

# Efficient time integration of the Maxwell-Klein-Gordon equation in the non-relativistic limit regime

Patrick Krämer, Katharina Schratz

CRC Preprint 2016/19, July 2016

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave  
phenomena

## Participating universities



**Universität Stuttgart**

EBERHARD KARLS  
UNIVERSITÄT  
TÜBINGEN



**Funded by**

**DFG**

ISSN 2365-662X

1 Efficient time integration of the Maxwell-Klein-Gordon  
2 equation in the non-relativistic limit regime

3 Patrick Krämer\*, Katharina Schratz

4 Karlsruhe Institute of Technology, Faculty of Mathematics, Englerstr. 2, 76131 Karlsruhe, Germany

---

5 **Abstract**

6 The Maxwell-Klein-Gordon equation describes the interaction of a charged particle with  
7 an electromagnetic field. Solving this equation in the non-relativistic limit regime, i.e.  
8 the speed of light  $c$  formally tending to infinity, is numerically very delicate as the so-  
9 lution becomes highly-oscillatory in time. In order to resolve the oscillations, standard  
10 numerical time integration schemes require severe time step restrictions depending on  
11 the large parameter  $c^2$ .

12 The idea to overcome this numerical challenge is to filter out the high frequencies  
13 explicitly by asymptotically expanding the exact solution with respect to the small pa-  
14 rameter  $c^{-2}$ . This allows us to reduce the highly-oscillatory problem to its corresponding  
15 non-oscillatory Schrödinger-Poisson limit system. On the basis of this expansion we are  
16 then able to construct efficient numerical time integration schemes, which do NOT suffer  
17 from any  $c$ -dependent time step restriction.

18 *Keywords:* Maxwell-Klein-Gordon, time integration, highly-oscillatory, wave equation,  
19 non-relativistic limit

---

20 **1. Introduction**

The Maxwell-Klein-Gordon (MKG) equation describes the motion of a charged particle in an electromagnetic field and the interactions between the field and the particle. The MKG equation can be derived from the linear Klein-Gordon (KG) equation

$$\left(\frac{\partial_t}{c}\right)^2 z - \nabla^2 z + c^2 z = 0 \tag{1}$$

by coupling the scalar field  $z(t, x) \in \mathbb{C}$  to the electromagnetic field via a so-called *minimal substitution* (cf. [17, 24, 25]), i.e.

$$\begin{aligned} \frac{\partial_t}{c} &\rightarrow \frac{\partial_t}{c} + i\frac{\Phi}{c} &=: D_0, \\ \nabla &\rightarrow \nabla - i\frac{\mathbf{A}}{c} &=: D_\alpha, \end{aligned} \tag{2}$$

---

\*Corresponding author

*Email addresses:* patrick.kraemer3@kit.edu (Patrick Krämer), katharina.schratz@kit.edu (Katharina Schratz)

21 where the electromagnetic field is represented by the real Maxwell potentials  $\Phi(t, x) \in \mathbb{R}$   
 22 and  $\mathcal{A}(t, x) \in \mathbb{R}^d$ .

We replace the operators  $\frac{\partial_t}{c}$  and  $\nabla$  in the KG equation (1) by their minimal substitution (2) such that in the so-called Coulomb gauge (cf. [1]), i.e. under the constraint  $\operatorname{div} \mathcal{A} \equiv 0$ , we obtain a KG equation coupled to the electromagnetic field as

$$\begin{cases} \left(\frac{\partial_t}{c} + i\frac{\Phi}{c}\right)^2 z - \left(\nabla - i\frac{\mathcal{A}}{c}\right)^2 z + c^2 z = 0, \\ \partial_{tt} \mathcal{A} - c^2 \Delta \mathcal{A} = c\mathcal{P}[\mathbf{J}], \\ -\Delta \Phi = \rho, \end{cases} \quad (3)$$

for some charge density  $\rho(t, x) \in \mathbb{R}$  and some current density  $\mathbf{J}(t, x) \in \mathbb{R}^d$ , where we define

$$\mathcal{P}[\mathbf{J}] := \mathbf{J} - \nabla \Delta^{-1} \operatorname{div} \mathbf{J}$$

23 the projection of  $\mathbf{J}$  onto its divergence-free part, i.e.  $\operatorname{div} \mathcal{P}[\mathbf{J}] \equiv 0$ .

Setting

$$\rho = \rho[z] := -\operatorname{Re} \left( i\frac{z}{c} \left( \frac{\partial_t}{c} - i\frac{\Phi}{c} \right) \bar{z} \right), \quad \mathbf{J} = \mathbf{J}[z] := \operatorname{Re} \left( iz \left( \nabla + i\frac{\mathcal{A}}{c} \right) \bar{z} \right), \quad (4)$$

where  $z$  solves (3), we find that  $\rho$  and  $\mathbf{J}$  satisfy the continuity equation

$$\partial_t \rho + \operatorname{div} \mathbf{J} = 0. \quad (5)$$

24 For notational simplicity in the following we may also write  $\rho(t, x)$ ,  $\mathbf{J}(t, x)$  instead of  
 25  $\rho[z(t, x)]$  and  $\mathbf{J}[z(t, x)]$ .

The definition of  $\rho$  and  $\mathbf{J}$  in (4) together with the constraint  $\operatorname{div} \mathcal{A}(t, x) \equiv 0$  yields the so-called *Maxwell-Klein-Gordon* equation in the Coulomb gauge

$$\begin{cases} \partial_{tt} z = -c^2(-\Delta + c^2)z + \Phi^2 z - 2i\Phi \partial_t z - iz \partial_t \Phi - 2ic\mathcal{A} \cdot \nabla z - |\mathcal{A}|^2 z, \\ \partial_{tt} \mathcal{A} = c^2 \Delta \mathcal{A} + c\mathcal{P}[\mathbf{J}], \quad \mathbf{J} = \operatorname{Re} (iz \overline{D_\alpha z}), \\ -\Delta \Phi = \rho, \quad \rho = -c^{-1} \operatorname{Re} (iz \overline{D_0 z}), \\ z(0, x) = \varphi(x), \quad D_0 z(0, x) = \sqrt{-\Delta + c^2} \psi(x), \\ \mathcal{A}(0, x) = A(x), \quad \partial_t \mathcal{A}(0, x) = cA'(x), \\ \int_{\mathbb{T}^d} \rho(0, x) dx = 0, \quad \int_{\mathbb{T}^d} \Phi(0, x) dx = 0. \end{cases} \quad (6a)$$

$$\begin{cases} \int_{\mathbb{T}^d} \rho(0, x) dx = 0, \quad \int_{\mathbb{T}^d} \Phi(0, x) dx = 0. \end{cases} \quad (6b)$$

26 Note that for practical implementation issues we assume *periodic boundary conditions*  
 27 (p.b.c.) in space in the above model, i.e.  $x \in \mathbb{T}^d$ . For simplicity we also assume that the  
 28 total charge  $Q(t) := (2\pi)^{-d} \int_{\mathbb{T}^d} \rho(t, x) dx$  at time  $t = 0$  is zero, i.e.  $Q(0) = 0$ . Also due to  
 29 the constraint  $\operatorname{div} \mathcal{A}(t, x) \equiv 0$  we assume that the initial data  $A, A'$  for  $\mathcal{A}$  are divergence-  
 30 free. Finally, the following assumption guarantees strongly well-prepared initial data.  
 31 However, approximation results also hold true under weaker initial assumptions, see for  
 32 instance [21].

33 **Assumption 1.** *The initial data  $\varphi, \psi, A, A'$  is independent of  $c$ .*

34 **Remark 1.** Note that the continuity equation (5) together with the initial assumption  
 35  $Q(0) = 0$  implies that for all  $t$  we have  $\int_{\mathbb{T}^d} \rho(t, x) dx = \int_{\mathbb{T}^d} \rho(0, x) dx = 0$ . This yields the  
 36 first condition in (6b).

37 **Remark 2.** Up to minor changes, all the results of this paper remain valid for Dirichlet  
 38 boundary conditions instead of periodic boundary conditions.

**Remark 3.** Note that the MKG system (6) is invariant under the gauge transform  
 $(z, \Phi, \mathcal{A}) \mapsto (z', \Phi', \mathcal{A}')$ , where for a suitable choice of  $\chi = \chi(t, x)$  we set

$$\Phi' := \Phi + \partial_t \chi, \quad \mathcal{A}' := \mathcal{A} - c \nabla \chi, \quad z' := z \exp(-i\chi),$$

39 i.e. if  $(z, \Phi, \mathcal{A})$  solves the MKG system (6) then also does  $(z', \Phi', \mathcal{A}')$  without modification  
 40 of the system (cf. [1, 11, 24, 25]). Henceforth, the second condition in (6b) holds  
 41 without loss of generality: If  $0 \neq (2\pi)^{-d} \int_{\mathbb{T}^d} \Phi(t, x) dx =: M(t) \in \mathbb{R}$ , we choose  $\chi$  as  
 42  $\chi(t, x) = \chi(t) = -(M(0) + \int_0^t M(\tau) d\tau)$ , such that (6b) is satisfied for  $\Phi'$ .

43 For more physical details on the derivation of the MKG equation, on Maxwell's po-  
 44 tentials, gauge theory formalisms and many more related topics we refer to [1, 11, 12,  
 45 17, 24, 25] and the references therein.

