A_\alpha, V_\alpha, \alpha\nu) \geq \alpha^2 C^{(N,1)}(A, V, \nu) - c(A, V) \alpha^{42/23} N^4$$

for some $c(A, V) \in \mathbb{R}$. Part (ii) now follows analogously to the proof of part (b) of Theorem 1.1 in [Wel15]. \square

4. Binding of Fröhlich polarons

We now give an example of an application of Theorem 1.1. Given external fields A and V , Theorem 1.1 combined with the inequality (1.10) and Remark 1.3 states that the ground state energy of a Fröhlich N -polaron may be estimated by

$$C^{(N,\alpha)}(A, V, \alpha\nu) - c(A_{\alpha^{-1}}, V_{\alpha^{-1}}) \alpha^{42/23} N^4 \leq E^{(N,\alpha)}(A, V, \alpha\nu) \leq C^{(N,\alpha)}(A, V, \alpha\nu)$$

for α large. This means that the Pekar-Tomasevich functional already gives a rough estimate on the ground state energy of the initial Fröhlich Hamiltonian. We define

$$\Delta E^{(N,\alpha)}(A, V, \alpha\nu) := \left[\min_{1 \leq k \leq N} (E^{(k,\alpha)} + E^{(N-k,\alpha)}) - E^{(N,\alpha)} \right](A, V, \alpha\nu).$$

If we further assume that A is linear, we may apply Theorem 1.1 in [AG14]. It states that for $\Delta C^{(N,\alpha)}(A, V, \alpha\nu)$, which is analogously defined to $\Delta E^{(N,\alpha)}(A, V, \alpha\nu)$, we have

$$\Delta C^{(N,1)}(A, 0, \nu) > 0$$

provided $2 < \nu < \nu_{A,N}$ for some $\nu_{A,N} > 2$. Note that [Lew11] proves an analogous result for $A = 0$.

Corollary. *Assume the vector potential A is linear (corresponding to a constant magnetic field) and $V = 0$. Then for any $N \in \mathbb{N}$ there is $\nu_{N,A} > 2$ such that for any $2 < \nu < \nu_{N,A}$ and $\alpha > 0$ large enough*

$$\Delta E^{(N,\alpha)}(A_\alpha, 0, \alpha\nu) > 0.$$

Moreover, if $A_k \in L^3_{loc}(\mathbb{R}^3)$ and $V \in L^{3/2}_{loc}(\mathbb{R}^3)$, then there is $\nu_N > 2$ such that for any $2 < \nu < \nu_N$ and any $\alpha > 0$ large enough we find

$$\Delta E^{(N,\alpha)}(A, V, \alpha\nu) > 0.$$

[Wel15] obtained this result without taking the Fermionic statistics into account. This corollary yields that in the strong coupling limit Fermionic Fröhlich multipolarons form a binding state.

A. Proof of Lemma 2.9

Before we turn to the actual proof we recall the following result given in [RS72], Theorem VIII.33:

A.1 Lemma. *Let A_1, A_2, \dots, A_k be self-adjoint operators on Hilbert spaces h_1, h_2, \dots, h_k and let A_j be essentially self-adjoint on a domain D_j for any $1 \leq j \leq k$. Then $A_1 + A_2 + \dots + A_k$ is essentially self-adjoint on $\bigotimes_{j=1}^k D_j$.*

This Lemma and the duality between closed quadratic forms and positive self-adjoint operators are the key ingredients of the following proof.

Proof of Lemma 2.9. Define \bar{q}_j to be the closure of q_j , $j \in \{1, 2\}$. Then

$$(\bar{q}_1 + \bar{q}_2)|_{D_1 \otimes D_2} = q_1 + q_2,$$

where $\bar{q}_1 + \bar{q}_2$ is analogously defined to $q_1 + q_2$. Next we define in virtue of Proposition 2.4 A_j to be the positive self-adjoint operator associated with \bar{q}_j , $j \in \{1, 2\}$. Then by Lemma A.1

