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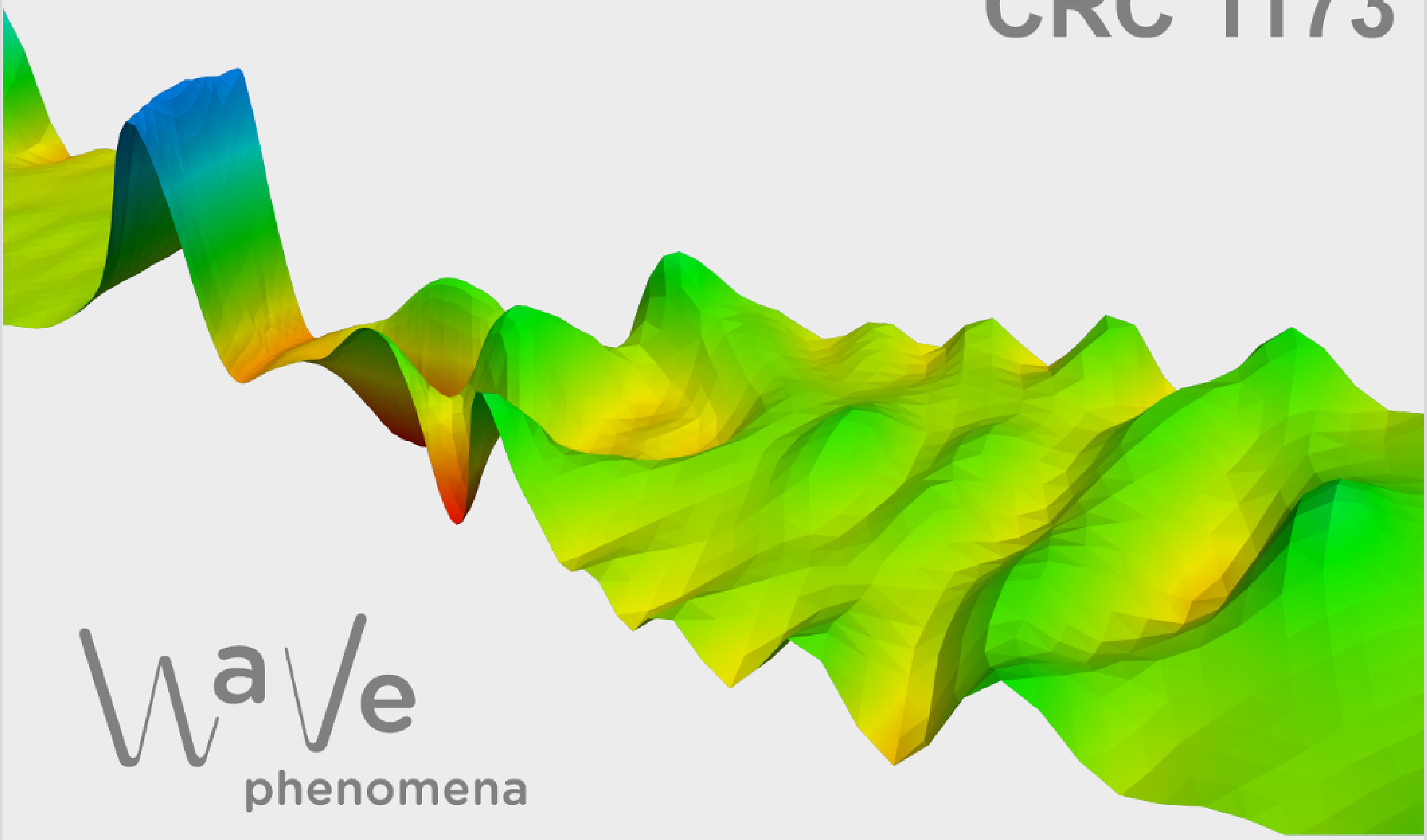

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Computation and Stability of Traveling Waves in Second Order Evolution Equations

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Abstract. The topic of this paper are nonlinear traveling waves occuring in a system of damped waves equations in one space variable. We extend the freezing method from first to second order equations in time. When applied to a Cauchy problem, this method generates a comoving frame in which the solution becomes stationary. In addition it generates an algebraic variable which converges to the speed of the wave, provided the original wave satisfies certain spectral conditions and initial perturbations are sufficiently small. We develop a rigorous theory for this effect by recourse to some recent nonlinear stability results for waves in first order hyperbolic systems. Numerical computations illustrate the theory for examples of Nagumo and FitzHugh-Nagumo type.

Key words. Systems of damped wave equations, traveling waves, nonlinear stability, freezing method, second order evolution equations, point spectra and essential spectra.
AMS subject classification. 65P40, 35L52, 47A25 (35B35, 35P30, 37C80).

1. Introduction

In this paper we study the numerical computation and stability of traveling waves in second order evolution equations. Our model system is a damped wave equation in one space dimension

$$Mu_{tt} + Bu_t = Au_{xx} + Cu_x + f(u),\ x \in \mathbb{R},\ t \geqslant 0, u(x,t) \in \mathbb{R}^m. \tag{1.1}$$

Here we use constant matrices $A, B, C, M \in \mathbb{R}^{m,m}$ and a sufficiently smooth nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$. Moreover, we require M to be invertible and $M^{-1}A$ to be real diagonalizable with positive eigenvalues (positive diagonalizable for short). This ensures that the principal part of equation (1.1) is well-posed.

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Our main concern are traveling wave solutions $u_\star : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$ of (1.1), i.e.

$$u_\star(x,t) = v_\star(x - \mu_\star t),\ x \in \mathbb{R},\ t \geqslant 0, \tag{1.2}$$

such that

$$\lim_{\xi \to \pm\infty} v_\star(\xi) = v_\pm \in \mathbb{R}^m \quad \text{and} \quad f(v_+) = f(v_-) = 0. \tag{1.3}$$

Here $v_\star : \mathbb{R} \to \mathbb{R}^m$ is a non-constant function and denotes the profile (or pattern) of the wave, $\mu_\star \in \mathbb{R}$ its translational velocity and $v_\pm$ its asymptotic states. The quantities $v_\star$ and $\mu_\star$ are generally unknown, explicit formulas are only availabe for very specific equations. As usual, a traveling wave $u_\star$ is called a traveling pulse if $v_+ = v_-$, and a traveling front if $v_+ \neq v_-$.
We have two main aims for this paper. First, we want to determine traveling wave solutions of (1.1) from second order boundary value problems and investigate their stability for the time-dependent problem. Second, we will generalize the method of freezing solutions of the Cauchy problem associated with (1.1), from first order to second order equations in time (cf. [4], [5]). The idea for approximating the traveling wave $u_\star$ is to determine the profile $v_\star$ and the velocity $\mu_\star$ simultaneously. For this purpose, let us transform (1.1) via $u(x,t) = v(\xi,t)$ with $\xi := x - \mu_\star t$ into a co-moving frame

$$Mv_{tt} + Bv_t = (A - \mu_\star^2 M)v_{\xi\xi} + 2\mu_\star M v_{\xi t} + (C + \mu_\star B)v_\xi + f(v),\ \xi \in \mathbb{R},\ t \geqslant 0. \tag{1.4}$$

Inserting (1.2) into (1.1) shows, that $v_\star$ is a stationary solution of (1.4), meaning that $v_\star$ solves the traveling wave equation

$$0 = (A - \mu_\star^2 M)v_{\star,\xi\xi}(\xi) + (C + \mu_\star B)v_{\star,\xi}(\xi) + f(v_\star(\xi)),\ \xi \in \mathbb{R}. \tag{1.5}$$

For equation (1.4) to be well-posed we will strengthen our assumptions to $M^{-1}A - \mu_\star^2 I_m$ being positive diagonalizable, which imposes a restriction on the possible wave speeds $\mu_\star$. There are basically two different ways of determining the profile $v_\star$ and the velocity $\mu_\star$ from the equations above. In the first approach one solves (1.5) as a boundary value problem for $v_\star, \mu_\star$ by truncating to a finite interval and using asymptotic boundary conditions as well as a scalar phase condition (see [6] for a survey). This method requires rather good initial approximations, but has the advantage of being applicable to unstable waves as well. The second approach is through simulation of (1.1) via the freezing method which transforms the orginal PDE (1.1) into a partial differential algebraic equation (PDAE). Its solutions converge to the unknown profile and the unknown velocity simultaneously, provided the initial data lie in the domain of attraction of a stable profile. In Section 2.1 below we will investigate this approach in more detail. For our numerical examples we will also employ and specify a well known relation of traveling waves for the hyperbolic system (1.1) to those of a parabolic system, cf. [13], [10] and Section 2.2.
We are also interested in nonlinear stability of traveling waves. Some far-reaching global stability results for scalar damped wave equations have been proved in [9], [10]. Here we consider local stability only. For a certain class of first-order evolution equations it is well-known, that spectral stability implies nonlinear stability, see [28], for example. Spectral stability of a traveling wave refers to the spectrum of the operator obtained by linearizing about the profile. In the case (1.1) the linearization of (1.4) at the wave profile $v_\star$ reads

$$Mv_{tt} + Bv_t - (A - \mu_\star^2 M)v_{\xi\xi} - 2\mu_\star M v_{\xi t} - (C + \mu_\star B)v_\xi - Df(v_\star)v = 0,\ \xi \in \mathbb{R},\ t \geqslant 0. \tag{1.6}$$

Applying separation of variables (or Laplace transform) to (1.6) via $v(\xi,t) = e^{\lambda t}w(\xi)$ leads us to the following quadratic eigenvalue problem

$$\mathcal{P}(\lambda)w = \left[\lambda^2 M + \lambda\left(B - 2\mu_\star M\frac{\partial}{\partial\xi}\right) - (A - \mu_\star^2 M)\frac{\partial^2}{\partial\xi^2} - (C + \mu_\star B)\frac{\partial}{\partial\xi} - Df(v_\star)\right]w = 0, \tag{1.7}$$

for the eigenfunction $w : \mathbb{R} \to \mathbb{C}^m$ and its associated eigenvalue $\lambda \in \mathbb{C}$ of $\mathcal{P}$. As usual $\mathcal{P}$ has the eigenvalue zero with associated eigenfunction $v_{\star,\xi}$ due to shift equivariance. If one requires this eigenvalue to be

simple and all other parts of the spectrum, both essential and point spectrum, to be strictly to the left of the imaginary axis, then one expects the traveling wave to be locally stable with asymptotic phase. This expectation will be confirmed in Section 4 by transforming to a first order hyperbolic system and using the extensive stability theory developed in [24],[25]. We will also transform the freezing approach and the spectral problem to the first order formulation. In this way we obtain a justification of the freezing approach, showing that the equilibrium $(v_\star, \mu_\star)$ of the freezing PDAE will be stable in the classical Lyapunov sense (w.r.t. appropriate norms) provided the conditions on spectral stability above are satisfied.
Section 3 is devoted to the study of the spectrum of the operator $\mathcal{P}$ from (1.7). While there is always the zero eigenvalue present, further isolated eigenvalues in the point spectrum are often determined by numerical computations (see [2] and the references therein for a variety of approaches). The essential spectrum can be analyzed by replacing $v_\star$ in $\mathcal{P}$ by its limits $v_\pm$ and the operator $\frac{\partial}{\partial\xi}$ by its Fourier symbol $i\omega, \omega \in \mathbb{R}$. The essential spectrum then contains all values $\lambda \in \mathbb{C}$ satisfying the dispersion relation

$$\det\left(\lambda^2 M + \lambda(B - 2i\omega\mu_\star M) + \omega^2(A - \mu_\star^2 M) - i\omega(C + \mu_\star B) - Df(v_\pm)\right) = 0 \tag{1.8}$$

for some $\omega \in \mathbb{R}$. In Section 3 we investigate the shape of these algebraic curves for two examples: a scalar equation with a nonlinearity of Nagumo type and a system of dimension two with nonlinearity of FitzHugh-Nagumo type. These examples will also be used for illustrating the effect of the freezing method from Section 2 when applied to the second order system (1.1).

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2. Freezing traveling waves in damped wave equations

In this section we extend the freezing method ([4], [5]) from first to second order evolution equations for the case of translational equivariance. A generalization to several space dimensions and more general symmetries will be discussed elsewhere.

2.1. Derivation of the partial differential algebraic equation (PDAE). Consider the Cauchy problem associated with (1.1)

$$Mu_{tt} + Bu_t = Au_{xx} + Cu_x + f(u), \quad x \in \mathbb{R},\, t \geq 0, \tag{2.1a}$$

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = v_0, \quad x \in \mathbb{R},\, t = 0, \tag{2.1b}$$

for some initial data $u_0, v_0 : \mathbb{R} \to \mathbb{R}^m$. Introducing new unknowns $\gamma(t) \in \mathbb{R}$ and $v(\xi,t) \in \mathbb{R}^m$ via the freezing ansatz

$$u(x,t) = v(\xi,t), \quad \xi := x - \gamma(t), \quad x \in \mathbb{R},\, t \geqslant 0, \tag{2.2}$$

we obtain (suppressing arguments)

$$u_t = -\gamma_t v_\xi + v_t, \quad u_{tt} = -\gamma_{tt} v_\xi + \gamma_t^2 v_{\xi\xi} - 2\gamma_t v_{\xi t} + v_{tt}. \tag{2.3}$$

Inserting this into (2.1a) leads to the equation

$$Mv_{tt} + Bv_t = (A - \gamma_t^2 M)v_{\xi\xi} + 2\gamma_t M v_{\xi t} + (\gamma_{tt} M + \gamma_t B + C)v_\xi + f(v), \quad \xi \in \mathbb{R},\, t \geqslant 0. \tag{2.4}$$

It is convenient to introduce the time-dependent functions $\mu_1(t) \in \mathbb{R}$ and $\mu_2(t) \in \mathbb{R}$ via

$$\mu_1(t) := \gamma_t(t), \quad \mu_2(t) := \mu_{1,t}(t) = \gamma_{tt}(t),$$

which transform (2.4) into the coupled PDE/ODE system

$$Mv_{tt} + Bv_t = (A - \mu_1^2 M)v_{\xi\xi} + 2\mu_1 M v_{\xi t} + (\mu_2 M + \mu_1 B + C)v_\xi + f(v), \quad \xi \in \mathbb{R},\, t \geqslant 0, \tag{2.5a}$$

$$\mu_{1,t} = \mu_2, \qquad t \geqslant 0, \tag{2.5b}$$

$$\gamma_t = \mu_1, \qquad t \geqslant 0. \tag{2.5c}$$

The quantity $\gamma(t)$ denotes the position, $\mu_1(t)$ the translational velocity and $\mu_2(t)$ the acceleration of the wave v at time t. Note that system (2.5a) may become ill-posed during integration if the leading matrix $M^{-1}A - \mu_1(t)^2 I_m$ loses positivity due to large values of $|\mu_1(t)|$. This may occur even if $M^{-1}A - \mu_\star^2 I_m$ is positive at the final speed $\mu_\star$ of the wave, cf. the numerical experiments in Section 2.3. We next specify initial data for the system (2.5) as follows,

$$v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \gamma(0) = 0. \tag{2.6}$$

Note that, requiring $\gamma(0) = 0$ and $\mu_1(0) = \mu_1^0$, the first equation in (2.6) follows from (2.2) and (2.1b), while the second condition in (2.6) can be deduced from (2.3), (2.1b), (2.5c). At first glance the initial value μ_1^0 can be taken arbitrarily and set to zero, for example. But, depending on the solver used, it can be advantageous to define the value μ_1^0 such that it is consistent with the algebraic constraint to be discussed below.

To compensate the extra variable μ_2 in the system (2.5), we impose an additional scalar algebraic constraint, also known as a phase condition, of the general form

$$\psi^{\mathrm{2nd}}(v, v_t, \mu_1, \mu_2) = 0, \ t \geqslant 0. \tag{2.7}$$

Together with (2.5) this will lead to a partial differential algebraic equation (PDAE). For the phase condition we require that it vanishes at the traveling wave solution

$$\psi^{\mathrm{2nd}}(v_\star, 0, \mu_\star, 0) = 0. \tag{2.8}$$

In essence, this condition singles out one element from the family of shifted profiles $v_\star(\cdot - \gamma), \gamma \in \mathbb{R}$.

In the following we discuss two possible choices for a phase condition:

Type 1: (**fixed phase condition**). Let $\hat{v} : \mathbb{R} \to \mathbb{R}^m$ denote a time-independent and sufficiently smooth template (or reference) function, e.g. $\hat{v} = u_0$. Then we consider the following fixed phase condition

$$\psi^{\mathrm{2nd}}_{\mathrm{fix},3}(v) := \langle v - \hat{v}, \hat{v}_\xi \rangle_{L^2} = 0, \ t \geqslant 0. \tag{2.9}$$

This condition is obtained from minimizing the L^2-distance of the shifted versions of v from the template $\hat{v}$ at each time instance

$$\rho(\gamma) := \|v(\cdot,t) - \hat{v}(\cdot - \gamma)\|^2_{L^2} = \|v(\cdot + \gamma, t) - \hat{v}(\cdot)\|^2_{L^2}.$$

The necessary condition for a local minimum to occur at $\gamma = 0$ is

$$0 \overset{!}{=} \left[\frac{d}{d\gamma} \langle v(\cdot,t) - \hat{v}(\cdot - \gamma), v(\cdot,t) - \hat{v}(\cdot - \gamma)\rangle_{L^2}\right]_{\gamma=0} = 2 \langle v(\cdot,t) - \hat{v}, \hat{v}_\xi \rangle_{L^2}, \ t \geqslant 0.$$

To reduce the index of the resulting PDAE, we differentiate (2.9) w.r.t. t and obtain

$$\psi^{\mathrm{2nd}}_{\mathrm{fix},2}(v_t) := \langle v_t, \hat{v}_\xi \rangle_{L^2} = 0, \ t \geqslant 0. \tag{2.10}$$

Finally, differentiating (2.10) once more w.r.t. t and using the differential equation (2.5a) yields the following condition

$$\begin{aligned} \psi^{\mathrm{2nd}}_{\mathrm{fix},1}(v, v_t, \mu_1, \mu_2) :=& \langle M^{-1} \left(-Bv_t + (A - \mu_1^2 M) v_{\xi\xi} + 2\mu_1 M v_{\xi t} + (\mu_1 B + C) v_\xi + f(v)\right), \hat{v}_\xi \rangle_{L^2} \\ &+ \mu_2 \langle v_\xi, \hat{v}_\xi \rangle_{L^2} = 0, \ t \geqslant 0. \end{aligned} \tag{2.11}$$

Note that equation (2.11) can be explicitly solved for μ_2, if the template $\hat{v}$ is chosen such that $\langle v_\xi, \hat{v}_\xi \rangle_{L^2} \neq 0$ for any $t \geqslant 0$.

The numbers $j = 1, 2, 3$ in the notation $\psi^{\mathrm{2nd}}_{\mathrm{fix},j}$ above indicate the index of the resulting PDAE (in a formal sense) as the minimum number of differentiations with respect to t, necessary to obtain an explicit differential equation for the unknowns (v, μ_1, μ_2) (cf. [14, Ch.1], [7, Ch.2]). In general, the value of this (differential) index may depend on the system formulation. For example, if we do not introduce μ_2, but omit (2.5b) from the system and replace μ_2 by $\mu_{1,t}$ in (2.5a), then we need only two differentiations to obtain an explicit differential equation for (v, μ_1). Hence the index is lowered by one (this methodology is described in the ODE setting in [7, Prop.2.5.3]).