46  
 47 Here we are interested in the so-called non-relativistic limit regime  $c \gg 1$  of the MKG  
 48 system (6). In this regime the numerical time integration becomes severely challenging  
 49 due to the highly-oscillatory behaviour of the solution. In order to resolve these high  
 50 oscillations standard numerical schemes require severe time step restrictions depending  
 51 on the large parameter  $c^2$ , which leads to a huge computational effort. This numerical  
 52 challenge has lately been extensively studied for the nonlinear Klein-Gordon (KG) equa-  
 53 tion, see [2, 3, 8, 14]. In particular it was pointed out that a Gautschi-type exponential  
 54 integrator only allows convergence under the constraint that the time step size is of order  
 55  $\mathcal{O}(c^{-2})$  (cf. [3]).

In this paper we construct numerical schemes for (6) which do not suffer from any  
 $c$ -dependent time step restriction. Our strategy is thereby similar to [2, 14] where the  
 Klein-Gordon equation is considered: In a first step we expand the exact solution into  
 a formal asymptotic expansion in terms of  $c^{-2}$  for  $z, \Phi$  and in terms of  $c^{-1}$  for  $\mathcal{A}$ .  
 This allows us to filter out the high oscillations in the solution explicitly. Therefore we  
 can break down the numerical task to only solving the corresponding non-oscillatory  
 Schrödinger-Poisson limit system. The latter can be carried out very efficiently without  
 imposing any CFL type condition on  $c$  nor the spatial grid size. This construction is  
 based on the *Modulated Fourier Expansion* (MFE) of the exact solution in terms of the  
 small parameter  $c^{-l}$ ,  $l \geq 1$ , see for instance [10, 14], [15, Chapter XIII] and the references  
 therein. However, as in [14] we control the expansion by computing the coefficients of the  
 MFE directly and in particular exploit the results in [6, 21] on the asymptotic behaviour  
 of the exact solution of the MKG equation (6). More precisely, formally the following  
 approximations hold

$$\begin{aligned} z(t, x) &= \frac{1}{2} (u_0(t, x) \exp(ic^2 t) + \bar{v}_0(t, x) \exp(-ic^2 t)) + \mathcal{O}(c^{-2}), \\ \mathcal{A}(t, x) &= \cos(c\sqrt{-\Delta}t)A(x) + \sqrt{-\Delta}^{-1} \sin(c\sqrt{-\Delta}t)A'(x) + \mathcal{O}(c^{-1}), \end{aligned} \tag{7}$$

where  $u_0$  and  $v_0$  solve the Schrödinger-Poisson (SP) system

$$\begin{cases} i\partial_t u_0 = \frac{1}{2}\Delta u_0 + \Phi_0 u_0, & u_0(0) = \varphi - i\psi, \\ i\partial_t v_0 = \frac{1}{2}\Delta v_0 - \Phi_0 v_0, & v_0(0) = \bar{\varphi} - i\bar{\psi}, \\ -\Delta\Phi_0 = -\frac{1}{4}(|u_0|^2 - |v_0|^2), & \int_{\mathbb{T}^d} \Phi_0(t, x) dx = 0. \end{cases} \quad (8)$$

**Remark 4.** The  $L^2$  conservation of  $u_0, v_0$  together with the choice  $Q(0) = 0$  yields that

$$\int_{\mathbb{T}^d} |u_0(t, x)|^2 - |v_0(t, x)|^2 dx = \int_{\mathbb{T}^d} |u_0(0, x)|^2 - |v_0(0, x)|^2 dx = 0.$$

56 Here we point out that in the asymptotic expansion (7) the highly-oscillatory nature  
 57 of the solution is only contained in the high-frequency terms  $\exp(\pm ic^2 t)$  and  $\cos(c\sqrt{-\Delta}t)$ ,  
 58  $\sin(c\sqrt{-\Delta}t)$ , respectively. In particular the SP system (8) does not depend on the large  
 59 parameter  $c$ . Henceforth, the expansion (7) allows us to derive an efficient and fast numerical  
 60 approximation without any  $c$ -dependent time step restriction: We only need to  
 61 solve the non-oscillatory SP system numerically and multiply the numerical approxima-  
 62 tions to the SP solution with the highly-oscillatory phases.

63 After a full discretization using for instance the second-order Strang splitting scheme  
 64 for the time discretization of the SP system (8) (see [20]) with time step size  $\tau$  and a  
 65 Fourier pseudospectral (FP) method for the space discretization with mesh size  $h$ , the  
 66 resulting numerical schemes then approximate the exact solution of the MKG equation  
 67 up to error terms of order  $\mathcal{O}(c^{-2} + \tau^2 + h^s)$  for  $z, \Phi$  and  $\mathcal{O}(c^{-1} + h^s)$  for  $\mathcal{A}$  respectively.

68 The main advantage here is that we can choose  $\tau$  and  $h$  independently of the large  
 69 parameter  $c$ . The value of  $s$  depends on the smoothness of the solution. We will discuss  
 70 the numerical scheme in more detail later on in Section 5.

71 **Remark 5.** Under additional smoothness assumptions on the initial data we can also  
 72 carry out the asymptotic expansion up to higher order terms in  $c^{-l}$ . In particular, every  
 73 term in this expansion can be easily computed numerically as the high oscillations can  
 74 be filtered out explicitly.

75 If we consider other boundary conditions, such as for example Dirichlet or Neumann  
 76 boundary conditions it may be favorable to use a finite element (FEM) space discretiza-  
 77 tion or a sine pseudospectral discretization method instead of the FP method. For details  
 78 on the convergence of a FEM applied to the MKG equation in the so-called temporal  
 79 gauge, see for instance [9] and references therein.

80 For further results on the construction of efficient methods on related Klein-Gordon  
 81 type equations in the non-relativistic limit regime we refer to [2-5, 8].

## 82 2. A priori bounds

We follow the strategy presented in [14, 21]: Firstly, we rewrite the MKG equation (6) as a first order system. Therefore, for a given  $c$  we introduce the operator

$$\langle \nabla \rangle_c := \sqrt{-\Delta + c^2},$$

83 which in Fourier space can be written as a diagonal operator  $(\langle \nabla \rangle_c)_{k\ell} = \delta_{k\ell} \sqrt{|k|^2 + c^2}$ ,  
84  $k, \ell \in \mathbb{Z}^d$ , where  $\delta_{k\ell}$  denotes the Kronecker symbol. By Taylor series expansion of  
85  $\sqrt{1+x}^{-1}$  we can easily see that for all  $k \in \mathbb{Z}^d$  there holds  $|(c \langle \nabla \rangle_c^{-1})_{kk}| \leq 1$ , i.e.  $c \langle \nabla \rangle_c^{-1}$   
86 is uniformly bounded with respect to  $c$ . In particular, there holds  $\|c \langle \nabla \rangle_c^{-1} u\|_s \leq \|u\|_s$ ,  
87 where  $\|\cdot\|_s$  denotes the standard Sobolev norm corresponding to the function space  
88  $H^s := H^s(\mathbb{T}^d, \mathbb{C})$ .

In order to rewrite the equation for  $z$  in (6) as a first order system we set

$$u = z - i \langle \nabla \rangle_c^{-1} D_0 z, \quad v = \bar{z} - i \langle \nabla \rangle_c^{-1} \overline{D_0 z}, \quad (9)$$

as proposed in [21]. By the definition of  $D_0 z = c^{-1}(\partial_t + i\Phi)z$  and since  $\Phi$  is real we have that  $z = \frac{1}{2}(u + \bar{v})$ . We define the abbreviations

$$\begin{aligned} \mathcal{N}_u[u, v, \Phi, \mathcal{A}] &:= -\frac{i}{2}(\Phi + \langle \nabla \rangle_c^{-1} \Phi \langle \nabla \rangle_c)u - \frac{i}{2}(\Phi - \langle \nabla \rangle_c^{-1} \Phi \langle \nabla \rangle_c)\bar{v} \\ &\quad + ic^{-1} \langle \nabla \rangle_c^{-1} \left( |\mathcal{A}|^2 \frac{1}{2}(u + \bar{v}) \right) - \langle \nabla \rangle_c^{-1} (\mathcal{A} \cdot \nabla(u + \bar{v})) \end{aligned} \quad (10)$$

89 and  $\mathcal{N}_v[u, v, \Phi, \mathcal{A}] := \mathcal{N}_u[v, u, -\Phi, -\mathcal{A}]$ . Differentiating  $u$  and  $v$  in (9) with respect to  $t$   
90 we obtain the system

$$\begin{cases} i\partial_t u = -c \langle \nabla \rangle_c u + i\mathcal{N}_u[u, v, \Phi, \mathcal{A}], & u(0) = \varphi - i\psi \\ i\partial_t v = -c \langle \nabla \rangle_c v + i\mathcal{N}_v[u, v, \Phi, \mathcal{A}], & v(0) = \bar{\varphi} - i\bar{\psi}, \\ -\Delta \Phi = \rho[u, v], \\ \partial_{tt} \mathcal{A} = c^2 \Delta \mathcal{A} + c\mathcal{P}[\mathcal{J}[u, v, \mathcal{A}]], & \mathcal{A}(0) = A, \partial_t \mathcal{A}(0) = cA' \end{cases} \quad (11)$$

where the definition of  $u(0), v(0)$  follows from the ansatz (9) together with the initial data  $\varphi, \psi, A, A'$  in (6). Furthermore since  $z = \frac{1}{2}(u + \bar{v})$  we have by (6) that

$$\begin{aligned} \rho[u, v] &= -\frac{1}{4} \operatorname{Re} \left( (u + \bar{v})c^{-1} \langle \nabla \rangle_c (\bar{u} - v) \right), \\ \mathcal{J}[u, v, \mathcal{A}] &= \frac{1}{4} \operatorname{Re} \left( i(u + \bar{v})\nabla(\bar{u} + v) - \frac{\mathcal{A}}{c} |u + \bar{v}|^2 \right). \end{aligned} \quad (12)$$