$$(A_1 + A_2, \mathcal{D}(A_1) \otimes \mathcal{D}(A_2))$$

is essentially self-adjoint. Next we note that due to the equality of the respective norms we have $\mathcal{D}(\sqrt{A_j}) = Q(q_j)$, see Proposition 2.5, and so D_j is a domain of essential self-adjointness for $\sqrt{A_j}$. In addition, it follows that $\mathcal{D}(A_j)$ is a dense subset of $Q(q_j)$. To see this, we recall an argument given in [BS87]. Indeed, assume there is some $\psi_0 \in Q(q_j)$ such that $\psi_0 \perp_{Q(q_j)} \mathcal{D}(A_j)$, so in particular $q_j(\phi, \psi_0) + \langle \phi, \psi_0 \rangle_{h_j} = 0$ for any $\phi \in \mathcal{D}(A_j)$. Including $q_j(\phi, \psi_0) = \langle A_j \phi, \psi_0 \rangle_{h_j}$, we find $\psi_0 \perp_{h_j} R(A_j + 1)$. Since A_j is self-adjoint and $0 \notin \sigma(A_j + 1)$, we obtain $R(A_j + 1) = h_j$ and thus $\psi_0 = 0$.

In the next step we will see that it is possible to define⁵

$$S := \sqrt{A_1 + A_2}$$

on $D_1 \otimes D_2$ by comparing it with

$$T := \sqrt{A_1} + \sqrt{A_2}.$$

Again by Lemma A.1 T is essentially self-adjoint on $D_1 \otimes D_2$ and also on $\mathcal{D}(A_1) \otimes \mathcal{D}(A_2)$.

Let $\phi \in \mathcal{D}(A_1) \otimes \mathcal{D}(A_2)$. We find

$$\begin{aligned} \|S\phi\|^2 &= \langle \overline{A_1 + A_2}\phi, \phi \rangle = \langle A_1\phi, \phi \rangle + \langle A_2\phi, \phi \rangle \\ &\leq \langle A_1\phi, \phi \rangle + 2\langle \sqrt{A_1}\phi, \sqrt{A_2}\phi \rangle + \langle A_2\phi, \phi \rangle = \|T\phi\|^2 \\ &\leq 2(\langle A_1\phi, \phi \rangle + \langle A_2\phi, \phi \rangle) \\ &= 2\|S\phi\|^2, \end{aligned}$$

i.e., $\|S\phi\| \leq \|T\phi\| \leq \sqrt{2}\|S\phi\|$ and so the operator norms are equivalent. Since for both operators, S and T , $\mathcal{D}(A_1) \otimes \mathcal{D}(A_2)$ is a domain of essential self-adjointness we deduce

$$\mathcal{D}(S) = \mathcal{D}(T).$$

For T is essentially self-adjoint on $D_1 \otimes D_2$, so is S . Next we want to establish that $\langle S \cdot, S \cdot \rangle|_{(D_1 \otimes D_2)^2}$ coincides with $q_1 + q_2$. We derive for $\phi, \psi \in \mathcal{D}(A_1) \otimes \mathcal{D}(A_2)$

$$\langle S\phi, S\psi \rangle = (\bar{q}_1 + \bar{q}_2)(\phi, \psi), \tag{A.1}$$

since $\langle S\phi, S\psi \rangle = \langle A_1\phi, \psi \rangle + \langle A_2\phi, \psi \rangle$ and \bar{q}_j is associated with A_j . Hence we find that the completion of $\mathcal{D}(A_1) \otimes \mathcal{D}(A_2)$ with respect to $\|\cdot\|_{\bar{q}_1 + \bar{q}_2}$ is equal to

$$\overline{\mathcal{D}(A_1) \otimes \mathcal{D}(A_2)}^{\|\cdot\|_S} = \mathcal{D}(S).$$

So Equation (A.1) holds for $\phi, \psi \in Q(A_1) \otimes Q(A_2) = \mathcal{D}(\sqrt{A_1}) \otimes \mathcal{D}(\sqrt{A_2}) \subseteq \mathcal{D}(T) = \mathcal{D}(S)$. In particular, Equation (A.1) then also holds on $D_1 \otimes D_2 \subseteq Q(A_1) \otimes Q(A_2)$ for which $\bar{q}_1 + \bar{q}_2$ coincides with $q_1 + q_2$. Since S is essentially self-adjoint on $D_1 \otimes D_2$, we then conclude that the completion of $D_1 \otimes D_2$ with respect to $\|\cdot\|_{q_1 + q_2}$ is equal to

$$\overline{D_1 \otimes D_2}^{\|\cdot\|_S} = \mathcal{D}(S).$$

Finally, this implies that $\langle S \cdot, S \cdot \rangle$ is the closure of $q_1 + q_2$ and therefore Lemma 2.9 follows. \square