Let us note that the index 2 formulation (2.10) and the index 1 formulation (2.11) enforce constraints on $\mu_1(0) = \mu_1^0$ and $\mu_2(0) = \mu_2^0$ in order to have consistent initial values. Setting $t = 0$ in (2.10) and using (2.6) yields the condition

$$\mu_1^0 \langle u_{0,\xi}, \hat{v}_\xi \rangle_{L^2} + \langle v_0, \hat{v}_\xi \rangle_{L^2} = 0, \tag{2.12}$$

from which μ_1^0 can be determined. Further, setting $t = 0$ in (2.11) and using (2.6) leads to an equation from which one can determine μ_2^0 from the remaining initial data

$$0 = \langle (\mu_1^0)^2 u_{0,\xi\xi} + 2\mu_1^0 v_{0,\xi} + M^{-1} \left(-Bv_0 + Au_{0,\xi\xi} + Cu_{0,\xi} + f(u_0)\right), \hat{v}_\xi \rangle_{L^2} + \mu_2^0 \langle u_{0,\xi}, \hat{v}_\xi \rangle_{L^2}. \tag{2.13}$$

Type 2: (**orthogonal phase condition**). The orthogonal phase conditions read as follows:

$$\psi^{\mathrm{2nd}}_{\mathrm{orth},2}(v, v_t) := \langle v_t, v_\xi \rangle_{L^2} = 0, \; t \geqslant 0, \tag{2.14}$$

$$\begin{aligned} \psi^{\mathrm{2nd}}_{\mathrm{orth},1}(v, v_t, \mu_1, \mu_2) :=& \langle M^{-1} \left(-Bv_t + (A - \mu_1^2 M) v_{\xi\xi} + 2\mu_1 M v_{\xi t} + (\mu_1 B + C) v_\xi + f(v)\right), v_\xi \rangle_{L^2} \\ &+ \langle v_t, v_{\xi t} \rangle_{L^2} + \mu_2 \langle v_\xi, v_\xi \rangle_{L^2} = 0, \; t \geqslant 0. \end{aligned} \tag{2.15}$$

For first order evolution equations, condition (2.14) has an immediate interpretation as a necessary condition for minimizing $\|v_t\|_{L^2}$ (cf. [4]). The same interpretation is possible here when applied to a proper formulation as a first order system, see (4.46) in Section 4. For the moment, our motivation is, that this condition expresses orthogonality of v_t to the vector v_ξ tangent to the group orbit $\{v(\cdot - \gamma) : \gamma \in \mathbb{R}\}$ at $\gamma = 0$. For a different kind of orthogonal phase condition that relies on the formulation as a first order system, see (4.45) in Section 4. The condition (2.14) leads to a PDAE of index 2 in the sense above. Differentiating (2.14) w.r.t. t and using (2.5a) implies (2.15) which yields a PDAE of index 1. Note that equation (2.15) can be explicitly solved for μ_2, provided that $\langle v_\xi, v_\xi \rangle_{L^2} \neq 0$ for any $t \geqslant 0$.

Similar to the type 1 phase condition, we obtain constraints for consistent initial values when setting $t = 0$ in (2.14), (2.15). Condition (2.14) leads to an equation for μ_1^0

$$0 = \mu_1^0 \langle u_{0,\xi}, u_{0,\xi} \rangle_{L^2} + \langle v_0, u_{0,\xi} \rangle_{L^2}, \tag{2.16}$$

while (2.15) gives an equation for μ_2^0

$$\begin{aligned} 0 =& \langle 2(\mu_1^0)^2 u_{0,\xi\xi} + 3\mu_1^0 v_{0,\xi} + M^{-1} \left(-Bv_0 + Au_{0,\xi\xi} + Cu_{0,\xi} + f(u_0)\right), u_{0,\xi} \rangle_{L^2} \\ &+ \langle v_0, v_{0,\xi} \rangle_{L^2} + \mu_1^0 \langle v_0, u_{0,\xi\xi} \rangle_{L^2} + \mu_2^0 \langle u_{0,\xi}, u_{0,\xi} \rangle_{L^2}. \end{aligned} \tag{2.17}$$

Let us summarize the system of equations obtained by the freezing method from the original Cauchy problem (2.1). Combining the differential equations (2.5), the initial data (2.6) and the phase condition (2.7), we arrive at the following PDAE to be solved numerically:

$$\begin{aligned} Mv_{tt} + Bv_t &= (A - \mu_1^2 M) v_{\xi\xi} + 2\mu_1 M v_{\xi,t} + (\mu_2 M + \mu_1 B + C) v_\xi + f(v), \\ \mu_{1,t} &= \mu_2, \quad \gamma_t = \mu_1, \end{aligned} \qquad t \geqslant 0, \tag{2.18a}$$

$$0 = \psi^{\mathrm{2nd}}(v, v_t, \mu_1, \mu_2), \qquad t \geqslant 0, \tag{2.18b}$$

$$v(\cdot, 0) = u_0, \quad v_t(\cdot, 0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \gamma(0) = 0. \tag{2.18c}$$

The system (2.18) depends on the choice of phase condition ψ^{2nd} and is to be solved for $(v, \mu_1, \mu_2, \gamma)$ with given initial data (u_0, v_0, μ_1^0). It consists of a PDE for v that is coupled to two ODEs for μ_1 and γ (2.18a) and an algebraic constraint (2.18b) which closes the system. A consistent initial value μ_1^0 for μ_1 is computed from the phase condition and the initial data (cf. (2.12), (2.16)). Further initialization of

the algebraic variable μ_2 is usually not needed for a PDAE-solver but can be provided if necessary (cf. (2.13), (2.17)).
The ODE for γ is called the reconstruction equation in [27]. It decouples from the other equations in (2.18) and can be solved in a postprocessing step. The ODE for μ_1 is the new feature of the PDAE for second order systems which does not appear with first order parabolic and hyperbolic equations, see for instance [5, 24, 4].
Finally, note that $(v,\mu_1,\mu_2)=(v_\star,\mu_\star,0)$ satisfies

$$\begin{aligned}0&=(A-\mu_\star^2 M)v_{\star,\xi\xi}(\xi)+(\mu_\star B+C)v_{\star,\xi}(\xi)+f(v_\star(\xi)),\ \xi\in\mathbb{R},\\0&=\mu_2,\\0&=\psi^{\mathrm{2nd}}(v_\star,0,\mu_\star,0),\end{aligned}$$

and hence is a stationary solution of (2.18a), (2.18b). Obviously, in this case we have $\gamma(t)=\mu_\star t$. For a stable traveling wave we expect that solutions (v,μ_1,μ_2,γ) of (2.18) show the limiting behavior

$$v(t)\to v_\star,\quad \mu_1(t)\to\mu_\star,\quad \mu_2(t)\to 0\quad \text{as}\quad t\to\infty,$$

provided the initial data are close to their limiting values. In Section 4 we will prove rigorous theorems that justify this expectation under suitable conditions.

2.2. Traveling waves related to parabolic equations. The following proposition shows an important relation between traveling waves (1.2) of the damped wave equation (1.1) with an invertible damping matrix $B\in\mathbb{R}^{m,m}$ and traveling waves

$$u_\star(x,t)=w_\star(x-c_\star t),\ x\in\mathbb{R},\ t\geqslant 0, \tag{2.19}$$

with $w_\star:\mathbb{R}\to\mathbb{R}^m$ such that $\lim_{\zeta\to\pm\infty}w_\star(\zeta)=w_\pm\in\mathbb{R}^m$ and $0\neq c_\star\in\mathbb{R}$, of the parabolic equation

$$Bu_t=\tilde{A}u_{xx}+\tilde{C}u_x+f(u),\ x\in\mathbb{R},\ t\geq 0. \tag{2.20}$$

The matrices $\tilde{A},\tilde{C}\in\mathbb{R}^{m,m}$ in (2.20) may differ from A,C from (1.1). This observation goes back to [13] and has also been used in [10]. Note that in this case $w_\star$ solves the traveling wave equation

$$0=\tilde{A}w_{\star,\zeta\zeta}+c_\star Bw_{\star,\zeta}+\tilde{C}w_{\star,\zeta}+f(w_\star),\ \zeta\in\mathbb{R}. \tag{2.21}$$

Proposition 2.1. (i) Let (2.19) be a traveling wave of the parabolic equation (2.20). Then for every $0\neq k\in\mathbb{R}$ and $A,C,M\in\mathbb{R}^{m,m}$, satisfying $\tilde{A}=k^2A-c_\star^2M$, $\tilde{C}=kC$, equation (1.2) with

$$v_\star(\xi)=w_\star(k\xi),\quad \mu_\star=\frac{c_\star}{k} \tag{2.22}$$

defines a traveling wave of the damped wave equation (1.1).
(ii) Conversely, let (1.2) be a traveling wave of (1.1). Then for every $0\neq k\in\mathbb{R}$ equation (2.19) with

$$w_\star(\zeta)=v_\star(\frac{\zeta}{k}),\quad c_\star=\mu_\star k \tag{2.23}$$

defines a traveling wave of (2.20) with $\tilde{A}=k^2(A-\mu_\star^2M)$, $\tilde{C}=kC$.

Proof. (i) Let $u_\star(x,t)=w_\star(x-c_\star t)$ from (2.19) be a traveling wave of (2.20), then $w_\star$ satisfies (2.21). Now, let $0\neq k\in\mathbb{R}$ and $A,M\in\mathbb{R}^{m,m}$ be such that $\tilde{A}=k^2A-c_\star^2M$ is satisfied and define $(v_\star,\mu_\star)$ as in (2.22). Then, $u_\star$ from (1.2)

$$u_\star(x,t)=v_\star(x-\mu_\star t)=w_\star(k(x-\mu_\star t))$$

satisfies

$$\begin{aligned}-Mu_{\star,tt}-Bu_{\star,t}+Au_{\star,xx}+Cu_{\star,x}+f(u_\star)&=-k^2\mu_\star^2Mw_{\star,\zeta\zeta}+k\mu_\star Bw_{\star,\zeta}+k^2Aw_{\star,\zeta\zeta}\\&\quad+kCw_{\star,\zeta}+f(w_\star)=\tilde{A}w_{\star,\zeta\zeta}+c_\star Bw_{\star,\zeta}+\tilde{C}w_{\star,\zeta}+f(w_\star)=0.\end{aligned}$$

(ii) Let $u_\star(x,t) = v_\star(x-\mu_\star t)$ from (1.2) be a traveling wave of (1.1), then $v_\star$ satisfies (1.5). Now, let $0 \neq k \in \mathbb{R}$ and define $\tilde{A} := k^2(A - \mu_\star^2 M) \in \mathbb{R}^{m,m}$, $\tilde{C} := kC$ and $(w_\star, c_\star)$ as in (2.23). Then, $u_\star$ from (2.19)

$$u_\star(x,t) = w_\star(x - c_\star t) = v_\star\left(\frac{x - c_\star t}{k}\right)$$

satisfies

$$\begin{aligned}-Bu_{\star,t} + \tilde{A}u_{\star,xx} + \tilde{C}u_{\star,x} + f(u_\star) =& \frac{c_\star}{k}Bv_{\star,\xi} + \frac{1}{k^2}\tilde{A}v_{\star,\xi\xi} + \frac{1}{k}\tilde{C}v_{\star,\xi} + f(v_\star) \\ =& (A - \mu_\star^2 M)v_{\star,\xi\xi} + \mu_\star Bv_{\star,\xi} + Cv_{\star,\xi} + f(v_\star) = 0.\end{aligned}$$

□

According to Proposition 2.1, any traveling wave (2.19) of the parabolic equation (2.20) leads to a traveling wave (1.2) of the damped wave equation (1.1) and vice versa.

Remark 2.2. *Note that the profiles $v_\star, w_\star$ and the velocities $\mu_\star, c_\star$ coincide if $k = 1$. In this case $\tilde{A} = A - c_\star^2 M$, and the matrices A and $\tilde{A}$ are different (provided $c_\star \neq 0$). If we insist on $A = \tilde{A}$ then the profiles will be different.*
In case $C = 0$ both equations (1.1), (2.20) share a symmetry property: if $v_\star(\xi)(\xi \in \mathbb{R}), c_\star$ resp. $w_\star(\zeta)(\zeta \in \mathbb{R}), \mu_\star$ is a traveling wave then so is the reflected pair $v_\star(-\xi)(\xi \in \mathbb{R}), -c_\star$ resp. $w_\star(-\zeta)(\zeta \in \mathbb{R}), -\mu_\star$. Thus, choosing $k < 0$ in (2.22) resp. (2.23) will not produce new waves other than those induced by reflection symmetry. Therefore, we will assume k to be positive in the following.
It is instructive to consider two limiting cases of the transformation (2.22) when a traveling wave $w_\star$ with velocity $c_\star \neq 0$ is given for the parabolic equation (2.20).
First assume $A = \tilde{A}$ and let $M \to 0$. Then the relation $\tilde{A} = k^2 A - c_\star^2 M$ implies $k \to 1$ and $v_\star \to w_\star$, $\mu_\star \to c_\star$. Thus the profile and the velocity of the traveling waves (1.2) of the system (1.1) converge to the correct limit in the parabolic case. Second, consider the scalar case, fix $A > 0$ and let $M \to \infty$. Then the relation $\tilde{A} = k^2 A - c_\star^2 M$ implies $k \to \infty$ and $\mu_\star = \frac{c_\star}{k} \to 0$. Thus a large value of M creates a slow wave for the system (1.1) which has steep gradients in its profile due to $v_{\star,\xi}(\xi) = k w_{\star,\zeta}(k\xi)$.

2.3. Applications and numerical examples.

In the following we consider two examples with nonlinearities of Nagumo and FitzHugh-Nagumo type. We use the mechanism from Proposition 2.1 to obtain traveling waves in these damped wave equations and then solve the PDAE (2.18) providing us with wave profiles, their positions, velocities and accelerations. All numerical computations in this paper were done with Comsol Multiphysics 5.2, [1]. Specific data of time and space discretization are given below.

Example 2.3 (Nagumo wave equation)**.** Consider the scalar parabolic Nagumo equation, [20], [21],

$$u_t = u_{xx} + f(u),\ x \in \mathbb{R},\ t \geq 0, \quad f(u) = u(1-u)(u-b), \tag{2.24}$$

with $u = u(x,t) \in \mathbb{R}$ and some fixed $b \in (0,1)$. It is well known that (2.24) has an explicit traveling front solution $u_\star(x,t) = w_\star(x - c_\star t)$ given by

$$w_\star(\zeta) = \left(1 + \exp\left(-\frac{\zeta}{\sqrt{2}}\right)\right)^{-1}, \quad c_\star = -\sqrt{2}\left(\frac{1}{2} - b\right),$$

with asymptotic states $w_- = 0$ and $w_+ = 1$. Note that $c_\star < 0$ if $b < \frac{1}{2}$ and $c_\star > 0$ if $b > \frac{1}{2}$. Proposition 2.1(i) implies that the corresponding Nagumo wave equation

$$\varepsilon u_{tt} + u_t = u_{xx} + u(1-u)(u-b),\ x \in \mathbb{R},\ t \geq 0, \tag{2.25}$$

has a traveling front solution $u_\star(x,t) = v_\star(x-\mu_\star t)$ given by

$$v_\star(\xi) = w_\star(k\xi), \quad \mu_\star = \frac{-\sqrt{2}\left(\frac{1}{2}-b\right)}{k}, \quad k = \left(1+2\varepsilon\left(\frac{1}{2}-b\right)^2\right)^{1/2}. \tag{2.26}$$

Figure 2.1 shows the time evolution of the traveling front solution u of (2.25) on the spatial domain $(-50,50)$ with homogeneous Neumann boundary conditions and initial data

$$u_0(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}, \quad v_0(x) = 0, \quad x \in (-50,50). \tag{2.27}$$

Further parameter values are $\varepsilon = b = \frac{1}{4}$. For the space discretization we used continuous piecewise linear finite elements with spatial stepsize $\triangle x = 0.1$. For the time discretization we used the BDF method of order 2 with absolute tolerance atol $= 10^{-3}$, relative tolerance rtol $= 10^{-2}$, temporal stepsize $\triangle t = 0.1$ and final time $T = 150$.

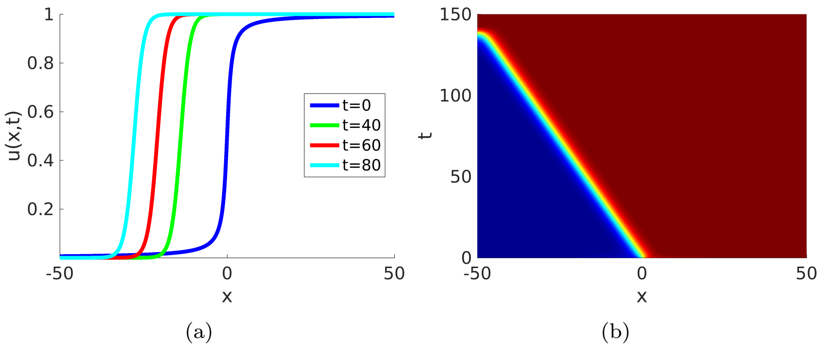


FIGURE 2.1. Traveling front of Nagumo wave equation (2.25) at different time instances (a) and its time evolution (b) for parameters $\varepsilon = b = \frac{1}{4}$.

Next we solve with the same data the frozen Nagumo wave equation resulting from (2.18)

$$\begin{aligned} &\varepsilon v_{tt} + v_t = (1-\mu_1^2\varepsilon)v_{\xi\xi} + 2\mu_1\varepsilon v_{\xi,t} + (\mu_2\varepsilon+\mu_1)v_\xi + v(1-v)(v-b), \\ &\mu_{1,t} = \mu_2, \quad \gamma_t = \mu_1, \end{aligned} \qquad t \geqslant 0, \tag{2.28a}$$

$$0 = \left\langle v_t(\cdot,t), \hat{v}_\xi \right\rangle_{L^2(\mathbb{R},\mathbb{R})}, \qquad t \geqslant 0, \tag{2.28b}$$

$$v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \gamma(0) = 0. \tag{2.28c}$$

Figure 2.2 shows the solution (v,μ_1,μ_2,γ) of (2.28) on the spatial domain $(-50,50)$ with homogeneous Neumann boundary conditions, initial data u_0, v_0 from (2.27), and reference function $\hat{v} = u_0$. For the computation we used the fixed phase condition $\psi^{\text{2nd}}_{\text{fix},2}(v_t)$ from (2.10) with consistent intial data μ_1^0, μ_2^0, c.f. (2.12) and (2.13). Note that $v_0 = 0$ from (2.27) implies $\mu_1^0 = 0$ according to (2.12). Then, inserting $\mu_1^0 = 0$, u_0, v_0 from (2.27), $\hat{v} = u_0$, $M = \varepsilon$, $A = B = 1$, $C = 0$ and f from (2.24) into (2.13), finally implies $\mu_2^0 = -1.0312$. The discretization data are taken as in the nonfrozen case. We emphasize that the initial data u_0, v_0 must be chosen sufficiently smooth since the PDEs (2.25) and (2.28a) are of hyperbolic-type. The diagrams show that after a very short transition phase the profile becomes stationary, the acceleration μ_2 converges to zero, and the speed μ_1 approaches to an asymptotic value $\mu_\star^{\text{num}}$ which is close to the exact value $\mu_\star \approx -0.34816$. We expect the error $|\mu_\star - \mu_\star^{\text{num}}| \to 0$ as the domain $(-R,R)$ grows and stepsizes tend to zero.

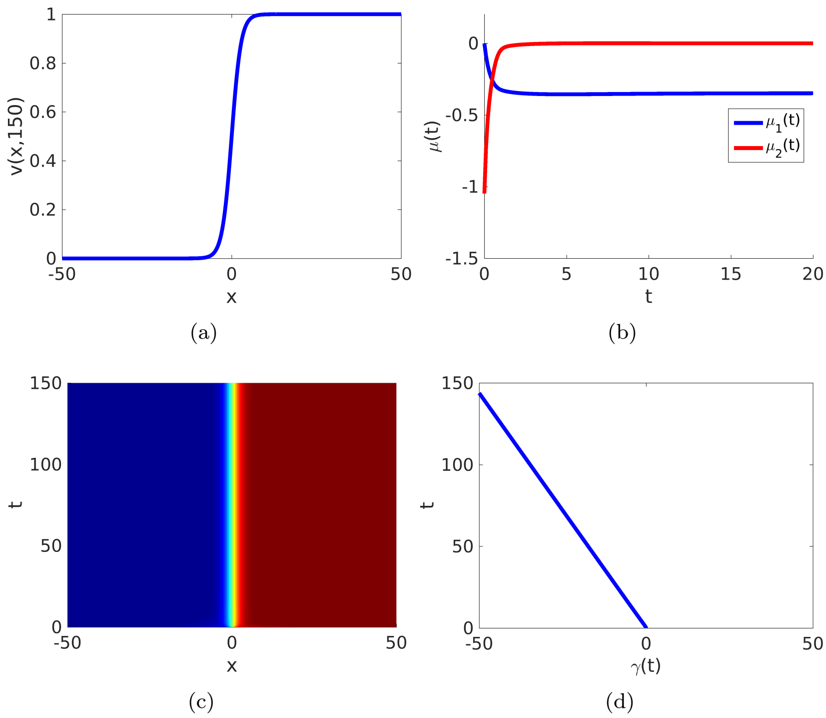


FIGURE 2.2. Solution of the frozen Nagumo wave equation (2.28): approximation of profile $v(x,150)$ (a), time evolutions of velocity μ_1 and acceleration μ_2 (b), of the profile v (c), and of the position γ (d) for parameters $\varepsilon = b = \frac{1}{4}$.

Figure 2.2(d) shows the function $\gamma(t), t \in [0,150]$, obtained by integrating the last equation in (2.28a). From its values one can still recover the position of the front in the original system (2.25). In particular, one can read off from Figure 2.2(d) that the wave hits the left boundary at $x = -50$ at time $t \approx 143.82$ (cf. Figure 2.1(b)).