Setting  $T_c(t) = \exp(ic \langle \nabla \rangle_c t)$  we can formulate the mild solutions of (11) as

$$\begin{aligned} u(t) &= T_c(t)u(0) + \int_0^t T_c(t-\tau) \mathcal{N}_u[u, v, \Phi, \mathcal{A}](\tau) d\tau, \\ v(t) &= T_c(t)v(0) + \int_0^t T_c(t-\tau) \mathcal{N}_v[u, v, \Phi, \mathcal{A}](\tau) d\tau, \\ \mathcal{A}(t) &= \cos(c \langle \nabla \rangle_0 t) \mathcal{A}(0) + (c \langle \nabla \rangle_0)^{-1} \sin(c \langle \nabla \rangle_0 t) \partial_t \mathcal{A}(0) \\ &\quad + \langle \nabla \rangle_0^{-1} \int_0^t \sin(c \langle \nabla \rangle_0 (t-\tau)) \mathcal{P}[\mathcal{J}[u, v, \mathcal{A}](\tau)] d\tau, \end{aligned} \quad (13)$$

where we define  $\exp(ic \langle \nabla \rangle_c t)w$ ,  $\cos(c \langle \nabla \rangle_0 t)w$  and  $c^{-1} \langle \nabla \rangle_0^{-1} \sin(c \langle \nabla \rangle_0 t)w$  for  $w \in H^s$  in Fourier space as follows: Let  $\hat{w}_k = (\mathcal{F}w)_k$  denote the  $k$ -th Fourier coefficient of  $w$ .

Then we have for all  $k \in \mathbb{Z}^d$

$$\begin{aligned} (\mathcal{F}[\exp(ic \langle \nabla \rangle_c t)w])_k &= \exp\left(ict\sqrt{|k|^2 + c^2}\right) \hat{w}_k, \\ (\mathcal{F}[\cos(c \langle \nabla \rangle_0 t)w])_k &= \cos(c|k|t) \hat{w}_k, \\ (\mathcal{F}[(c \langle \nabla \rangle_0)^{-1} \sin(c \langle \nabla \rangle_0 t)w])_k &= t \operatorname{sinc}(c|k|t) \hat{w}_k. \end{aligned}$$

Since the Fourier transform is an isometry in  $H^s$  it follows easily, that the operators  $\cos(c \langle \nabla \rangle_0 t)$  and  $\sin(c \langle \nabla \rangle_0 t)$  are uniformly bounded with respect to  $c$  and that  $\exp(ic \langle \nabla \rangle_c t)$  is an isometry in  $H^s$ , i.e. for all  $w \in H^s$  and for all  $t \in \mathbb{R}$  we have

$$\|\exp(ic \langle \nabla \rangle_c t)w\|_s = \|w\|_s, \quad \|\cos(c \langle \nabla \rangle_0 t)w\|_s \leq \|w\|_s, \quad \left\| \frac{\sin(c \langle \nabla \rangle_0 t)}{c \langle \nabla \rangle_0} w \right\|_s \leq t \|w\|_s. \quad (14)$$

As the nonlinearities  $\mathcal{N}_u$  and  $\mathcal{N}_v$  in the system (11) involve products of  $u, v, \Phi, \mathcal{A}$  we will exploit the standard bilinear estimates in  $H^s$ : For  $s > d/2$  we have

$$\|uv\|_s \leq C_s \|u\|_s \|v\|_s \quad (15)$$

91 for some constant  $C_s$  depending only on  $s$  and  $d$ .

In the following we assume that  $s > d/2$ . By representation in Fourier space we see that for  $w \in H^{s'}$ ,  $s' = \max\{s, s+m\}$ ,  $m \in \mathbb{Z}$  there holds

$$\|\langle \nabla \rangle_1^m w\|_s \leq C_{s,m} \|w\|_{s+m}. \quad (16)$$

Thus, (15) and (16) yield for  $w \in H^s, \Phi \in H^{s+2}$

$$\begin{aligned} \left\| \langle \nabla \rangle_c^{-1} (\Phi \langle \nabla \rangle_c w) \right\|_s &\leq C_1 \left\| \langle \nabla \rangle_c^{-1} (\Phi \langle \nabla \rangle_0 w) \right\|_s + C_2 \left\| c \langle \nabla \rangle_c^{-1} (\Phi w) \right\|_s \\ &\leq C \|\Phi\|_{s+2} \|w\|_s, \end{aligned} \quad (17)$$

since (16) implies that for all  $\tilde{w} \in H^s$  and  $c \geq 1$  we find a constant  $C$  such that

$$\left\| \langle \nabla \rangle_c^{-1} \tilde{w} \right\|_s \leq \left\| \langle \nabla \rangle_1^{-1} \tilde{w} \right\|_s \leq C \|\tilde{w}\|_{s-1}.$$

After a short calculation we find that for  $u_j, v_j, \mathcal{A}_j \in H^s, \Phi_j \in H^{s+2}$ ,  $j = 1, 2$  there holds, with  $\mathcal{N} = \mathcal{N}_u$  and  $\mathcal{N} = \mathcal{N}_v$  respectively, that

$$\begin{aligned} &\|\mathcal{N}[u_1, v_1, \Phi_1, \mathcal{A}_1] - \mathcal{N}[u_2, v_2, \Phi_2, \mathcal{A}_2]\|_s \\ &\leq K_{\mathcal{N}}(\|u_1 - u_2\|_s + \|v_1 - v_2\|_s + \|\Phi_1 - \Phi_2\|_{s+2} + \|\mathcal{A}_1 - \mathcal{A}_2\|_s) \end{aligned}$$

and

$$\begin{aligned} &\left\| \langle \nabla \rangle_0^{-1} (\mathcal{J}[u_1, v_1, \mathcal{A}_1] - \mathcal{J}[u_2, v_2, \mathcal{A}_2]) \right\|_s \\ &\leq K_{\mathcal{J}}(\|u_1 - u_2\|_s + \|v_1 - v_2\|_s + \|\mathcal{A}_1 - \mathcal{A}_2\|_s), \end{aligned}$$

92 where the constants  $K_{\mathcal{N}}$  and  $K_{\mathcal{J}}$  only depend on  $\|u_j\|_s, \|v_j\|_s, \|\Phi_j\|_{s+2}, \|\mathcal{A}_j\|_s, j = 1, 2$ .

Together with (14) a standard fix point argument now implies immediately local well-posedness in  $H^s$ ,  $s > d/2$  (see for instance [13, Theorem III.7]), i.e. for initial data  $u(0), v(0), \mathcal{A}(0) \in H^s, \partial_t \mathcal{A}(0) \in H^{s-1}$  there exists  $T_s > 0$  and a constant  $B_s > 0$  such that

$$\|u(t)\|_s + \|v(t)\|_s + \|\Phi(t)\|_{s+2} + \|\mathcal{A}(t)\|_s \leq B_s, \quad \forall t \in [0, T_s]. \quad (18)$$

93 For local and global well-posedness results on the MKG equation in other gauges, e.g. in  
94 Lorentz gauge, and low regularity spaces we refer to [18, 21, 26] and references therein.



95 **3. Formal asymptotic expansion**

96 In this section we formally derive the Schrödinger-Poisson system (8) as the non-  
 97 relativistic limit of the MKG equation (6), i.e. we formally motivate the expansion (7).  
 98 For a detailed rigorous analysis in low regularity spaces we refer to [6, 21] and references  
 99 therein; results on asymptotics of related systems such as the Maxwell-Dirac system can  
 100 be found in [7, 21].

On the  $c$ -independent finite time interval  $[0, T]$  we now look, at first formally, for a solution  $(u, v, \Phi, \mathcal{A})$  of (6) in the form of a Modulated Fourier expansion (cf. [15, Chapter XIII]), i.e. we make the ansatz

$$\begin{aligned} u(t, x) = U(t, \theta, x) &= \sum_{n=0}^{\infty} c^{-2n} U_n(t, \theta, x), & v(t, x) = V(t, \theta, x) &= \sum_{n=0}^{\infty} c^{-2n} V_n(t, \theta, x), \\ \Phi(t, x) = \tilde{\Phi}(t, \theta, x) &= \sum_{n=0}^{\infty} c^{-2n} \Phi_n(t, \theta, x), & \mathcal{A}(t, x) = \mathfrak{A}(t, \sigma, x) &= \sum_{n=0}^{\infty} c^{-n} \mathcal{A}_n(t, \sigma, x), \end{aligned} \quad (19)$$

101 where  $\sigma = ct$ ,  $\theta = c^2 t$  are fast time scales which are used to separate the high oscillations  
 102 from the slow time dependency of the solution. Next we apply the so-called method of  
 103 multiple scales to  $U, V, \tilde{\Phi}$  and  $\mathfrak{A}$ , where the idea is to treat the time scales  $t, \sigma$  and  $\theta$   
 104 as independent variables. This allows us to derive a sequence of equations for the MFE  
 105 coefficients  $U_n, V_n, \Phi_n, \mathcal{A}_n$ ,  $n \geq 0$  and henceforth to determine the asymptotic expansion  
 106 (19). For more details on the method of multiple scales and perturbation theory we refer  
 107 to [19, 22, 23].