B. Tensor product of \mathcal{C}_c^∞ -spaces

The goal of this Section is to show that

$$\overline{\bigotimes_{j=1}^N \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^2)}^{\text{H}_A^1(\mathbb{R}^{3N}; \mathbb{C}^{2^N})} \cong \mathcal{C}_c^\infty(\mathbb{R}^{3N}; \mathbb{C}^{2^N}). \tag{B.1}$$

⁵By \bar{A} we denote the closure of an essentially self-adjoint operator A .

We will instead prove the following lemma. Then (B.1) follows by induction or directly with a proof that is analogous to the proof of the lemma.

B.1 Lemma. *Let $a_j \in L_{loc}^2(\mathbb{R}^{d_1+d_2})$, $1 \leq j \leq k_1 k_2$. We have (up to an isomorphism)*

$$\overline{\mathcal{C}_c^\infty(\mathbb{R}^{d_1}; \mathbb{C}^{k_1}) \otimes \mathcal{C}_c^\infty(\mathbb{R}^{d_2}; \mathbb{C}^{k_2})}^{H_a^1(\mathbb{R}^{d_1+d_2}; \mathbb{C}^{k_1 k_2})} \supseteq \mathcal{C}_c^\infty(\mathbb{R}^{d_1+d_2}; \mathbb{C}^{k_1 k_2}).$$

Proof. Take $f \in \mathcal{C}_c^\infty(\mathbb{R}^{d_1+d_2}; \mathbb{C}^{k_1 k_2})$. Without loss of generality suppose that $\text{supp}(f) \subseteq [-\frac{1}{4}, \frac{1}{4}]^{d_1+d_2}$. Since for $g \in \mathcal{C}_c^\infty(\mathbb{R}^{d_1}; \mathbb{C}^{k_1}) \otimes \mathcal{C}_c^\infty(\mathbb{R}^{d_2}; \mathbb{C}^{k_2})$ we compute

$$\|f - g\|_{H_a^1(\mathbb{R}^{d_1+d_2}; \mathbb{C}^{k_1 k_2})}^2 = \sum_{j=1}^{k_1 k_2} \|f_j - g_j\|_{H_a^1(\mathbb{R}^{d_1+d_2})}^2,$$

where we write $g = (g_1, \dots, g_{k_1 k_2})^T \in \mathcal{C}_c^\infty(\mathbb{R}^{d_1+d_2}; \mathbb{C}^{k_1 k_2})$, it suffices to only consider one component of f .

So fix $1 \leq j \leq k_1 k_2$. Let $(c_n)_n$ be a sequence satisfying

$$\int_{[-1,1]} c_n(1-x^2)^n dx = 1$$

for any $n \in \mathbb{N}$ and define

$$q_n^{(d)} : \mathbb{R}^d \rightarrow \mathbb{R}, \quad x \mapsto \prod_{j=1}^d c_n(1-x_j^2)^n.$$

We then introduce

$$Q_n : \mathcal{C}_0^1([-\frac{1}{2}, \frac{1}{2}]^{d_1+d_2}) \rightarrow \mathcal{C}^1([-\frac{1}{2}, \frac{1}{2}]^{d_1+d_2}), \quad g \mapsto [(q_n^{(d_1)} \otimes q_n^{(d_2)}) \star \tilde{g}]|_{[-\frac{1}{2}, \frac{1}{2}]^{d_1+d_2}},$$

where \tilde{g} is the trivial extension of g to the whole of $\mathbb{R}^{d_1+d_2}$. One can easily check that $(Q_n)_n$ yields an approximate identity (cf. Theorem 7.26, [Rud64]). This then means that $Q_n(f) \rightarrow f$ in $\mathcal{C}^1([-\frac{1}{2}, \frac{1}{2}]^{d_1+d_2})$. In addition to the argument given in [Rud64], we also use

$$\partial_i(Q_n(g)) = Q_n(\partial_i g) \quad \text{for any } 1 \leq i \leq d_1 + d_2$$

to establish convergence in the \mathcal{C}^1 -norm. We even have $\mathcal{R}(Q_n) \subseteq \mathcal{P}^{(d_1+d_2)}$, where $\mathcal{P}^{(d_1+d_2)}$ denotes the polynomials in $d_1 + d_2$ variables.

