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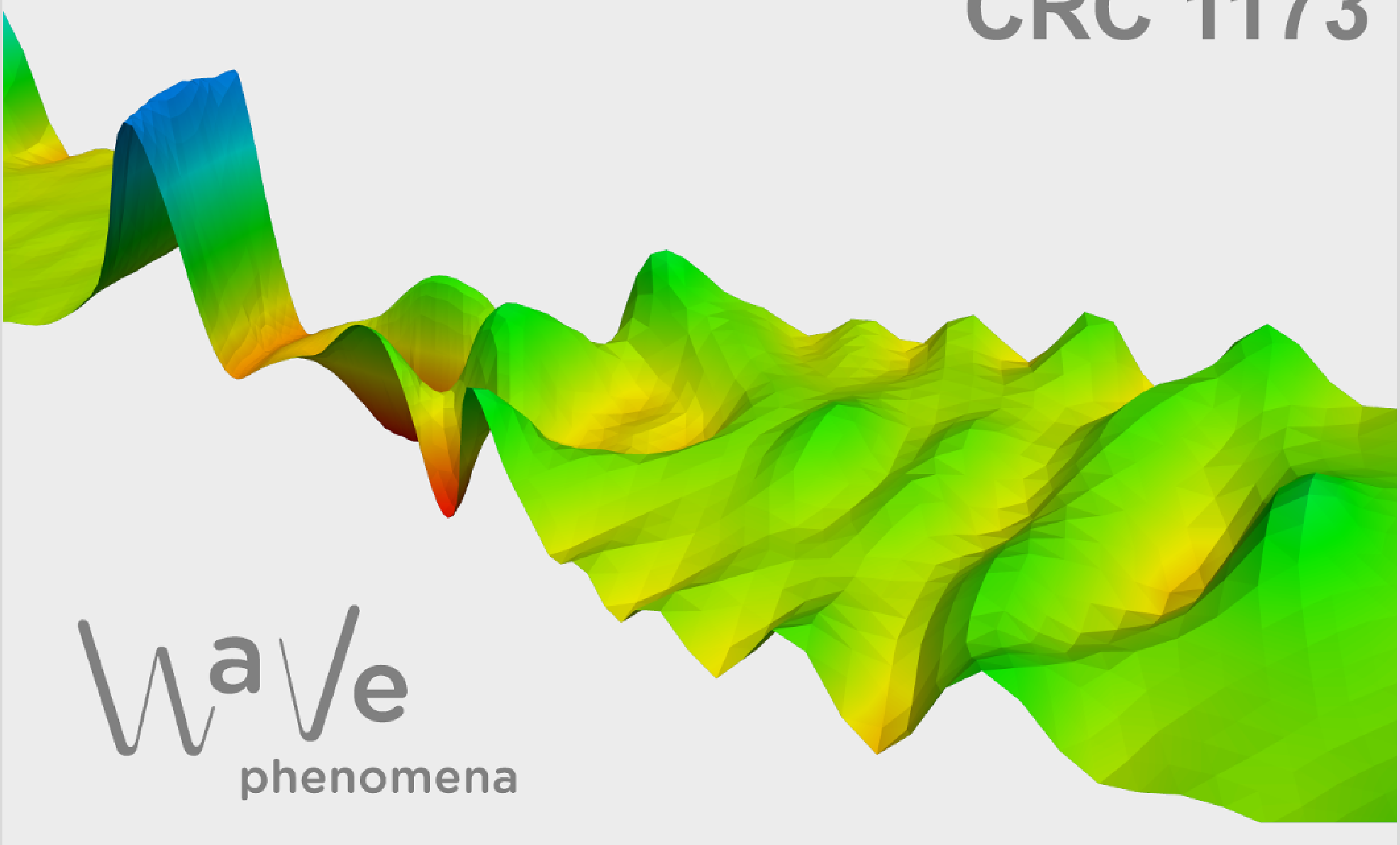

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EXISTENCE OF DISPERSION MANAGEMENT SOLITONS FOR GENERAL NONLINEARITIES

MI-RAN CHOI, DIRK HUNDERTMARK, YOUNG-RAN LEE

Abstract. We give a proof of existence of solitary solutions of the dispersion management equation for positive and zero average dispersion for a large class of nonlinearities. These solutions are found as minimizers of nonlinear and nonlocal variational problems which are invariant under a large non compact group. Our proof of existence of minimizers is rather direct and avoids the use of Lions' concentration compactness argument. The existence of dispersion managed solitons is shown under very mild conditions on the dispersion profile and the nonlinear polarization of optical active medium, which cover all physically relevant cases for the dispersion profile and a large class of nonlinear polarizations.

Contents

1. Introduction

1.1. The variational problems. We show the existence of minimizers for a family of nonlocal and nonlinear variational problems

$$E_\lambda^{d_{\rm av}} := \inf\left\{H(f) : \|f\|^2 = \lambda\right\}, \tag{1.1}$$

where $\lambda > 0$, the average dispersion $d_{\rm av} \geq 0$, $\|f\|^2 = \int_{\mathbb{R}} |f|^2\,dx$, the Hamiltonian takes the form

$$H(f) := \frac{d_{\rm av}}{2}\|f'\|^2 - N(f), \tag{1.2}$$

and the nonlocal nonlinearity is given by

$$N(f) := \iint_{\mathbb{R}^2} V(|T_r f(x)|)dx\psi(r)dr. \tag{1.3}$$

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Here $V : [0, \infty) \to \mathbb{R}$ is a suitable nonlinear potential and $T_r = e^{ir\partial_x^2}$ is the solution operator of the free Schrödinger equation in one dimension. The function ψ is the density of a probability measure and is assumed to be in suitable L^p-spaces.

If $d_{\mathrm{av}} > 0$ then, strictly speaking, the infimum in (1.1) is taken over all f with additionally $f \in H^1(\mathbb{R})$, the usual Sobolev space of square integrable functions whose distributional derivative f' is also square integrable. One can recover our formulation (1.1) by setting $\|f'\| := \infty$ if $f \in L^2 \setminus H^1$.

Our interest in these variational problems stems from the fact that the minimizers of (1.1) are the building blocks for (quasi-)periodic breather type solutions, the so-called dispersion management solitons, of the dispersion managed nonlinear Schrödinger equation. The dispersion management solitons have attracted a lot of interest in the development of ultrafast longhaul optical data transmission fibers. So far, it has mainly been studied for a Kerr-type nonlinearity, i.e., the special case where $V(a) = a^4$. The purpose of this work is to extend our previous existence results from [11] to a large class of nonlinearities V and also to positive average dispersion. We address the connection of the above variational problems with nonlinear optics later in Section 1.2.

The standard approach to show the existence of a minimizer of (1.1) is to identify it as the strong limit of a suitable minimizing sequence, that is, a sequence $(f_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R})$ with $\|f_n\|^2 = \lambda$ and $E_\lambda^{d_{\mathrm{av}}} = \lim H(f_n)$. The catch is that the above variational problem is invariant under translations of $L^2(\mathbb{R})$ if $d_{\mathrm{av}} > 0$ and under translations and boosts, that is, shifts in Fourier space, if $d_{\mathrm{av}} = 0$. This invariance under a large non-compact group of transformations leads to a loss of compactness since minimizing sequences can easily converge weakly to zero. The usual strategy to compensate for such a loss of compactness is Lion's concentration compactness method. In a previous paper, [11], we used an alternative approach, which for the special nonlinearity $V(a) = a^4$ and vanishing average dispersion directly showed that modulo the natural symmetries of the problem, minimizing sequences stay compact. The tools were very much tailored to the special type of Kerr nonlinearity. This paper extends our approach from [11] to a much more general setting. This extension is by no means straightforward, see Section 2 and Remark 1.3.

Our main assumptions on the nonlinear potential $V : \mathbb{R}_+ \to \mathbb{R}_+$ are

A1) $V(a) = q(a)a$ for $a \geq 0$ with $q : \mathbb{R}_+ \to \mathbb{R}_+$ continuous on $\mathbb{R}_+$, differentiable on $(0, \infty)$, and $q(0) = 0$. Moreover, there exist $\gamma_1,\ \gamma_2 \in \mathbb{R}$ with $2 \leq \gamma_1 \leq \gamma_2 \leq 6$ such that

$$q'(a) \lesssim a^{\gamma_1 - 2} + a^{\gamma_2 - 2}$$

for all $a > 0$.

A2) There exists $\gamma_0 > 2$ such that for all $\rho \geq 1$ and $a > 0$

$$V(\rho a) \geq \rho^{\gamma_0} V(a). \tag{1.4}$$

A3) If $d_{\mathrm{av}} > 0$, there exist $\varepsilon > 0$ and $2 \leq \kappa < 6$ such that

$$V(a) \gtrsim a^\kappa \quad \text{for all } 0 < a \leq \varepsilon. \tag{1.5}$$

If $d_{\mathrm{av}} = 0$, there exists $\varepsilon > 0$ such that$V(a) > 0$ for all $0 < a < \varepsilon$.

Above, we use the convention $f \lesssim g$, if there exists a finite constant $C > 0$ such that $f \leq Cg$. Our existence results are

Theorem 1.1 (Existence for positive average dispersion)**.** *Assume $d_{\mathrm{av}} > 0$, $2 \leq \gamma_1 \leq \gamma_2 \leq 6$, V obeys the assumptions A1) through A3), and $\psi \in L^{\frac{4}{6-\gamma_2}}$ has compact support. Then for any $\lambda > 0$, there exists a minimizer for the variational problem* (1.1)*. This minimizer is also a weak solution of the dispersion management equation* (1.13) *for some Lagrange multiplier ω.*

We have a similar existence result in the case of $d_{\mathrm{av}} = 0$ where we need only slightly stronger L^p assumptions on the density ψ.

Theorem 1.2 (Existence for zero average dispersion)**.** *Assume $d_{\mathrm{av}} = 0$, $2 < \gamma_1 \le \gamma_2 < 6$, V obeys the assumptions A1) through A3), and the density ψ has compact support and $\psi \in L^{\frac{4}{6-\gamma_2}+\delta}$ for arbitrarily small $\delta > 0$. Then for any $\lambda > 0$, there exists a minimizer for the variational problem (1.1). This minimizer is also a weak solution of the dispersion management equation (1.13) for some Lagrange multiplier ω.*

Remark 1.3. These two theorems extend our previous existence result in [11] to a large class of nonlinearities. As we will see below in Section 1.2, in particular, Lemma 1.4, for the application to dispersion management, it is quite natural to assume that ψ has a compact support. Hence even in the case of a Kerr nonlinearity, where $V(a) \sim a^4$, i.e., $\gamma_1 = \gamma_2 = 4$, the above two theorems strongly improve our result in [11] in terms of scales of L^p spaces: In [11], we needed that $\psi \in L^4$, whereas now with $\gamma_2 = 4$, one sees that $\psi \in L^2$ is enough for positive average dispersion and for vanishing average dispersion we only need $L^{2+\delta}$ for arbitrarily small $\delta > 0$.

For the Kerr nonlinearity, the smoothness and decay of the minimizers has been studied in [3] and [10] for the simplest case of an alternating dispersion profile given by $d_0(t) = \mathbf{1}_{[0,1)} - \mathbf{1}_{[1,2)}$ and extended to more general dispersion profiles in [9]. In the more general setting discussed in this paper the smoothness and decay of solitary solutions is an open problem.