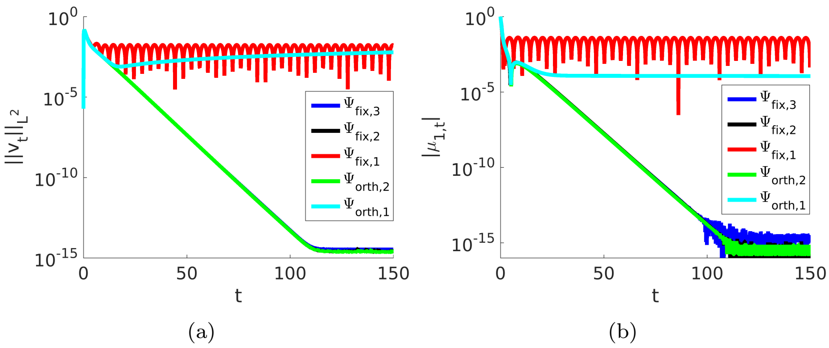


FIGURE 2.3. Comparison of the phase conditions for the frozen Nagumo wave equation (2.28): time evolution of $\|v_t\|_{L^2}$ (a) and $|\mu_{1,t}|$ (b) for parameters $\varepsilon = b = \frac{1}{4}$.

If we replace the phase condition $\psi^{\text{2nd}}_{\text{fix},2}$ in (2.28) by $\psi^{\text{2nd}}_{\text{fix},3}$ or $\psi^{\text{2nd}}_{\text{orth},2}$, we obtain very similar results as those from Figure 2.2. The profile again becomes stationary, the acceleration μ_2 converges to zero, and the speed μ_1 approaches to an asymptotic value. Since we expect $v_t(t) \to 0$ and $\mu_{1,t}(t) \to 0$ as $t \to \infty$, we use these quantities as an indicator checking whether the solution has reached its equilibrium. Figure 2.3 shows the time evolution of $\|v_t\|_{L^2}$ and $|\mu_{1,t}|$ when solving (2.28) for different phase conditions. Both index one formulations do not provide us convergence against an equilibrium. This seems somewhat suprising and may be caused by the solver or the settings used. Further investigations have shown that the consistency condition for μ_2^0 does not really affect the numerical results for the different phase conditions. Therefore, in the next example we avoid to compute the complicated expression for μ_2^0 and consider directly the expected limiting value $\mu_2^0 = 0$. By contrast the consistency condition for μ_1^0 is crucial since otherwise we may loose positivity of the leading term $M^{-1}A - \mu_1^2 I_m$.

Example 2.4 (FitzHugh-Nagumo wave system)**.** Consider the parabolic 2-dimensional FitzHugh-Nagumo system, [8],

$$u_t = \tilde{A}u_{xx} + f(u),\ x \in \mathbb{R},\ t \geq 0, \quad \tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix},\ f(u) = \begin{pmatrix} u_1 - \frac{1}{3}u_1^3 - u_2 \\ \phi(u_1 + a - bu_2) \end{pmatrix}, \tag{2.29}$$

with $u = u(x,t) \in \mathbb{R}^2$ and positive parameters $\rho, a, b, \phi \in \mathbb{R}$. There are no explicit traveling waves available, but it is known, that (2.29) has traveling pulse solutions at parameter values $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$, $b = 0.8$ with approximately

$$w_{\pm} = \begin{pmatrix} -1.19941 \\ -0.62426 \end{pmatrix}, \quad c_\star = -0.7892, \tag{2.30}$$

and traveling front solutions for the same ρ, a, ϕ but $b = 3$ with asymptotic states and velocity approximately given by

$$w_- = \begin{pmatrix} 1.18779 \\ 0.62923 \end{pmatrix}, \quad w_+ = \begin{pmatrix} -1.56443 \\ -0.28814 \end{pmatrix}, \quad c_\star = -0.8557.$$

Applying Proposition 2.1(i) with $M = \varepsilon I_2$ for some $\varepsilon > 0$, requires the equality

$$\begin{pmatrix} 1 + c_\star^2\varepsilon & 0 \\ 0 & \rho + c_\star^2\varepsilon \end{pmatrix} = \tilde{A} + c_\star^2 M = k^2 A = k^2 \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

which leads to $A_{12} = A_{21} = 0$, $1 + c_\star^2\varepsilon = k^2 A_{11}$ and $\rho + c_\star^2\varepsilon = k^2 A_{22}$. Setting $A_{11} := 1$, Proposition 2.1(i) implies that for the above choices of ρ, a, ϕ, b the corresponding FitzHugh-Nagumo wave system

$$Mu_{tt} + Bu_t = Au_{xx} + \begin{pmatrix} u_1 - \frac{1}{3}u_1^3 - u_2 \\ \phi(u_1 + a - bu_2) \end{pmatrix},\ x \in \mathbb{R},\ t \geq 0, \tag{2.31}$$

with

$$M = \varepsilon I_2, \quad B = I_2, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\rho + c_\star^2\varepsilon}{1 + c_\star^2\varepsilon} \end{pmatrix}, \quad k = \sqrt{1 + c_\star^2\varepsilon}, \quad \varepsilon > 0, \quad \rho, c_\star \text{ given}$$

has a traveling pulse (or a traveling front) solution with a scaled profile $v_\star$ and velocity $\mu_\star = \frac{c_\star}{k}$.
In the following we show the computations for the traveling pulse. Results for the traveling front are very similar and are not displayed here. In the frozen and the nonfrozen case, we choose parameter values $\varepsilon = 10^{-2}$, $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$ and $b = 0.8$ and discretize space and time as in Example 2.3. Figure 2.4 shows the time evolution of the traveling pulse solution $u = (u_1, u_2)^T$ of (2.31) on the spatial domain $(-50, 50)$ with homogeneous Neumann boundary conditions. The initial data are

$$u_0(x) = \begin{pmatrix} \frac{1}{\pi}\arctan(x) + \frac{1}{2} \\ 0 \end{pmatrix} + v_{\pm}, \quad v_0(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.32}$$

where $v_{\pm} = w_{\pm}$ is the asymptotic state from (2.30).

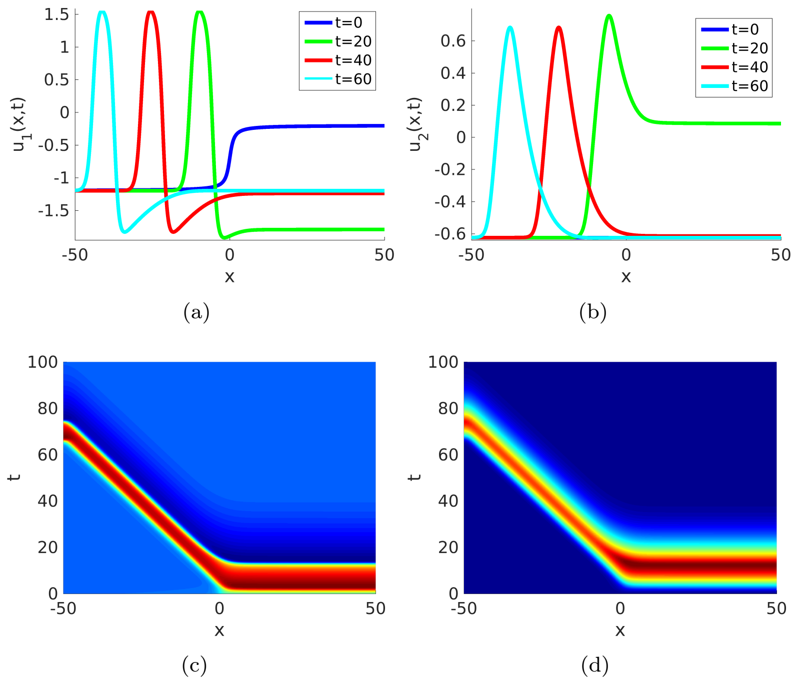


FIGURE 2.4. Traveling pulse of FitzHugh-Nagumo wave system (2.31) at different time instances for u_1 (a) and u_2 (b), as well as their time evolutions (c), (d) for parameters $\varepsilon = 10^{-2}$, $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$ and $b = 0.8$.

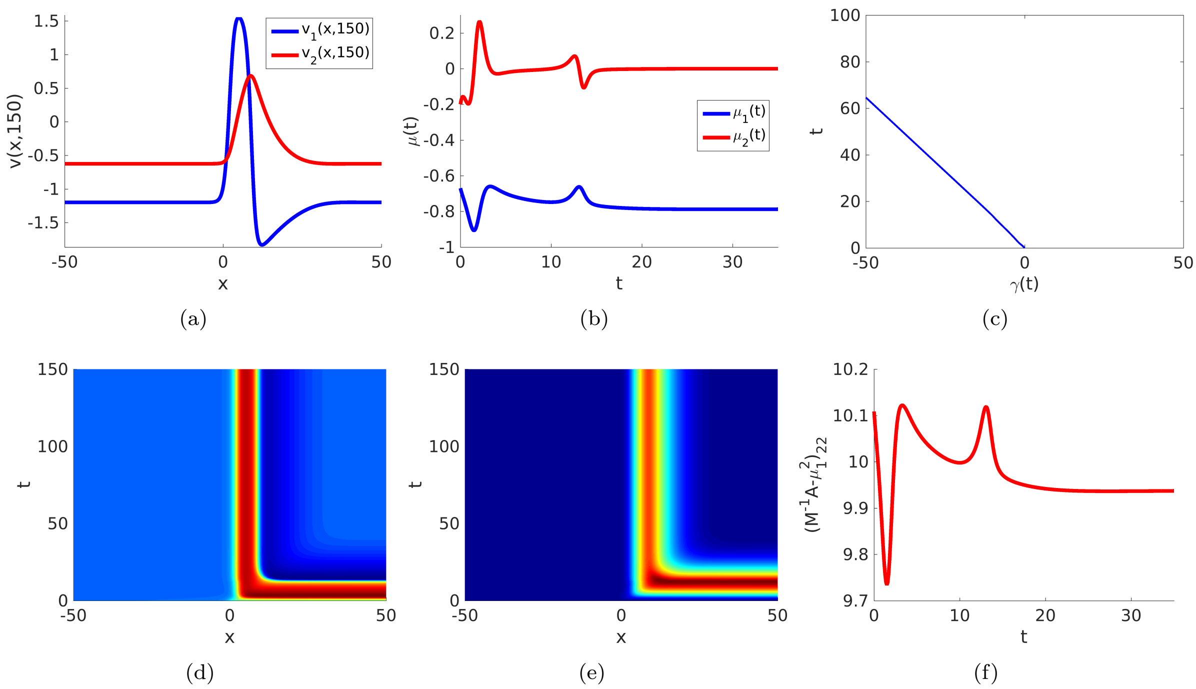
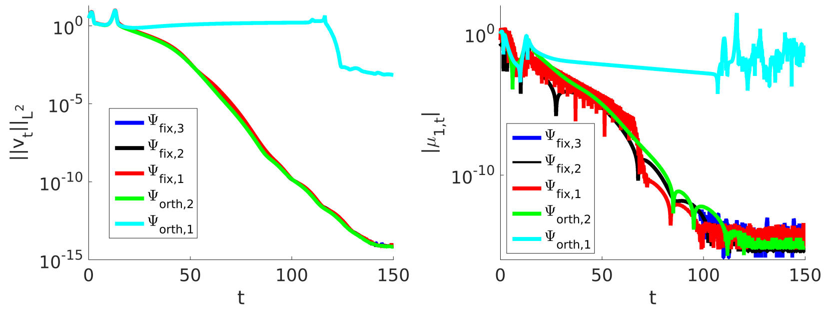


FIGURE 2.5. Solution of the frozen FitzHugh-Nagumo wave system (2.18): approximation of profile components $v_1(x, 150)$, $v_2(x, 150)$ (a), time evolutions of velocity μ_1 and acceleration μ_2 (b), of the position γ (c), of the profile's components v_1 (d) and v_2 (e), and of the diffusion matrix component $(M^{-1}A - \mu_1^2 I_2)_{22}$ (f) for parameters $\varepsilon = 10^{-2}$, $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$ and $b = 0.8$.

Next consider for the same parameter values the corresponding frozen FitzHugh-Nagumo wave system (2.18). Figure 2.5 shows the solution $(v, \mu_1, \mu_2, \gamma)$ of (2.18) on the spatial domain $(-50, 50)$, with homogeneous Neumann boundary conditions, initial data u_0, v_0 from (2.32), and reference function $\hat{v} = u_0$. For the computation we used again the fixed phase condition $\psi^{\text{2nd}}_{\text{fix},2}(v_t)$ from (2.10) with consistent intial data for μ_1^0. Note that $v_0 = 0$ from (2.32) implies $\mu_1^0 = 0$ according to (2.12). Further we used the initial value $\mu_2^0 = 0$ which does not satisfy the consistency condition from (2.13). Time and space discretization are done as in the nonfrozen case. Again the profile quickly stabilizes and the velocity and the acceleration reach their asymptotic values. Figure 2.5(f) shows the time evolution for the smallest diagonal entry of the diffusion matrix. Since the function $(M^{-1}A - \mu_1(t)^2)_{22}$ does not fall below zero, the diffusion matrix keep positivity during the whole computation.



(a) (b)

FIGURE 2.6. Comparison of the phase conditions for the frozen FitzHugh-Nagumo wave system (2.18): $\|v_t\|_{L^2}$ (a) and $|\mu_{1,t}|$ (b) for parameters $\varepsilon = 10^{-2}$, $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$ and $b = 0.8$.

Finally, Figure 2.6 shows that similar results are obtained if we replace the phase condition $\psi^{\text{2nd}}_{\text{fix},2}$ in frozen FitzHugh-Nagumo wave system (2.18) by $\psi^{\text{2nd}}_{\text{fix},3}$, $\psi^{\text{2nd}}_{\text{orth},2}$, or even by $\psi^{\text{2nd}}_{\text{fix},1}$. We emphasize that in this example the fixed phase condition of index 1 provides us good results, while the index 1 formulation of the orthogonal phase condition $\psi^{\text{2nd}}_{\text{orth},1}$ does not yield the expected convergence against the relative equilibrium.

3. Spectra and eigenfunctions of traveling waves

In this section we study the spectrum of the quadratic operator polynomial (cf. (1.7))

$$\mathcal{P}(\lambda) := \lambda^2 P_2 + \lambda P_1 + P_0, \quad \lambda \in \mathbb{C}. \tag{3.1}$$

Here the differential operators P_j are defined by

$$P_2 = M, \quad P_1 = B - 2\mu_\star M \frac{\partial}{\partial \xi}, \quad P_0 = -(A - \mu_\star^2 M)\frac{\partial^2}{\partial \xi^2} - (C + \mu_\star B)\frac{\partial}{\partial \xi} - Df(v_\star),$$

and $v_\star$ and $\mu_\star$ denote the profile and the velocity of a traveling wave solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ of (1.1). Note that P_j is a differential operator of order $2-j$ for $j = 0, 1, 2$. In the following we recall some standard notion of point and essential spectrum for operator polynomials.

Definition 3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be complex Banach spaces and let $\mathcal{P}(\lambda) = \sum_{j=0}^q P_j \lambda^j$, $\lambda \in \mathbb{C}$ be an operator polynomial with linear continuous coefficients $P_j : Y \to X$, $j = 0, \dots, q$.*

(a) *A value* $\lambda \in \mathbb{C}$ *lies in the resolvent set* $\rho(\mathcal{P})$ *if* $\mathcal{P}(\lambda) : Y \to X$ *is bijective and has bounded inverse* $\mathcal{P}(\lambda)^{-1} : X \to Y$*. The inverse* $\mathcal{P}(\lambda)^{-1}$ *is called the resolvent of* $\mathcal{P}$ *at* λ*, and* $\sigma(\mathcal{P}) := \mathbb{C}\backslash\rho(\mathcal{P})$ *is called the spectrum of* $\mathcal{P}$*.*

(b) $\lambda_0 \in \sigma(\mathcal{P})$ *is called isolated if there is* $\varepsilon > 0$ *such that* $\lambda \in \rho(\mathcal{P})$ *for all* $\lambda_0 \neq \lambda \in \mathbb{C}$ *with* $|\lambda - \lambda_0| < \varepsilon$*.*

(c) *An element* $\lambda_0 \in \sigma(\mathcal{P})$ *satisfying* $\mathcal{P}(\lambda_0)y = 0$ *for some* $y \neq 0$ *is called an eigenvalue of* $\mathcal{P}$ *and* y *is called an eigenfunction. The eigenvalue* λ_0 *has finite multiplicity if* $\mathcal{P}(\lambda_0)$ *has finite dimensional kernel and if there is a maximum number* $n \in \mathbb{N}$*, for which polynomials* $y(\lambda) = \sum_{j=0}^{r}(\lambda - \lambda_0)^j y_j$ *exist in* Y *satisfying*

$$y_0 \neq 0, \quad (\mathcal{P}y)^{(\nu)}(\lambda_0) = 0, \quad \nu = 0, \ldots, n-1, \quad (\mathcal{P}y)^{(n)}(\lambda_0) \neq 0.$$

(d) $\lambda \in \mathbb{C}$ *is called a normal point of* $\mathcal{P}$ *if*

$$\lambda \in \rho(\mathcal{P}) \quad \text{or} \quad \lambda \in \sigma_{\text{point}}(\mathcal{P}) := \{\lambda \in \sigma(\mathcal{P}) : \ \lambda \text{ is isolated eigenvalue of finite multiplicity}\}.$$

The set $\sigma_{\text{point}}(\mathcal{P})$ *is called the point spectrum of* $\mathcal{P}$*.*

(e) *The essential spectrum of* $\mathcal{P}$ *is defined by*

$$\sigma_{\text{ess}}(\mathcal{P}) := \{\lambda \in \mathbb{C} : \ \lambda \text{ is not a normal point of } \mathcal{P}\}.$$

Remark 3.2. *(i) By the inverse mapping theorem it is enough for* $\lambda \in \rho(\mathcal{P})$ *to require* $\mathcal{P}(\lambda)$ *to be bijective in (a). Concerning (c) we refer to* [17], [19], [18] *for more details on root polynomials, geometric, and algebraic multiplicities. The definition of essential spectrum follows* [15].

(ii) An eigenvalue λ_0 *of* $\mathcal{P}$ *is called simple, if the kernel* $\mathcal{N}(\mathcal{P})$ *of* $\mathcal{P}$ *is one-dimensional,* $\mathcal{N}(\mathcal{P}) = \text{span}(y_0) \neq \{0\}$ *and for all polynomials* $y(\lambda)$ *in* Y *with* $y(\lambda_0) = y_0$ *hold*

$$\mathcal{P}(\lambda_0)y(\lambda_0) = 0, \ (\mathcal{P}y)'(\lambda_0) \neq 0.$$

For the concrete case of $\mathcal{P}$ *given by* (3.1) *and* $\lambda_0 = 0$*, simplicity of the zero eigenvalue means* $\mathcal{N}(P_0) = \text{span}(y_0)$ *and* $P_1 y_0 \notin \mathcal{R}(P_0)$*.*

By definition the spectrum $\sigma(\mathcal{P})$ of $\mathcal{P}$ can be decomposed into its point spectrum and its essential spectrum

$$\sigma(\mathcal{P}) = \sigma_{\text{ess}}(\mathcal{P}) \,\dot{\cup}\, \sigma_{\text{point}}(\mathcal{P}).$$

The function spaces underlying the definition of spectra will be subspaces of $L^2(\mathbb{R}, \mathbb{R}^m)$. But in this section our calculations will be purely formal (without reference to a specific function space) and we postpone the rigorous justification to Section 4.

3.1. Point spectrum on the imaginary axis. Applying $\frac{d}{d\xi}$ to the traveling wave equation, cf. (1.5),

$$0 = (A - \mu_\star^2 M)v_{\star,\xi\xi}(\xi) + (C + \mu_\star B)v_{\star,\xi}(\xi) + f(v_\star(\xi)), \ \xi \in \mathbb{R} \tag{3.2}$$

leads to the equation

$$0 = -\,(A - \mu_\star^2 M)v_{\star,\xi\xi\xi}(\xi) - (C + \mu_\star B)v_{\star,\xi\xi}(\xi) - Df(v_\star(\xi))v_{\star,\xi}(\xi) = P_0 v_{\star,\xi}(\xi), \ \xi \in \mathbb{R},$$

provided that $v_\star \in C^3(\mathbb{R}, \mathbb{R}^m)$ and $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$. This shows that $w = v_{\star,\xi}$ solves the quadratic eigenvalue problem $\mathcal{P}(\lambda)w = 0$ for $\lambda = 0$, and $w = v_{\star,\xi}$ is an eigenfunction if the wave profile $v_\star$ is nontrivial (i.e. not constant). This behavior is to be expected since the original equation is equivariant with respect to the shift, and the spatial derivative $\frac{d}{d\xi}$ is the generator of shift equivariance. Summarizing, we obtain the following result:

Proposition 3.3 (Point spectrum of traveling waves)**.** Let $v_\star \in C^3(\mathbb{R}, \mathbb{R}^m)$, $\mu_\star$ be a nontrivial classical solution of (3.2) and $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$. Then $\lambda = 0$ is an eigenvalue with eigenfunction $v_{\star,\xi}$ of the quadratic eigenvalue problem $\mathcal{P}(\lambda)w = 0$. In particular, $0 \in \sigma_{\text{point}}(\mathcal{P})$.