Now take $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, $\chi|_{[-\frac{1}{3}, \frac{1}{3}]} = 1$, $\chi|_{[-\frac{1}{2}, \frac{1}{2}]^c} = 0$ and $|\chi'| \leq 7$. Define $\eta_m(x) := \prod_{i=1}^{d_m} \chi(x_i)$, $x = (x_1, \dots, x_{d_m}) \in \mathbb{R}^{d_m}$, $m \in \{1, 2\}$. For convenience we abbreviate $K := [-\frac{1}{2}, \frac{1}{2}]^{d_1+d_2}$. Then choose $n \in \mathbb{N}$ sufficiently large such that

$$\|Q_n(f_j|_K) - f_j|_K\|_{\mathcal{C}^1(K)} < \frac{\varepsilon}{\kappa_1}$$

for a suitable choice of κ_1 yet to determine. By abuse of notation we will identify Q_n with its (obvious) extension to $\mathcal{C}_0^1(\mathbb{R}^{d_1+d_2})$. Next we define

$$f_j^{(n)} := (\eta_1 \otimes \eta_2) \cdot Q_n(f_j).$$

Since we have $\mathcal{R}(Q_n) \subseteq \mathcal{P}^{(d_1+d_2)}$, it follows that $f_j^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{d_1}) \otimes \mathcal{C}_c^\infty(\mathbb{R}^{d_2})$. We find

$$\|f_j^{(n)} - f_j\|_{\mathcal{C}^1(\mathbb{R}^{d_1+d_2})} < \frac{\varepsilon}{\kappa_2}$$

for some κ_2 depending on κ_1 . Finally, we arrive at

$$\begin{aligned} \|f_j^{(n)} - f_j\|_{H_A^1(\mathbb{R}^{d_1+d_2})}^2 &= \|(f_j^{(n)} - f_j)\|_2^2 + \|(-i\nabla + a)(f_j^{(n)} - f_j)\|_2^2 \\ &\leq \|f_j^{(n)} - f_j\|_{\mathcal{C}^1}^2 + 2\Re(\langle -i\nabla(f_j^{(n)} - f_j), a(f_j^{(n)} - f_j) \rangle) \\ &\quad + \|a(f_j^{(n)} - f_j)\|_2^2 \\ &\leq 2(1 + \|a\mathbf{1}_K\|_2^2) \|f_j^{(n)} - f_j\|_{\mathcal{C}^1}^2 \\ &\leq \frac{2(1 + \|a\mathbf{1}_K\|_2^2)}{\kappa_2^2} \varepsilon^2 < \varepsilon^2, \end{aligned}$$

where the last inequality holds for an obvious choice of κ_2 , and thus of κ_1 . This concludes the proof. \square

C. Auxiliary results for the localization of a multipolaron

Proof of Equation (3.3). Let $X^{(N)} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N}$. Then we have

$$\begin{aligned} P(X^{(N)}) &= \int \left(\sum_{\pi \in \mathcal{S}_N} \prod_{i=1}^N \chi(x_i - y_{\pi(i)}) \right)^2 dY = \int \left(\sum_{\pi \in \mathcal{S}_N} \prod_{i=1}^N \chi(x_{\pi(i)} - y_i) \right)^2 dY \\ &= \sum_{\pi, \pi' \in \mathcal{S}_N} \prod_{i=1}^N \int \chi(x_{\pi(i)} - y_i) \chi(x_{\pi'(i)} - y_i) dY \\ &= N! \sum_{\pi \in \mathcal{S}_N} \int \prod_{i=1}^N \chi(x_{\pi(i)} - y_i) \chi(x_i - y_i) dY = N! \sum_{\pi \in \mathcal{S}_N} \prod_{i=1}^N M_{i\pi(i)} \\ &= N! \cdot \text{Per}(M). \end{aligned}$$