The strategy of the proofs of our Existence Theorems 1.1 and 1.2 is as follows: Due to the bound (2.18) from Lemma 2.13, the main building blocks, for which one has to develop suitable space-time bounds, turn out to be of the form given in Definition 2.4. We develop the necessary estimates for this in Section 2.1 and their consequences for the nonlinear and nonlocal potential in Section 2.2. Strict subadditivity of the energy is done in Section 3 and the necessary tightness bound, modulo the symmetries of the problem, together with the proofs of Theorems 1.1 and 1.2, are established in Section 4. Our proofs for strictly positive average dispersion rely on some very useful space-time bounds for coherent states, see Lemma B.3, which are new and proven in Appendix B

1.2. **The connection with nonlinear optics.** Our main motivation for studying (1.1) comes from the fact that the minimizer of the variational problem is related to breather-type solutions of the dispersion managed nonlinear Schrödinger equation

$$i\partial_t u = -d(t)\partial_x^2 u - g(|u|)u, \tag{1.6}$$

where the dispersion $d(t)$ is parametrically modulated and $P(u) = g(|u|)u$ is the nonlinear interaction due to the polarizability of the glass fiber cable. In nonlinear optics (1.6) describes the evolution of a pulse in a frame moving with the group velocity of the signal through a glass fiber cable, see [20]. As a *warning*: with our choice of notation the variable t denotes the position along the glass fiber cable and x the (retarded) time. Hence $d(t)$ is *not varying in time* but denotes indeed a dispersion *varying along* the optical cable. For physical reasons it would not be a strong restriction to assume that d is piecewise constant, but we will not make this assumption in this paper. By symmetry, one assumes that P is odd and $P(0) = 0$ can always be enforced by adding a constant term. Most often one makes a Taylor series expansion, keeping just the lowest order nontrivial term leads to $P(u) \simeq |u|^2 u$, the Kerr nonlinearity, but we will not make this approximation.

The dispersion management idea, i.e., the possibility to periodically manage the dispersion by putting alternating sections with positive and negative dispersion together in an optical glass-fiber cable to compensate for dispersion of the signal was predicted by Lin, Kogelnik, and Cohen already in 1980, see [15], and then implemented by Chraplyvy and Tkach for which they received the Marconi prize in 2009. See the reviews [21, 22] and the references cited in [11] for a discussion of the dispersion management technique.

The periodic modulation of the dispersion can be modeled by the ansatz

$$d(t) = \varepsilon^{-1} d_0(t/\varepsilon) + d_{\mathrm{av}}. \tag{1.7}$$

Here $d_{\mathrm{av}} \geq 0$ is the average component and d_0 its mean zero part which we assume to have period L. For small ε the equation (1.7) describes a fast strongly varying dispersion which corresponds to the regime of *strong* dispersion management.

A technical complication is the fact that (1.6) is a non-autonomous equation. We seek to rewrite (1.6) into a more convenient form in order to find breather type solutions. Let $D(t) = \int_0^t d_0(r)\, dr$ and note that as long as d_0 is locally integrable and has period L with mean zero, D is also periodic with period L. Furthermore, $T_r = e^{ir\partial_x^2}$ is a unitary operator and thus the unitary family $t \mapsto T_{D(t/\varepsilon)}$ is periodic with period εL. Making the ansatz $u(t,x) = (T_{D(t/\varepsilon)} v(t,\cdot))(x)$ in (1.6), a short calculation shows

$$i\partial_t v = -d_{\mathrm{av}} \partial_x^2 v - T^{-1}_{D(t/\varepsilon)} \big[P(T_{D(t/\varepsilon)} v) \big] \tag{1.8}$$

which is equivalent to (1.6) and still a non-autonomous equation.

For small ε, that is, in the regime of strong dispersion management, $T_{D(t/\varepsilon)}$ is fast oscillating in the variable t, hence the solution v is expected to evolve on two widely separated time-scales, a slowly evolving part v_{slow} and a fast, oscillating part with a small amplitude. Analogously to Kapitza's treatment of the unstable pendulum which is stabilized by fast oscillations of the pivot, see [13], the effective equation for the slow part v_{slow} was derived by Gabitov and Turitsyn [5, 6] for the special case of a Kerr nonlinearity. It is given by integrating the fast oscillating term containing $T_{D(t/\varepsilon)}$ over one period in t,

$$\begin{aligned} i\partial_t v_{\mathrm{slow}} &= -d_{\mathrm{av}} \partial_x^2 v_{\mathrm{slow}} - \frac{1}{\varepsilon L} \int_0^{\varepsilon L} T^{-1}_{D(r/\varepsilon)} \big[P(T_{D(r/\varepsilon)} v) \big]\, dr \\ &= -d_{\mathrm{av}} \partial_x^2 v_{\mathrm{slow}} - \frac{1}{L} \int_0^{L} T^{-1}_{D(r)} \big[P(T_{D(r)} v) \big]\, dr. \end{aligned} \tag{1.9}$$

This averaging procedure leading to (1.9) was rigorously justified in [23] for suitable dispersion profiles d_0 in the case of a Kerr nonlinearity. The averaged equation is autonomous and stationary solutions of (1.9) can be found by making the ansatz

$$v_{\mathrm{slow}}(t,x) = e^{-i\omega t} f(x). \tag{1.10}$$

Before doing so, it turns out to be advantageous to rewrite the nonlocal nonlinear term in (1.9): Define a measure $\mu(B)$ by setting $\mu(B) := \frac{1}{L}\int_0^L \mathbf{1}_B(D(r))\, dr$ for any Lebesgue measurable set $B \subset \mathbb{R}$. Since $\mu(B) \geq 0$ and $\mu(\mathbb{R}) = \frac{1}{L}\int_0^L \mathbf{1}_{\mathbb{R}}(D(r))\, dr = \frac{1}{L}\int_0^L dr = 1$, one sees that μ is a probability measure. Since μ is the image measure of normalized Lebesgue measure on $[0, L]$ under D, we can rewrite (1.9) as

$$i\partial_t v_{\mathrm{slow}} = -d_{\mathrm{av}} \partial_x^2 v_{\mathrm{slow}} - \int_{\mathbb{R}} T_r^{-1} \big[P(T_r v) \big]\, \mu(dr). \tag{1.11}$$

The simplest case of dispersion management, $L = 2$, $d_0 = 1$ on $[0, 1)$ and $d_0 = -1$ on $[1, 2)$, i.e., $d_0 = \mathbf{1}_{[0,1)} - \mathbf{1}_{[1,2)}$, which is the case most studied in the literature, corresponds to the measure μ having density $\mathbf{1}_{[0,1]}$, the uniform distribution on $[0, 1]$. For the general case, we gather some basic properties of the probability measure μ in the following Lemma. For its proof, which for some parts uses the co-area formula from geometric measure theory [1, 4], see [11].

Lemma 1.4 (Lemma 1.4 in [11]). *Assume that the dispersion profile d_0 is locally integrable. Then* (i) *the probability measure μ has compact support.*
(ii) *If the set $\{d_0 = 0\}$ has zero Lebesgue measure, then μ is absolutely continuous with respect to Lebesgue measure.*
(iii) *If furthermore d_0 changes sign finitely many times on $[0, L]$ and is bounded away from zero then μ has a bounded density ψ.*
(iv) *Moreover, if d_0 changes sign finitely many times on $[0, L]$ and for some $p > 1$*

$$\int_0^L |d_0(s)|^{1-p}\,ds < \infty,$$

then μ has a density $\psi \in L^p$. More precisely, we have the bound

$$\|\psi\|_{L^p} \lesssim \Big(\int_0^L |d_0(s)|^{1-p}\,ds\Big)^{\frac{1}{p}} \tag{1.12}$$

where the implicit constant depends only on the number of sign changes of d_0 and the period L.

As explained in [11], the bound (1.12) is quite natural and sharp. Plugging (1.10) into (1.11), we see that f should solve

$$\omega f = -d_{\mathrm{av}} f'' - \int_{\mathbb{R}} T_r^{-1}\big[P(T_r f)\big]\,\mu(dr), \tag{1.13}$$

which is a nonlocal nonlinear eigenvalue equation for f. Testing (1.13) with suitable test functions g one gets the weak formulation

$$\omega\langle g, f\rangle = d_{\mathrm{av}}\langle g', f'\rangle - \langle g, \int_{\mathbb{R}} T_r^{-1}\big[P(T_r f)\big]\,\mu(dr)\rangle$$

where $\langle h_1, h_2\rangle$ is the scalar product on $L^2(\mathbb{R})$ given by $\int_{\mathbb{R}} \overline{h_1(x)} h_2(x)\,dx$. Exchanging integrals, a formal calculation, using the unicity of T_r, yields

$$\langle g, \int_{\mathbb{R}} T_r^{-1}\big[P(T_r f)\big]\,\mu(dr)\rangle = \int_{\mathbb{R}} \langle T_r g, P(T_r f)\rangle\,\mu(dr)$$

and one arrives at the weak formulation of (1.13) in the form

$$\omega\langle g, f\rangle = d_{\mathrm{av}}\langle g', f'\rangle - \int_{\mathbb{R}} \langle T_r g, P(T_r f)\rangle\,\mu(dr), \tag{1.14}$$

supposed to hold for any g in the Sobolev space $H^1(\mathbb{R})$.

Using the formula from Lemma 4.6 for the derivative of the nonlocal nonlinearity $N(f)$ from (1.3), one sees that (1.14) is the weak form of the Euler-Lagrange equation associated to the energy $H(f)$ given in (1.2), as long as $V'(|T_r f|)\mathrm{sgn}(T_r f) = P(T_r f)$. This is the case if

$$V'(a) = g(a)a = P(a) \quad \text{for all } a > 0,$$

i.e., V is the antiderivative of the polarizability P,

$$V(a) := \int_0^a P(s)\,ds.$$

In this case, any minimizer of the associated constrained minimization problem (1.1) will be, up to some minor technicalities, a weak solution of (1.13) for some choice of Lagrange multiplier ω, as long as the variational problem (1.1) admits minimizers. In particular, combining Theorems 1.1 and 1.2 with Lemma 1.4 one sees that (1.13) has a non trivial weak solution under the condition that the assumptions A1)–A3) hold for the antiderivative of P and that the dispersion profile d_0 changes signs finitely many times and obeys

$$d_0^{-1} \in L^{\frac{4}{6-\gamma_2}-1} = L^{\frac{\gamma_2-2}{6-\gamma_2}} \tag{1.15}$$

for positive average dispersion $d_{\mathrm{av}} > 0$ and

$$d_0^{-1} \in L^{\frac{\gamma_2-2}{6-\gamma_2}+\delta} \tag{1.16}$$

for arbitrarily small $\delta > 0$ in the singular limit case of zero average dispersion. This allows for a large class of dispersion profiles d_0, covering all physically relevant cases.

2. Nonlinear estimates

2.1. **Fractional Bilinear Estimates.** In this paper, the nonlocal nonlinearity is not a pure power, thus the multilinear estimates from [11] cannot be used anymore. First, we gather the estimates which will be used in the proof of fat–tail propositions, Propositions 4.1 and 4.2, which are crucial for the existence proof in this paper. The core of the argument will be suitable splitting bounds on the nonlocal nonlinearity $N(f)$ from (1.3) given in Proposition 2.15. For this, inspired by the splitting Lemma 2.13 for V, one needs certain *fractional linear* bounds on the building blocks from Definition 2.4.

Since $T_r = e^{ir\partial_x^2}$ is the solution operator for the free Schrödinger equation in dimension one, we can express $T_r f$ for any nice f, for example, in the Schwartz class, as follows:

$$T_r f(x) = \frac{1}{\sqrt{4\pi i r}} \int_{\mathbb{R}} e^{i\frac{|x-y|^2}{4r}} f(y)dy \tag{2.1}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\eta} e^{-ir\eta^2} \hat{f}(\eta)d\eta, \tag{2.2}$$

where $\hat{f}$ is the Fourier transform of f given by

$$\hat{f}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\eta} f(x)dx.$$

As a first step, we note that, for ψ in suitable L^p spaces, certain space time norms of $T_r f$ are bounded.