As usual, further isolated eigenvalues are difficult to detect analytically, and we refer to the extensive literature on solving quadratic eigenvalue problems and on locating zeros of the so-called Evans function, see e.g. [2], [29].

Example 3.4 (Nagumo wave equation)**.** Recall from Example 2.3 that the Nagumo wave equation

$$\varepsilon u_{tt} + u_t = u_{xx} + u(1-u)(u-b),\; x\in\mathbb{R},\; t\geqslant 0,\; 0<b<1,\; \varepsilon>0$$

has an explicit traveling front solution $u_\star(x,t) = v_\star(\xi)$, $\xi = x-\mu_\star t$, with $v_\star$ and $\mu_\star$ from (2.26), i.e. $v_\star$ and $\mu_\star$ solve the associated traveling wave equation

$$0 = (1-\mu_\star^2\varepsilon)v_{\star,\xi\xi}(\xi) + \mu_\star v_{\star,\xi}(\xi) + v_\star(\xi)\left(1-v_\star(\xi)\right)\left(v_\star(\xi)-b\right),\; \xi\in\mathbb{R}.$$

The quadratic eigenvalue problem for the linearization reads as follows,

$$\varepsilon\left(\lambda-\mu_\star\frac{\partial}{\partial\xi}\right)^2 w(\xi) + \left(\lambda-\mu_\star\frac{\partial}{\partial\xi}\right)w(\xi) - w_{\xi\xi}(\xi) + \left(3v_\star^2(\xi)-2(b+1)v_\star(\xi)-b\right)w(\xi) = 0.$$

With k from (2.26), it has the solution

$$\lambda = 0,\quad w(\xi) = v_{\star,\xi}(\xi) = \frac{k}{\sqrt{2}}\exp\left(-\frac{k\xi}{\sqrt{2}}\right)\left(1+\exp\left(-\frac{k\xi}{\sqrt{2}}\right)\right)^{-2},\quad \xi\in\mathbb{R}.$$

3.2. Essential spectrum and dispersion relation of traveling waves. To detect the essential spectrum of $\mathcal{P}$ from (3.1) it is enough to discuss the constant coefficient operators obtained by letting $\xi\to\pm\infty$ in the coefficient $Df(v_\star(\xi))$ of P_0:

$$\begin{aligned}\mathcal{P}^{\pm}(\lambda) :=& \lambda^2 P_2 + \lambda P_1 + P_0^{\pm},\quad \lambda\in\mathbb{C}\\ P_0^{\pm} =& -(A-\mu_\star^2 M)\frac{\partial^2}{\partial\xi^2} - (C+\mu_\star B)\frac{\partial}{\partial\xi} - Df(v_\pm).\end{aligned}$$

We seek bounded solutions v of $\mathcal{P}^{\pm}(\lambda)v = 0$ by the Fourier ansatz $v(\xi) = e^{i\omega\xi}w, w\in\mathbb{C}^m, |w| = 1$ and arrive at the following quadratic eigenvalue problem

$$\mathcal{A}_{\pm}(\lambda,\omega)w = \left(\lambda^2 A_2 + \lambda A_1(\omega) + A_0^{\pm}(\omega)\right)w = 0$$

with matrices

$$A_2 = M,\quad A_1(\omega) = B - 2i\omega\mu_\star M,\quad A_0^{\pm}(\omega) = \omega^2(A-\mu_\star^2 M) - i\omega(C+\mu_\star B) - Df(v_\pm). \tag{3.3}$$

Every $\lambda\in\mathbb{C}$ satisfying the dispersion relation

$$\det\left(\lambda^2 A_2 + \lambda A_1(\omega) + A_0^{\pm}(\omega)\right) = 0 \tag{3.4}$$

for some $\omega\in\mathbb{R}$, i.e. for v_+ or for v_-, belongs to the essential spectrum of $\mathcal{P}$. A proof of this statement is obtained in the standard way by cutting off $v(\xi)$ at $\xi\notin[n,2n]$ resp. $\xi\notin[-2n,-n]$ and letting $n\to\infty$. Then this contradicts the continuity of the resolvent at λ in appropriate function spaces (see Section 4.2 for more details). Summarizing, we have the following result:

Proposition 3.5 (Essential spectrum of traveling waves)**.** Let $f\in C^1(\mathbb{R}^m,\mathbb{R}^m)$ with $f(v_\pm) = 0$ for some $v_\pm\in\mathbb{R}^m$. Let $v_\star\in C^2(\mathbb{R},\mathbb{R}^m)$, $\mu_\star$ be a nontrivial classical solution of (3.2) satisfying $v_\star(\xi)\to v_\pm$ as $\xi\to\pm\infty$. Then, the set

$$\sigma_{\mathrm{disp}}(\mathcal{P}) := \{\lambda\in\mathbb{C}\mid \lambda \text{ satisfies (3.4) for some } \omega\in\mathbb{R}\}$$

belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{P})$ of $\mathcal{P}$.

In the general matrix case it is not easy to analyze the shape of the algebraic set $\sigma_{\mathrm{disp}}(\mathcal{P})$, since (3.4) amounts to finding the zeroes of a polynomial of degree $2m$. In view of the stability results in Theorem 4.20 and Theorem 4.24 our main interest is in finding a spectral gap, i.e. a constant $\beta > 0$ such that

$$\operatorname{Re}\lambda \le -\beta < 0, \quad \text{for all} \quad \lambda \in \sigma_{\mathrm{disp}}(\mathcal{P}). \tag{3.5}$$

We discuss this question for three special cases with $C = 0$.

(i) **Parabolic case.** In the case $M = 0$, $B = I_m$ the dispersion relation (3.4) reads

$$\det\left(\tilde{\lambda} I_m + \omega^2 A - Df(v_\pm)\right) = 0, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star, \tag{3.6}$$

and the corresponding eigenvalue problem may be written as

$$\tilde{\lambda} w = -\left(\omega^2 A - Df(v_\pm)\right) w, \quad 0 \ne w \in \mathbb{C}^m, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star. \tag{3.7}$$

Let us assume positivity of A and $-Df(v_\pm)$ in the sense that

$$\operatorname{Re} w^H A w > 0, \quad \operatorname{Re} w^H Df(v_\pm) w < 0 \quad \text{for all } w \in \mathbb{C}^m. \tag{3.8}$$

Multiplying (3.7) by w^H and taking the real part, shows that the solutions $\tilde{\lambda}$ of (3.6) have negative real parts and the gap is guaranteed. This is still true if A is nonnegative but has zero eigenvalues. Note that in this case, equation (2.1) is of mixed hyperbolic-parabolic type and the nonlinear stability theory becomes considerably more involved, see [26].

(ii) **Undamped hyperbolic case**. In case $B = 0$, $M = I_m$, the dispersion relation (3.4) reads

$$\det\left(\tilde{\lambda}^2 I_m + \omega^2 A - Df(v_\pm)\right) = 0, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star$$

Whenever $\lambda \in \mathbb{C}, \omega \in \mathbb{R}$ solve this system, so does the pair $-\lambda, -\omega$. Hence, the eigenvalues lie either on the imaginary axis or on both sides of the imaginary axis. Therefore, a spectral gap cannot exist. In this case one can only expect stability of the wave (but not asymptotic stability), and we refer to the local stability theory developed in [11],[12] (see also [16] for a recent account). Note that in this case the positivity assumption (3.8) only guarantees $\operatorname{Re}\tilde{\lambda}^2 < 0$, i.e. $\frac{\pi}{4} < |\arg(\lambda - i\omega\mu_\star)| < \frac{3\pi}{4}$ for all eigenvalues $\lambda \in \sigma(\mathcal{A}(\cdot,\omega))$.

(iii) **Scalar case.** It is instructive to discuss the dispersion relation (3.4) in the scalar case with $M = 1$ and real numbers $a, \eta, \delta > 0$

$$\tilde{\lambda}^2 + \eta\tilde{\lambda} + a\omega^2 + \delta = 0, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star. \tag{3.9}$$

This case occurs with the Nagumo wave equation below. The solutions are

$$\lambda = i\omega\mu_\star - \frac{\eta}{2} \pm \left(\frac{\eta^2}{4} - \delta - \omega^2 a\right)^{1/2}, \quad \omega \in \mathbb{R}.$$

If $\eta^2 \le 4\delta$, then all solutions λ of (3.9) lie on the vertical line $\operatorname{Re}\lambda = -\frac{\eta}{2} < 0$. A computation shows that they cover this line under the assumption $\mu_\star^2 < a$, which is precisely the condition ensuring wellposedness of the damped wave equation that belongs to (3.9). If $\eta^2 > 4\delta$ then the solutions λ of (3.9) lie again on this line (resp. cover it if $\mu_\star^2 < a$) for values $|\omega| \ge \omega_0 := (\frac{1}{a}(\frac{\eta^2}{4} - \delta))^{1/2}$. But for values $|\omega| \le \omega_0$ they form the ellipse

$$\frac{(\operatorname{Re}\lambda + \frac{\eta}{2})^2}{p_1^2} + \frac{(\operatorname{Im}\lambda)^2}{p_2^2} = 1, \quad \text{with semiaxes} \quad p_1 = a^{1/2}\omega_0, \ \ p_2 = |\mu_\star|\omega_0. \tag{3.10}$$

The rightmost point of the ellipse $-\beta := -\frac{\eta}{2} + \left(\frac{\eta^2}{4} - \delta\right)^{1/2}$ is still negative and therefore can be taken for the spectral gap (3.5).

Example 3.6 (Spectrum of Nagumo wave equation)**.** As in Example 2.3 consider the Nagumo wave equation (2.24) with coefficients

$$M = \varepsilon > 0, \quad A = B = 1, \quad C = 0.$$

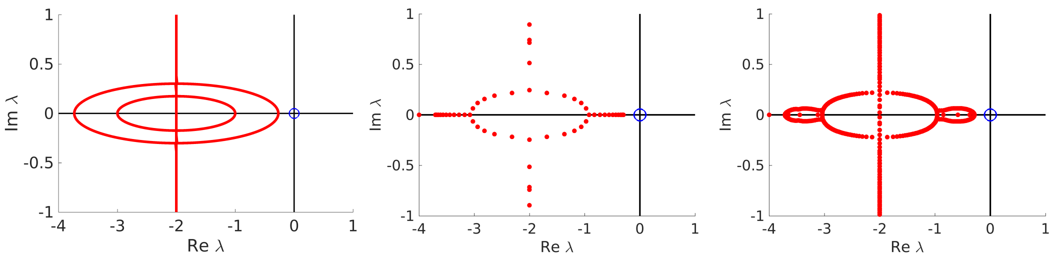
There is a traveling front solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with $v_\star$, $\mu_\star$ from (2.26). With the asymptotic states $v_+ = 1$, $v_- = 0$ and $f'(v_+) = b - 1$, $f'(v_-) = -b$ from (2.24), we find the dispersion relation

$$\varepsilon\tilde{\lambda}^2 + \tilde{\lambda} + \omega^2 + b = 0 \quad \text{or} \quad \varepsilon\tilde{\lambda}^2 + \tilde{\lambda} + \omega^2 - b + 1 = 0.$$

The scalar case discussed above applies with the settings $\eta = \frac{1}{\varepsilon} = a$, $\delta_\pm = -\frac{f'(v_\pm)}{\varepsilon}$. Thus the subset $\sigma_{\text{disp}}(\mathcal{P})$ of the essential spectrum lies on the union of the line $\operatorname{Re}\lambda = -\frac{1}{2\epsilon}$ and possibly two ellipses defined by (3.10) with $\omega_0 = \omega_\pm = \left(\frac{1}{4\varepsilon} + f'(v_\pm)\right)^{1/2}$. The ellipse belonging to v_+ occurs if $1 - b < \frac{1}{4\varepsilon}$, and the one belonging to v_- occurs if $b < \frac{1}{4\varepsilon}$. Since $0 < b < 1$ both ellipses show up in $\sigma_{\text{disp}}(\mathcal{P})$ if $\varepsilon \leq \frac{1}{4}$. In any case, there is a gap beween the essential spectrum and the imaginary axis in the sense of (3.5) with

$$\beta = \frac{1}{2\varepsilon}\left(1 - (1 - 4\varepsilon\max(b, 1-b))^{1/2}\right).$$

Figure 3.1(a) shows part of the spectrum of the Nagumo wave guaranteed by our propositions at parameter values $\varepsilon = b = \frac{1}{4}$. It is subdivided into point spectrum (blue circle) determined by Proposition 3.3 and essential spectrum (red lines) determined by Proposition 3.5. There may be further isolated eigenvalues. The numerical spectrum of the Nagumo wave equation on the spatial domain $[-R, R]$ equipped with periodic boundary conditions is shown in Figure 3.1(b) for $R = 50$ and in Figure 3.1(c) for $R = 400$. Each of them consists of the approximations of the point spectrum (blue circle) and of the essential spectrum (red dots). The missing line inside the ellipse in Figure 3.1(b) gradually appears numerically when enlarging the spatial domain, see Figure 3.1(c). The second ellipse only develops on even larger domains.



(a) (b) (c)

FIGURE 3.1. Essential spectrum of the Nagumo wave equation for parameters $\varepsilon = b = \frac{1}{4}$ (a) and the numerical spectrum on the spatial domain $[-R, R]$ for $R = 50$ (b) and $R = 400$ (c).

Example 3.7 (Spectrum of FitzHugh-Nagumo wave system)**.** As shown in Example 2.4, the FitzHugh-Nagumo wave system (2.29) with coefficient matrices

$$M = \varepsilon I_2, \quad B = I_2, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\rho + c_\star^2\varepsilon}{1 + c_\star^2\varepsilon} \end{pmatrix}, \quad C = 0$$

and parameters

$$\rho = 0.1, \quad \phi = 0.08, \quad a = 0.7, \quad b = 0.8, \quad \varepsilon > 0 \tag{3.11}$$

has a traveling pulse solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with

$$\mu_\star = \frac{c_\star}{k}, \quad k = \sqrt{1 + c_\star^2 \varepsilon}, \quad c_\star \approx -0.7892.$$

The profile $v_\star$ connects the asymptotic state $v_\pm$ from (2.30) with itself, i.e. $v_\star(\xi) \to v_\pm$ as $\xi \to \pm\infty$. The profile $v_\star$ and the velocity $\mu_\star$ are obtained from the simulation performed in Example 2.4. The FitzHugh-Nagumo nonlinearity f from (2.29) satisfies

$$f(v_\pm) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad Df(v_\pm) = \begin{pmatrix} 1 - \left(v_\pm^{(1)}\right)^2 & -1 \\ \phi & -b\phi \end{pmatrix}.$$

The dispersion relation for the FitzHugh-Nagumo pulse states that every $\lambda \in \mathbb{C}$ satisfying

$$\det \begin{pmatrix} \varepsilon\lambda^2 + p\lambda + q_1 & 1 \\ -\phi & \varepsilon\lambda^2 + p\lambda + q_2 \end{pmatrix} = 0. \tag{3.12}$$

for some $\omega \in \mathbb{R}$ belongs to $\sigma_{\mathrm{ess}}(\mathcal{P})$, where we used the abbreviations

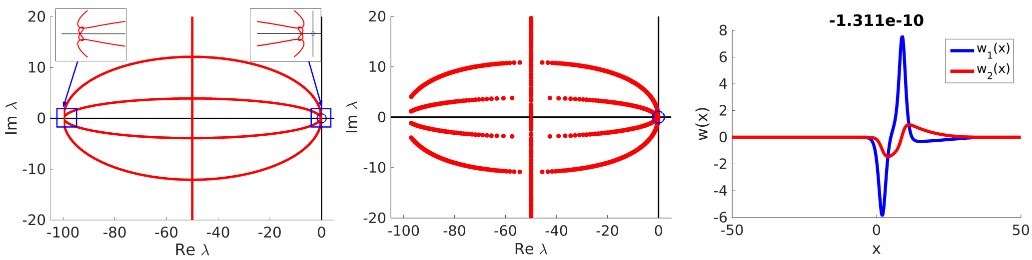
$$p = 1 - 2i\omega\mu_\star\varepsilon, \; q_1 = \omega^2(1 - \mu_\star^2\varepsilon) - i\omega\mu_\star - (1 - (v_\pm^{(1)})^2), \; q_2 = \omega^2\left(\frac{\rho + c_\star^2\varepsilon}{1 + c_\star^2\varepsilon} - \mu_\star^2\varepsilon\right) - i\omega\mu_\star + b\phi.$$

Note that (3.12) leads to the quartic problem

$$0 = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

with ω-dependent coefficients

$$a_4 = \varepsilon^2, \quad a_3 = 2\varepsilon p, \quad a_2 = \varepsilon(q_1 + q_2) + p^2, \quad a_1 = p(q_1 + q_2), \quad a_0 = q_1 q_2 + \phi.$$



(a) (b) (c)

FIGURE 3.2. Essential spectrum of the FitzHugh-Nagumo wave system for parameters from (3.11) and $\varepsilon = 10^{-2}$ (a), the numerical spectrum (b) and both components of the eigenfunction belonging to $\lambda \approx 0$ (c).

Instead of this we solved numerically the quadratic eigenvalue problem (3.12). In this way we obtain analytical information about the spectrum of the FitzHugh-Nagumo pulse shown in Figure 3.2(a) (red lines) for $\varepsilon = 10^{-2}$. Again part of the point spectrum (blue circle) is determined by Proposition 3.3 and the essential spectrum (red lines) by Proposition 3.5. Zooming into the essential spectrum shows that the parabola-shaped structure contains at both ends a loop which is already known from the first order limit case, see [3]. From these results it is obvious that there is again a spectral gap to the imaginary axis, but we have no analytic expression for this gap. The numerical spectrum for periodic boundary conditions is shown in Figure 3.2(b). It consists of the approximations of the point spectrum (blue circle) and of the essential spectrum (red dots). Figure 3.2(c) shows the approximation of both components w_1 and w_2 of the eigenfunction $w(x) \approx v_{\star,x}(x)$ belonging to the small eigenvalue $\lambda = 1.311 \cdot 10^{-10}$ which approximates the eigenvalue 0. Note that an approximation of $v_\star = (v_{\star,1}, v_{\star,2})^T$ was provided in Figure 2.5(a).

4. First order systems and stability of traveling waves

In this section we transform the original second order damped wave equation (1.1) into a first order system of double size. To the first order system we then apply stability results from [25] and derive asymptotic stability of traveling waves for the original second order problem and the second order freezing method. The nonlinearity f is assumed to be sufficiently smooth. To be more precise, we impose the following condition.

Assumption 4.1. *The function $f : \mathbb{R}^m \to \mathbb{R}^m$ satisfies $f \in C^3(\mathbb{R}^m, \mathbb{R}^m)$.*

4.1. Transformation to first order system. For the transformation of the m-dimensional damped wave equation (1.1) into a $2m$-dimensional hyperbolic first order system we require the following well-posedness condition.

Assumption 4.2. *The matrix $M \in \mathbb{R}^{m,m}$ is invertible and $M^{-1}A$ is positive diagonalizable.*

Assumption 4.2 implies that there is a (not necessarily unique) positive diagonalizable matrix $N \in \mathbb{R}^{m,m}$ satisfying $N^2 = M^{-1}A$. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_m > 0$ denote the real positive eigenvalues of N.
Introducing $U = (U_1, U_2)^\top \in \mathbb{R}^{2m}$ with $U_i = U_i(x,t) \in \mathbb{R}^m$ via

$$U_1 = u, \quad U_2 = u_t - Nu_x,$$

we transform (1.1) into the first order system

$$U_t = EU_x + F(U), \tag{4.1}$$

with $E \in \mathbb{R}^{2m,2m}$ and $F : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ given by

$$E = \begin{pmatrix} N & 0 \\ M^{-1}(C - BN) & -N \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 & I_m \\ 0 & -M^{-1}B \end{pmatrix} U + \begin{pmatrix} 0 \\ M^{-1}f(U_1) \end{pmatrix}. \tag{4.2}$$

Thus we write the second-order Cauchy problem (2.1) as a first-order Cauchy problem for (4.1),

$$U_t = EU_x + F(U), \quad U(\cdot, 0) = U_0 := \begin{pmatrix} u_0 \\ v_0 - Nu_{0,x} \end{pmatrix}, \quad x \in \mathbb{R}. \tag{4.3}$$

We emphasize that system (4.1) is diagonalizable hyperbolic. More precisely, there is an invertible matrix $T \in \mathbb{R}^{2m,2m}$ (see Remark 4.3 below), so that the change of variables $W = T^{-1}U$ transforms (4.1) into diagonal hyperbolic form

$$W_t = \Lambda_E W_x + G(W), \quad \Lambda_E = \operatorname{diag}(\lambda_1, \ldots, \lambda_m, -\lambda_m, \ldots, -\lambda_1), \; G(W) = T^{-1}F(TW). \tag{4.4}$$

This implies local well-posedness of the Cauchy problem (4.3). Moreover, in Sections 4.3 and 4.4 we carefully apply the stability results from [25], which are originally formulated for diagonal hyperbolic systems of the form (4.4).