We start off by plugging the ansatz (19) into (11) and obtain for  $W = (U, V)^T$  the equation

$$\partial_t W + c^2 \partial_\theta W = ic \langle \nabla \rangle_c W + \begin{pmatrix} \mathcal{N}_u(U, V, \tilde{\Phi}, \mathfrak{A}) \\ \mathcal{N}_v(U, V, \tilde{\Phi}, \mathfrak{A}) \end{pmatrix} \quad (20)$$

with initial condition

$$U(0, 0, x) = \varphi(x) - i\psi(x), \quad V(0, 0, x) = \overline{\varphi(x)} - i\overline{\psi(x)} \quad (21)$$

and an equation for  $\mathfrak{A}$  in terms of  $t$  and  $\sigma$ , i.e.

$$\partial_{tt} \mathfrak{A} + 2c \partial_\sigma \partial_t \mathfrak{A} + c^2 \partial_{\sigma\sigma} \mathfrak{A} = c^2 \Delta \mathfrak{A} + c \mathcal{P} [J[U, V, \mathfrak{A}]] \quad (22)$$

with initial condition

$$(\mathfrak{A}(0, 0, x), (\partial_t + c \partial_\sigma) \mathfrak{A}(0, 0, x)) = (\mathcal{A}(0, x), \partial_t \mathcal{A}(0, x)).$$

For the potential  $\tilde{\Phi}$  we find the equation

$$-\Delta \tilde{\Phi} = -\frac{1}{4} \operatorname{Re} ((U + \bar{V}) c^{-1} \langle \nabla \rangle_c (\bar{U} - V)). \quad (23)$$

In the next step we expand the operators  $\langle \nabla \rangle_c$  and  $\langle \nabla \rangle_c^{-1}$  into their Taylor series expansion. For  $w$  sufficiently smooth we have

$$c \langle \nabla \rangle_c w = (c^2 - \frac{1}{2} \Delta - c^{-2} \frac{1}{8} \Delta^2 + \sum_{n \geq 2} \alpha_{n+1} c^{-2n} (-\Delta)^{n+1}) w. \quad (24)$$

Similarly, we find

$$c \langle \nabla \rangle_c^{-1} w = (1 + c^{-2} \frac{1}{2} \Delta + \sum_{n \geq 2} \beta_n c^{-2n} (-\Delta)^n) w. \quad (25)$$

Now (24) and (25) yield for  $\Psi, w \in H^{s+2}$

$$\langle \nabla \rangle_c^{-1} \Psi \langle \nabla \rangle_c w = \Psi w + \mathcal{O}(c^{-2} [\Delta, \Psi] w) \quad (26)$$

108 in the sense of the  $H^s$  norm and where  $[A, B] := AB - BA$  denotes the commutator of  
109 the operators  $A$  and  $B$ , i.e.  $[\Delta, \Psi] w = \Delta(\Psi w) - \Psi(\Delta w)$ .

Since  $\varphi$  and  $\psi$  are independent of  $c$ , the ansatz (19) yields by (21) that

$$\begin{aligned} U_0(0, 0, x) &= \varphi(x) - i\psi(x), & U_n(0, 0, x) &= 0, n \geq 1, \\ V_0(0, 0, x) &= \overline{\varphi(x) - i\psi(x)}, & V_n(0, 0, x) &= 0, n \geq 1. \end{aligned} \quad (27)$$

110 Now the idea is to compare the coefficients of the left- and right-hand side of (20) with  
111 respect to equal powers of  $c$  by plugging the ansatz (19) and the expansions (24), (25)  
112 and (26) into the equation. This finally yields a sequence of partial differential equations  
113 at each order of  $c$ .

At order  $c^2$  we obtain

$$\begin{cases} (\partial_\theta - i)U_0(t, \theta, x) = 0, \\ (\partial_\theta - i)V_0(t, \theta, x) = 0, \end{cases}$$

which allows solutions of the form

$$U_0(t, \theta, x) = \exp(i\theta)u_0(t, x), \quad V_0(t, \theta, x) = \exp(i\theta)v_0(t, x) \quad (28)$$

114 where  $u_0, v_0$  will be determined in the next step.

Plugging (28) into (23) we obtain the first term  $\Phi_0$  in the expansion (19) of  $\tilde{\Phi}$  as the solution of the Poisson equation

$$-\Delta \Phi_0(t, \theta, x) = -\Delta \Phi_0(t, x) = -\frac{1}{4} (|u_0(t, x)|^2 - |v_0(t, x)|^2). \quad (29)$$

At order  $c^0$  we use (28) and obtain the equations

$$\begin{cases} (\partial_\theta - i)U_1(t, \theta, x) = \exp(i\theta) \left( -\partial_t u_0(t, x) - \frac{i}{2} \Delta u_0(t, x) - i\Phi_0(t, x)u_0(t, x) \right) \\ (\partial_\theta - i)V_1(t, \theta, x) = \exp(i\theta) \left( -\partial_t v_0(t, x) - \frac{i}{2} \Delta v_0(t, x) + i\Phi_0(t, x)v_0(t, x) \right). \end{cases} \quad (30)$$

Since  $\exp(i\theta)$  lies in the kernel of the operator  $(\partial_\theta - i)$  and since  $u_0, v_0, \Phi_0$  are independent of  $\theta$ , we demand  $u_0$  and  $v_0$  to satisfy

$$\begin{cases} i\partial_t u_0(t, x) = \frac{1}{2} \Delta u_0(t, x) + \Phi_0(t, x)u_0(t, x), \\ i\partial_t v_0(t, x) = \frac{1}{2} \Delta v_0(t, x) - \Phi_0(t, x)v_0(t, x), \end{cases} \quad (31)$$

115 with initial data  $u_0(0, x) = \varphi(x) - i\psi(x)$ , and  $v_0(0, x) = \overline{\varphi(x) - i\psi(x)}$ .

As  $u_0, v_0$  satisfy (31), we can proceed as above: (30) allows solutions of the form

$$U_1(t, \theta, x) = \exp(i\theta)u_1(t, x), \quad V_1(t, \theta, x) = \exp(i\theta)v_1(t, x),$$

116 where we can determine  $u_1$  and  $v_1$  by considering the equation arising at order  $c^{-2}$ . In  
117 the same way the coefficients  $U_n, V_n, n \geq 2$  can be obtained.

In this paper we will only treat the expansion (19) up to its first term at order  $c^0$ . Therefore, in the following we set

$$z_0(t, x) = \frac{1}{2}(\exp(ic^2t)u_0(t, x) + \exp(-ic^2t)\bar{v}_0(t, x)). \quad (32)$$

Then, by the above procedure we know that at least formally the approximation

$$\|z(t, x) - z_0(t, x)\|_s \leq Kc^{-2}$$

118 holds for sufficiently smooth data. In Section 4 below we will state the precise regularity  
119 assumptions and give the ideas of the convergence proof. For a rigorous analysis we refer  
120 to [6, 21] and references therein.

Next we repeat the same procedure with equation (22) for the MFE coefficients of  $\mathcal{A}$ . As  $\mathcal{A}$  is a real vector field we look for real coefficients  $\mathcal{A}_n, n \geq 0$ . At order  $c^2$  we find the homogeneous equation

$$(\partial_{\sigma\sigma} - \Delta)\mathcal{A}_0(t, \sigma, x) = 0, \quad (33)$$

which allows solutions of the form

$$\mathcal{A}_0(t, \sigma, x) = \cos(\sigma\sqrt{-\Delta})a_0(t, x) + \sqrt{-\Delta}^{-1} \sin(\sigma\sqrt{-\Delta})b_0(t, x) \quad (34)$$

121 with some  $a_0, b_0$  that will be determined in the next step.

The equation arising from the comparison of the terms at order  $c^1$  reads

$$(\partial_{\sigma\sigma} - \Delta)\mathcal{A}_1 = -2\partial_\sigma\partial_t\mathcal{A}_0 + \frac{1}{4}\mathcal{P} [\operatorname{Re}(i(U_0 + \bar{V}_0)\nabla(\bar{U}_0 + V_0))].$$

As the term

$$\partial_\sigma\partial_t\mathcal{A}_0(t, \sigma, x) = -\sin(\sigma\sqrt{-\Delta})\sqrt{-\Delta}\partial_t a_0(t, x) + \cos(\sigma\sqrt{-\Delta})\partial_t b_0(t, x)$$

lies in the kernel of the operator  $(\partial_{\sigma\sigma} - \Delta)$  we demand by the same argumentation as before that  $\partial_\sigma\partial_t\mathcal{A}_0(t, \sigma, x) = 0$ . This in particular implies that  $\partial_t a_0(t, x) = 0$  and  $\partial_t b_0(t, x) = 0$ . Hence  $\partial_t\mathcal{A}_0(t, \sigma, x) \equiv 0$  and we find

$$\mathcal{A}_0(t, \sigma, x) = \mathcal{A}_0(\sigma, x) \quad \text{and} \quad a_0(t, x) = a_0(x), \quad b_0(t, x) = b_0(x).$$

At  $\sigma = 0$  we find  $a_0(x) = \mathcal{A}_0(0, x)$  and by differentiation of  $\mathcal{A}_0$  with respect to  $\sigma$  we obtain  $b_0(x) = \partial_\sigma\mathcal{A}_0(0, x)$ . The data  $\mathcal{A}_0(0, x)$  and  $\partial_\sigma\mathcal{A}_0(0, x)$  are again determined via comparison of coefficients: the initial data of  $\mathcal{A}$  in (6) are given as

$$\mathcal{A}(0, x) = A(x), \quad \partial_t\mathcal{A}(0, x) = cA'(x),$$

where  $A, A'$  do not depend on  $c$ . Hence, the formal asymptotic expansion

$$\mathcal{A}(t=0, x) = \mathcal{A}_0(\sigma=0, x) + \sum_{n \geq 1} c^{-n} \mathcal{A}_n(t=0, \sigma=0, x)$$

yields that

$$a_0(x) = \mathcal{A}_0(0, x) = A(x). \quad (35)$$

Since

$$cA'(x) = \partial_t \mathcal{A}(0, x) \simeq (\partial_t + c\partial_\sigma) \mathcal{A}(0, 0, x) = c\partial_\sigma \mathcal{A}_0(0, x) + \sum_{n \geq 1} c^{-n} (\partial_t + c\partial_\sigma) \mathcal{A}_n(0, 0, x)$$

we choose

$$b_0(x) = \partial_\sigma \mathcal{A}_0(0, x) = A'(x). \quad (36)$$

Finally by (34), (35) and (36) we obtain the first term of the expansion as

$$\mathcal{A}_0(t, x) = \cos(ct\sqrt{-\Delta})A(x) + (c\sqrt{-\Delta})^{-1} \sin(ct\sqrt{-\Delta})cA'(t, x). \quad (37)$$

122 We remark that at this point we can explicitly evaluate the first term  $\mathcal{A}_0(t, x)$  of the  
123 MFE of  $\mathcal{A}$  for all  $t \in [0, T]$ .