\square

Proof of inequality (3.9). We will repeat the steps given in the proof of Lemma 4.1 in [LL05]. First of all, we compute

$$\begin{aligned} \nabla_{x_j} [G(\cdot, Y^{(N)}) P(\cdot)^{-1/2}] (X^{(N)}) &= P(X^{(N)})^{-1/2} \nabla_{x_j} (G(\cdot, Y^{(N)})) (X^{(N)}) \\ &\quad - \frac{G(X^{(N)}, Y^{(N)}) (\nabla_{x_j} P)(X^{(N)})}{2P(X^{(N)})^{3/2}}. \end{aligned}$$

Employing $(\nabla_{x_j} P)(X^{(N)}) = 2 \int G(X^{(N)}, Y^{(N)}) [\nabla_{x_j} G(\cdot, Y^{(N)})] (X^{(N)}) dY^{(N)}$ together with $P(X^{(N)}) = \int G(X^{(N)}, Y^{(N)})^2 dY^{(N)}$, we obtain

$$\begin{aligned} F_j(X^{(N)}) &= P(X^{(N)})^{-1} \int |[\nabla_{x_j} G(\cdot, Y^{(N)})] (X^{(N)})|^2 dY^{(N)} - \frac{|(\nabla_{x_j} P)(X^{(N)})|^2}{4P(X^{(N)})^2} \\ &\leq P(X^{(N)})^{-1} \int |[\nabla_{x_j} G(\cdot, Y^{(N)})] (X^{(N)})|^2 dY^{(N)}, \end{aligned} \tag{C.1}$$

where we dropped the last term since we are only interested in an upper bound on F_j . Next we rewrite G by first defining

$$\mu_k(X^{(N,\hat{j})}, Y^{(N,\hat{k})}) := \sum_{\pi \in \mathcal{S}_{N-1}^{\hat{j}, \hat{k}}} \prod_{l \neq j} \chi(x_l - y_{\pi(l)}),$$

where $Z^{(N,\hat{l})} = (z_1, z_2, \dots, \hat{z}_l, \dots, z_N)$ and $\mathcal{S}_{N-1}^{\hat{j}, \hat{k}}$ denotes the set of bijections from $\{1; \dots; \hat{j}; \dots; N\}$ into $\{1; \dots; \hat{k}; \dots; N\}$. We further set $a_k^{(j)}(X^{(N)}, Y^{(N)}) := \chi(x_j - y_k) \mu_k(X^{(N,\hat{j})}, Y^{(N,\hat{k})})$ and obtain

$$G(X^{(N)}, Y^{(N)}) = \sum_{k=1}^N \chi(x_j - y_k) \mu_k(X^{(N,\hat{j})}, Y^{(N,\hat{k})}) = \sum_{k=1}^N a_k^{(j)}(X^{(N)}, Y^{(N)}).$$

This then yields

$$\begin{aligned} \int |\nabla_{x_j} G|^2 dY^{(N)} &= \sum_{k,l=1}^N \int \nabla_{x_j} a_k^{(j)} \cdot \nabla_{x_j} a_l^{(j)} dY^{(N)} \leq N \sum_{k=1}^N \int |\nabla_{x_j} a_k^{(j)}|^2 dY^{(N)} \\ &= N \sum_{k=1}^N \int |(\nabla \chi)(x_j - y_k)|^2 dy_k \int \mu_k(X^{(N,\hat{j})}, Y^{(N,\hat{k})})^2 dY^{(N,\hat{k})}. \end{aligned} \quad (\text{C.2})$$

As in the proof of (3.3), one can easily show

$$\begin{aligned} M_{jj} \int \mu_k(X^{(N,\hat{j})}, Y^{(N,\hat{k})})^2 dY^{(N,\hat{k})} &= (N-1)! \cdot M_{jj} \tilde{M}_{jj} \leq (N-1)! \cdot \text{Per}(M) \\ &\stackrel{(3.3)}{=} P(X^{(N)})/N, \end{aligned} \quad (\text{C.3})$$

where \tilde{M}_{jj} is set to be the cofactor of M_{jj} in the permanent of M . Recall $M_{jj} \tilde{M}_{jj} \leq \text{Per}(M)$ due to $M_{mn} \geq 0$. Since $M_{jj} = 1$ by the choice of χ , from (C.1), (C.2) and (C.3) it follows that

$$F_j(X^{(N)}) \leq \sum_{k=1}^N \int |\nabla \chi|^2 dx = N \int |\nabla \chi|^2 dx$$

as desired. □

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