Remark 4.3. *For completeness we set up a suitable transformation of the form $T = T_1T_2$. The first matrix T_1 transforms E from (4.2) into block-diagonal form*

$$\Lambda^E_{\text{block}} = T_1^{-1}ET_1, \quad \Lambda^E_{\text{block}} = \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix}, \quad T_1 = \begin{pmatrix} I_m & 0 \\ Z & I_m \end{pmatrix},$$

where $Z \in \mathbb{R}^{m,m}$ is the unique solution of the Sylvester equation $NZ+ZN = M^{-1}(C-BN)$. Uniqueness follows from $\sigma(N)\cap\sigma(-N) = \emptyset$. The second transformation T_2 uses the diagonalization of N via $Y^{-1}NY = \Lambda_N = \mathrm{diag}(\lambda_1,\ldots,\lambda_m)$, and a permutation P which reverses the order of m components

$$\Lambda_E = T_2^{-1}\Lambda_{\mathrm{block}}^E T_2, \quad \Lambda_E = \begin{pmatrix}\Lambda_N & 0\\ 0 & -P^{-1}\Lambda_N P\end{pmatrix}, \quad T_2 = \begin{pmatrix} Y & 0\\ 0 & YP\end{pmatrix}. \tag{4.5}$$

Next consider a traveling wave of (4.1), i.e. a solution $U_\star : \mathbb{R}\times[0,\infty)\to\mathbb{R}^{2m}$ of the form

$$U_\star(x,t) = V_\star(x-\mu_\star t),\ x\in\mathbb{R},\ t\geqslant 0, \tag{4.6}$$

which satisfies

$$\lim_{\xi\to\pm\infty} V_\star(\xi) = V_\pm \in \mathbb{R}^m \quad \text{and} \quad F(V_+) = F(V_-) = 0 \tag{4.7}$$

for some non-constant profile $V_\star : \mathbb{R}\to\mathbb{R}^{2m}$ and velocity $\mu_\star\in\mathbb{R}$. Inserting this ansatz into (4.1) shows that the pair $(V_\star,\mu_\star)$ solves the first order traveling wave equation

$$0 = (E+\mu_\star I_{2m})V_{\star,\xi} + F(V_\star). \tag{4.8}$$

In Section 2.1 we observed, that the damped wave equation in co-moving coordinates with velocity μ_1 is well-posed if the leading matrix $M^{-1}A-\mu_1^2$ is positive at each time instance. Therefore, we extend Assumption 4.2 as follows.

Assumption 4.4. *The velocity $\mu_\star\in\mathbb{R}$ of the traveling wave satisfies $\mu_\star^2 < \sigma(M^{-1}A) = \sigma(N^2)$.*

Under this assumption the matrices $N\pm\mu_\star I_m$ are invertible. A simple computation then proves the following equivalence.

Lemma 4.5. *Let Assumptions 4.1, 4.2, 4.4 be satisfied.*

(i) If $(v_\star,\mu_\star)\in C_b^2(\mathbb{R},\mathbb{R}^m)\times\mathbb{R}$ is a solution of the second order traveling wave equation (1.5), then

$$V_\star := \begin{pmatrix} v_\star \\ -(N+\mu_\star I_m)v_{\star,\xi}\end{pmatrix} \tag{4.9}$$

belongs to $C_b^2(\mathbb{R},\mathbb{R}^m)\times C_b^1(\mathbb{R},\mathbb{R}^m)$, and $(V_\star,\mu_\star)$ solves the first order traveling wave equation (4.8).

(ii) Conversely, if $(V_\star,\mu_\star)\in C_b^1(\mathbb{R},\mathbb{R}^{2m})\times\mathbb{R}$ is a solution of (4.8), then $v_\star := V_{\star,1}\in C_b^2(\mathbb{R},\mathbb{R}^m)$, and $(v_\star,\mu_\star)$ is a solution of (1.5).

In the following we work with the first order system (4.1) and consider a traveling wave (4.6),(4.7) with profile $V_\star$ satisfying

$$V_\star\in C_b^1(\mathbb{R},\mathbb{R}^{2m}), \quad V_{\star,\xi}\in H^3(\mathbb{R},\mathbb{R}^m). \tag{4.10}$$

Under the Assumptions 4.1, 4.2 and 4.4 we discuss the regularity of $V_\star$ as well as the solvability of the Cauchy problem (4.3).

Since $E+\mu_\star I_{2m}$ is invertible, we can multiply (4.8) from the left by $(E+\mu_\star I_{2m})^{-1}$ and obtain that the solution $V_\star$ of the traveling wave equation (4.8) satisfies

$$V_\star\in C_b^4(\mathbb{R},\mathbb{R}^{2m}) \text{ and } V_{1,\star}\in C_b^5(\mathbb{R},\mathbb{R}^m). \tag{4.11}$$

In Section 4.3 below, we formulate suitable conditions for the asymptotic states $v_\pm\in\mathbb{R}^m$ of the traveling wave solution of the second order problem which directly imply the property $V_{\star,\xi}\in H^3(\mathbb{R},\mathbb{R}^{2m})$ in (4.10) for $V_\star$ from (4.9).

Let us turn to the Cauchy problem (4.3). A crucial role is played by the following spaces

$$\mathcal{CH}^k(J;\mathbb{R}^n) = \bigcap_{j=0}^{k} C^{k-j}\big(J,H^j(\mathbb{R},\mathbb{R}^n)\big), \quad J\subseteq\mathbb{R} \text{ interval},\ k\in\mathbb{N}_0,\ n\in\mathbb{N},$$

which are standard in the theory of hyperbolic PDEs. Assume that $V_{\star,\xi} \in H^k(\mathbb{R},\mathbb{R}^{2m})$ and the initial data U_0 in (4.3) belongs to the affine space $V_\star + H^k(\mathbb{R},\mathbb{R}^{2m})$ for some $k = 1,2,3$. If the solution U of (4.3) is written as a perturbation of $V_\star$, i.e. $U(\cdot,t) = V_\star(\cdot) + \tilde{U}(\cdot,t)$, then (4.1) and (4.8) formally imply

$$\tilde{U}_t = E(V_\star + \tilde{U})_x + F(V_\star + \tilde{U}) = E\tilde{U}_x + \tilde{F}(x,\tilde{U}) - \mu_\star V_{\star,x}$$

with $\tilde{F}(x,u) := F\big(V_\star(x)+u\big) - F\big(V_\star(x)\big)$. The term $\mu_\star V_{\star,x}$ belongs to $L^1_{\mathrm{loc}}([0,\infty);H^k(\mathbb{R},\mathbb{R}^{2m}))$ since we assumed $V_{\star,\xi} \in H^k(\mathbb{R},\mathbb{R}^{2m})$. Moreover, Assumption 4.1 and (4.11) imply that $\tilde{F} \in C^3(\mathbb{R}\times\mathbb{R}^{2m},\mathbb{R}^{2m})$, $\tilde{F}(x,0) = 0$ and $\tilde{F}$ is bounded on $\mathbb{R}\times K$ for any compact set $K \subset \mathbb{R}^{2m}$. From the standard hyperbolic theory for semilinear first order systems, e.g. [23, Sect. 6], we obtain existence of solutions. A precise statement of existence and maximal continuation of solutions is given in the following Lemma 4.6.

Lemma 4.6. *Let the Assumptions 4.1, 4.2, 4.4 hold and let $V_\star \in C_b^1(\mathbb{R},\mathbb{R}^m)$. Given a number $k \in \{1,2,3\}$ with $V_{\star,\xi} \in H^k(\mathbb{R},\mathbb{R}^{2m})$ and initial data $U_0 \in V_\star + H^k(\mathbb{R},\mathbb{R}^{2m})$, then there exists a unique maximally extended solution $U \in V_\star + \mathcal{CH}^k([0,T);\mathbb{R}^{2m})$ of (4.3). Moreover,*

$$T = \infty \quad \text{or} \quad \lim_{t\nearrow T} \|U(t) - V_\star\|_{H^k} = \infty. \tag{4.12}$$

Corresponding statements hold true for any semilinear hyperbolic system of a similar structure and we refer to Lemma 4.6 in all these cases.

Let us apply the preceding theory to the second order Cauchy problem (2.1). We require the following:

Assumption 4.7. *The pair $(v_\star,\mu_\star) \in C_b^2(\mathbb{R},\mathbb{R}^m)\times\mathbb{R}$ satisfies $v_{\star,\xi} \in H^3(\mathbb{R},\mathbb{R}^m)$ and is a solution of the second order traveling wave equation (1.5) with*

$$\lim_{\xi\to\pm\infty} v_\star(\xi) = v_\pm \in \mathbb{R}^m, \quad f(v_\pm) = 0, \quad v_{\star,\xi} \not\equiv 0.$$

First observe that $u_0 \in v_\star + H^2(\mathbb{R},\mathbb{R}^m)$ and $v_0 \in H^1(\mathbb{R},\mathbb{R}^m)$ implies $U_0 \in V_\star + \big(H^2(\mathbb{R},\mathbb{R}^m)\times H^1(\mathbb{R},\mathbb{R}^m)\big)$, since $V_{\star,2} = -(N+\mu_\star I_m)v_{\star,\xi} \in H^3(\mathbb{R},\mathbb{R}^m)$. Taking the special structure $U = (U_1,U_2)^\top$ of problem (4.3) and $U_{0,1} \in v_\star + H^2(\mathbb{R},\mathbb{R}^m)$ into account, we will show higher regularity for the first component U_1 of the solution with this initial data than guaranteed by Lemma 4.6. This is specified in the next lemma. The key is not only to use the transformation $(u, u_t - Nu_x)$ of (2.1) but also the transformation $(u, u_t + Nu_x)$, which leads to

$$\widetilde{U}_t = \widetilde{E}\widetilde{U}_x + F(\widetilde{U}), \qquad \widetilde{E} = \begin{pmatrix} -N & 0 \\ M^{-1}(C+BN) & N \end{pmatrix}, \quad F \text{ from (4.2)}, \tag{4.13a}$$

$$\widetilde{U}(\cdot,0) = \begin{pmatrix} u_0 \\ v_0 + Nu_{0,x} \end{pmatrix} \in \widetilde{V}_\star + \big(H^2(\mathbb{R},\mathbb{R}^m)\times H^1(\mathbb{R},\mathbb{R}^m)\big), \tag{4.13b}$$

where

$$\widetilde{V}_\star = (v_\star, (N-\mu_\star I_m)v_{\star,\xi})^\top \tag{4.14}$$

is the profile of a traveling wave of (4.13a), traveling with velocity $\mu_\star$.

Lemma 4.8. *Let the Assumptions 4.1, 4.2, 4.4, 4.7 be satisfied. Moreover, define $V_\star$ by (4.9) and $\widetilde{V}_\star$ by (4.14). Let $u_0 \in v_\star + H^2(\mathbb{R},\mathbb{R}^m)$, $v_0 \in H^1(\mathbb{R},\mathbb{R}^m)$, and $U = (U_1,U_2)^\top \in V_\star + \mathcal{CH}^1([0,T);\mathbb{R}^{2m})$ denotes the unique maximally extended solution of (4.3), and $\widetilde{U} = (\widetilde{U}_1,\widetilde{U}_2)^\top \in \widetilde{V}_\star + \mathcal{CH}^1([0,\widetilde{T}),\mathbb{R}^{2m})$ denotes the unique maximally extended solution of (4.13).*

With the definition $T_0 = \min(T,\widetilde{T})$ then hold $U_1|_{[0,T_0)} = \widetilde{U}_1|_{[0,T_0)}$ and

$$U_1|_{[0,T_0)} \in v_\star + \mathcal{CH}^2([0,T_0);\mathbb{R}^m), \quad U_2|_{[0,T_0)} \in \mathcal{CH}^1([0,T_0);\mathbb{R}^m). \tag{4.15}$$

Moreover,

$$T_0 = \infty \quad \text{or} \quad \lim_{t\nearrow T_0} \|U_1(t) - v_\star\|_{H^2} + \|U_{1,t}(t)\|_{H^1} = \infty. \tag{4.16}$$

Proof. First note that Assumption 4.7 and the conditions on u_0, v_0 show that the assumptions of Lemma 4.6 are satisfied for the system (4.3) with $k=1$. Thus we have a unique maximally extended solution $U \in V_\star + \mathcal{CH}^1([0,T);\mathbb{R}^{2m})$ of (4.3). Similarly, we have a unique maximally extended solution $\widetilde{U} \in \widetilde{V}_\star + \mathcal{CH}^1([0,\widetilde{T});\mathbb{R}^{2m})$ of (4.13), cf. Lemma 4.6. The initial data for (4.3) and (4.13) are related by $\widetilde{U}(\cdot,0) = U(\cdot,0) + \big(0,\, 2NU_{1,x}(\cdot,0)\big)$.
In order to show $U_1|_{[0,T_0)} = \tilde{U}_1|_{[0,T_0)}$ in $v_\star + \mathcal{CH}^1([0,T_0),\mathbb{R}^m)$ we consider $W := (\widetilde{U}_1 - U_1, \widetilde{U}_1 + U_1, \widetilde{U}_2 - U_2, \widetilde{U}_2 + U_2)^\top$, $W \in W_\star + \mathcal{CH}^1([0,T_0),\mathbb{R}^{4m})$ with $W_\star = (0, 2v_\star, 2Nv_{\star,\xi}, -2\mu_\star v_{\star,\xi})^\top$, which solves the semilinear hyperbolic Cauchy problem

$$
\begin{aligned}
W_t &= \begin{pmatrix} 0 & -N & 0 & 0 \\ -N & 0 & 0 & 0 \\ M^{-1}C & M^{-1}BN & 0 & N \\ M^{-1}BN & M^{-1}C & N & 0 \end{pmatrix} W_x + \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -M^{-1}B & 0 \\ 0 & 0 & 0 & -M^{-1}B \end{pmatrix} W + H(W), \\
W(\cdot,0) &= (0, 2u_0, 2Nu_{0,x}, 2v_0)^\top \in W_\star + H^1(\mathbb{R},\mathbb{R}^{4m})
\end{aligned} \tag{4.17}
$$

with

$$
H(W) = \Big(0, 0, M^{-1}\big(f(\tfrac{W_1+W_2}{2}) - f(\tfrac{W_2-W_1}{2})\big), M^{-1}\big(f(\tfrac{W_1+W_2}{2}) + f(\tfrac{W_2-W_1}{2})\big)\Big)^\top.
$$

The system (4.17) has a unique maximally extended solution (cf. Lemma 4.6) which belongs to $W_\star + \mathcal{CH}^1([0,T_4);\mathbb{R}^{4m})$. We denote this maximally extended solution again by W and observe $T_4 \geq T_0$ and

$$
W|_{[0,T_0)} = (\widetilde{U}_1 - U_1, \widetilde{U}_1 + U_1, \widetilde{U}_2 - U_2, \widetilde{U}_2 + U_2)^\top.
$$

We show $W_1 = 0$ by constructing a second solution of (4.17) which has the form $(0,\widetilde{W})^\top$. Let

$$
\widetilde{W} = (\widetilde{W}_2, \widetilde{W}_3, \widetilde{W}_4)^\top \in \widetilde{W}_\star + \mathcal{CH}^1([0,T_3);\mathbb{R}^{3m}), \quad \widetilde{W}_\star = (2v_\star, 2Nv_{\star,\xi}, -2\mu_\star v_{\star,\xi})^\top
$$

be the unique maximally extended solution (cf. Lemma 4.6) of the hyperbolic system

$$
\begin{aligned}
\widetilde{W}_t &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & N \\ M^{-1}C & N & 0 \end{pmatrix} \widetilde{W}_x + \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & -M^{-1}B \end{pmatrix} \widetilde{W} + \begin{pmatrix} 0 \\ 0 \\ 2M^{-1}f(\frac{\widetilde{W}_2}{2}) \end{pmatrix}, \\
\widetilde{W}(0) &= (2u_0, 2Nu_{0,x}, 2v_0)^\top.
\end{aligned} \tag{4.18}
$$

From the first equation in (4.18) we obtain $\widetilde{W}_{2,t} = \widetilde{W}_4$, and we claim that this implies

$$
\widetilde{W}_{2,x} \in C^1([0,T_3), L^2(\mathbb{R}^m)), \quad \frac{d}{dt}\widetilde{W}_{2,x} = \widetilde{W}_{4,x}.
$$

In fact, commuting derivatives is allowed, since $\widetilde{W}_2 \in 2v_\star + \mathcal{CH}^1([0,T_3);\mathbb{R}^m)$ entails $\widetilde{W}_{2,x} \in C^1([0,T_3), H^{-1})$ and therefore, for all $\varphi \in H^1(\mathbb{R},\mathbb{R}^m)$,

$$
\frac{d}{dt}\langle \widetilde{W}_{2,x}, \varphi\rangle_{L^2} = -\frac{d}{dt}\langle \widetilde{W}_2, \varphi_x\rangle_{L^2} = -\langle \widetilde{W}_{2,t}, \varphi_x\rangle_{L^2} = -\langle \widetilde{W}_4, \varphi_x\rangle_{L^2} = \langle \widetilde{W}_{4,x}, \varphi\rangle_{L^2}.
$$

From the second equation in (4.18) we have $\frac{d}{dt}(\widetilde{W}_3 - N\widetilde{W}_{2,x}) = 0$ in $C^0([0,T_3), L^2)$ which together with $\widetilde{W}_3(0) = 2Nu_{0,x} = N\widetilde{W}_{2,x}(0)$ implies $\widetilde{W}_3 = N\widetilde{W}_{2,x}$ in $C^0([0,T_3), L^2)$. With these equalities it is straightforward to verify that $(0,\widetilde{W})^\top$ solves (4.17) in $[0,T_3)$.
Uniqueness of solutions shows $T_3 \leq T_4$ and

$$
W_1 = 0, W_3 = NW_{2,x}, W_4 = W_{2,t} \text{ in } [0,T_3). \tag{4.19}
$$

Actually, it holds $T_3 = T_4$. To see this assume $T_3 < T_4$. By uniqueness follows $\widetilde{W}(t) = (W_2(t), W_3(t), W_4(t))^\top$ for all $0 \le t < T_3$, so that

$$\sup_{0\le t<T_3} \|\widetilde{W}(t) - \widetilde{W}_\star\|_{H^1} = \sup_{0\le t<T_3} \|W(t) - W_\star\|_{H^1} < \infty.$$

This contradicts the maximality of T_3 (cf. (4.12)), and hence $T_3 = T_4 \ge T_0 = \min(T, \widetilde{T})$ follows. Moreover, (4.19) shows $U_1 = \widetilde{U}_1$ in $[0, T_0)$, and using this and the definition of W, we obtain

$$U_{1,t} = \tfrac{1}{2}W_{2,t} = \tfrac{1}{2}W_4 \in -\mu_\star v_{\star,\xi} + \mathcal{CH}^1([0,T_0);\mathbb{R}^m) = \mathcal{CH}^1([0,T_0);\mathbb{R}^m), \tag{4.20}$$

$$U_{1,x} = \tfrac{1}{2}W_{2,x} = \tfrac{1}{2}N^{-1}W_3 \in v_{\star,\xi} + \mathcal{CH}^1([0,T_0);\mathbb{R}^m) = \mathcal{CH}^1([0,T_0);\mathbb{R}^m), \tag{4.21}$$

which proves (4.15).
Finally, we prove the alternative (4.16). In case $T_0 = \infty$ or $T_0 = T < \infty$, this statement follows from (4.12). If $T > T_0 = \widetilde{T}$, we first note that $T_4 = T_0$, since otherwise $T_4 > T_0$ and

$$\begin{aligned}
\sup_{0\le t<\widetilde{T}} \|\widetilde{U}_1(t) - v_\star\|_{H^1} &= \sup_{0\le t<\widetilde{T}} \|W_1(t) + U_1(t) - v_\star\|_{H^1} \\
&\le \sup_{0\le t<\widetilde{T}} \|W_1(t)\|_{H^1} + \sup_{0\le t<\widetilde{T}} \|U_1(t) - v_\star\|_{H^1} < \infty, \\
\sup_{0\le t<\widetilde{T}} \|\widetilde{U}_2(t) + (-N + \mu_\star I_m)v_{\star,\xi}\|_{H^1} &= \sup_{0\le t<\widetilde{T}} \|W_3(t) + U_2(t) + (-N + \mu_\star I_m)v_{\star,\xi}\|_{H^1} \\
&\le \sup_{0\le t<\widetilde{T}} \|W_3(t) - 2Nv_{\star,\xi}\|_{H^1} + \sup_{0\le t<\widetilde{T}} \|U_2 + (N + \mu_\star I_m)v_{\star,\xi}\|_{H^1} < \infty,
\end{aligned}$$

which contradicts the maximality of $\widetilde{T} = T_0$ (cf. (4.12)). From $W_1 = 0$ in $[0, T_0)$ and $T_0 = \widetilde{T}$ follows

$$\sup_{0\le t<\widetilde{T}} \|\widetilde{U}_1(t) - v_\star\|_{H^1} = \sup_{0\le t<\widetilde{T}} \|U_1(t) - v_\star\|_{H^1} < \infty.$$

This also implies $\sup_{0\le t<T_0} \|W_2(t) - 2v_\star\|_{H^1} < \infty$, so that the maximality of $T_4 = T_0$ (cf. (4.12)) shows

$$\lim_{t\nearrow T_0} \left(\|W_3(t) - 2Nv_{\star,\xi}\|_{H^1} + \|W_4(t) + 2\mu_\star v_{\star,\xi}\|_{H^1}\right) = \infty.$$

Now we use (4.20) and (4.21) to obtain for all $0 \le t < T_0$ the estimate

$$\begin{aligned}
\|U_1(t) - v_\star\|_{H^2} + \|U_{1,t}(t)\|_{H^1} &\ge \|U_{1,x}(t) - v_{\star,x}\|_{H^1} + \|U_{1,t}(t)\|_{H^1} \ge \left\|\tfrac{1}{2}N^{-1}W_3(t) - v_{\star,x}\right\|_{H^1} \\
&+ \left\|\tfrac{1}{2}W_4(t)\right\|_{H^1} \ge \text{const}\left(\|W_3(t) - 2Nv_{\star,x}\|_{H^1} + \|W_4(t) + 2\mu_\star v_{\star,x}\|_{H^1} - \|2\mu_\star v_{\star,x}\|_{H^1}\right)
\end{aligned}$$

with a constant const $=$ const$(N) > 0$. Considering the limit $t \nearrow T_0$ finishes the proof of (4.16). □

Remark 4.9. *The proof of Lemma 4.8 is based on coupling the two different transformations $(u, u_t \pm Nu_x)$ of (2.1) which lead to (4.3) and (4.13). Of course, it is no surprise that the first components of the solutions to both systems coincide, anticipating that the first and second order versions are equivalent. But note, that a priori the first component of the solutions of (4.3) or (4.13) is not smooth enough, to be a solution of (2.1). The above technique of doubling the system will be repeatedly used for the stability proofs in Sections 4.3 and 4.4.*

With Lemma 4.8 at hand, an equivalence statement for solutions of (2.1) and (4.3) follows.