Lemma 2.1. *Let $f \in L^2(\mathbb{R})$, $2 \le q \le 6$ and $\psi \in L^{\frac{4}{6-q}}(\mathbb{R})$. Then*

$$\|T_r f\|^q_{L^q(\mathbb{R}^2, dx\psi dr)} \lesssim \|\psi\|_{L^{\frac{4}{6-q}}(\mathbb{R})} \|f\|^q. \tag{2.3}$$

Proof. Interpolating between 2 and 6 using the Hölder inequality, we get

$$\iint_{\mathbb{R}^2} |T_r f|^q dx\psi dr = \iint_{\mathbb{R}^2} \left(|T_r f|^{\frac{2(6-q)}{4}} \psi\right) \left(|T_r f|^{\frac{6(q-2)}{4}}\right) dxdr$$
$$\le \left(\iint_{\mathbb{R}^2} |T_r f|^2 \psi^{\frac{4}{6-q}}\, dxdr\right)^{\frac{6-q}{4}} \left(\iint_{\mathbb{R}^2} |T_r f|^6\, dxdr\right)^{\frac{q-2}{4}}.$$

Since T_r is unitary on $L^2(\mathbb{R})$,

$$\iint_{\mathbb{R}^2} |T_r f|^2 \psi^{\frac{4}{6-q}}\, dxdr = \|f\|^2 \int_{\mathbb{R}} \psi^{\frac{4}{6-q}}\, dr$$

and the one-dimensional Strichartz inequality [8, 12, 19] gives

$$\iint_{\mathbb{R}^2} |T_r f|^6\, dxdr \le S_1^6 \|f\|^6$$

and so (2.3) follows. ∎

To take advantage of the fact that an interaction term containing the product of two terms of the form $T_r f_1$ and $T_r f_2$ is typically small if the functions $\hat{f}_1$ and $\hat{f}_2$ have separated supports, we need

Lemma 2.2 (Fractional bilinear estimate)**.** *Let* $2 \le p < 3$ *and* $f_1,\ f_2 \in L^2(\mathbb{R})$ *whose Fourier transforms have separated supports, say* $s = \mathrm{dist}(\mathrm{supp}\,\hat{f}_1, \mathrm{supp}\,\hat{f}_2) > 0$*. Then*

$$\|T_r f_1 T_r f_2\|_{L^p(\mathbb{R}^2,\, dxdr)} \lesssim \frac{1}{s^{(3-p)/p}} \|f_1\|\|f_2\|. \tag{2.4}$$

Remark 2.3. The bound (2.4) is a well-known bilinear estimate for $p = 2$, see [2]. For readers' convenience, we give a proof of (2.4) for any $2 \le p < 3$. As the proof shows, (2.4) holds also for $p = 3$, without any support condition on $\hat{f}_1$ and $\hat{f}_2$.

Proof. Using (2.2), we get

$$T_r f_1(x) T_r f_2(x) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{ix(\eta_1+\eta_2)-ir(\eta_1^2+\eta_2^2)} \hat{f}_1(\eta_1)\hat{f}_2(\eta_2) d\eta_1 d\eta_2.$$

Doing the change of variables $a = \eta_1+\eta_2$, $b = \eta_1^2+\eta_2^2$, with Jacobian $J = \frac{\partial(a,b)}{\partial(\eta_1,\eta_2)} = 2(\eta_2-\eta_1)$ and introducing

$$F(a,b) := \frac{1}{|J|}\hat{f}_1(\eta_1(a,b))\hat{f}_2(\eta_2(a,b))\mathbf{1}_{[0,\infty)}(b)$$

one sees

$$T_r f_1(x) T_r f_2(x) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{ixa-irb} F(a,b)\, dadb,$$

that is, up to sign in one of the variables, $T_r f_1(x) T_r f_2(x)$ is the space-time Fourier transform of F. Since $p \ge 2$, one can apply the Hausdorff-Young inequality, which reduces to Plancherel's identity for $p = 2$, to get

$$\|T_r f_1 T_r f_2\|_{L^p(\mathbb{R}\times\mathbb{R}, dxdr)} \le \|F\|_{L^{p'}(\mathbb{R}^2, dadb)}$$

with p' the dual index to p. Undoing the above change of variables, one sees

$$\|F\|_{L^{p'}(\mathbb{R}^2,dadb)} = 2^{-1/p}\left(\iint_{\mathbb{R}^2} \frac{1}{|\eta_2-\eta_1|^{p'-1}} |\hat{f}_1(\eta_1)\hat{f}_2(\eta_2)|^{p'}\, d\eta_1 d\eta_2\right)^{1/p'}. \tag{2.5}$$

If $p = p' = 2$, we use $|\eta_2 - \eta_1| \ge s$ on the support of the product $\hat{f}_1\hat{f}_2$ to get

$$\|F\|_{L^2(\mathbb{R}^2,dadb)} \lesssim \frac{1}{\sqrt{s}}\|\hat{f}_1\|\|\hat{f}_1\|$$

which concludes the proof for $p = 2$, since the Fourier transform is an isometry on L^2.

Since $3/2 < p' < 2$, one can use the Hardy-Littlewood-Sobolev inequality to see

$$\begin{aligned}(2.5) &\le \frac{1}{s^{2-3/p'}}\left(\iint_{\mathbb{R}^2} \frac{|\hat{f}_1(\eta_1)|^{p'}|\hat{f}_2(\eta_2)|^{p'}}{|\eta_2-\eta_1|^{2-p'}} d\eta_1 d\eta_2\right)^{\frac{1}{p'}} \\ &\lesssim \frac{1}{s^{(3-p)/p}}\|\hat{f}_1\|\|\hat{f}_2\|\end{aligned}$$

which yields (2.4) for $2 < p < 3$. ∎

The following will be the building blocks for our bounds on the nonlocal nonlinear potential, see (2.18).

Definition 2.4. For any $\gamma \geq 2$, define

$$M_\psi^\gamma(f_1, f_2) := \iint_{\mathbb{R}^2} |T_r f_1||T_r f_2|(|T_r f_1| + |T_r f_2|)^{\gamma-2}\, dx\psi dr. \tag{2.6}$$

Remark 2.5. At first $M_\psi^\gamma(f_1, f_2)$ is defined only when f_1, f_2 are nice Schwartz functions. We will shortly see that for ψ in certain L^p spaces, $M_\psi^\gamma(f_1, f_2)$ can be extended to all $f_1, f_2 \in L^2$ by density of Schwartz functions in L^2.

Proposition 2.6. *Let $2 \leq \gamma \leq 6$ and $\psi \in L^{\frac{4}{6-\gamma}}$. Then*

$$M_\psi^\gamma(f_1, f_2) \lesssim \|f_1\|\|f_2\|(\|f_1\| + \|f_2\|)^{\gamma-2} \tag{2.7}$$

where the implicit constant depends only on the $L^{\frac{4}{6-\gamma}}$ norm of ψ.

Proof. Using Hölder's inequality for 3 functions with exponents γ, γ, and $\gamma/(\gamma-2)$ one has

$$M_\psi^\gamma(f_1, f_2) \leq \|T_r f_1\|_{L^\gamma(\mathbb{R}^2, dx\psi dr)} \|T_r f_2\|_{L^\gamma(\mathbb{R}^2, dx\psi dr)} \||T_r f_1| + |T_r f_2|\|_{L^\gamma(\mathbb{R}^2, dx\psi dr)}^{\gamma-2}$$

Applying the triangle inequality and Lemma 2.1 completes the proof. ∎

Proposition 2.7. *Let $s = \operatorname{dist}(\operatorname{supp} \hat{f}_1, \operatorname{supp} \hat{f}_2) > 0$.*
If $2 < \gamma < 6$, $\tau > 1$ and $\psi \in L^{\beta(\gamma,\tau)}$, then

$$M_\psi^\gamma(f_1, f_2) \lesssim s^{-\alpha(\gamma,\tau)} \|f_1\|\|f_2\|(\|f_1\| + \|f_2\|)^{\gamma-2}, \tag{2.8}$$

where $\alpha(\gamma, \tau) := \min\{\frac{\gamma-2}{6\tau}, \frac{6-\gamma}{2\tau}\}$ and $\beta(\gamma, \tau) := \frac{4}{6-\gamma-2\alpha(\gamma,\tau)}$.

Remark 2.8. Note that $\beta(\gamma, \tau)$ is only slightly bigger than $\frac{4}{6-\gamma}$ since $\alpha(\gamma, \tau) > 0$ tends to zero as $\tau \to \infty$ and that it is increasing in γ. So we loose only an epsilon, by choosing τ large enough, with respect to the bound from Proposition 2.6.

Proof. Let $0 < \alpha < 1$ to be chosen later and write

$$M_\psi^\gamma(f_1, f_2) = \iint_{\mathbb{R}^2} \left\{(|T_r f_1||T_r f_2|)^{1-2\alpha}\psi\right\} \left\{|T_r f_1||T_r f_2|\right\}^{2\alpha} \left\{(|T_r f_1| + |T_r f_2|)^{\gamma-2}\right\} dxdr.$$

Now use Hölder's inequality for 3 functions with exponents p_1, $\frac{1}{\alpha}$, and, $\frac{6}{\gamma-2}$, where

$$\frac{1}{p_1} = 1 - \alpha - \frac{\gamma-2}{6} = \frac{8-\gamma-6\alpha}{6}$$

to see that

$$M_\psi^\gamma(f_1, f_2) \leq \left(\iint_{\mathbb{R}^2} |T_r f_1 T_r f_2|^{\frac{6(1-2\alpha)}{8-\gamma-6\alpha}} \psi^{\frac{6}{8-\gamma-6\alpha}}\, dxdr\right)^{\frac{8-\gamma-6\alpha}{6}}$$
$$\|T_r f_1 T_r f_2\|_{L^2(\mathbb{R}^2, dxdr)}^{2\alpha} \||T_r f_1| + |T_r f_2|\|_{L^6(\mathbb{R}^2, dxdr)}^{\gamma-2}.$$

Up to a constant, the third factor is bounded by $(\|f_1\| + \|f_2\|)^{\gamma-2}$, using the triangle and Strichartz inequalities. Using Lemma 2.2, the second factor is bounded by

$$\|T_r f_1 T_r f_2\|_{L^2(\mathbb{R}^2, dxdr)}^{2\alpha} \lesssim s^{-\alpha}\|f_1\|^{2\alpha}\|f_2\|^{2\alpha}.$$

For the first factor, we note that with the help of the Cauchy-Schwarz inequality one gets

$$\iint_{\mathbb{R}^2} |T_r f_1 T_r f_2|^{\frac{6(1-2\alpha)}{8-\gamma-6\alpha}} \psi^{\frac{6}{8-\gamma-6\alpha}}\, dxdr$$
$$\leq \left(\iint_{\mathbb{R}^2} |T_r f_1|^{\frac{12(1-2\alpha)}{8-\gamma-6\alpha}} \psi^{\frac{6}{8-\gamma-6\alpha}}\, dxdr\right)^{1/2} \left(\iint_{\mathbb{R}^2} |T_r f_2|^{\frac{12(1-2\alpha)}{8-\gamma-6\alpha}} \psi^{\frac{6}{8-\gamma-6\alpha}}\, dxdr\right)^{1/2}.$$

In order to use Lemma 2.1 for this, we need to have $2 \le q \le 6$ with $q = \frac{12(1-2\alpha)}{8-\gamma-6\alpha}$. This is equivalent to $6\alpha < 8-\gamma$, $6\alpha \le \gamma - 2$ and $2\alpha \le 6-\gamma$.