Collecting the results in (29), (31) and (37) yields the non-relativistic limit Schrödinger-Poisson system as in [21], i.e.

$$\begin{cases} i\partial_t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{2}\Delta \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \Phi_0 \begin{pmatrix} u_0 \\ -v_0 \end{pmatrix}, & \begin{pmatrix} u_0(0) \\ v_0(0) \end{pmatrix} = \begin{pmatrix} \varphi - i\psi \\ \bar{\varphi} - i\bar{\psi} \end{pmatrix}, \\ -\Delta \Phi_0 = -\frac{1}{4}(|u_0|^2 - |v_0|^2), & \int_{\mathbb{T}^d} \Phi_0(t, x) dx = 0. \\ \mathcal{A}_0(t, x) = \cos(ct\sqrt{-\Delta})A(x) + (c\sqrt{-\Delta})^{-1} \sin(ct\sqrt{-\Delta})cA'(x). \end{cases} \quad (38)$$

124 The numerical advantage of the above approximation lies in the fact that compared to  
125 the challenging highly-oscillatory MKG system (6), the SP system (38) can be solved  
126 very efficiently (for example by applying a Strang splitting method, see [20]), without  
127 imposing any CFL type condition on  $c$  nor the spatial discretization parameter  $h$ . Details  
128 will be given in Section 5 below.

#### 129 4. Error bounds

130 In the following, let  $(u, v, \mathcal{A}, \Phi)$  denote the solution of the first order MKG system  
131 (11) and let  $(u_0, v_0, \Phi_0, \mathcal{A}_0)$  denote the solution of the corresponding limit system (8)  
132 with initial data  $\varphi, \psi, A, A'$ , where the limit vector potential  $\mathcal{A}_0$  is given by (37).

133 The following Theorem states rigorous error bounds on the asymptotic approxima-  
134 tions  $z_0, \Phi_0$  and  $\mathcal{A}_0$  towards  $z, \Phi$  and  $\mathcal{A}$ , where  $z_0$  is defined in (32). For a detailed  
135 analysis and bounds in low regularity spaces we refer to [6, 21]. Here, we will only  
136 outline the ideas of the proof.

**Theorem 1** (cf. [6, 21]). *Let  $s > d/2$  and let  $\varphi, \psi \in H^{s+4}, A \in H^{s+1}, A' \in H^s$ . Then there exists a  $T > 0$  such that for all  $t \in [0, T]$  and  $c \geq 1$  there holds*

$$\begin{aligned} \|z(t) - z_0(t)\|_s + \|\Delta(\Phi(t) - \Phi_0(t))\|_s &\leq c^{-2}(1 + K_{\Phi}^T)b(T)\exp(\lambda(T)) \\ \|\mathcal{A}(t) - \mathcal{A}_0(t)\|_s &\leq c^{-1}(K_{\mathcal{A}.1}^T + TK_{\mathcal{A}.2}^T), \end{aligned}$$

where

$$b(t) = M_0^T + tM_1^T + t^2M_2^T, \quad \lambda(t) = M_3^T + tM_4^T$$

with constants  $K_{\Phi}^T, K_{\mathcal{A}.1}^T, K_{\mathcal{A}.2}^T, M_0^T, \dots, M_4^T$  only depending on  $\|\varphi\|_{s+4}, \|\psi\|_{s+4}, \|A\|_{s+1}, \|A'\|_s$  as well as on

$$K = \sup_{\tau \in [0, T]} \{ \|\mathcal{A}(\tau)\|_s + \|u(\tau)\|_{s+2} + \|v(\tau)\|_{s+2} + \|u_0(\tau)\|_{s+4} + \|v_0(\tau)\|_{s+4} \}.$$

We outline the ideas in the proof in several steps. Note that since

$$z(t) = \frac{1}{2}(u(t) + \bar{v}(t)) \quad \text{and} \quad z_0(t) = \frac{1}{2}(\exp(ic^2t)u_0(t) + \exp(-ic^2t)\bar{v}_0(t))$$

the triangle inequality allows us to break down the problem as follows:

$$\|z(t) - z_0(t)\|_s \leq \|u(t) - \exp(ic^2t)u_0(t)\|_s + \|v(t) - \exp(ic^2t)v_0(t)\|_s =: \mathcal{R}(t). \quad (39)$$

137 We start with the following proposition.

**Proposition 1** (cf. [21]). *Under the assumptions of Theorem 1 for all  $t \in [0, T]$  there holds that*

$$\|\Delta(\Phi(t) - \Phi_0(t))\|_s \leq c^{-2}K_{\Phi.1}^T + K_{\Phi.2}^T\mathcal{R}(t),$$

138 where  $K_{\Phi.1}^T, K_{\Phi.2}^T$  depend on  $\sup_{\tau \in [0, T]} \{ \|u(\tau)\|_{s+2} + \|v(\tau)\|_{s+2} + \|u_0(\tau)\|_s + \|v_0(\tau)\|_s \}$ .

139 *Proof.* The idea of the proof is to write down the representation of  $\Delta\Phi$  and  $\Delta\Phi_0$  given  
140 in (11) and (38). Using the expansion (25) and adding "zeros" in terms of  $\exp(ic^2t)u_0(t)$   
141 and  $\exp(ic^2t)v_0(t)$  yields the result.  $\square$

**Proposition 2** (cf. [21]). *Under the assumptions of Theorem 1 for all  $t \in [0, T]$  there holds that*

$$\|\mathcal{A}(t) - \mathcal{A}_0(t)\|_s \leq c^{-1}(K_{\mathcal{A}.1}^T + tK_{\mathcal{A}.2}^T) + M^T \int_0^t \mathcal{R}(\tau)d\tau,$$

142 where  $M^T$  depends on  $\sup_{\tau \in [0, T]} \{ \|u(\tau)\|_s + \|v(\tau)\|_s + \|u_0(\tau)\|_{s+1} + \|v_0(\tau)\|_{s+1} \}$  and where  
143 the dependency of  $K_{\mathcal{A}.1}^T, K_{\mathcal{A}.2}^T$  on the solutions is stated in Theorem 1.

*Proof.* The idea of the proof is to replace  $\mathcal{A}(t)$  by its mild formulation given in (13). The difference  $\mathcal{A} - \mathcal{A}_0$  then only involves an integral term over the current density  $\mathcal{P}[\mathbf{J}[u, v, \mathcal{A}]]$ . We introduce the limit current density as  $\mathbf{J}_0[u_0, v_0](t) = \text{Re}(iz_0\nabla\bar{z}_0)$ . Now adding "zeros" in terms of  $\mathbf{J}_0[u_0, v_0]$  gives an integral term involving the difference

$$\|\mathbf{J}[u, v, \mathcal{A}](\tau) - \mathbf{J}_0[u_0, v_0](\tau)\|_s = \mathcal{O}(c^{-1}) + K\mathcal{R}(\tau)$$

for some constant  $K$  not depending on  $c$ , and another integral term involving

$$\langle \nabla \rangle_0^{-1} \sin(c \langle \nabla \rangle_0 (t - \tau)) \mathcal{P}[\mathbf{J}_0[z_0](\tau)].$$

144 Integration by parts then yields the assertion.  $\square$

145

The above propositions allow us to prove Theorem 1 as follows:

*Proof of Theorem 1.* Note that both terms in  $\mathcal{R}(t)$  (see (39)) can be estimated in exactly the same way. Thus, we only establish a bound on  $\|u(t) - \exp(ic^2t)u_0(t)\|_s$ . The main tool thereby is to exploit that the operators  $T_c(t) = \exp(ic\langle\nabla\rangle_c t)$  and  $T_0(t) = \exp(-i\frac{1}{2}\Delta t)$  are isometries in  $H^s$ . Expanding  $\exp(i(-c\langle\nabla\rangle_c + c^2 - \frac{1}{2}\Delta)t)$  into its Taylor series yields with the aid of (24) that

$$\|T_c(t)w - T_0(t)\exp(ic^2t)\tilde{w}\|_s \leq \|w - \tilde{w}\|_s + \mathcal{O}(c^{-2}t\|\tilde{w}\|_{s+4}). \quad (40)$$

Note that the mild solutions of (38) read

$$\begin{aligned} u_0(t) &= T_0(t)u_0(0) - i \int_0^t T_0(t-\tau)\Phi_0(\tau)u_0(\tau)d\tau, \\ v_0(t) &= T_0(t)v_0(0) + i \int_0^t T_0(t-\tau)\Phi_0(\tau)v_0(\tau)d\tau. \end{aligned} \quad (41)$$

As  $u(0) = u_0(0)$ , the mild formulation of  $u$  and  $u_0$  given in (13) and (41) together with (40) thus imply that

$$\begin{aligned} \|u(t) - \exp(ic^2t)u_0(t)\|_s &\leq c^{-2}tK \|u_0(0)\|_{s+4} \\ &+ \left\| \int_0^t T_c(t-\tau)\mathcal{N}_u[u, v, \Phi, \mathcal{A}](\tau) + i \exp(ic^2t)T_0(t-\tau)\Phi_0(\tau)u_0(\tau)d\tau \right\|_s, \end{aligned} \quad (42)$$

146 where  $\mathcal{N}_u[u, v, \Phi, \mathcal{A}]$  is defined in (10).