Theorem 4.10. *Let the Assumptions 4.1, 4.2, 4.4, 4.7 be satisfied and define $V_\star$ by (4.9) and $\widetilde{V}_\star$ by (4.14). Furthermore, let $u_0 \in v_\star + H^2(\mathbb{R},\mathbb{R}^m)$ and $v_0 \in H^1(\mathbb{R},\mathbb{R}^m)$.*

(i) *If $u \in v_\star + \mathcal{CH}^2([0,T_0);\mathbb{R}^m)$ solves (2.1) in $[0,T_0)$, then $U = (u, u_t - Nu_x)^\top \in V_\star + \mathcal{CH}^1([0,T_0);\mathbb{R}^{2m})$ solves (4.3) in $[0,T_0)$ and $\widetilde{U} = (u, u_t + Nu_x)^\top \in \widetilde{V}_\star + \mathcal{CH}^1([0,T_0);\mathbb{R}^{2m})$ solves (4.13) in $[0,T_0)$.*

(ii) *Conversely, assume* $U \in V_\star + \mathcal{CH}^1([0,T);\mathbb{R}^{2m})$ *solves* (4.3) *in* $[0,T)$ *and* $\widetilde{U} \in \widetilde{V}_\star + \mathcal{CH}^1([0,\widetilde{T});\mathbb{R}^{2m})$ *solves* (4.13) *in* $[0,\widetilde{T})$ *and define* $T_0 := \min(T,\widetilde{T})$*. Then*

$$u := U_1|_{[0,T_0)} = \widetilde{U}_1|_{[0,T_0)} \in v_\star + \mathcal{CH}^2([0,T_0);\mathbb{R}^m)$$

solves (2.1) *in* $[0,T_0)$*. Moreover,* $U \in V_\star + \big(\mathcal{CH}^2([0,T_0);\mathbb{R}^m) \times \mathcal{CH}^1([0,T_0);\mathbb{R}^m)\big)$.

Proof. Part (i) follows by construction of the first order systems. For part (ii) note, that U satisfies the higher regularity by Lemma 4.8. Using this, it is easy to check from (4.3) that $u = U_1|_{[0,T_0)}$ solves (2.1). □

We now apply the freezing method to the first order system (4.3). First, we introduce new unknowns $\gamma(t) \in \mathbb{R}$ and $V(\xi,t) \in \mathbb{R}^{2m}$ via the freezing ansatz

$$U(x,t) = V(\xi,t), \quad \xi := x - \gamma(t), \quad x \in \mathbb{R},\, t \geq 0. \tag{4.22}$$

This formally leads to

$$V_t = (E + \mu I_{2m})V_\xi + F(V), \tag{4.23a}$$

$$\gamma_t = \mu, \tag{4.23b}$$

$$V(\cdot,0) = V_0 := U_0 = (u_0, v_0 - N u_{0,\xi})^\top, \quad \gamma(0) = 0, \tag{4.23c}$$

with E and F from (4.2). In (4.23) we introduced the time-dependent function $\mu(t) \in \mathbb{R}$ for convenience. As before, equation (4.23b) decouples and can be solved in a postprocessing step. One needs an additional algebraic constraint to compensate the extra variable μ. Suitable choices corresponding to the phase conditions in Section 2.1 are discussed in Section 4.4. To relate the second order freezing equation (2.5) and the first order version (4.23), we omit the introduction of μ_2 in (2.5) and write it in the form

$$Mv_{tt} + Bv_t = (A - \mu^2 M)v_{\xi\xi} + 2\mu M v_{\xi t} + (\mu_t M + \mu B + C)v_\xi + f(v), \tag{4.24a}$$

$$\gamma_t = \mu, \tag{4.24b}$$

$$v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + \mu(0)u_{0,\xi}, \quad \gamma(0) = 0. \tag{4.24c}$$

Transforming (4.24) into a first order system by introducing $V = (V_1,V_2)^\top \in \mathbb{R}^{2m}$ via

$$V_1 = v, \quad V_2 = v_t - (N + \mu I_m)v_\xi,$$

we again find the system (4.23). As a consequence we obtain the equivalence of the freezing systems for the first and the second order formulation. For this result it will not be necessary to set up the freezing system for the companion first order formulation (4.13a).

Theorem 4.11. *Let the Assumptions 4.1, 4.2, 4.4, 4.7 be satisfied and define* $V_\star$ *by* (4.9)*. Moreover, let* $u_0 \in v_\star + H^2(\mathbb{R},\mathbb{R}^m)$*,* $v_0 \in H^1(\mathbb{R},\mathbb{R}^m)$ *and* $\mu \in C^1([0,T);\mathbb{R})$ *for some* $T > 0$*. Then the following statements hold.*

(i) *If* $v \in v_\star + \mathcal{CH}^2([0,T_0);\mathbb{R}^m)$*,* $\gamma \in C^2([0,T_0);\mathbb{R})$ *solve* (4.24) *in* $[0,T_0)$*, then the functions* $V = (v, v_t - (N + \mu I_m)v_\xi)^\top \in V_\star + \mathcal{CH}^1([0,T_0);\mathbb{R}^{2m})$*,* $\gamma \in C^2([0,T_0);\mathbb{R})$ *solve* (4.23) *in* $[0,T_0)$*.*

(ii) *Conversely, let* $V \in V_\star + \mathcal{CH}^1([0,T);\mathbb{R}^{2m})$*,* $\gamma \in C^2([0,T);\mathbb{R})$ *solve* (4.23) *in* $[0,T)$*, let* $[0,\widetilde{T})$ *be the interval of existence of the maximally extended solution to* (4.13)*, and define* $T_0 = \min(T,\widetilde{T})$*. Then* $v = V_1|_{[0,T_0)} \in v_\star + \mathcal{CH}^2([0,T_0);\mathbb{R}^m)$ *and* $\gamma \in C^2([0,T_0);\mathbb{R})$ *solve* (4.24) *in* $[0,T_0)$*.*

Proof. For part (i) note that the function v solves the co-moving equation (4.24) if and only if $u(x,t) = v(x-\gamma(t),t)$ solves the original Cauchy problem (2.1). By Theorem 4.10 (i) $U = (u, u_t - Nu_x)$ solves the Cauchy problem (4.3) in $[0,T_0)$. Finally, U solves (4.3) in $[0,T_0)$ if and only if $V(x,t) = U(x+\gamma(t),t)$ solves (4.23) in $[0,T_0)$.

For (ii) note that the function V solves (4.23) in $[0,T)$ if and only if $U(x,t) = V(x-\gamma(t),t)$ satisfies $U \in V_\star + \mathcal{CH}^1([0,T);\mathbb{R}^{2m})$ and solves (4.3) in $[0,T)$. Then $u := U_1|_{[0,T_0)} \in v_\star + \mathcal{CH}^2([0,T_0);\mathbb{R}^m)$ solves (2.1) in $[0,T_0)$ by Theorem 4.10 (ii). Hence v,γ with v given by $v(x,t) = u(x+\gamma(t),t) = U_1(x+\gamma(t),t) = V_1(x,t)$ solves (4.24) in $[0,T_0)$. □

The results of this section can be summarized in the following diagram:

$$
\begin{array}{ccc}
u \text{ solves } (2.1) & \xleftrightarrow[\text{Thm. } 4.10]{U = \begin{pmatrix} u \\ u_t - N u_x \end{pmatrix}} & U \text{ solves } (4.3) \\
\text{freezing ansatz } (2.2)\ \ u(x,t) = v(x-\gamma(t),t)\ \updownarrow & & \updownarrow\ \text{freezing ansatz } (4.22)\ \ U(x,t) = V(x-\gamma(t),t) \\
(v,\gamma,\mu_1) \text{ solves } (2.18) & \xleftrightarrow[\text{Thm. } 4.11]{V = \begin{pmatrix} v \\ v_t - (N+\mu_1 I_m) v_\xi \end{pmatrix}} & (V,\gamma,\mu_1) \text{ solves } (4.23)
\end{array}
$$

4.2. Spectral properties of second and first order problems. We now make the statements from Section 3 precise and relate the spectral properties of the original second order problem (1.1) and the first order problem (4.1). As base space we always consider a subspace of L^2. Throughout this section we impose without further notice Assumptions 4.1, 4.2, 4.4, 4.7 from Section 4.1 and let $V_\star$ be defined by (4.9).

By Definition 3.1, the spectral problem for the second order problem (1.1), considered in a co-moving frame, is given by the solvability properties of

$$\mathcal{P}(\lambda) : H^2(\mathbb{R},\mathbb{C}^m) \to L^2(\mathbb{R},\mathbb{C}^m), \quad \text{defined by } (3.1).$$

The analog for the first order formulation (4.1) is the first order differential operator

$$
\begin{aligned}
&\mathcal{P}_{1\text{st}}(\lambda) : H^1(\mathbb{R},\mathbb{C}^m) \times H^1(\mathbb{R},\mathbb{C}^m) \to L^2(\mathbb{R},\mathbb{C}^m) \times L^2(\mathbb{R},\mathbb{C}^m), \text{ given by} \\
&\mathcal{P}_{1\text{st}}(\lambda) = \lambda I_{2m} - (E + \mu_\star I_{2m})\partial_\xi - DF(V_\star), \quad DF(V_\star) = \begin{pmatrix} 0 & I_m \\ M^{-1}Df(v_\star) & -M^{-1}B \end{pmatrix},
\end{aligned} \tag{4.25}
$$

which is obtained as above by linearizing (4.1) in the co-moving frame about the traveling wave $V_\star$. It is well-known (e.g. [28]) that the dispersion relation yields (at least a part of) the essential spectrum for first order problems. Namely, the dispersion curves, given as the algebraic set

$$\sigma_{\text{disp}}(\mathcal{P}_{1\text{st}}) = \left\{ \lambda \in \mathbb{C} : \det\big(\lambda \tilde{A}_1 + \tilde{A}_0^\pm(\omega)\big) = 0 \text{ for some } \omega \in \mathbb{R}, + \text{ or } - \right\}, \tag{4.26}$$

where $\tilde{A}_1 = I_{2m}$ and $\tilde{A}_0^\pm(\omega) = -i\omega\big(E + \mu_\star I_{2m}\big) - DF(V_\pm)$, belong to $\sigma_{\text{ess}}(\mathcal{P}_{1\text{st}})$. Recall the dispersion set (3.4) for the original second order problem

$$\sigma_{\text{disp}}(\mathcal{P}) = \left\{ \lambda \in \mathbb{C} : \det\big(\lambda^2 A_2 + \lambda A_1(\omega) + A_0^\pm(\omega)\big) = 0 \text{ for some } \omega \in \mathbb{R}, + \text{ or } - \right\}, \tag{4.27}$$

with $A_0^\pm, A_1, A_2$ given in (3.3). Because for fixed $\omega \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and any choice of $+$ or $-$ the equation $\big(\lambda \tilde{A}_1 + \tilde{A}_0^\pm(\omega)\big)W = 0$, $W = (w_1,w_2) \in \mathbb{C}^{2m}$ implies $w_2 = \lambda w_1 - i\omega(N + \mu_\star I_m)w_1$, the equivalence

$$\big(\lambda^2 A_2 + \lambda A_1(\omega) + A_0^\pm(\omega)\big)w = 0 \;\Leftrightarrow\; \big(\lambda \tilde{A}_1 + \tilde{A}_0^\pm(\omega)\big) \begin{pmatrix} w \\ \lambda w - i\omega(N + \mu_\star I_m)w \end{pmatrix} = 0$$

for $w \in \mathbb{C}^m$ yields the following proposition.

Proposition 4.12. The two sets $\sigma_{\text{disp}}(\mathcal{P}_{1\text{st}})$ and $\sigma_{\text{disp}}(\mathcal{P})$ coincide.

The following theorem now collects statements that relate the spectral properties of the first order operator $\mathcal{P}_{1\text{st}}$ and the second order operator $\mathcal{P}$. For the stability analysis it is crucial that the spectrum lies to the right of the dispersion set $\sigma_{\text{disp}}(\mathcal{P})$. Therefore, we define ρ_+ as the connected component of $\mathbb{C} \setminus \sigma_{\text{disp}}(\mathcal{P})$ which contains some positive semi-axis $[\lambda,\infty)$.

Theorem 4.13. *The operators* $\mathcal{P} : H^2(\mathbb{R},\mathbb{C}^m) \to L^2(\mathbb{R},\mathbb{C}^m)$ *and* $\mathcal{P}_{1\text{st}} : H^1(\mathbb{R},\mathbb{C}^{2m}) \to L^2(\mathbb{R},\mathbb{C}^{2m})$ *satisfy:*

(i) $\sigma_{\text{disp}}(\mathcal{P}) \subseteq \sigma_{\text{ess}}(\mathcal{P}) \cap \sigma_{\text{ess}}(\mathcal{P}_{1\text{st}})$.
(ii) $\dim \mathcal{N}(\mathcal{P}(\lambda)) = \dim \mathcal{N}(\mathcal{P}_{1\text{st}}(\lambda))$ *for all* $\lambda \in \mathbb{C}$.
(iii) $\rho(\mathcal{P}) \supseteq \rho(\mathcal{P}_{1\text{st}})$.
(iv) *The operator* $\mathcal{P}_{1\text{st}}(\lambda)$ *is Fredholm of index zero for all* $\lambda \in \rho_+$.
(v) *The spectra of* $\mathcal{P}_{1\text{st}}$ *and* $\mathcal{P}$ *coincide in* ρ_+, *i.e.* $\sigma(\mathcal{P}) \cap \rho_+ = \sigma(\mathcal{P}_{1\text{st}}) \cap \rho_+$.
(vi) *A point* $\lambda \in \rho_+$ *is a simple eigenvalue of* $\mathcal{P}$ *if and only if it is a simple eigenvalue of* $\mathcal{P}_{1\text{st}}$.

Proof. (i) The inclusion $\sigma_{\text{disp}}(\mathcal{P}) \subseteq \sigma_{\text{ess}}(\mathcal{P}_{1\text{st}})$ is well-known and by a standard cut-off process one proves that $\sigma_{\text{disp}}(\mathcal{P})$ belongs to the essential spectrum $\sigma_{\text{ess}}(\mathcal{P})$.

(ii) This follows from the observation, that for $\lambda \in \mathbb{C}$ holds

$$\phi \in \mathcal{N}(\mathcal{P}(\lambda)) \quad \Leftrightarrow \quad \begin{pmatrix} \phi \\ \lambda\phi - (N + \mu_\star I)\phi_\xi \end{pmatrix} \in \mathcal{N}(\mathcal{P}_{1\text{st}}(\lambda)) \tag{4.28}$$

and every element $V = (v_1, v_2)^\top \in \mathcal{N}(\mathcal{P}_{1\text{st}}(\lambda))$ is of the form $(v_1, \lambda v_1 - (N + \mu_\star I_m)v_{1,\xi})^\top$ with $v_1 \in H^2(\mathbb{R},\mathbb{C}^m)$.

(iii) Let $\lambda \in \rho(\mathcal{P}_{1\text{st}})$. Then $\mathcal{P}(\lambda)$ is one-to-one by (ii). Now let $h \in L^2$ be arbitrary and let $V = (v_1, v_2)^\top \in H^1 \times H^1$ solve $\mathcal{P}_{1\text{st}}(\lambda)V = \begin{pmatrix} 0 \\ M^{-1}h \end{pmatrix}$. The differential equation implies $\lambda v_1 - (N + \mu_\star I)v_{1,\xi} - v_2 = 0$, so that $v_1 \in H^2$ by Assumption 4.4, and a simple calculation shows $\mathcal{P}(\lambda)v_1 = h$, so that $\mathcal{P}(\lambda)$ is onto. Therefore, $\mathcal{P}(\lambda)^{-1} : L^2 \to H^2$ exists and is bounded by the inverse operator theorem.

(iv) Note that $\mathcal{P}_{1\text{st}}(\lambda)V = H$ in L^2 is equivalent to

$$\big(\partial_\xi - Q(\lambda,\xi)\big)V = B_0^{-1}H, \quad \text{in } L^2(\mathbb{R},\mathbb{C}^{2m}),$$

where $Q(\lambda,\xi) = B_0^{-1}\big(\lambda I_{2m} - DF(V_\star(\xi))\big)$, $B_0 = E + \mu_\star I_{2m}$. The limits $\lim_{\xi\to\pm\infty} Q(\lambda,\xi) =: Q(\lambda)_\pm$ exist and are hyperbolic for $\lambda \notin \sigma_{\text{disp}}(\mathcal{P})$. Then a result of Palmer [22] shows that $\mathcal{P}_{1\text{st}}(\lambda)$ is Fredholm of index $\dim E_+^s - \dim E_-^s$, where E_+^s is the stable subspace of $Q(\lambda)_+$ and E_-^s is the stable subspace of $Q(\lambda)_-$. Considering $Q(\lambda)_\pm$ for $\lambda \gg 0$, one finds that $Q(\lambda)_+$ and $Q(\lambda)_-$ both have m eigenvalues with positive real part and m eigenvalues with negative real part (counting multiplicity). Therefore, the index is 0 in ρ_+, because the index is constant in connected components of $\mathbb{C} \setminus \sigma_{\text{disp}}(\mathcal{P})$.

(v) The inclusion "$\subseteq$" follows from (iii). For the inclusion "$\supseteq$" let $\lambda \in \rho(\mathcal{P}) \cap \rho_+$. Then $\mathcal{P}_{1\text{st}}(\lambda)$ is one-to-one by (ii) and also onto by (iv), hence it has bounded inverse.