Moreover, we need

$$\psi^{\frac{6}{8-\gamma-6\alpha}} \in L^{\frac{4}{6-q}} = L^{\frac{4(8-\gamma-6\alpha)}{6(6-\gamma-2\alpha)}}$$

hence

$$\psi \in L^{\frac{4}{6-\gamma-2\alpha}}.$$

Now we come to the choice of α: In order to guarantee that $0 < \alpha < 1$, $6\alpha < 8-\gamma$, $6\alpha \le \gamma - 2$, and $2\alpha \le 6-\gamma$, we take any $\tau > 1$ and put $\alpha := \alpha(\gamma, \tau)$. Then one checks that α obeys the above bounds and so this finishes the proof. ∎

Lemma 2.9 (Duality). *Define*

$$\widetilde{\psi}(s) := \frac{1}{(2|s|)^{\frac{6-\gamma}{2}}}\psi\big(-\frac{1}{4s}\big) \tag{2.9}$$

for $s \neq 0$. *Then*

$$M^\gamma_\psi(f_1, f_2) = M^\gamma_{\widetilde{\psi}}(\check{f}_1, \check{f}_2) \tag{2.10}$$

where $\check{f}$ *is the inverse Fourier transform of* f*.*

Remark 2.10. Of course, the definition of $\widetilde{\psi}$ depends on γ, but we drop this dependence in our notation, for simplicity. For $2 \le \gamma \le 6$, Proposition 2.6 yields a natural a priori bound on $M^\gamma_\psi(f_1, f_2)$ which depends on the $L^{\frac{4}{6-\gamma}}$ norm of ψ. It is an easy exercise to check that $\|\psi\|_{L^{\frac{4}{6-\gamma}}} = \|\widetilde{\psi}\|_{L^{\frac{4}{6-\gamma}}}$, so Proposition 2.6 and the duality expressed in (2.10) are consistent.

Proof. Without loss of generality, assume that f_1 and f_2 are Schwartz functions for the calculations below. Defining $u_j(r,x) := (T_r f_j)(x)$ and $\check{u}_j(r,x) := (T_r \check{f}_j)(x)$, $j = 1,2$, using the explicit form of the free time evolution (2.1) for $u_j(r,x)$, and expanding the square, one sees

$$u_j(r,x) = \frac{1}{\sqrt{2ir}} e^{i\frac{x^2}{4r}} \check{u}_j\Big(\frac{-1}{4r}, \frac{-x}{2r}\Big)$$

which is often called pseudo-conformal invariance of the free Schrödinger evolution. Then

$$M^\gamma_\psi(f_1, f_2)$$
$$= \iint_{\mathbb{R}^2} \frac{\left|\check{u}_1\Big(\frac{-1}{4r}, \frac{-x}{2r}\Big)\right| \left|\check{u}_2\Big(\frac{-1}{4r}, \frac{-x}{2r}\Big)\right| \left(\left|\check{u}_1\Big(\frac{-1}{4r}, \frac{-x}{2r}\Big)\right| + \left|\check{u}_2\Big(\frac{-1}{4r}, \frac{-x}{2r}\Big)\right|\right)^{\gamma-2}}{(2|r|)^{\gamma/2}} dx\psi(r)dr. \tag{2.11}$$

Doing first the change of variables $x = -2ry$, $dx = 2|r|dy$ and then $r = -1/(4s)$ with $dr = (2|s|)^{-2}\, ds$, yields

$$(2.11) = \iint_{\mathbb{R}^2} \frac{|\check{u}_1(s,y)|\,|\check{u}_2(s,y)|\,(|\check{u}_1(s,y)| + |\check{u}_2(s,y)|)^{\gamma-2}}{(2|s|)^{\frac{6-\gamma}{2}}} dy\psi(-\frac{1}{4s})ds$$

which completes the proof. ∎

This duality is a convenient tool in the proof of the analogue of Proposition 2.7 when the functions f_1 and f_2 have separated supports.

Proposition 2.11. *Let* $s = \operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) > 0$.
If $2 < \gamma < 6$, $\tau > 1$ *and* $\psi \in L^{\beta(\gamma,\tau)}(|r|^{\alpha(\gamma,\tau)\beta(\gamma,\tau)}dr)$, *then*

$$M^\gamma_\psi(f_1, f_2) \lesssim s^{-\alpha(\gamma,\tau)} \|f_1\| \|f_2\| (\|f_1\| + \|f_2\|)^{\gamma-2}. \tag{2.12}$$

Proof. Given the duality expressed in Lemma 2.9 this is now simple: We have

$$M_\psi^\gamma(f_1, f_2) = M_{\widetilde{\psi}}^\gamma(\check{f}_1, \check{f}_2)$$

and note that the assumption on the separation of the supports of f_1 and f_2 means, of course, that $\check{f}_1$ and $\check{f}_2$ have separated Fourier support, so Proposition 2.7 applies to $M_{\widetilde{\psi}}^\gamma(\check{f}_1, \check{f}_2)$ as long as $\widetilde{\psi}$ is in the correct L^p space. A short calculation shows

$$\|\widetilde{\psi}\|_{L^p(dr)}^p = \int_{\mathbb{R}} (2|r|)^{\frac{p(6-\gamma)}{2}-2} |\psi(r)|^p\, dr$$

and (2.12) follows by choosing $p = \beta(\gamma, \tau)$. ∎

To handle the cases with $\gamma = 2$ or $\gamma = 6$ for positive average dispersion, we need a fractional bilinear estimate for M_ψ^γ in H^1 as follows.

Proposition 2.12 (H^1 bilinear estimate)**.** *Let $2 \le \gamma \le 6$ and $\psi \in L^{\frac{4}{6-\gamma}}(\mathbb{R})$ with compact support. Then for any $f_1, f_2 \in H^1(\mathbb{R})$ with $s = \mathrm{dist}(\mathrm{supp}\, f_1, \mathrm{supp}\, f_2) > 0$,*

$$M_\psi^\gamma(f_1, f_2) \lesssim s^{-1} \|f_1\|_{H^1} \|f_2\|_{H^1} (\|f_1\| + \|f_2\|)^{\gamma-2}, \tag{2.13}$$

where the implicit constant depends only on the support and the $L^{\frac{4}{6-\gamma}}$ norm of ψ.

Proof. Using Hölder's inequality, one has

$$M_\psi^\gamma(f_1, f_2) \le \|T_r f_1 T_r f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2, dx\psi dr)} \||T_r f_1| + |T_r f_2|\|_{L^\gamma(\mathbb{R}^2, dx\psi dr)}^{\gamma-2} \tag{2.14}$$

and with the triangle inequality and Lemma 2.1, we have

$$\||T_r f_1| + |T_r f_2|\|_{L^\gamma(\mathbb{R}^2, dx\psi dr)} \lesssim \|f_1\| + \|f_2\| \tag{2.15}$$

when $\psi \in L^{\frac{4}{6-\gamma}}$.

To bound the first factor, we use the positive operators $P_L^{\le}$ and $P_L^{>}$ from Lemma B.3 for suitably chosen $L > 0$. Although they are not projection operators, we think of $P_L^{\le}$ as 'projecting' onto frequencies localized to $\lesssim L$ and $P_L^{>}$ as 'projecting' onto large frequencies $\gtrsim L$. At the same time, the supports of $P_L^{\le} f_1$ and $P_L^{>} f_2$ will still be essentially separated. See Lemma B.2 and B.3 in appendix B for the properties of $P_L^{\le}$ and $P_L^{>}$ which we will need.

Since $P_L^{\le} + P_L^{>} = \mathbf{1}$ on $L^2(\mathbb{R})$ by Lemma B.2, we can use the triangle inequality and the linearity of T_r to split

$$\begin{aligned} &\|T_r f_1 T_r f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} \\ &\qquad \le \|T_r P_L^{>} f_1 T_r f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} + \|T_r P_L^{\le} f_1 T_r P_L^{>} f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} \\ &\qquad\quad + \|T_r P_L^{\le} f_1 T_r P_L^{\le} f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)}. \end{aligned} \tag{2.16}$$

The Cauchy–Schwarz inequality and Lemma 2.1 yield

$$\begin{aligned} \|T_r P_L^{>} f_1 T_r f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} &\le \|T_r P_L^{>} f_1\|_{L^\gamma(\mathbb{R}^2,\, dx\psi dr)} \|T_r f_2\|_{L^\gamma(\mathbb{R}^2,\, dx\psi dr)} \\ &\lesssim \|P_L^{>} f_1\| \|f_2\| \lesssim L^{-1} \|f_1\|_{H^1} \|f_2\|, \end{aligned}$$

where we use (B.14). Switching the roles of f_1 and f_2, using in addition that $P_L^{\le} \le \mathbf{1}$, shows

$$\|T_r P_L^{\le} f_1 T_r P_L^{>} f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} \lesssim L^{-1} \|f_1\| \|f_2\|_{H^1}.$$

To bound the last term of the right hand side in (2.16), we note that (B.16) shows

$$\|T_r P_L^{\le} f_1 T_r P_L^{\le} f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} \le \|\psi\|_{L^1}^{2/\gamma} \sup_{|r| \le T} \|T_r P_L^{\le} f_1 T_r P_L^{\le} f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}, dx)}$$

$$\leq \|\psi\|_{L^1}^{2/\gamma} A_R L^2\, e^{2L^2/\gamma B_{\gamma/2,R} s^2}\, \|f_1\|\|f_2\|,$$

with $R > 0$ chosen such that supp $\psi \subset [-R, R]$ and the constants A_R and $B_{\gamma/2,R}$ from Lemma B.3. Therefore

$$\|T_r f_1 T_r f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} \lesssim \left[L^2 e^{2L^2/\gamma - B_{\gamma/2,R} s^2} + L^{-1}\right] \|f_1\|_{H^1}\|f_2\|_{H^1}$$

for any $L \geq 0$. Choosing $2L^2 = B_{\gamma/2,R} s^2$, we get

$$\|T_r f_1 T_r f_2\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2,\, dx\psi dr)} \lesssim s^{-1} \|f_1\|_{H^1}\|f_2\|_{H^1},$$

and using this with (2.15) in (2.14) proves Proposition 2.12. ∎

2.2. **Splitting the Nonlocal Nonlinear Potential.** Recall the nonlinear potential $V : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $V(a) = q(a)a$ for $a \geq 0$ with $q : \mathbb{R}_+ \to \mathbb{R}_+$ continuous on $\mathbb{R}_+$ and differentiable on $(0, \infty)$ with $q(0) = 0$. Recall also that for $2 \leq \gamma_1 \leq \gamma_2 \leq 6$,

$$q'(a) \lesssim a^{\gamma_1 - 2} + a^{\gamma_2 - 2}$$

for all $a > 0$.

Lemma 2.13. *With V as above,*

$$V(a) \lesssim a^{\gamma_1} + a^{\gamma_2} \tag{2.17}$$

for all $a \geq 0$ and

$$V(|z+w|) - V(|z|) - V(|w|) \lesssim |z||w| \left((|z| + |w|)^{\gamma_1 - 2} + (|z| + |w|)^{\gamma_2 - 2}\right) \tag{2.18}$$

for all $z, w \in \mathbb{C}$.