Our aim is now to express the integral term in (42) as a term of type

$$\mathcal{O}(c^{-2}) + \int_0^t \mathcal{R}(\tau)d\tau,$$

147 which will allow us to conclude the assertion by Gronwall's lemma. Therefore we consider  
148 each term in  $\mathcal{N}_u[u, v, \Phi, \mathcal{A}]$  separately.

By (25) and (26) we find after a short calculation that

$$\|\mathcal{N}_u[u, v, \Phi, \mathcal{A}] + i\Phi u + \langle\nabla\rangle_c^{-1}(\mathcal{A} \cdot \nabla(u + \bar{v}))\|_s \leq Kc^{-2},$$

where  $K = K(\|\Phi\|_{s+2}, \|u\|_{s+2}, \|v\|_{s+2}, \|\mathcal{A}\|_s)$ . Thus, using (40) we can bound the integral term in (42) as follows:

$$\begin{aligned} &\left\| \int_0^t T_c(t-\tau)\mathcal{N}_u[u, v, \Phi, \mathcal{A}](\tau) + i \exp(ic^2t)T_0(t-\tau)\Phi_0(\tau)u_0(\tau)d\tau \right\|_s \\ &\leq Kc^{-2}t \sup_{\tau \in [0, t]} \|\Phi_0(\tau)u_0(\tau)\|_{s+4} + \int_0^t \|\Phi(\tau)u(\tau) - \Phi_0(\tau)\exp(ic^2\tau)u_0(\tau)\|_s d\tau \\ &+ \left\| \int_0^t T_c(t-\tau)\langle\nabla\rangle_c^{-1}(\mathcal{A}(\tau) \cdot \nabla(u(\tau) + \bar{v}(\tau)))d\tau \right\|_s. \end{aligned} \quad (43)$$

149 The latter term can be bounded up to a term of order  $\mathcal{O}(c^{-2}) + \int_0^t \mathcal{R}(\tau)d\tau$  by insert-  
150 ing "zeros" in terms of  $\mathcal{A}_0(\tau)$ ,  $\exp(ic^2\tau)u_0(\tau)$  and  $\exp(ic^2\tau)v_0(\tau)$  and then applying  
151 integration by parts with respect to  $\tau$  and applying Proposition 2.

Furthermore we can estimate  $\|\Phi(\tau)u(\tau) - \Phi_0(\tau)\exp(ic^2\tau)u_0(\tau)\|_s$  as

$$\|\Phi u - \Phi_0 \exp(ic^2\tau)u_0\|_s \leq C(\|\Phi - \Phi_0\|_s \|u\|_s + \|\Phi_0\|_s \|u - \exp(ic^2\tau)u_0\|_s)$$

such that by Proposition 1 we find that

$$\|\Phi u - \Phi_0 \exp(ic^2\tau)u_0\|_s \leq c^{-2}C_1 + C_2\mathcal{R}(\tau),$$

152 where the constants  $C_1$  and  $C_2$  depend on the same data as the constants in the assertion  
153 of Proposition 1.

Plugging the above bounds into (42) yields that

$$\mathcal{R}(t) \leq c^{-2}(M_0^T + M_1^T t + M_2^T t^2) + (M_3^T + tM_4^T) \int_0^t \mathcal{R}(\tau) d\tau$$

which by Gronwall's Lemma implies the desired bound

$$\mathcal{R}(t) \leq c^{-2}b(T) \exp(\lambda(T)), \quad \forall t \in [0, T]. \quad (44)$$

154 The results on  $\Phi_0(t)$  and  $\mathcal{A}_0(t)$  follow the line of argumentation by using (44) in the  
155 results of Proposition 1 and Proposition 2.  $\square$

## 156 5. Construction of numerical schemes

157 In this section we construct an efficient and robust numerical scheme for the highly-  
158 oscillatory MKG equation (6) in the non-relativistic limit regime, i.e. for  $c \gg 1$ . In order  
159 to overcome any  $c$ -dependent time step restriction we exploit the limit approximation  
160 (38) derived in Section 3.

### 161 5.1. The numerical scheme and its error

We consider the MKG equation (6) in the Coulomb gauge in the non-relativistic limit regime  $c \gg 1$

$$\begin{cases} \partial_{tt}z = -c^2 \langle \nabla \rangle_c^2 z + \Phi^2 z - 2i\Phi \partial_t z - iz \partial_t \Phi - 2ic\mathcal{A}\nabla z - |\mathcal{A}|^2 z, \\ \partial_{tt}\mathcal{A} = c^2 \Delta \mathcal{A} + c\mathcal{P}[J], \quad J = \text{Re}(iz \overline{D_\alpha z}), \\ -\Delta \Phi = \rho, \quad \rho = -c^{-1} \text{Re}(iz \overline{D_0 z}), \end{cases} \quad (45a)$$

$$\begin{cases} z(0, x) = \varphi(x), \quad D_0 z(0, x) = \sqrt{-\Delta + c^2} \psi(x), \\ \mathcal{A}(0, x) = A(x), \quad \partial_t \mathcal{A}(0, x) = cA'(x), \\ \int_{\mathbb{T}^d} \rho(0, x) dx = 0, \quad \int_{\mathbb{T}^d} \Phi(t, x) dx = 0 \end{cases} \quad (45b)$$

equipped with periodic boundary conditions, i.e.  $x \in \mathbb{T}^d = [-\pi, \pi]^d$ . In the previous sections we derived the corresponding SP limit system (cf. (38))

$$\begin{cases} i\partial_t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{2}\Delta \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \Phi_0 \begin{pmatrix} u_0 \\ -v_0 \end{pmatrix}, \quad \begin{pmatrix} u_0(0) \\ v_0(0) \end{pmatrix} = \begin{pmatrix} \varphi - i\psi \\ \overline{\varphi} - i\overline{\psi} \end{pmatrix}, \\ -\Delta \Phi_0 = -\frac{1}{4}(|u_0|^2 - |v_0|^2), \quad \int_{\mathbb{T}^d} \Phi_0(t, x) dx = 0, \\ \mathcal{A}_0(t, x) = \cos(ct\sqrt{-\Delta})A(x) + (c\sqrt{-\Delta})^{-1} \sin(ct\sqrt{-\Delta})cA'(x) \end{cases} \quad (46)$$

162 which will now allow us to derive an efficient numerical time integration scheme: Since the  
 163 SP system (46) does not depend on the large parameter  $c$  we can solve it very efficiently;  
 164 in particular without any  $c$ -depending time step restriction. Multiplying the numerical  
 165 approximations of the non-oscillatory functions  $u_0$  and  $v_0$  with the high frequency terms  
 166  $\exp(\pm ic^2 t)$  then gives a good approximation to the exact solution, see Theorem 2 below  
 167 for the detailed description. In particular this approach allows us to overcome any  $c$ -  
 168 dependent time step restriction.

**Time discretization:** We carry out the numerical time integration of the Schrödinger-Poisson system

$$\begin{cases} i\partial_t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{2}\Delta \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \Phi_0 \begin{pmatrix} u_0 \\ -v_0 \end{pmatrix}, & \begin{pmatrix} u_0(0) \\ v_0(0) \end{pmatrix} = \begin{pmatrix} \varphi - i\psi \\ \bar{\varphi} - i\bar{\psi} \end{pmatrix}, \\ -\Delta\Phi_0 = -\frac{1}{4}(|u_0|^2 - |v_0|^2), & \int_{\mathbb{T}^d} \Phi_0(t, x) dx = 0. \end{cases} \quad (47)$$

with an exponential Strang splitting method (cf. [20]), where we naturally split the system into the kinetic part

$$i\partial_t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{2}\Delta \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \quad (\text{T})$$

with the exact flow  $\varphi_T^t(u_0(0), v_0(0))$  and the potential part

$$\begin{cases} i\partial_t \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \Phi_0 \begin{pmatrix} u_0 \\ -v_0 \end{pmatrix}, \\ -\Delta\Phi_0 = -\frac{1}{4}(|u_0|^2 - |v_0|^2), & \int_{\mathbb{T}^d} \Phi_0(t, x) dx = 0, \end{cases} \quad (\text{P})$$

with the exact flow  $\varphi_P^t(u_0(0), v_0(0))$ . The Strang splitting approximation to the exact flow  $\varphi^t(u_0(0), v_0(0)) = \varphi_{T+P}^t(u_0(0), v_0(0))$  of the SP system (47) at time  $t_n = n\tau, n = 0, 1, 2, \dots$  with time step size  $\tau$  is then given by

$$\varphi^{t_n}(u_0(0), v_0(0)) \approx \left( \varphi_T^{\tau/2} \circ \varphi_P^\tau \circ \varphi_T^{\tau/2} \right)^n (u_0(0), v_0(0)). \quad (48)$$

169 We can solve the kinetic subproblem (T) in Fourier space exactly in time. In subproblem  
 170 (P) we can show that the modulus of  $u_0$  and  $v_0$  are constant in time, i.e.  $|u_0(t)|^2 =$   
 171  $|u_0(0)|^2$  and  $|v_0(t)|^2 = |v_0(0)|^2$ , and thence also  $\Phi_0$  is constant in time, i.e.  $\Phi_0(t) = \Phi_0(0)$ .  
 172 Thus, we can also solve the potential subproblem (P) exactly in time.