(vi) Recall that λ is a simple eigenvalue of $\mathcal{P}$ if and only if there exists $\phi_0 \in H^2 \setminus \{0\}$ with $\mathcal{N}(\mathcal{P}(\lambda)) = \text{span}\{\phi_0\}$ and $\big(2\lambda M + (B - 2\mu_\star M\partial_\xi)\big)\phi_0 \notin \mathcal{R}(\mathcal{P}(\lambda))$, see Remark 3.2. Because of (ii) and (4.28) it thus suffices to show for $\phi \in H^2$,

$$\big(2\lambda M + (B - 2\mu_\star M\partial_\xi)\big)\phi \in \mathcal{R}(\mathcal{P}(\lambda)) \quad \Leftrightarrow \quad \begin{pmatrix} \phi \\ \lambda\phi - (N + \mu_\star)\phi_\xi \end{pmatrix} \in \mathcal{R}\big(\mathcal{P}_{1\text{st}}(\lambda)\big).$$

But this easily follows from

$$\left\{\begin{matrix} \mathcal{P}(\lambda)v_1 & = & \big(2\lambda M + (B - 2\mu_\star M\partial_\xi)\big)\phi, \\ v_2 & = & \lambda v_1 - (N + \mu_\star)v_{1,\xi} - \phi \end{matrix}\right\} \Leftrightarrow \mathcal{P}_{1\text{st}}(\lambda)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \lambda\phi - (N + \mu_\star)\phi_\xi \end{pmatrix}.$$

□

Remark 4.14. *Adapting the proof of* [22, Lemma 4.2] *shows that we can consider the operator* $\mathcal{P}_{1\text{st}}(\lambda) : H^1 \times H^1 \to L^2 \times L^2$ *likewise as an operator* $\widetilde{\mathcal{P}}_{1\text{st}}(\lambda) : H^2 \times H^1 \to H^1 \times L^2$ *and that one operator is*

Fredholm if and only if the other one is, and that their indices are the same. The identity (4.28) *then shows* $\dim \mathcal{N}(\widetilde{\mathcal{P}}_{1st}(\lambda)) = \dim \mathcal{N}(\mathcal{P}(\lambda))$*. Moreover, the following implications hold:*

$$h \in L^2 \setminus \mathcal{R}(\mathcal{P}(\lambda)) \Rightarrow \begin{pmatrix} 0 \\ M^{-1}h \end{pmatrix} \in (H^1 \times L^2) \setminus \mathcal{R}(\widetilde{\mathcal{P}}_{1\mathrm{st}}(\lambda)),$$

$$\begin{pmatrix} r \\ s \end{pmatrix} \in (H^1 \times L^2) \setminus \mathcal{R}(\widetilde{\mathcal{P}}_{1\mathrm{st}}(\lambda)) \Rightarrow Ms + \lambda Mr - \mu_\star Mr_\xi - Br \in L^2 \setminus \mathcal{R}(\mathcal{P}(\lambda)).$$

Therefore, if $\widetilde{\mathcal{P}}_{1\mathrm{st}}(\lambda)$ *is Fredholm of index* l *then so is* $\mathcal{P}(\lambda)$ *with the same index* l*. In particular,* $\mathcal{P}(\lambda)$ *is Fredholm of index* 0 *for* $\lambda \in \rho_+$ *by Theorem 4.13 (iv).*

4.3. Stability of traveling waves. In this section we employ the results from [25] on first order hyperbolic systems to obtain nonlinear stability of traveling waves in nonlinear damped wave equations. As in Section 4.2 we continue to impose the general Assumptions 4.1, 4.2, 4.4, 4.7 and let $V_\star$ be defined by (4.9) and $\widetilde{V}_\star$ by (4.14) throughout the section. The main result in [25] infers nonlinear stability from spectral stability under suitable conditions. Therefore, we impose the following assumption on the spectrum of the linearized operator.

Assumption 4.15. *There is* $\delta > 0$*, such that* $\mathrm{Re}\,(\sigma_{\mathrm{disp}}(\mathcal{P})) < -\delta$ *for the dispersion set from* (4.27)*.*

This condition ensures that the domain ρ_+ consists of normal points, and it also guarantees that the eigenfunction $v_{\star,\xi}$ corresponding to the eigenvalue 0 lies in suitable function spaces.

Lemma 4.16. *Under the additional Assumption 4.15 the profile* $v_\star$ *of the traveling wave satisfies* $v_\star \in C_b^5(\mathbb{R}, \mathbb{R}^m)$ *and* $v_{\star,\xi} \in H^4(\mathbb{R}, \mathbb{R}^m)$*.*

Proof. Recall from Lemma 4.5 that $V_\star$ solves the traveling wave equation (4.8) and satisfies the smoothness (4.11). Therefore, it remains to prove the second assertion. Differentiating (4.8) with respect to ξ shows that $V_{\star,\xi}$ is a solution of

$$0 = (E + \mu_\star I_{2m})V_\xi + DF(V_\star)V, \tag{4.29}$$

which we write as $V_\xi - Q(\xi)V = 0$ with $Q(\xi) = (E + \mu_\star I_{2m})^{-1} DF(V_\star)$. Assumption 4.7 implies that the limit matrices $\lim_{\xi\to\pm\infty} Q(\xi) = Q_\pm$ exist. Moreover, as in the proof of Theorem 4.13 (iv) these matrices have m stable and m unstable eigenvalues due to Assumption 4.15. Therefore, every bounded solution of $V_\xi - Q(\xi)V = 0$ converges exponentially fast to 0 as $\xi \to \pm\infty$. This implies $V_{\star,\xi}(\xi) \to 0$ exponentially fast as $\xi \to \pm\infty$ and, in particular, $V_{\star,\xi} \in L^2$. The same is true for the second and third derivative with respect to ξ as follows by differentiating (4.29) two more times. Thus we have $V_{\star,\xi} \in H^3(\mathbb{R}, \mathbb{R}^{2m})$. The structure of E and F reveals that $v_{\star,\xi} = V_{\star,1,\xi}$ is a matrix multiple of $V_{\star,2}$ and the regularity $v_{\star,\xi} \in H^4(\mathbb{R}, \mathbb{R}^m)$ follows. □

Equation (4.29) and the regularity proved in Lemma 4.16 shows that $V_{\star,\xi} \in \mathcal{N}(\mathcal{P}_{1st}(0))$ and therefore, by (4.28), $v_{\star,\xi}$ is an eigenfunction of the quadratic operator polynomial $\mathcal{P}$ to the eigenvalue 0. In addition to the Assumption 4.15 on the location of the dispersion curves, we also need that $\mathcal{P}$ has no further eigenvalues except 0 in a certain right half plane.

Assumption 4.17. *The eigenvalue* 0 *of* $\mathcal{P}$ *is simple and there is no other eigenvalue of* $\mathcal{P}$ *with real part greater than* $-\delta$ *with* δ *given by Assumption 4.15.*

The stability theorem for traveling waves and also for the freezing method, stated in [25, Thm. 2.5] are formulated for diagonal hyperbolic systems of the form

$$W_t = \Lambda W + G(W) \tag{4.30}$$

under the following conditions

(i) The matrix Λ is diagonal, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_l)$ with $\lambda_1 \geq \ldots \geq \lambda_l$.

(ii) The nonlinearity G belongs to $C^3(\mathbb{R}^l, \mathbb{R}^l)$.
(iii) There exists a traveling wave solution $W_\star$ of (4.30) with velocity $\mu_\star$ such that $W_\star \in C_b^1$, $W_{\star,\xi} \in H^2$ and $G(W_\star) \in L^2$.
(iv) The matrix valued function $Z(\xi) = DG\big(W_\star(\xi)\big)$ belongs to C_b^1, its limits $\lim_{\xi\to\pm\infty} Z(\xi) = Z_\pm$ exist and $\lim_{\xi\to\pm\infty} Z'(\xi) = 0$.
(v) The matrix $\Lambda + \mu_\star I_l \in \mathbb{R}^{l,l}$ is invertible.
(vi) There is $\delta > 0$ such that $\operatorname{Re}\{s \in \mathbb{C} : s \in \sigma\big(i\omega(\Lambda + \mu_\star I_l) + Z_\pm\big)$ for some $\omega \in \mathbb{R}\} \le -\delta$.
(vii) The point spectrum of $P = (\Lambda + \mu_\star I_l)\partial_\xi + Z(\cdot) : H^1 \to L^2$ satisfies $\sigma_{\text{point}}(P) \cap \{\operatorname{Re} s > -\delta\} = \{0\}$, and 0 is an algebraically simple eigenvalue.

It is easy to see that a transformation of the form $U = TW$, $T \in \mathbb{R}^{l,l}$ invertible, does not have an influence on (ii)–(vii). Therefore, instead of (i) it suffices to require that the problem can be diagonalized by a constant transformation. For completeness we collect the arguments in the following lemma.

Lemma 4.18. *Under the additional spectral Assumptions 4.15 and 4.17 the system* (4.3) *can be transformed into a diagonal hyperbolic system, which satisfies conditions* (i)–(vii).

Proof. As in Section 4.1 there is a matrix $T \in \mathbb{R}^{2m,2m}$ so that $W = T^{-1}U$ transforms (4.3) into a diagonal hyperbolic form which satisfies (i) with Λ_E from (4.5) instead of Λ.
The nonlinearity $G : \mathbb{R}^{2m} \to \mathbb{R}^{2m}, G(W) = T^{-1}F(TW)$ is of class $C^3(\mathbb{R}^{2m}, \mathbb{R}^{2m})$ by Assumption 4.1 so that (ii) holds. Property (iii) is satisfied by the profile $W_\star = T^{-1}V_\star$ because of Assumption 4.7 and because of Lemma 4.16 together with Assumption 4.1 on f. Moreover, these assumptions also imply (iv). Property (v) holds for $\Lambda_E + \mu_\star I_{2m}$ by Assumption 4.4. For (vi) note $Z(\xi) = T^{-1}DF(V_\star(\xi))T$, so that

$$T\Big(i\omega(\Lambda_E + \mu_\star I_{2m}) + Z_\pm\Big)T^{-1} = i\omega(E + \mu_\star I_{2m}) + DF(V_\pm).$$

Therefore, (vi) is equivalent to $\operatorname{Re}\lambda \le -\delta$ for all $\lambda \in \sigma_{\text{disp}}(\mathcal{P}_{1\text{st}})$, and this follows by Proposition 4.12 from Assumption 4.15. Finally, for (vii) note that $\mathcal{P}_{1\text{st}} = TPT^{-1}$, where T is considered as the continuous and invertible transformation on L^2 (or H^k, $k = 1, 2$) given by $W \mapsto TW$. Then (vii) holds if and only if it holds for $\mathcal{P}_{1\text{st}}$, and the latter is implied by Assumption 4.17 according to Theorem 4.13. $\square$

Summarizing, the stability result [25, Thm. 2.5] applies and yields stability of traveling waves for the first order system (4.3). This will be stated in the next Lemma 4.19 which also includes the corresponding statement for the first order system (4.13). The reason is that both formulations will be used for the proof of the stability of traveling waves in damped wave equations, see Theorem 4.20 below.

Lemma 4.19. *Let the assumptions of Lemma 4.18 hold. Then for every $0 < \eta < \delta$ there is $\rho_0 > 0$ so that for all $U_0 \in V_\star + H^2(\mathbb{R}, \mathbb{R}^{2m})$ with $\|U_0 - V_\star\|_{H^2} \le \rho_0$ the Cauchy problem (4.1) with initial data $U(0) = U_0$ has a unique global solution $U \in V_\star + \mathcal{CH}^1([0,\infty); \mathbb{R}^{2m})$. Moreover, there is $\varphi_\infty = \varphi_\infty(U_0) \in \mathbb{R}$ and $C = C(\eta, \rho_0) > 0$ with*

$$|\varphi_\infty| \le C\|U_0 - V_\star\|_{H^2},$$
$$\|U(t) - V_\star(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} \le C\|U_0 - V_\star\|_{H^2} e^{-\eta t} \quad \forall t \ge 0.$$

Similarly, for all $\widetilde{U}_0 \in \widetilde{V}_\star + H^2(\mathbb{R}, \mathbb{R}^{2m})$, with $\|\widetilde{U}_0 - \widetilde{V}_\star\|_{H^2} \le \rho_0$ the Cauchy problem (4.13a) with initial data $\widetilde{U}(0) = \widetilde{U}_0$ has a unique global solution $\widetilde{U} \in \widetilde{V}_\star + \mathcal{CH}^1([0,\infty); \mathbb{R}^{2m})$ and there is $\widetilde{\varphi}_\infty = \widetilde{\varphi}_\infty(\widetilde{U}_0) \in \mathbb{R}$ and $\widetilde{C} = \widetilde{C}(\eta)$ with

$$|\widetilde{\varphi}_\infty| \le \widetilde{C}\|\widetilde{U}_0 - \widetilde{V}_\star\|_{H^2},$$
$$\|\widetilde{U}(t) - \widetilde{V}_\star(\cdot - \mu_\star t - \widetilde{\varphi}_\infty)\|_{H^1} \le \widetilde{C}\|\widetilde{U}_0 - \widetilde{V}_\star\|_{H^2} e^{-\eta t} \; \forall t \ge 0.$$

This lemma is the key to our main stability theorem for the original second order equation.

Theorem 4.20 (Stability with asymptotic phase)**.** *Let the regularity Assumption 4.1 and the well-posedness Assumption 4.2 hold, assume the existence of a traveling wave Assumption 4.7 with well-posedness in the co-moving frame Assumption 4.4. Furthermore, let the spectral stability Assumptions 4.15 and 4.17 hold. Then for all* $0 < \eta < \delta$ *there is* $\rho > 0$ *so that for all* $u_0 \in v_\star + H^3(\mathbb{R}, \mathbb{R}^m)$, $v_0 \in -\mu_\star v_{\star,\xi} + H^2(\mathbb{R}, \mathbb{R}^m)$ *with*

$$\|u_0 - v_\star\|_{H^3}^2 + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}^2 \le \rho^2, \tag{4.31}$$

the Cauchy problem (2.1) has a unique global solution $u \in v_\star + \mathcal{CH}^2([0,\infty);\mathbb{R}^m)$*. Moreover, there exist* $\varphi_\infty = \varphi(u_0, v_0)$ *and* $C = C(\eta, \rho)$ *with*

$$|\varphi_\infty| \le C\Big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\Big) \tag{4.32}$$

and

$$\begin{aligned} &\|u(t) - v_\star(\cdot - \mu_\star t - \varphi_\infty)\|_{H^2} + \|u_t + \mu_\star v_{\star,\xi}(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} \\ &\qquad \le C\Big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\Big)e^{-\eta t} \quad \forall t \ge 0. \end{aligned} \tag{4.33}$$

Proof. By Theorem 4.10 we can argue with the Cauchy problems to the first order formulations. Let us first consider the Cauchy problem (4.3), obtained by the transformation $U = (u, u_t - Nu_x)^\top$, which by Lemma 4.5 has the traveling wave solution $V_\star$ with velocity $\mu_\star$. For the initial data $U(0) = U_0 = (u_0\,,\, v_0 - Nu_{0,\xi})^\top$ we have with ρ small

$$\begin{aligned} \|U(0) - V_\star\|_{H^2}^2 \le& 3\big(\|u_0 - v_\star\|_{H^2}^2 + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}^2 + \|-Nu_{0,\xi} + Nv_{\star,\xi}\|_{H^2}^2\big) \\ \le& \text{const}\big(\|u_0 - v_\star\|_{H^3}^2 + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}^2\big) \le \text{const}\,\rho^2 \le \rho_0^2. \end{aligned} \tag{4.34}$$

For ρ sufficiently small, only depending on η, Lemma 4.19 applies and shows that there exist constants $\varphi_\infty = \varphi_\infty(u_0, v_0)$ and $C = C(\eta, \rho) > 0$ with

$$|\varphi_\infty| \le C\big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\big), \tag{4.35}$$

$$\begin{aligned} &\|U_1(t) - v_\star(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} + \|U_2(t) + (N + \mu_\star I_m)v_{\star,\xi}(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} \\ &\qquad \le C\big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\big)e^{-\eta t} \quad \forall t \ge 0. \end{aligned} \tag{4.36}$$

Similarly, the transformation $\widetilde{U} = (u, u_t + Nu_x)^\top$ leads to the Cauchy problem (4.13) with initial condition $\widetilde{U}(0) = \widetilde{U}_0 = (u_0, v_0 + Nu_{0,\xi})^\top$, which by Lemma 4.5 has the traveling wave solution $\widetilde{V}_\star$ with velocity $\mu_\star$. For the initial data of (4.13) we have for ρ small

$$\begin{aligned} \|\widetilde{U}(0) - \widetilde{V}_\star\|_{H^2}^2 \le& 3\big(\|u_0 - v_\star\|_{H^2}^2 + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}^2 + \|Nu_{0,\xi} - Nv_{\star,\xi}\|_{H^2}^2\big) \\ \le& \text{const}\big(\|u_0 - v_\star\|_{H^3}^2 + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}^2\big) \le \text{const}\,\rho^2 \le \rho_0^2. \end{aligned} \tag{4.37}$$

Lemma 4.19 applies again and thus there exists $\widetilde{\varphi}_\infty = \widetilde{\varphi}_\infty(u_0, v_0)$ and $\widetilde{C} = \widetilde{C}(\eta, \rho)$ with

$$|\widetilde{\varphi}_\infty| \le \widetilde{C}\big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\big), \tag{4.38}$$

$$\begin{aligned} &\|\widetilde{U}_1(t) - v_\star(\cdot - \mu_\star t - \widetilde{\varphi}_\infty)\|_{H^1} + \|\widetilde{U}_2(t) - (N - \mu_\star I_m)v_{\star,\xi}(\cdot - \mu_\star t - \widetilde{\varphi}_\infty)\|_{H^1} \\ &\qquad \le \widetilde{C}\big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\big)e^{-\eta t} \quad \forall t \ge 0. \end{aligned} \tag{4.39}$$

Under these restrictions on the initial data, both existence times satisfy $T = \widetilde{T} = \infty$ and hence $T_0 = \infty$ in Lemma 4.8. Thus we have $U_1 \equiv \widetilde{U}_1 =: u$, so that (4.36) and (4.39) imply

$$\|v_\star(\cdot - \mu_\star t - \varphi_\infty) - v_\star(\cdot - \mu_\star t - \widetilde{\varphi}_\infty)\|_{H^1} = \|v_\star(\cdot - \varphi_\infty) - v_\star(\cdot - \widetilde{\varphi}_\infty)\|_{H^1} \le Ce^{-\eta t}.$$

Therefore, $v_\star(\cdot-\varphi_\infty)=v_\star(\cdot-\widetilde{\varphi}_\infty)$ holds and we conclude $\varphi_\infty=\widetilde{\varphi}_\infty$, since a nontrivial period of $v_\star$ will contradict Assumption 4.7. Furthermore, the differential equations (4.1) and (4.13a) imply the relations $U_2(t)=u_t(t)-Nu_x(t)$ and $\widetilde{U}_2(t)=u_t(t)+Nu_x(t)$, which we use to obtain

$$\begin{aligned}\|u_t(t)+&\mu_\star v_{\star,\xi}(\cdot-\mu_\star t-\varphi_\infty)\|_{H^1}\\ =&\Big\|\frac{1}{2}\big(U_2(t)+\widetilde{U}_2(t)\big)-\frac{1}{2}\big(V_{\star,2}(\cdot-\mu_\star t-\varphi_\infty)+\widetilde{V}_{\star,2}(\cdot-\mu_\star t-\varphi_\infty)\big)\Big\|_{H^1}\\ \leq& C(\|u_0-v_\star\|_{H^3}+\|v_0+\mu_\star v_{\star,\xi}\|_{H^2})e^{-\eta t}\quad\forall t\geq 0,\end{aligned}\tag{4.40}$$

$$\begin{aligned}\|u_x(t)+&v_{\star,\xi}(\cdot-\mu_\star t-\varphi_\infty)\|_{H^1}\\ \leq&|N^{-1}|\Big\|\frac{1}{2}\big(\widetilde{U}_2(t)-U_2(t)\big)-\frac{1}{2}\big(\widetilde{V}_{\star,2}(\cdot-\mu_\star t-\varphi_\infty)-V_{\star,2}(\cdot-\mu_\star t-\varphi_\infty)\big)\Big\|_{H^1}\\ \leq& C|N^{-1}|(\|u_0-v_\star\|_{H^3}+\|v_0+\mu_\star v_{\star,\xi}\|_{H^2})e^{-\eta t}\quad\forall t\geq 0.\end{aligned}\tag{4.41}$$

This finishes the proof of (4.33). □

4.4. Lyapunov stability of the freezing method. Prior to the stability analysis of the freezing method, let us briefly comment on some phase conditions for the first order system (4.23) which can be used, to cope with the additional degree of freedom. We discuss how standard choices for the first order system relate to the phase conditions introduced in (2.9) and (2.14) for the original second order problem (2.5).