Proof. Integrating the bound for q' gives $q(a) \lesssim a^{\gamma_1 - 1} + a^{\gamma_2 - 1}$ which implies (2.17). For the second claim, note that by the triangle inequality

$$\begin{aligned} V(|z+w|) - V(|z|) - V(|w|) &= q(|z+w|)|z+w| - q(|z|)|z| - q(|w|)|w| \\ &\leq |z|\big(q(|z+w|) - q(|z|)\big) + |w|\big(q(|z+w|) - q(|w|)\big). \end{aligned}$$

If $|z+w| \geq |z|$, using the assumption on q' and the triangle inequality, one sees

$$\begin{aligned} q(|z+w|) - q(|z|) &= \int_{|z|}^{|z+w|} q'(s)\, ds \lesssim \left[|z+w|^{\gamma_1 - 2} + |z+w|^{\gamma_2 - 2}\right](|z+w| - |z|) \\ &\leq \left[(|z| + |w|)^{\gamma_1 - 2} + (|z| + |w|)^{\gamma_2 - 2}\right]|w|. \end{aligned}$$

Similarly, if $|z+w| \leq |z|$ then

$$\begin{aligned} q(|z+w|) - q(|z|) &\lesssim \left[|z|^{\gamma_1 - 2} + |z|^{\gamma_2 - 2}\right](|z| - |z+w|) \\ &\leq \left[(|z| + |w|)^{\gamma_1 - 2} + (|z| + |w|)^{\gamma_2 - 2}\right]|w|. \end{aligned}$$

Switching z and w, one also has

$$q(|z+w|) - q(|w|) \lesssim \left[(|z| + |w|)^{\gamma_1 - 2} + (|z| + |w|)^{\gamma_2 - 2}\right]|z|$$

and therefore (2.18) follows. ∎

Recall

$$N(f) := \iint_{\mathbb{R}^2} V(|T_r f(x)|)\, dx\psi dr \tag{2.19}$$

and we get the following estimate of N immediately from (2.17) and Lemma 2.1.

Proposition 2.14 (Boundedness)**.** *Let $2 \le \gamma_1 \le \gamma_2 \le 6$ and $\psi \in L^1 \cap L^{\frac{4}{6-\gamma_2}}$. Then for all $f \in L^2(\mathbb{R})$*

$$N(f) \lesssim \|f\|^{\gamma_1} + \|f\|^{\gamma_2}, \tag{2.20}$$

where the implicit constant depends only on the L^1 and $L^{\frac{4}{6-\gamma_2}}$ norms of ψ.

Proposition 2.15 (Splitting)**.** (i) *Assume $2 \le \gamma_1 \le \gamma_2 \le 6$ and $\psi \in L^1 \cap L^{\frac{4}{6-\gamma_2}}$. Then*

$$N(f_1 + f_2) - N(f_1) - N(f_2) \lesssim \|f_1\|\|f_2\| \left(1 + \|f_1\|^4 + \|f_2\|^4\right). \tag{2.21}$$

(ii) *Assume $2 < \gamma_1 \le \gamma_2 < 6$ and $\tau > 1$. Then*

$$N(f_1 + f_2) - N(f_1) - N(f_2) \lesssim \max\{1, s\}^{-\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}} \|f_1\|\|f_2\| \left(1 + \|f_1\|^4 + \|f_2\|^4\right) \tag{2.22}$$

if $\psi \in L^1 \cap L^{\beta(\gamma_2,\tau)}$ and $s = \mathrm{dist}(\mathrm{supp}\, \hat{f}_1, \mathrm{supp}\, \hat{f}_2) > 0$, or $\psi \in L^{\beta(\gamma_2,\tau)}$ has compact support and $s = \mathrm{dist}(\mathrm{supp}\, f_1, \mathrm{supp}\, f_2) > 0$.

(iii) *Assume $2 \le \gamma_1 \le \gamma_2 \le 6$. Then*

$$N(f_1 + f_2) - N(f_1) - N(f_2) \lesssim \max\{1, s\}^{-1} \|f_1\|_{H^1} \|f_2\|_{H^1} \left(1 + \|f_1\|^4 + \|f_2\|^4\right) \tag{2.23}$$

if $\psi \in L^{\frac{4}{6-\gamma_2}}$ has compact support and $s = \mathrm{dist}(\mathrm{supp}\, f_1, \mathrm{supp}\, f_2) > 0$.

Proof. Because of Lemma 2.13, we have

$$\begin{aligned} N(f_1 &+ f_2) - N(f_1) - N(f_2) \\ &= \iint_{\mathbb{R}^2} \Big[V(|T_r f_1(x) + T_r f_2(x)|) - V(|T_r f_1(x)|) - V(|T_r f_2(x)|) \Big]\, dx \psi dr \\ &\lesssim M_\psi^{\gamma_1}(f_1, f_2) + M_\psi^{\gamma_2}(f_1, f_2). \end{aligned} \tag{2.24}$$

So (2.23) follows from Proposition 2.12, noting also that

$$(a + b)^{\gamma_1 - 2} + (a + b)^{\gamma_2 - 2} \lesssim 1 + a^4 + b^4,$$

for all $a, b \ge 0$, as long as $\psi \in L^{\frac{4}{6-\gamma_1}} \cap L^{\frac{4}{6-\gamma_2}}$. Since ψ has compact support, this is the same as requiring $\psi \in L^{\frac{4}{6-\gamma_2}}$. Similarly, (2.21) follows from Proposition 2.6 as long as $\psi \in L^{\frac{4}{6-\gamma_1}} \cap L^{\frac{4}{6-\gamma_2}}$. Since $\psi \in L^1$ this condition reduces to $\psi \in L^1 \cap L^{\frac{4}{6-\gamma_2}}$.

For the proof of (2.22), we first assume $s = \mathrm{dist}(\mathrm{supp}\, \hat{f}_1, \mathrm{supp}\, \hat{f}_2) > 0$. Clearly, Proposition 2.7 shows

$$M_\psi^\gamma(f_1, f_2) \lesssim s^{-\alpha(\gamma,\tau)} \|f_1\|\|f_2\| (\|f_1\| + \|f_2\|)^{\gamma-2}$$

for any $2 < \gamma < 6$ and $\tau > 1$, as long as $\psi \in L^{\beta(\gamma,\tau)}$.
Thus (2.22) follows from (2.24) as long as $\psi \in L^{\beta(\gamma_1,\tau)} \cap L^{\beta(\gamma_2,\tau)}$. Noting

$$1 < \beta(\gamma_1, \tau) \le \beta(\gamma_2, \tau) \quad \text{and} \quad L^1 \cap L^{\beta(\gamma_2,\tau)} \subset L^{\beta(\gamma_1,\tau)} \cap L^{\beta(\gamma_2,\tau)}$$

finishes the proof of (2.22) when $\hat{f}_1$ and $\hat{f}_2$ have separated supports.

If $s = \mathrm{dist}(\mathrm{supp}\, f_1, \mathrm{supp}\, f_2) > 0$, we make the simple observation that for any compactly supported ψ one has

$$\psi \in L^p \Rightarrow \psi \in L^p(|r|^a\, dr) \cap L^1$$

for any weight $|r|^a$ with $a \ge 0$ and $p \ge 1$. With this observation, the above proofs carry over to the case that the functions f_1 and f_2 have separated supports, using now Proposition 2.11 instead of Proposition 2.7. ∎

3. Strict subadditivity of the ground state energy

Recall that for $d_{\rm av} \geq 0$

$$H(f) = \frac{d_{\rm av}}{2}\|f'\|^2 - N(f)$$

and

$$E_\lambda^{d_{\rm av}} = \inf\left\{H(f) : \|f\|^2 = \lambda\right\}.$$

Recall also that if $f \in L^2 \setminus H^1$ we set $\|f'\| = \infty$, so the infimum in the definition of $E_\lambda^{d_{\rm av}}$ is over all $f \in H^1$ with fixed L^2 norm if $d_{\rm av} > 0$.

In this section, we will give an a-priori bound on the ground-state energy which will be an essential ingredient in the construction of strongly convergent minimizing sequences.

Lemma 3.1. *Let $2 \leq \gamma_1 \leq \gamma_2 \leq 6$ and $\psi \in L^1 \cap L^{\frac{4}{6-\gamma_2}}$. Then, for every $\lambda \geq 0$*

$$E_\lambda^{d_{\rm av}} \gtrsim -(\lambda^{\gamma_1/2} + \lambda^{\gamma_2/2}),$$

where the implicit constant depends only on the L^1 and $L^{\frac{4}{6-\gamma_2}}$ norms of ψ. In particular, the variational problem is well-posed.

Moreover, if $V(a) > 0$ for every $a > 0$, then $E_\lambda^0 < 0$ for any $\lambda > 0$. If there exists $\varepsilon > 0$ and $2 \leq \kappa < 6$ such that $V(a) \gtrsim a^\kappa$ for all $0 \leq a \leq \varepsilon$ and ψ is away from zero on the support of ψ, then $E_\lambda^{d_{\rm av}} < 0$ for any $d_{\rm av}, \lambda > 0$.

Proof. The first part follows immediately from $H(f) \geq -N(f)$ and Proposition 2.14. If $V(a) > 0$ for all $a > 0$, then

$$0 = N(f) = \iint_{\mathbb{R}^2} V(|T_r f(x)|)\, dx\psi dr$$

implies $|T_r f(x)| = 0$ for almost all x and almost all $r \in \operatorname{supp}\psi$. Thus for almost all $r \in \operatorname{supp}\psi$ one has $\|f\| = \|T_r f\| = 0$, so f vanishes. Thus if $\|f\|^2 = \lambda > 0$ then $E_\lambda^0 \leq -N(f) < 0$.

To show $E_\lambda^{d_{\rm av}} < 0$ for $d_{\rm av} > 0$, we fix $\lambda > 0$ and find a suitable Gaussian function f satisfying $\|f\|^2 = \lambda$ and $H(f) < 0$. Recall that there exist $\varepsilon > 0$ and $2 \leq \kappa < 6$ such that

$$V(a) \gtrsim a^\kappa \quad \text{for all } 0 < a \leq \varepsilon. \tag{3.1}$$

If the Gaussian test function f is given by

$$f(x) = A_0 e^{-\frac{x^2}{\sigma_0}} \quad \text{with} \quad \sigma_0 > 0,$$

then $(T_r f)(x) = A(r)e^{-\frac{x^2}{\sigma(r)}}$ with $\sigma(r) = \sigma_0 + 4ir$ and $A(r) = A_0\sqrt{\sigma_0}/\sqrt{\sigma(r)}$ as in (B.18). Note that

$$|T_r f(x)| = |A(r)|\, e^{-\frac{\sigma_0 x^2}{|\sigma(r)|^2}} \leq |A(r)| \leq |A_0|.$$

Choosing

$$|A_0| = \left(\frac{2\lambda^2}{\pi\sigma_0}\right)^{1/4}$$

yields $\|f\|^2 = \lambda$ and $\|f'\|^2 = \lambda/\sigma_0$. To apply (3.1), we consider σ_0 large enough so that

$$|T_r f(x)| \leq |A_0| = \left(\frac{2\lambda^2}{\pi\sigma_0}\right)^{1/4} < \epsilon.$$