173 **Space discretization:** For the space discretization we choose a Fourier pseudospec-  
 174 tral method with  $N$  grid points (or frequencies respectively), i.e. we choose a mesh size  
 175  $h = 2\pi/N$  and grid points  $x_j = -\pi + jh, j = 0, \dots, N-1$ . We then denote the discretized  
 176 spatial operators by  $\Delta_h$  and  $\nabla_h$  respectively.

177 **Full discretization:** The fully discrete numerical scheme can then be implemented  
 178 efficiently using the Fast Fourier transform (FFT).

179 This ansatz allows us to state the following convergence result on the approximation  
 180 of the MKG system (45) in the non-relativistic limit regime:

**Theorem 2.** Consider the MKG (45) on the torus  $\mathbb{T}^d$ . Fix  $s'_1, s'_2, s > d/2$  and let  $\varphi, \psi \in H^{s+r_1}(\mathbb{T}^d), A, A' \in H^{s+r_2}(\mathbb{T}^d)$  with  $r_1 = \max\{4, s'_1\}, r_2 = \max\{2, s'_2\}$ . Then



there exist  $T, C, h_0, \tau_0 > 0$  such that the following holds: Let us define the numerical approximation of the the first-order approximation term  $z_0(t)$  at time  $t_n = n\tau$  through

$$z_0^{n,h} := \frac{1}{2} \left( u_0^{n,h} \exp(ic^2 t_n) + \bar{v}_0^{n,h} \exp(-ic^2 t_n) \right),$$

where  $u_0^{n,h}, v_0^{n,h}$  denote the numerical approximation to the solutions  $u_0(t_n), v_0(t_n)$  of the limit system (46) obtained by the Fourier Pseudospectral Strang splitting scheme (48) with mesh size  $h \leq h_0$  and time step  $\tau \leq \tau_0$ . Furthermore let  $\Phi_0^{n,h}$  denote the numerical approximation to  $\Phi_0(t_n)$  given through the discrete Poisson equation

$$-\Delta_h \Phi_0^{n,h} := -\frac{1}{4} \left( \left| u_0^{n,h} \right|^2 - \left| v_0^{n,h} \right|^2 \right). \quad (49)$$

Also let

$$\mathcal{A}_0^{n,h} = \cos \left( ct_n \sqrt{-\Delta_h} \right) A_h + \left( c \sqrt{-\Delta_h} \right)^{-1} \sin \left( ct_n \sqrt{-\Delta_h} \right) c A'_h$$

181 denote the numerical approximation to  $\mathcal{A}_0(t_n)$ , where  $A_h, A'_h$  are the evaluations of  $A$   
182 and  $A'$  on the grid points.

Then, the following convergence towards the exact solution of the MKG equation (45) holds for all  $t_n \in [0, T]$  and  $c \geq 1$  :

$$\begin{aligned} \left\| z(t_n) - z_0^{n,h} \right\|_s + \left\| \Delta \Phi(t_n) - \Delta_h \Phi_0^{n,h} \right\|_s &\leq C \left( \tau^2 + h^{s'_1} + c^{-2} \right), \\ \left\| \mathcal{A}(t_n) - \mathcal{A}_0^{n,h} \right\|_s &\leq C \left( h^{s'_2} + c^{-1} \right). \end{aligned}$$

*Proof.* The proof follows the same ideas as the proof of [14, Theorem 3]. The triangle inequality yields

$$\begin{aligned} \left\| z(t_n) - z_0^{n,h} \right\|_s &\leq \left\| z(t_n) - z_0(t_n) \right\|_s + \left\| z_0(t_n) - z_0^{n,h} \right\|_s, \\ \left\| \Delta \Phi(t_n) - \Delta_h \Phi_0^{n,h} \right\|_s &\leq \left\| \Delta(\Phi(t_n) - \Phi_0(t_n)) \right\|_s + \left\| \Delta \Phi_0(t_n) - \Delta_h \Phi_0^{n,h} \right\|_s, \\ \left\| \mathcal{A}(t_n) - \mathcal{A}_0^{n,h} \right\|_s &\leq \left\| \mathcal{A}(t_n) - \mathcal{A}_0(t_n) \right\|_s + \left\| \mathcal{A}_0(t_n) - \mathcal{A}_0^{n,h} \right\|_s. \end{aligned} \quad (50)$$

183 Theorem 1 allows us to bound the first term in each of the inequalities above in order  $c^{-2}$   
184 and  $c^{-1}$ , respectively. Henceforth, we only need to derive bounds on the numerical errors  
185 of the Fourier Pseudospectral Strang splitting scheme approximating the SP system.

**Error in  $z_0^{n,h}$ :** Note that

$$\begin{aligned} \left\| z_0(t_n) - z_0^{n,h} \right\|_s &\leq \left\| \exp(ic^2 t)(u_0(t_n) - u_0^{n,h}) \right\|_s + \left\| \exp(-ic^2 t)(\bar{v}_0(t_n) - \bar{v}_0^{n,h}) \right\|_s \\ &\leq \left\| u_0(t_n) - u_0^{n,h} \right\|_s + \left\| v_0(t_n) - v_0^{n,h} \right\|_s \\ &\leq C(\tau^2 + h^{s'_1}). \end{aligned}$$

186 The latter follows for sufficiently smooth solutions (i.e. if  $u_0, v_0 \in H^r$ ,  $r = s + s'_1$ ) by the  
187 convergence bound on the Strang splitting applied to the Schrödinger-Poisson system  
188 derived in [20].

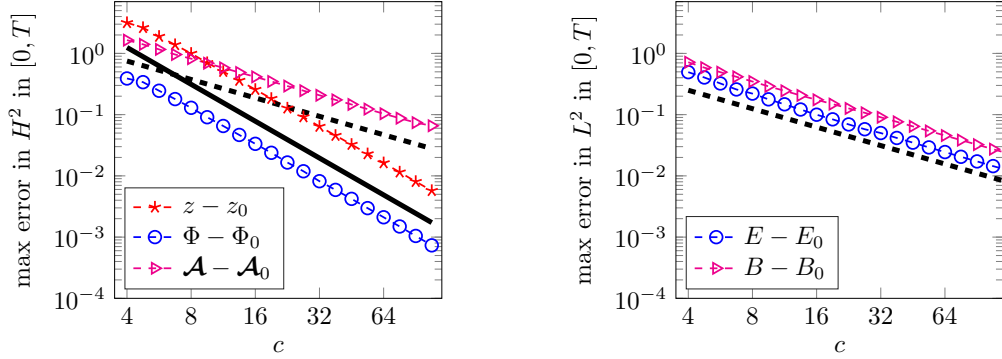


Figure 1: **Left:**  $H^2$  error of the numerical limit approximation  $(z_0^{n,h}, \Phi_0^{n,h}, \mathcal{A}_0^{n,h})$  to the exact solution. **Right:**  $L^2$  error of the numerical approximations  $E_0^{n,h}, B_0^{n,h}$  to the electromagnetic field. The reference solution  $(z, \Phi, \mathcal{A})$  was computed with a Gautschi-type exponential integrator with time step size  $\tau = 2^{-22} \approx 10^{-7}$ . The black dashed line with slope  $-1$  and the black solid line with slope  $-2$  represent the order  $\mathcal{O}(c^{-1})$  and  $\mathcal{O}(c^{-2})$  respectively.

**Error in  $\Phi_0^{n,h}$ :** By (46) and (49) we obtain that

$$\left\| \Delta \Phi_0(t_n) - \Delta_h \Phi_0^{n,h} \right\|_s \leq M \left( \left\| u_0(t_n) - u_0^{n,h} \right\|_s + \left\| v_0(t_n) - v_0^{n,h} \right\|_s \right) \leq C(\tau^2 + h^{s'_1}).$$

**Error in  $\mathcal{A}_0^{n,h}$ :** As  $\mathcal{A}_0$  is explicitly given in time we do not have any time discretization error. Only the error by the Fourier pseudospectral method comes into play which yields that

$$\left\| \mathcal{A}_0(t_n) - \mathcal{A}_0^{n,h} \right\|_s \leq Ch^{s'_2},$$

189 if the exact solution is smooth enough, i.e. if  $\mathcal{A}_0 \in H^{\tilde{r}}$ ,  $\tilde{r} = s + s'_2$ .

190 Collecting the results yields the assertion. □

## 191 5.2. Numerical results

192

193 In this section we numerically underline the sharpness of the theoretical results derived  
194 in the previous sections.