The setting is the same as in Sections 4.1 and 4.2. In particular, we impose Assumptions 4.1, 4.2, 4.4, 4.7 from Section 4.1 without further notice and let $V_\star$ be defined by (4.9) and $\widetilde{V}_\star$ by (4.14).

We begin with **fixed phase conditions (Type 1)**. Let $\widehat{V}=(\widehat{V}_1,\widehat{V}_2)^\top\in V_\star+H^1(\mathbb{R},\mathbb{R}^{2m})$ be a given template. Then, with the functional $\Psi\in\big(L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m)\big)'$ given by $\Psi\big(\binom{w_1}{w_2}\big)=\int_\mathbb{R}\widehat{V}_{1,\xi}^\top w_1+\widehat{V}_{2,\xi}^\top w_2\,d\xi$, we define a fixed phase condition by

$$\Psi(V-\widehat{V})=\int_\mathbb{R}\widehat{V}_{1,\xi}^\top\big(V_1-\widehat{V}_1\big)+\widehat{V}_{2,\xi}^\top\big(V_2-\widehat{V}_2\big)\,d\xi,\quad t\geq 0\tag{4.42}$$

as considered in [25]. If we choose $\widehat{V}=(\widehat{v},0)^\top$, $\widehat{v}\in v_\star+H^1(\mathbb{R},\mathbb{R}^m)$, (4.42) becomes

$$\Psi(V-\widehat{V})=\int_\mathbb{R}\widehat{v}_\xi^\top\big(V_1-\widehat{v}\big)\,d\xi=\int_\mathbb{R}\widehat{v}_\xi^\top\big(v-\widehat{v}\big)\,d\xi,\quad t\geq 0\tag{4.43}$$

where we used $v=V_1$ from Theorem 4.11. Observe that (4.43) is precisely the fixed phase condition (2.9) for the original second order problem (2.5).

For the derivation of **orthogonal phase conditions (Type 2)** we require that at each time instance the algebraic variable $\mu(t)$ is chosen, such that $\|V_t(t)\|_{L^2}$ is minimized. A necessary condition is

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\mu}\Big(\big\|(N+\mu I_m)V_{1,\xi}+V_2\big\|_{L^2}^2+\big\|M^{-1}(C-BN)V_{1,\xi}-(N-\mu I_m)V_{2,\xi}-M^{-1}BV_2+M^1f(V_1)\big\|_{L^2}^2\Big)\\ &\qquad=\langle V_{1,t},V_{1,\xi}\rangle+\langle V_{2,t},V_{2,\xi}\rangle=0,\quad t\geq 0.\end{aligned}\tag{4.44}$$

To obtain a suitable form for the second order problem (2.5) we insert the relations $V_1=v$ and $V_2=V_{1,t}-(N+\mu I_m)V_{1,\xi}$ from Theorem 4.11 into (4.44) and define $\mu_1=\mu$. This yields the side constraint

$$0=\langle v_t,v_\xi\rangle+\langle v_{tt}-(N+\mu_1 I_m)v_{t\xi}-\mu_2 v_\xi,v_{t\xi}-(N+\mu_1 I_m)v_{\xi\xi}\rangle,\quad t\geq 0,\tag{4.45}$$

for the second order system (2.5). We call (4.45) the *generalized orthogonal phase condition*.

Finally, if we minimize $\|V_{1,t}(t)\|_{L^2}$ with respect to $\mu(t)$ at each time instance instead of $\|V_t(t)\|_{L^2}=\|(V_{1,t}(t),V_{2,t}(t))\|_{L^2}$, we obtain the necessary condition

$$\frac{1}{2}\frac{d}{d\mu}\big\|(N+\mu I_m)V_{1,\xi}+V_2\big\|_{L^2}^2=\langle V_{1,t},V_{1,\xi}\rangle=\langle v_t,v_\xi\rangle=0,\quad t\geq 0,\tag{4.46}$$

where we used $V_1 = v$ from Theorem 4.11. This is precisely the orthogonal phase condition (2.14).

Remark 4.21. *Due to the choice of* $N = (M^{-1}A)^{\frac{1}{2}}$ *there is some arbitrariness present in the phase condition* (4.45) *when used for the original second order problem. The arbitrariness disappears with condition* (4.46) *which seems to be more natural for the original damped wave equation* (2.5).

In essence, all phase conditions, considered above, can be written in the abstract form

$$\psi^{\text{1st}}(V,\mu) = 0, \quad t \geq 0, \tag{4.47}$$

which covers all phase conditions from Section 2.1 and all other standard choices for the first order system.

Combining the differential equations (4.23a), (4.23b), the initial data (4.23c) and the phase condition (4.47), we arrive at the freezing system for the first order formulation of the damped wave equation,

$$\begin{aligned} V_t &= (E + \mu I_{2m})V_\xi + F(V), \\ \gamma_t &= \mu, \end{aligned} \qquad t \geq 0, \tag{4.48a}$$

$$0 = \psi^{\text{1st}}(V,\mu), \qquad t \geq 0, \tag{4.48b}$$

$$V(\cdot,0) = V_0 = (u_0, v_0 - Nu_{0,\xi})^\top, \quad \gamma(0) = 0. \tag{4.48c}$$

The initial datum for μ is given by the hidden constraint of this PDAE-system.

Before we state our main stability result for freezing traveling waves in damped wave equations, let us clarify the notion of solution to the freezing equations (2.18) and (4.48) that we use.

Definition 4.22. (i) *A tuple* $(v,\mu_1,\mu_2,\gamma) \in (v_\star + \mathcal{CH}^2([0,T);\mathbb{R}^m)) \times C^1([0,T)) \times C([0,T)) \times C^2([0,T))$ *is called a solution of* (2.18) *in* $[0,T)$ *if the equalities* (2.18a) *hold in* $L^2(\mathbb{R},\mathbb{R}^m) \times \mathbb{R} \times \mathbb{R}$ *for all* $t \in [0,T)$, (2.18b) *holds pointwise for all* $t \in [0,T)$, *and the equalities* (2.18c) *hold in* $H^2(\mathbb{R},\mathbb{R}^m) \times H^1(\mathbb{R},\mathbb{R}^m) \times \mathbb{R} \times \mathbb{R}$.

(ii) *A tuple* $(V,\mu,\gamma) \in (V_\star + \mathcal{CH}^1([0,T);\mathbb{R}^{2m})) \times C([0,T)) \times C^1([0,T))$ *is called a solution of* (4.48) *if the equalities* (4.48a) *hold in* $L^2(\mathbb{R},\mathbb{R}^{2m}) \times \mathbb{R}$ *for all* $t \in [0,T)$, (4.48b) *holds pointwise for all* $t \in [0,T)$, *and the equalities* (4.48c) *hold in* $H^1(\mathbb{R},\mathbb{R}^{2m}) \times \mathbb{R}$.

The stability (in the sense of Lyapunov) for the freezing method cannot be expected for all choices of the phase condition. For example the orthogonal phase condition (2.14) is invariant under spatial translations such that steady states of (2.18) with $\psi^{\text{2nd}} = \psi^{\text{2nd}}_{\text{orth},2}$ come in one-parameter families of equilibria. Henceforth we restrict to the fixed phase condition (2.9) for which we require the following non-degeneracy condition.

Assumption 4.23. *The template function* $\hat{v} : \mathbb{R} \to \mathbb{R}^m$ *belongs to* $v_\star + H^1(\mathbb{R},\mathbb{R}^m)$ *and satisfies*

$$\langle \hat{v} - v_\star, \hat{v}_\xi \rangle_{L^2} = 0, \tag{4.49a}$$

$$\langle v_{\star,\xi}, \hat{v}_\xi \rangle_{L^2} \neq 0. \tag{4.49b}$$

Condition (4.49a) implies that (2.8) holds for the fixed phase condition (2.9), so that $(v_\star, \mu_\star, 0)$ is a stationary solution of (2.18a), (2.18b) (skipping the γ-equation needed for reconstruction only).

Now we are ready to prove asymptotic stability (in the sense of Lyapunov) of the steady state $(v_\star, \mu_\star, 0)$ for the freezing system (2.18) that belongs to the second order damped wave equation.

Theorem 4.24 (Stability of the freezing method)**.** *Let the regularity Assumption 4.1 and the well-posedness Assumption 4.2 hold, assume the existence of a traveling wave Assumption 4.7 with well-posedness in the co-moving frame Assumption 4.4. Furthermore, let the spectral stability Assumptions 4.15 and 4.17 hold and let the template satisfy the non-degeneracy Assumption 4.23. Then for all* $0 < \eta < \delta$ *there is* $\rho > 0$ *so that for all* $u_0 \in v_\star + H^3(\mathbb{R},\mathbb{R}^m)$, $v_0 \in H^2(\mathbb{R},\mathbb{R}^m)$ *with*

$$\|u_0 - v_\star\|^2_{H^3} + \|v_0 + \mu_\star v_{\star,x}\|^2_{H^2} \leq \rho^2 \tag{4.50}$$

problem (2.18) *with phase condition* $\psi^{\rm 2nd}(v,v_t,\mu_1,\mu_2)=\langle v-\hat{v},\hat{v}_\xi\rangle_{L^2}$ *and consistent initial value* μ_1^0 *from* (2.12) *has a unique global solution* (v,μ_1,μ_2,γ). *Moreover, there exists some* $C=C(\rho,\eta)>0$ *such that the following exponential stability estimate holds*

$$\|v(t)-v_\star\|_{H^2}+\|v_t(t)\|_{H^1}+|\mu_1(t)-\mu_\star|\le C\big(\|u_0-v_\star\|_{H^3}+\|v_0+\mu_\star v_{\star,\xi}\|_{H^2}\big)\,e^{-\eta t}\quad\forall t\ge 0. \tag{4.51}$$

The proof builds on the following lemma, which shows that the original second order version (2.18) and the first order version (4.48) of the freezing method for traveling waves in (1.1) are equivalent.

Lemma 4.25. *Let Assumptions 4.1, 4.2, 4.4, 4.7 be satisfied, let* $\widehat{v}\in v_\star+H^1(\mathbb{R},\mathbb{R}^m)$ *and consider the fixed phase condition* $\psi^{\rm 1st}(V,\mu)=\langle V_1-\widehat{v},\widehat{v}_\xi\rangle_{L^2}$ *for* (2.18) *and* (4.48). *Further let* $u_0\in v_\star+H^2(\mathbb{R},\mathbb{R}^m)$, $v_0\in H^1(\mathbb{R},\mathbb{R}^m)$. *Then the following statements hold.*

(i) *If a tuple* (v,μ_1,μ_2,γ) *solves* (2.18) *in some interval* $[0,T_0)$ *and satisfies* $\langle v_\xi(t),\widehat{v}_\xi\rangle_{L^2}\neq 0$ *for all* $t\in[0,T_0)$, *then the tuple* (V,μ_1,γ) *with* $V=(v,v_t-(N+\mu_1 I_m)v_\xi)^\top$ *solves* (4.48) *in* $[0,T_0)$.

(ii) *Conversely, assume* (V,μ,γ) *solves* (4.48) *in some interval* $[0,T)$ *such that* $\langle V_{1,\xi}(t),\widehat{v}_\xi\rangle_{L^2}\neq 0$ *for all* $t\in[0,T)$, *let* $[0,\widetilde{T})$ *be the interval of existence for the maximally extended solution of* (4.13), *and define* $T_0=\min(T,\widetilde{T})$. *Then* $v=V_1\in v_\star+\mathcal{CH}^2([0,T_0);\mathbb{R}^m)$, $\gamma\in C^2([0,T_0))$ *and* (v,μ,μ_t,γ) *solves* (2.18) *in* $[0,T_0)$.

Proof. Assertion (i) is an obvious consequence of Theorem 4.11.
As for assertion (ii), note that $t\mapsto\gamma(t)=\int_0^t\mu(\tau)\,d\tau$ is in $C^1([0,T))$ by Definition 4.22, so that the function U given by $U(x,t)=V\big(x-\gamma(t),t\big)$ belongs to $V_\star+\mathcal{CH}^1([0,T);\mathbb{R}^{2m})$ and solves (4.3) in $[0,T)$. By Lemma 4.8, $U_1|_{[0,T_0)}\in v_\star+\mathcal{CH}^2([0,T_0);\mathbb{R}^m)$, hence

$$V_{1,\xi}\in\mathcal{CH}^1([0,T);\mathbb{R}^m)\quad\text{for}\quad V_{1,\xi}(t)=U_{1,x}\big(\cdot+\gamma(t),t\big),t\in[0,T). \tag{4.52}$$

Because of $\langle V_1(t)-\hat{v},\hat{v}_\xi\rangle_{L^2}=0$ for all $t\in[0,T)$ and $V_1\in\hat{v}+\mathcal{CH}^1([0,T);\mathbb{R}^{2m})$, the phase condition can be differentiated with respect to time, which yields

$$0=\langle V_{1,t},\hat{v}_\xi\rangle_{L^2}=\langle(N+\mu I_m)V_{1,\xi}+V_2,\hat{v}_\xi\rangle_{L^2}\quad\forall t\in[0,T). \tag{4.53}$$

Equation (4.53) can be solved for μ by our assumption $\langle V_{1,\xi}(t),\hat{v}_\xi\rangle_{L^2}\neq 0, t\in[0,T)$. Then (4.52) and $V_2\in\mathcal{CH}^1([0,T);\mathbb{R}^m)$ imply $\mu\in C^1([0,T))$, $\gamma\in C^2([0,T))$. Using $V(\xi,t)=U\big(\xi+\gamma(t),t\big)$ thus yields $V\in V_\star+\big(\mathcal{CH}^2([0,T);\mathbb{R}^m)\times\mathcal{CH}^1([0,T);\mathbb{R}^m)\big)$. Hence, Theorem 4.11 applies and proves the assertion. ☐

Proof of Theorem 4.24. Let $\eta<\delta$ be given. Instead of (2.18), consider the first order problem (4.48) without the γ-equation,

$$V_t=(E+\mu I_{2m})V_\xi+F(V),\quad 0=\langle\hat{v}_\xi,V_1-\hat{v}\rangle_{L^2},\quad V(0)=(u_0,v_0-Nu_{0,\xi})^\top. \tag{4.54}$$

For this problem all assumptions needed for the stability theorem [25, Thm. 2.3] are satisfied, see Lemma 4.18, the discussion of (i)–(vii) in Section 4.3, and Assumption 4.23. Moreover,

$$\|V(0)-V_\star\|_{H^2}^2\le(1+2|N|^2)\|u_0-v_\star\|_{H^3}^2+2\|v_0+\mu_\star v_{\star,\xi}\|_{H^2}^2\le\text{const}\rho^2.$$

Therefore, by [25, Thm. 2.3] there is $\rho=\rho(\eta)>0$, so that (4.54) has a unique global solution (V,μ), $V\in V_\star+\mathcal{CH}^1([0,\infty);\mathbb{R}^{2m})$, $\mu\in C([0,\infty))$, with

$$\|V_1(t)-V_{\star,1}\|_{H^1}+\|V_2(t)-V_{\star,2}\|_{H^1}+|\mu(t)-\mu_\star|\le C\|V(0)-V_\star\|_{H^2}e^{-\eta t},\quad\forall t\ge 0. \tag{4.55}$$

Let $\gamma(t):=\int_0^t\mu(\tau)\,d\tau$, then (V,μ,γ) solves (4.48) in $[0,\infty)$. Decreasing $\rho>0$ even further, we can achieve

$$\langle V_{1,\xi}(t),\hat{v}_\xi\rangle_{L^2}\neq 0\quad\forall t\in[0,\infty)$$

by using (4.49b), the equality $\langle V_{1,\xi}(t),\hat{v}_\xi\rangle_{L^2}=\langle V_{1,\xi}(t)-v_{\star,\xi},\hat{v}_\xi\rangle_{L^2}+\langle v_{\star,\xi},\hat{v}_\xi\rangle_{L^2}$, and the estimate $\|V_{1,\xi}(t)-v_{\star,\xi}\|_{L^2}\le\text{const}\rho e^{-\eta t}$.

Moreover, the unique maximally extended solution U to (4.3) is given by $U(x,t) = V(x-\gamma(t),t)$, and it exists for all $t \geq 0$. Finally, we consider the freezing version of the companion system (4.13) obtained by setting $\widetilde{V} = (v, v_t + (N - \widetilde{\mu} I_m)v_\xi)^\top$,

$$\widetilde{V}_t = (\widetilde{E} + \widetilde{\mu} I_{2m})\widetilde{V}_\xi + F(\widetilde{V}), \quad 0 = \langle \hat{v}_\xi, \widetilde{V}_1 - \hat{v}\rangle_{L^2}, \quad \widetilde{V}(0) = (u_0, v_0 + N u_{0,\xi})^\top. \tag{4.56}$$

As before,

$$\|\widetilde{V}(0) - \widetilde{V}_\star\|_{H^2}^2 \leq (1 + 2|N|^2)\|u_0 - v_\star\|_{H^3}^2 + 2\|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}^2 \leq \text{const}\rho^2,$$

and the same arguments as above show that the stability result [25, Thm. 2.3] applies to (4.56). Thus there exists a unique global solution $(\widetilde{V}, \widetilde{\mu})$ of (4.56) and

$$\|\widetilde{V}_1(t) - \widetilde{V}_{\star,1}\|_{H^1} + \|\widetilde{V}_2(t) - \widetilde{V}_{\star,2}\|_{H^1} + |\widetilde{\mu}(t) - \mu_\star| \leq C\|\widetilde{V}(0) - \widetilde{V}_\star\|_{H^2} e^{-\eta t}, \quad \forall t \geq 0. \tag{4.57}$$

Let $\widetilde{\gamma}(t) := \int_0^t \widetilde{\mu}(\tau)\,d\tau$, then the solution $\widetilde{U}$ to (4.13) is given by $\widetilde{U}(x,t) = \widetilde{V}(x - \widetilde{\gamma}(t), t)$, and it exists for all $t \geq 0$. Again, by decreasing ρ further, we can arrange $\langle \widetilde{V}_{1,\xi}(t), \hat{v}_\xi\rangle_{L^2} \neq 0$ for all $t \in [0,\infty)$. Then Lemma 4.25 applies twice with $T_0 = T = \widetilde{T} = \infty$ to yield the equivalence of solutions to (4.54) and (2.18), and of solutions to (4.56) and (2.18). The equivalence statements hold in the sense of Lemma 4.25, i.e.

$$V_1 = \widetilde{V}_1 = v,\ \mu = \widetilde{\mu} = \mu_1,\ V_2 = v_t - (N + \mu I_m)v_\xi,\ \widetilde{V}_2 = v_t + (N - \mu I_m)v_\xi \quad \text{for all } t \geq 0.$$

These relations and the estimates (4.55), (4.57) imply for all $t \geq 0$

$$\begin{aligned} \|v(t) - v_\star\|_{H^2} \leq& \|v(t) - v_\star\|_{H^1} + \|v_\xi(t) - v_{\star,\xi}\|_{H^1} \\ =& \|V_1(t) - V_{\star,1}\|_{H^1} + \left\|N^{-1}(\widetilde{V}_2(t) - V_2(t) - \widetilde{V}_{\star,2} + V_{\star,2})\right\|_{H^1} \\ \leq& \big(1 + |N^{-1}|\big)\Big(\|V_1(t) - V_{\star,1}\|_{H^1} + \|V_2(t) - V_{\star,2}\|_{H^1} + \|\widetilde{V}_2(t) - \widetilde{V}_{\star,2}\|_{H^1}\Big) \\ \leq& \text{const}\Big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\Big)e^{-\eta t}, \end{aligned} \tag{4.58}$$

and also

$$\begin{aligned} \|v_t\|_{H^1} \leq& \Big(\|v_t - \mu v_\xi + \mu_\star v_{\star,\xi}\|_{H^1} + \|\mu v_\xi - \mu_\star v_{\star,\xi}\|_{H^1}\Big) \\ \leq& \frac{1}{2}\|V_2 - V_{\star,2} + \widetilde{V}_2 - \widetilde{V}_{\star,2}\|_{H^1} + |\mu|\|v - v_\star\|_{H^2} + |\mu - \mu_\star|\|v_{\star,\xi}\|_{H^1} \\ \leq& \text{const}\Big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}\Big)e^{-\eta t}. \end{aligned} \tag{4.59}$$

In the last inequality of (4.59), we used (4.55) and (4.57) for the first and last summand while the middle summand is estimated by (4.58) and the bound (4.55) for the factor $|\mu|$. Combining (4.55), (4.58) and (4.59) and recalling $\mu = \mu_1$ finishes the proof. □

Remark 4.26. *The stability result* [25, Thm. 2.3] *shows that the solution of the first order freezing system* (4.54) *is unique, and because of the equivalence of solutions from Lemma 4.25, unique solvability transfers to the original second order freezing system* (2.18) *with* (2.18b) *given by* $\langle v - \hat{v}, \hat{v}_\xi\rangle_{L^2} = 0$.

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