Then

$$N(f) = \iint_{\mathbb{R}^2} V(|T_r f(x)|) dx\psi dr \gtrsim \iint_{\mathbb{R}^2} |T_r f(x)|^\kappa dx\psi dr$$
$$= |A_0|^\kappa \sigma_0^{\frac{\kappa}{2}} \int_{\mathbb{R}} \frac{\psi(r)}{|\sigma(r)|^{\frac{\kappa}{2}}} \int_{\mathbb{R}} e^{-\frac{\kappa\sigma_0 x^2}{|\sigma(r)|^2}} dxdr = |A_0|^\kappa \sigma_0^{\frac{\kappa-1}{2}} \left(\frac{\pi}{\kappa}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \frac{\psi(r)}{|\sigma(r)|^{\frac{\kappa-2}{2}}}\, dr$$
$$= \left(\frac{2\lambda^2}{\pi}\right)^{\frac{\kappa}{4}} \left(\frac{\pi}{\kappa}\right)^{1/2} \sigma_0^{-\frac{\kappa-2}{4}} \int_{\mathbb{R}} \frac{\psi(r)}{[1+(4r/\sigma_0)^2]^{\frac{\kappa-2}{4}}}\, dr.$$

Thus, the energy of this Gaussian test function is bounded above by

$$H(f) \le \frac{d_{\mathrm{av}}\lambda}{2\sigma_0}\left[1 - \frac{C}{d_{\mathrm{av}}\lambda\kappa^{1/2}}\left(\frac{2\lambda^2}{\pi}\right)^{\frac{\kappa}{4}} \sigma_0^{\frac{6-\kappa}{4}} \int_{\mathbb{R}} \frac{\psi(r)}{(1+(4r/\sigma_0)^2)^{\frac{\kappa-2}{4}}}\, dr\right]$$

for some constant C. So, using a large enough σ_0, we get $H(f) < 0$ since $2 \le \kappa < 6$ and

$$\int_{\mathbb{R}} \frac{\psi(r)}{[1+(4r/\sigma_0)^2]^{\frac{\kappa-2}{4}}}\, dr \to \|\psi\|_{L^1}$$

as $\sigma_0 \to \infty$ by Lebesgue's dominated convergence theorem. ∎

Recall that there exists $\gamma_0 > 2$ such that for all $\rho \ge 1$ and $a > 0$

$$V(\rho a) \ge \rho^{\gamma_0} V(a). \tag{3.2}$$

This will be the main input the following strict subadditivity of $E_\lambda^{d_{\mathrm{av}}}$, which in turn will be crucial in the proof of Propositions 4.1 and 4.2.

Proposition 3.2 (Strict Subadditivity)**.** *Let $\lambda > 0$, $0 < \delta < \lambda/2$, and $\lambda_1,\ \lambda_2 \ge \delta$ with $\lambda_1 + \lambda_2 \le \lambda$. Then*

$$E_{\lambda_1}^{d_{\mathrm{av}}} + E_{\lambda_2}^{d_{\mathrm{av}}} \ge \left[1 - (2^{\frac{\gamma_0}{2}} - 2)\left(\frac{\delta}{\lambda}\right)^{\frac{\gamma_0}{2}}\right] E_\lambda^{d_{\mathrm{av}}},$$

for $\gamma_0 > 2$ as in (3.2).

Proof. First we show that for all $\lambda > 0$ and $0 < \mu \le 1$

$$E_{\mu\lambda}^{d_{\mathrm{av}}} \ge \mu^{\frac{\gamma_0}{2}} E_\lambda^{d_{\mathrm{av}}}. \tag{3.3}$$

First, with $\widetilde{\lambda} = \mu\lambda$ and $\mu = \rho^{-1}$, we see that inequality (3.3) is equivalent to

$$E_{\rho\tilde{\lambda}}^{d_{\mathrm{av}}} \le \rho^{\frac{\gamma_0}{2}} E_{\tilde{\lambda}}^{d_{\mathrm{av}}} \quad \text{ for all } \rho \ge 1,\ \tilde{\lambda} > 0. \tag{3.4}$$

Given $f \in H^1(\mathbb{R})$, or $f \in L^2(\mathbb{R})$ if $d_{\mathrm{av}} = 0$, with $\|f\|^2 = \lambda$ and $\rho \ge 1$, we get

$$N(\rho^{1/2} f) = \iint_{\mathbb{R}^2} V(\rho^{1/2}|T_r f(x)|)\, dx\psi dr \ge \rho^{\frac{\gamma_0}{2}} N(f)$$

from (3.2), $\|\rho^{1/2} f\|^2 = \rho\lambda$, and

$$H(\rho^{1/2} f)\ \le \rho \frac{d_{\mathrm{av}}}{2}\|f'\|^2 - \rho^{\frac{\gamma_0}{2}} N(f) \le \rho^{\frac{\gamma_0}{2}} H(f),$$

which proves inequality (3.4).

Let $\lambda_1 = \mu_1\lambda$ and $\lambda_2 = \mu_2\lambda$ with $\mu_1 + \mu_2 \le 1$ and $\mu_1,\ \mu_2 \ge \delta/\lambda$. From (3.3), we get

$$E_{\lambda_1}^{d_{\mathrm{av}}} + E_{\lambda_2}^{d_{\mathrm{av}}} = E_{\mu_1\lambda}^{d_{\mathrm{av}}} + E_{\mu_2\lambda}^{d_{\mathrm{av}}} \ge (\mu_1^{\frac{\gamma_0}{2}} + \mu_2^{\frac{\gamma_0}{2}}) E_\lambda^{d_{\mathrm{av}}}. \tag{3.5}$$

Without loss of generality, we may assume that $\mu_1 \le \mu_2$. Note that

$$\begin{aligned} 1 \ge (\mu_1+\mu_2)^{\frac{\gamma_0}{2}} &= \mu_1^{\frac{\gamma_0}{2}} + \mu_2^{\frac{\gamma_0}{2}} + (\mu_1+\mu_2)^{\frac{\gamma_0}{2}} - \mu_1^{\frac{\gamma_0}{2}} - \mu_2^{\frac{\gamma_0}{2}} \\ &= \mu_1^{\frac{\gamma_0}{2}} + \mu_2^{\frac{\gamma_0}{2}} + \mu_1^{\frac{\gamma_0}{2}} \left[\left(1+\frac{\mu_2}{\mu_1}\right)^{\frac{\gamma_0}{2}} - 1 - \left(\frac{\mu_2}{\mu_1}\right)^{\frac{\gamma_0}{2}} \right] \\ &\ge \mu_1^{\frac{\gamma_0}{2}} + \mu_2^{\frac{\gamma_0}{2}} + \mu_1^{\frac{\gamma_0}{2}} (2^{\frac{\gamma_0}{2}} - 2) \\ &\ge \mu_1^{\frac{\gamma_0}{2}} + \mu_2^{\frac{\gamma_0}{2}} + (2^{\frac{\gamma_0}{2}} - 2)\left(\frac{\delta}{\lambda}\right)^{\frac{\gamma_0}{2}}, \end{aligned}$$

where we have used that the function $t \mapsto (1+t)^{\frac{\gamma_0}{2}} - 1 - t^{\frac{\gamma_0}{2}}$ is increasing on $[1,\infty)$. Therefore, we get

$$\mu_1^{\frac{\gamma_0}{2}} + \mu_2^{\frac{\gamma_0}{2}} \le 1 - (2^{\frac{\gamma_0}{2}} - 2)\left(\frac{\delta}{\lambda}\right)^{\frac{\gamma_0}{2}}.$$

Therefore, multiplying this to $E_\lambda^{d_{\mathrm{av}}} < 0$ completes the proof due to (3.5). ■

4. The existence proof

The following propositions are the key propositions for the proof of the existence of a minimizer here. First, we introduce notations. For $s > 0$ and $0 < \alpha \le 1$, define

$$G_\alpha(s) := \left[(s+1)^{\frac{2\alpha}{1+2\alpha}} - 1\right]^{-1/2}. \tag{4.1}$$

Note that G_α is a decreasing function on $(0,\infty)$ which vanishes at infinity, which is important for us, and

$$\lim_{s\to 0^+} G_\alpha(s) = \infty \tag{4.2}$$

which is of less importance. Moreover, for $x \in \mathbb{R}$, let $x_+ := \max\{x, 0\}$.

Proposition 4.1 (Fat-tail for positive average dispersion). *Assume $d_{\mathrm{av}} > 0$, $2 \le \gamma_1 \le \gamma_2 \le 6$ and $\psi \in L^{\frac{4}{6-\gamma_2}}$ has compact support. Let $\lambda > 0$, $f \in H^1$ with $\|f\|^2 = \lambda$, and $0 < \delta < \lambda/2$, and $a, b \in \mathbb{R}$ with*

$$\int_{-\infty}^a |f(x)|^2 dx \ge \delta \ \text{ and } \ \int_b^\infty |f(x)|^2 dx \ge \delta \tag{4.3}$$

then

$$H(f) \ge \left[1 - (2^{\frac{\gamma_0}{2}} - 2)\left(\frac{\delta}{\lambda}\right)^{\frac{\gamma_0}{2}}\right] E_\lambda^{d_{\mathrm{av}}} - C(1+\lambda^2)\|f\|^2_{H^1(\mathbb{R})} G_1\left((b-a-1)_+\right), \tag{4.4}$$

where the constant C depends only on the support and the $L^{\frac{4}{6-\gamma_2}}$ norm of ψ.

We have a similar bound in the case of vanishing average dispersion.

Proposition 4.2 (Fat-tail for zero average dispersion). *Assume $d_{\mathrm{av}} = 0$, $2 < \gamma_1 \le \gamma_2 < 6$ and $\psi \in L^{\beta(\gamma_2,\tau)}$ has compact support. Let $\lambda > 0$, $f \in L^2$ with $\|f\|^2 = \lambda$, and $0 < \delta < \lambda/2$, and $a, b \in \mathbb{R}$ with either*

$$\int_{-\infty}^a |f(x)|^2 dx \ge \delta \ \text{ and } \ \int_b^\infty |f(x)|^2 dx \ge \delta \tag{4.5}$$

or

$$\int_{-\infty}^a |\widehat{f}(\eta)|^2\, d\eta \ge \delta \ \text{ and } \ \int_b^\infty |\widehat{f}(\eta)|^2\, d\eta \ge \delta, \tag{4.6}$$

then

$$H(f) \geq \left[1 - (2^{\frac{\gamma_0}{2}} - 2)\left(\frac{\delta}{\lambda}\right)^{\frac{\gamma_0}{2}}\right] E_\lambda^0 - C\lambda(1+\lambda^2) G_{\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}}((b-a-1)_+) \quad (4.7)$$

where the constant C depends only on the support and the $L^{\beta(\gamma_2,\tau)}$ norm of ψ.

Proof of Proposition 4.1. If $b-a \leq 1$, (4.4) holds immediately since its right hand side is $-\infty$ by (4.2). So now we assume that $b-a > 1$. Let a' and b' be arbitrary numbers satisfying $a \leq a' < b' \leq b$ and $b'-a' \geq 1$, which we will suitably choose later. The estimate of $\|f'\|^2$ is based on a one-dimensional version of the well-known IMS localization formula

$$\|f'\|^2 = \sum_j \langle (\xi_j f)', (\xi_j f)' \rangle - \sum_j \langle f, |\xi_j'|^2 f \rangle \quad (4.8)$$

for any collection of functions $\{\xi_j\}$ which are smooth, $0 \leq \xi_j \leq 1$, and $\sum_j \xi_j^2 = 1$. To construct such a partition which suits our needs, consider smooth functions $\{\chi_j\}$ that satisfy

i) $0 \leq \chi_j \leq 1$ for $j = -1, 0, 1$.

ii) $\displaystyle\sum_{j=-1}^{1} \chi_j^2 = 1$.

iii) $\operatorname{supp}\chi_0 \subset [-\frac{1}{2}, \frac{1}{2}]$, $\chi_0 = 1$ on $[-\frac{1}{4}, \frac{1}{4}]$,
$\operatorname{supp}\chi_{-1} \subset (-\infty, -\frac{1}{4}]$, $\chi_{-1} = 1$ on $(-\infty, -\frac{1}{2}]$,
$\operatorname{supp}\chi_1 \subset [\frac{1}{4}, \infty)$, $\chi_1 = 1$ on $[\frac{1}{2}, \infty)$.