For the MKG equation (45) on the two-dimensional torus, i.e.  $d = 2$ ,  $(x, y)^T \in \mathbb{T}^2 = [-\pi, \pi]^2$ , we choose the initial data  $\varphi, \psi, A, A'$  as

$$\begin{aligned} \tilde{\varphi}(x, y) &= \sin(y) + \cos(x) + i(\cos(2x) + \sin(y)), & \varphi &= \tilde{\varphi} / \|\tilde{\varphi}\|_{L^2}, \\ \tilde{\psi}(x, y) &= \cos(x) + \sin(2y) + i \cos(2x) \sin(y), & \psi &= \tilde{\psi} / \|\tilde{\psi}\|_{L^2}, \\ \tilde{A}(x, y) &= (\partial_y V_1(x, y), -\partial_x V_1(x, y))^T, & A &= \tilde{A} / \|\tilde{A}\|_{L^2}, \\ \tilde{A}'(x, y) &= c(\partial_y V_2(x, y), -\partial_x V_2(x, y))^T, & A' &= \tilde{A}' / \|\tilde{A}'\|_{L^2}, \end{aligned}$$

where

$$V_1(x, y) = \sin(x) \cos(y) + \sin(2y) + \cos(x), \quad V_2(x, y) = \sin(y) + \cos(2x).$$

195 It is easy to check that  $\operatorname{div} A = 0$  and  $\operatorname{div} A' = 0$ . Furthermore the initial data satisfy  
196 Remark 1, i.e.  $\int_{\mathbb{T}^d} \rho(0, x) dx = 0$ , where  $\rho(0) = -\operatorname{Re}(i\varphi \langle \nabla \rangle_c \bar{\psi})$ . We simulate the limit  
197 solution on the time interval  $t \in [0, T = 1]$  with a time step size  $\tau = 2^{-10} \approx 10^{-3}$ , and a  
198 spatial grid with  $N = 128$  grid points in both dimensions and measure the maximal error  
199 of the limit approximation  $(z_0, \Phi_0, \mathcal{A}_0)$  on the time interval  $[0, T]$  in the  $H^2$  norm. A  
200 reference solution of the MKG equation (45) is obtained with an adapted Gautschi-type  
201 exponential integrator, as proposed in [16] for highly-oscillatory ODEs or in [3] for the  
202 nonlinear Klein-Gordon equation. Thereby a very small time step size  $\tau_{\text{ref}}$  satisfying  
203 the CFL condition  $\tau_{\text{ref}} \leq c^{-2}h$  is necessary. Fig. 1 verifies the theoretical convergence  
204 bounds stated in Theorem 2. We furthermore observe numerically that also the electric  
205 field  $E_0^{n,h} := -c^{-1}\partial_t \mathcal{A}_0^{n,h} - \nabla_h \Phi_0^{n,h}$  and the magnetic field  $B_0^{n,h} := \nabla_h \times \mathcal{A}_0^{n,h}$  show a  
206  $c^{-1}$  convergence towards  $E = -c^{-1}\partial_t \mathcal{A} - \nabla \Phi$  and  $B = \nabla \times \mathcal{A}$  in  $L^2$ , respectively.

## 207 6. Conclusion

208 In order to derive an efficient and accurate numerical method for solving the MKG  
209 equation in the non-relativistic limit regime  $c \gg 1$  we followed the idea of a formal  
210 asymptotic expansion of the exact solution  $(z, \Phi, \mathcal{A})$  in terms of  $c^{-2}$  and  $c^{-1}$ , respectively.  
211 This allowed us to reduce the numerical effort of solving the highly-oscillatory MKG  
212 system under severe time step restrictions  $\tau = \mathcal{O}(c^{-2})$  to solving the corresponding  
213 non-oscillatory Schrödinger-Poisson (SP) limit system. The latter can be carried out  
214 very efficiently and in particular independently of the large parameter  $c$ . We obtained a  
215 numerical approximation  $(u_0^{n,h}, v_0^{n,h}, \Phi_0^{n,h})$  to the solution  $(u_0, v_0, \Phi_0)$  of the SP system  
216 at time  $t_n = n\tau$  by solving the SP system via an exponential Strang splitting method  
217 with time step  $\tau$  in time together with a Fourier pseudospectral method for the space  
218 discretization on a grid with mesh size  $h$ . In particular  $\tau$  and  $h$  do not depend on the large  
219 parameter  $c$ . The numerical approximations  $z_0^{n,h}, \Phi_0^{n,h}, \mathcal{A}_0^{n,h}$  then satisfy error bounds  
220 of order  $\mathcal{O}(c^{-2} + \tau^2 + h^{s'})$  and  $\mathcal{O}(c^{-1} + h^{s'})$  respectively. We underlined the sharpness  
221 of the error bound with numerical experiments. For practical implementation issues  
222 we assumed periodic boundary conditions. Up to minor changes all the results of this  
223 paper remain valid for Dirichlet boundary conditions and different spatial discretization  
224 schemes.

## 225 Acknowledgement

226 We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft  
227 (DFG) through CRC 1173. We furthermore thank Dirk Hundertmark and Claire Scheid  
228 for fruitful discussions on this topic.

## 229 References

- 230 [1] Aitchison, I. J. R., Hey, A. J. G., 1993. Gauge theories in particle physics : a practical introduction,  
231 2nd Edition. Graduate student series in physics. Inst. of Physics Publ., Bristol [u.a.].  
232 [2] Bao, W., Cai, Y., Zhao, X., 2014. A uniformly accurate multiscale time integrator pseudospectral  
233 method for the Klein-Gordon equation in the nonrelativistic limit regime. SIAM J. Numer. Anal.  
234 52 (5), 2488–2511.  
235 [3] Bao, W., Dong, X., 2012. Analysis and comparison of numerical methods for the Klein–Gordon  
236 equation in the nonrelativistic limit regime. Numerische Mathematik 120 (2), 189–229.

- 237 [4] Bao, W., Li, X.-G., 2004. An efficient and stable numerical method for the Maxwell-Dirac system.  
238 J. Comput. Phys. 199 (2), 663–687.
- 239 [5] Bao, W., Yang, L., 2007. Efficient and accurate numerical methods for the Klein-Gordon-  
240 Schrödinger equations. J. Comput. Phys. 225 (2), 1863–1893.
- 241 [6] Bechouche, P., Mauser, N. J., Selberg, S., 2004. Nonrelativistic limit of Klein-Gordon-Maxwell to  
242 Schrödinger-Poisson. Amer. J. Math. 126 (1), 31–64.
- 243 [7] Bechouche, P., Mauser, N. J., Selberg, S., 2005. On the asymptotic analysis of the Dirac-Maxwell  
244 system in the nonrelativistic limit. J. Hyperbolic Differ. Equ. 2 (1), 129–182.
- 245 [8] Chartier, P., Crouseilles, N., Lemou, M., Méhats, F., 2015. Uniformly accurate numerical schemes  
246 for highly oscillatory Klein-Gordon and nonlinear Schrödinger equations. Numer. Math. 129 (2),  
247 211–250.
- 248 [9] Christiansen, S. H., Scheid, C., 2011. Convergence of a constrained finite element discretization of  
249 the Maxwell Klein Gordon equation. ESAIM Math. Model. Numer. Anal. 45 (4), 739–760.
- 250 [10] Cohen, D., Gauckler, L., Hairer, E., Lubich, C., 2015. Long-term analysis of numerical integra-  
251 tors for oscillatory hamiltonian systems under minimal non-resonance conditions. BIT Numerical  
252 Mathematics 55 (3), 705–732.
- 253 [11] Deumens, E., 1986. The Klein-Gordon-Maxwell nonlinear system of equations. Physica D: Nonlinear  
254 Phenomena 18 (1), 371 – 373.
- 255 [12] Eder, G., 1967. Elektrodynamik. BI-Hochschultaschenbücher ; 233/233a. Bibliogr. Inst., Mannheim.
- 256 [13] Faou, E., 2012. Geometric numerical integration and Schrödinger equations. Zurich lectures in  
257 advanced mathematics. European Mathematical Society, Zürich.
- 258 [14] Faou, E., Schratz, K., 2014. Asymptotic preserving schemes for the Klein-Gordon equation in the  
259 non-relativistic limit regime. Numer. Math. 126 (3), 441–469.
- 260 [15] Hairer, E., Lubich, C., Wanner, G., 2006. Geometric numerical integration : structure-preserving  
261 algorithms for ordinary differential equations, 2nd Edition. Springer series in computational math-  
262 ematics ; 31. Springer, Berlin.
- 263 [16] Hochbruck, M., Lubich, C., 1999. A Gautschi-type method for oscillatory second-order differential  
264 equations. Numer. Math. 83 (3), 403–426.
- 265 [17] Jackson, J. D., 2006. Klassische Elektrodynamik, 4th Edition. de Gruyter, Berlin [u.a.].
- 266 [18] Keel, M., Roy, T., Tao, T., 2011. Global well-posedness of the Maxwell-Klein-Gordon equation  
267 below the energy norm. Discrete Contin. Dyn. Syst. 30 (3), 573–621.
- 268 [19] Kirrmann, P., Schneider, G., Mielke, A., 1992. The validity of modulation equations for extended  
269 systems with cubic nonlinearities. Proc. Roy. Soc. Edinburgh Sect. A 122 (1-2), 85–91.
- 270 [20] Lubich, C., 2008. On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger  
271 equations. Math. Comp. 77 (264), 2141–2153.
- 272 [21] Masmoudi, N., Nakanishi, K., 2003. Nonrelativistic limit from Maxwell-Klein-Gordon and Maxwell-  
273 Dirac to Poisson-Schrödinger. Int. Math. Res. Not. (13), 697–734.
- 274 [22] Murdock, J. A., 1991. Perturbations. A Wiley-Interscience Publication. John Wiley & Sons Inc.,  
275 New York, theory and methods.
- 276 [23] Newell, A. C., 1985. Solitons in mathematics and physics. Vol. 48 of CBMS-NSF Regional Con-  
277 ference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM),  
278 Philadelphia, PA.
- 279 [24] Schwabl, F., 2007. Quantenmechanik (QM I) : Eine Einführung, 7th Edition. Springer-  
280 LehrbuchSpringerLink : Bücher. Springer Berlin Heidelberg, Berlin, Heidelberg.
- 281 [25] Schwabl, F., 2008. Quantenmechanik für Fortgeschrittene : (QM II), 5th Edition. Springer-  
282 Lehrbuch. Springer, Berlin.
- 283 [26] Selberg, S., Tesfahun, A., 2010. Finite-energy global well-posedness of the Maxwell-Klein-Gordon  
284 system in Lorenz gauge. Comm. Partial Differential Equations 35 (6), 1029–1057.