Let

$$\xi_j(x) = \chi_j\left(\frac{x - \frac{1}{2}(a'+b')}{b'-a'}\right) \quad \text{for } j = -1, 0, 1.$$

Since χ_j' is bounded, we see that for some constant $C_1 > 0$

$$\sum_{j=-1}^{1} |\xi_j'|^2 \leq \frac{C_1}{(b'-a')^2}.$$

Plugging this into (4.8) yields

$$\begin{aligned} \|f'\|^2 &\geq \|(\xi_{-1}f)'\|^2 + \|(\xi_0 f)'\|^2 + \|(\xi_1 f)'\|^2 - \frac{C_1\|f\|^2}{(b'-a')^2} \\ &\geq \|(\xi_{-1}f)'\|^2 + \|(\xi_1 f)'\|^2 - \frac{C_1\|f\|^2}{(b'-a')^2}. \end{aligned} \quad (4.9)$$

Now we set $f_j := \xi_j f$ for $j = -1, 1$ and $f_0 := f - f_1 - f_{-1} = (1 - \xi_{-1} - \xi_1)f$, where we note that f_0 is defined differently from f_{-1} and f_1!

Obviously, $\|f_j\| \leq \|f\|$ for $j = -1, 1$, and since the supports of ξ_{-1} and ξ_1 are disjoint also $|f_0| \leq |f|$, hence $\|f_0\| \leq \|f\|$.

Set $h := f_{-1} + f_1$. Then $f = f_0 + h$ and the bound (2.21) from Proposition 2.15 shows

$$N(f) - N(f_0) - N(h) \lesssim \|f_0\|\|h\| \left(1 + \|f_0\|^4 + \|h\|^4\right)$$

and using Proposition 2.14, we have

$$N(f_0) \lesssim \|f_0\|^2 + \|f_0\|^6,$$

and combining the above two bounds we arrive at

$$N(f) - N(h) \lesssim \|f_0\|\|f\|(1 + \|f\|^4) \quad (4.10)$$

where we used $\|f_0\|, \|h\| \le \|f\|$.

Since f_{-1} and f_1 have supports separated by at least $(b'-a')/2$, (2.23) gives

$$\begin{aligned} N(h) - N(f_{-1}) - N(f_1) &\lesssim (b'-a')^{-1}\|f_{-1}\|_{H^1}\|f_1\|_{H^1}\left(1+\|f_{-1}\|^4+\|f_1\|^4\right) \\ &\lesssim (b'-a')^{-1}\|f\|_{H^1}^2\left(1+\|f\|^4\right) \end{aligned} \tag{4.11}$$

where we also used that, because of our assumption that $b'-a' \ge 1$, the bound $\|f_j\|_{H^1} \lesssim \|f\|_{H^1}$ holds, where the implicit constant does not depend on a' and b'.

Combining (4.10) and (4.11), we get

$$N(f) - N(f_{-1}) - N(f_1) \lesssim \left(\|f_0\|\|f\| + \frac{\|f\|_{H^1}^2}{b'-a'}\right)\left(1+\|f\|^4\right) \tag{4.12}$$

so when combined with (4.9), this yields

$$H(f) - H(f_{-1}) - H(f_1) \gtrsim -\left[\frac{\|f\|^2}{(b'-a')^2} + \left(\|f_0\|\|f\| + \frac{\|f\|_{H^1}^2}{b'-a'}\right)\left(1+\|f\|^4\right)\right]. \tag{4.13}$$

To choose a' and b', we use a continuous version of the pigeon hole principle, as in our previous work [11]: Let $1 \le l \le b-a$ and note that

$$\int_a^{b-l}\int_y^{y+l} |f(x)|^2 dx dy \le \int_a^b \int_{x-l}^x |f(x)|^2 dy dx \le l\|f\|^2. \tag{4.14}$$

Moreover, by the mean value theorem, there exists $y' \in (a, b-l)$ such that

$$(b-a-l)\int_{y'}^{y'+l} |f(x)|^2 dx = \int_a^{b-l}\int_y^{y+l} |f(x)|^2 dx d\eta.$$

Thus, since f_0 has support in $[a', b']$ and $|f_0| \le |f|$, choosing $a' = y'$ and $b' = y'+l$ in the previous identity together with (4.14) gives $l = b'-a'$ and

$$\|f_0\|^2 \le \|f\mathbf{1}_{[a',b']}\|^2 \le \frac{l}{b-a-l}\|f\|^2.$$

Plugging this into (4.13) yields

$$\begin{aligned} H(f) - H(f_{-1}) - H(f_1) &\gtrsim -\left[\frac{\|f\|^2}{l^2} + \left(\left(\frac{l}{b-a-l}\right)^{1/2}\|f\|^2 + \frac{\|f\|_{H^1}^2}{l}\right)\left(1+\|f\|^4\right)\right] \\ &\ge -\|f\|_{H^1}^2\left(1+\|f\|^4\right)\left[\frac{1}{l^2} + \left(\frac{l}{b-a-l}\right)^{1/2} + \frac{1}{l}\right]. \end{aligned}$$

Since $\|f\|^2 = \lambda$, $\|f_j\| \ge \delta$, $j = -1, 1$ and $\|f_{-1}\|^2 + \|f_1\|^2 \le \lambda$, by Proposition 3.2,

$$H(f) - \left[1 - (2^{\frac{\gamma_0}{2}} - 2)\left(\frac{\delta}{\lambda}\right)^{\frac{\gamma_0}{2}}\right] E_\lambda^{d_{\text{av}}} \gtrsim -\|f\|_{H^1}^2\left(1+\lambda^2\right)\left[\frac{1}{l^2} + \left(\frac{l}{b-a-l}\right)^{1/2} + \frac{1}{l}\right] \tag{4.15}$$

for any $0 < \delta < \lambda/2$ and all $1 \le l \le b-a$. Now we choose $l = \sqrt[3]{b-a}$. Then $1 \le l \le b-a$ since $b-a \ge 1$, and

$$\max\left\{\frac{1}{l^2}, \left(\frac{l}{b-a-l}\right)^{1/2}, \frac{1}{l}\right\} = \left(\frac{l}{b-a-l}\right)^{1/2} = \left(\frac{1}{(b-a)^{2/3}-1}\right)^{1/2} = G_1((b-a-1)_+)$$

which completes the proof. ∎

Proof of Proposition 4.2. Since its proof is very analogous to that of Proposition 4.1, let us mention only the things which need to be changed: In the case of zero average dispersion, the energy contains no $\|f'\|^2$ term, hence we do not need to use smooth cut-offs, that is, we can use $f = f_{-1} + f_0 + f_1$ where we set $f_{-1} = f\mathbf{1}_{(-\infty,a')}$, $f_0 = f\mathbf{1}_{[a',b']}$ and $f_1 = f\mathbf{1}_{(b',\infty)}$, and similarly for $\hat{f}$.

We can then simply repeat the argument in the proof of (4.13), again using (2.21) but now combined with (2.22) instead of (2.23), to see that

$$\begin{aligned} H(f) - H(f_{-1}) - H(f_1) &\gtrsim -\left(\|f_0\|\|f\| + \frac{\|f\|^2}{(b'-a')^{\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}}}\right)\left(1+\|f\|^4\right) \\ &\geq -\lambda\left(1+\lambda^2\right)\left[\left(\frac{l}{b-a-l}\right)^{1/2} + \frac{1}{l^{\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}}}\right] \end{aligned} \tag{4.16}$$

with the only restriction that $l = b' - a' \geq 1$.

If $0 < b - a \leq 1$, we note that (4.7) trivially holds since the right hand side equals $-\infty$. So let $b - a > 1$. We choose $l := (b-a)^{\frac{1}{1+2\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}}}$. Then $1 < l < b - a$ and $l^{1+2\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}} = b - a > b - a - l > 0$ hence

$$\left(\frac{l}{b-a-l}\right)^{1/2} \geq \frac{1}{l^{\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}}}.$$

This together with (4.16) and our choice of $G_{\min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\}}((b-a-1)_+)$, which satisfies $0 < \min\{\alpha(\gamma_1,\tau),\alpha(\gamma_2,\tau)\} \leq 1$, finishes the proof. ∎

Since the function G_α is decreasing on $\mathbb{R}_+$ and vanishes at infinity, similar results to Proposition 2.4 in [11] follow from Propositions 4.1 and 4.2.

Proposition 4.3 (Tightness for Positive Average Dispersion)**.** *Let $(f_n)_n \subset H^1(\mathbb{R})$ be a minimizing sequence for the variational problem (1.1) for $d_{\text{av}} > 0$ with $\lambda = \|f_n\|^2 > 0$. Then there exists $K < \infty$ such that, for any $L > 0$,*

$$\sup_{n\in\mathbb{N}} \int_{|\eta|>L} |\hat{f}_n(\eta)|^2 \, d\eta \leq \frac{K}{L^2} \tag{4.17}$$

i.e., the sequence is tight in Fourier space. Moreover, there exist shifts y_n such that

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|>R} |f_n(x-y_n)|^2 dx = 0. \tag{4.18}$$

Proposition 4.4 (Tightness for Zero Average Dispersion)**.** *Let $(f_n)_n \subset L^2(\mathbb{R})$ be a minimizing sequence for the variational problem (1.1) for $d_{\text{av}} = 0$ with $\lambda = \|f_n\|^2 > 0$. Then there exist shifts y_n and boosts ξ_n such that*

$$\lim_{L\to\infty} \sup_{n\in\mathbb{N}} \int_{|\eta-\xi_n|>L} |\widehat{f_n}(\eta)|^2 \, d\eta = 0. \tag{4.19}$$

and

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|x-y_n|>R} |f_n(x)|^2 \, dx = 0, \tag{4.20}$$

Proof of Proposition 4.3. Let $(f_n)_n$ be a minimizing sequence. Inequality (2.20) shows

$$\|f_n'\|^2 \lesssim H(f_n) + \lambda^{\gamma_1/2} + \lambda^{\gamma_2/2}$$

and since $H(f_n) \to E_\lambda < 0$, we see that

$$K := \sup_{n\in\mathbb{N}} \|f_n'\|^2 < \infty.$$

Thus, for every $n \in \mathbb{N}$ and $L > 0$, we obtain

$$\int_{|\eta|>L} |\hat{f}_n(\eta)|^2 \, d\eta \leq \int_{|\eta|>L} \frac{|\eta|^2}{L^2} |\hat{f}_n(\eta)|^2 \, d\eta \leq \int_{\mathbb{R}} \frac{|\eta|^2}{L^2} |\hat{f}_n(\eta)|^2 \, d\eta \leq \frac{K}{L^2}$$

and so (4.17).

To prove the second bound, we follow the argument of [11] closely. We give some details for the readers' convenience. Define $a_{n,\delta}$ and $b_{n,\delta}$ by

$$a_{n,\delta} := \inf \left\{ a \in \mathbb{R} : \int_{-\infty}^{a} |f_n(x)|^2 \, dx \geq \delta \right\}$$

and

$$b_{n,\delta} := \sup \left\{ b \in \mathbb{R} : \int_{b}^{\infty} |f_n(x)|^2 \, dx \geq \delta \right\}.$$

Note that the measure $|f_n(x)|^2 \, dx$ is absolutely continuous with respect to Lebesgue measure and hence

$$\int_{-\infty}^{a_{n,\delta}} |f_n(x)|^2 \, dx = \delta \quad \text{and} \quad \int_{b_{n,\delta}}^{\infty} |f_n(x)|^2 \, dx = \delta.$$

Furthermore $\delta \mapsto a_{n,\delta}$ and $\delta \mapsto b_{n,\delta}$ are monotone, more precisely, for $0 < \delta_2 < \delta_1 < \lambda/2$ one has $a_{n,\delta_2} \leq a_{n,\delta_1}$ and $b_{n,\delta_2} \geq b_{n,\delta_1}$. Let $R_{n,\delta} := b_{n,\delta} - a_{n,\delta}$ and note that the above monotonicity yields $R_{n,\delta_2} \geq R_{n,\delta_1}$ for $0 < \delta_2 < \delta_1 < \lambda/2$. Lastly, for some fixed $0 < \delta_0 < \lambda/2$ put

$$y_n := \frac{b_{n,\delta_0} + a_{n,\delta_0}}{2} \in [a_{n,\delta_0}, b_{n,\delta_0}].$$

In particular, $a_{n,\delta} \leq a_{n,\delta_0} \leq y_n \leq b_{n,\delta_0} \leq b_{n,\delta}$ for all $0 < \delta \leq \delta_0$. This implies

$$b_{n,\delta} - y_n \leq b_{n,\delta} - a_{n,\delta} = R_{n,\delta} \quad \text{and} \quad y_n - a_{n,\delta} \leq b_{n,\delta} - a_{n,\delta} = R_{n,\delta} \tag{4.21}$$

Now assume that

$$R_\delta := \sup_{n \in \mathbb{N}} R_{n,\delta} < \infty \tag{4.22}$$

for $0 < \delta \leq \delta_0$ and put $R_\delta := R_{\delta_0}$ for $\delta_0 < \delta < \lambda/2$. Then (4.21) yields

$$\int_{|x-y_n|>R_\delta} |f_n(x)|^2 \, dx \leq \int_{-\infty}^{a_{n,\delta}} |f_n(x)|^2 \, dx + \int_{b_{n,\delta}}^{\infty} |f_n(x)|^2 \, dx = 2\delta.$$

for all $0 < \delta \leq \delta_0$ but the same bound also holds when $\delta_0 < \delta < \lambda/2$ since in this case

$$\int_{|x-y_n|>R_\delta} |f_n(x)|^2 \, dx = \int_{|x-y_n|>R_{\delta_0}} |f_n(x)|^2 \, dx \leq 2\delta_0 < 2\delta.$$

It remains to show (4.22): Using $b = b_{n,\delta}$ and $a = a_{n,\delta}$, rearranging (4.4) from Proposition 4.1 yields

$$E_\lambda^{d_{\mathrm{av}}} - (2^{\frac{\gamma_0}{2}} - 2) \left(\frac{\delta}{\lambda} \right)^{\frac{\gamma_0}{2}} E_\lambda^{d_{\mathrm{av}}} - H(f_n) \leq C(1 + \lambda^2) \|f\|_{H^1(\mathbb{R})}^2 G_1\left((R_{n,\delta} - 1)_+ \right).$$

Thus, since $H(f_n) \to E_\lambda^+ < 0$,

$$0 < -(2^{\frac{\gamma_0}{2}} - 2) \left(\frac{\delta}{\lambda} \right)^{\frac{\gamma_0}{2}} E_\lambda^{d_{\mathrm{av}}} \leq C(1 + \lambda^2) \|f\|_{H^1(\mathbb{R})}^2 \liminf_{n \to \infty} G_1\left((R_{n,\delta} - 1)_+ \right).$$

Since G_1 is monotone decreasing, we get

$$G_1((\limsup_{n \to \infty} R_{n,\delta} - 1)_+) = \liminf_{n \to \infty} G_1((R_{n,\delta} - 1)_+) > 0$$

and so

$$\limsup_{n\to\infty} R_{n,\delta} < \infty. \tag{4.23}$$

Hence (4.22) holds. ∎

Proof of Proposition 4.4. Using the fact that the function G_α is monotone decreasing, the proof is virtually identical to the proof of (4.18) in Proposition 4.3 and Proposition 2.4 in [11] ∎

To prove Theorems 1.1 and 1.2, we need one more result for the continuity of the nonlinear functional $N(f)$.

Lemma 4.5. *If $0 \le \psi \in L^1 \cap L^{\frac{4}{6-\gamma_2}}$ then the functional $N : L^2(\mathbb{R}) \to \mathbb{R}$ given by*

$$L^2(\mathbb{R}) \ni f \mapsto N(f) = \iint_{\mathbb{R}^2} V(|T_r f|) dx\psi dr$$

is locally Lipshitz continuous.

Proof. Note that

$$V(|z|) - V(|w|) = |z|(q(|z|) - q(|w|)) + (|z| - |w|)q(|w|)$$

and so, using the assumption on q', we get

$$\begin{aligned}
&|V(|z|) - V(|w|)| \le |z|\,|q(|z|) - q(|w|)| + ||z| - |w|||q(|w|)| \\
&= |z| \left| \int_{|w|}^{|z|} q'(s)ds \right| + ||z| - |w|||q(|w|)| \\
&\lesssim |z|||z| - |w||(|z|^{\gamma_1-2} + |z|^{\gamma_2-2} + |w|^{\gamma_1-2} + |w|^{\gamma_2-2}) + ||z| - |w||\,(|w|^{\gamma_1-1} + |w|^{\gamma_2-1}) \\
&\lesssim |z - w|(|z|^{\gamma_1-1} + |z|^{\gamma_2-1} + |w|^{\gamma_1-1} + |w|^{\gamma_2-1}).
\end{aligned}$$

Thus, for any $f, g \in L^2(\mathbb{R})$, using Hölder inequality with γ_j and $\frac{\gamma_j}{\gamma_j-1}$, we get

$$\begin{aligned}
|N(f) - N(g)| &\lesssim \sum_{j=1}^{2} \iint_{\mathbb{R}^2} |T_r f - T_r g|(|T_r f|^{\gamma_j-1} + |T_r g|^{\gamma_j-1}) dx\psi dr \\
&\le \sum_{j=1}^{2} \|T_r(f-g)\|_{L^{\gamma_j}(\mathbb{R}^2, dx\psi dr)} \||T_r f|^{\gamma_j-1} + |T_r g|^{\gamma_j-1}\|_{L^{\frac{\gamma_j}{\gamma_j-1}}(\mathbb{R}^2, dx\psi dr)}.
\end{aligned} \tag{4.24}$$

Applying Lemma 2.1 for the first factor and the triangle inequality with Lemma 2.1 for the second factor, which requires $\psi \in L^{\frac{4}{6-\gamma_j}}$, yields

$$(4.24) \lesssim \|f - g\|(\|f\|^{\gamma_1-1} + \|g\|^{\gamma_1-1} + \|f\|^{\gamma_2-1} + \|g\|^{\gamma_2-1}).$$

Note that

$$L^1 \cap L^{\frac{4}{6-\gamma_2}} \subset L^{\frac{4}{6-\gamma_1}} \cap L^{\frac{4}{6-\gamma_2}}$$

which completes the proof. ∎

Lemma 4.6. *If $0 \le \psi \in L^1 \cap L^{\frac{4}{6-\gamma_2}}$ then for any $f \in L^2(\mathbb{R})$ the functional N as above is differentiable with derivative*

$$L^2(\mathbb{R}) \ni h \mapsto DN(f)[h] = \int_{\mathbb{R}} \mathrm{Re}\left\langle T_r h, \left[V'(|T_r f|)\mathrm{sgn}(T_r f)\right]\right\rangle \psi dr.$$

Proof. Let $f \in L^2(\mathbb{R})$ and $\epsilon \neq 0$. Fix any $h \in L^2(\mathbb{R})$ and the quotient of N is

$$\begin{aligned}\frac{N(f+\epsilon h)-N(f)}{\epsilon} &= \frac{1}{\epsilon}\left[\iint_{\mathbb{R}^2} V(|T_r(f+\epsilon h)|) - V(|T_r f|)dx\psi dr\right] \\ &= \frac{1}{\epsilon}\iint_{\mathbb{R}^2}\int_0^1 \frac{d}{ds}V(|T_r(f+s\epsilon h)|)dsdx\psi dr. \end{aligned} \tag{4.25}$$

By straightforward calculations, we obtain

$$\frac{d}{ds}V(|T_r(f+s\epsilon h)|) = V'(|T_r(f+s\epsilon h)|)\frac{\epsilon(T_r f\overline{T_r h} + T_r h\overline{T_r f} + 2s\epsilon|T_r h|^2)}{2|T_r(f+s\epsilon h)|}$$

and thus

$$(4.25) = \iint_{\mathbb{R}^2}\int_0^1 V'(|T_r(f+s\epsilon h)|)\frac{T_r f\overline{T_r h} + T_r h\overline{T_r f} + 2s\epsilon|T_r h|^2}{2|T_r(f+s\epsilon h)|}dsdx\psi dr.$$

By Lebesgue's dominated convergence theorem, letting $\epsilon \to 0$, we get

$$DN(f)[h] = \iint_{\mathbb{R}^2}\int_0^1 V'(|T_r f|)\frac{\mathrm{Re}(T_r f\overline{T_r h})}{|T_r f|}dsdx\psi dr = \iint_{\mathbb{R}^2} V'(|T_r f|)\frac{\mathrm{Re}(T_r f\overline{T_r h})}{|T_r f|}dx\psi dr$$

which completes the proof. ∎

Now we are ready to prove the existence of a minimizer of (1.1).

Proof of Theorems 1.1 and 1.2. First, we give the proof for the existence of a minimizer of (1.1) for $d_{\mathrm{av}} > 0$. Let $(f_n)_n \subset H^1(\mathbb{R})$ be a minimizing sequence of the variational problem (1.1) for $d_{\mathrm{av}} > 0$. First, applying Proposition 4.3, there exist shifts y_n such that for the shifted sequence h_n, $h_n(x) := f_n(x - y_n)$ for $x \in \mathbb{R}$, we have

$$\lim_{R\to\infty}\sup_{n\in\mathbb{N}}\int_{|x|>R}|h_n(x)|^2dx = 0. \tag{4.26}$$

On the Fourier side, shifts correspond to modulations with $e^{iy_n\eta}$, so for the shifted sequence Proposition 4.3 also yields that there exists $K_1 < \infty$ such that for any $L > 0$

$$\sup_{n\in\mathbb{N}}\int_{|\eta|>L}|\hat{h}_n(\eta)|^2\,d\eta \le \frac{K_1}{L^2}. \tag{4.27}$$

Thus, by translation invariance of the minimization problem, the shifted sequence is a minimizing sequence with $\|h_n\|^2 = \|f_n\|^2 = \lambda$ which implies the strong convergence of the sequence $(h_n)_n$ in $L^2(\mathbb{R})$, see, for example, [11]. Let

$$f := \lim_{n\to\infty} h_n \tag{4.28}$$

in $L^2(\mathbb{R})$. By the strong convergence in $L^2(\mathbb{R})$, clearly we get $\|f\|^2 = \lambda > 0$. Let $K_2 := \lambda + K_1$. Then $\|f_n\|^2_{H^1(\mathbb{R})} \le K_2$ for all $n \in \mathbb{N}$. We infer $\|h_n\|^2_{H^1(\mathbb{R})} = \|f_n\|^2_{H^1(\mathbb{R})} \le K_2 < \infty$ for all $n \in \mathbb{N}$, i.e., the sequence $(h_n)_n$ is bounded in $H^1(\mathbb{R})$. Since H^1 is a Hilbert space, this shows that there is a subsequence, which, by a slight abuse of notation, we will continue to denote by $(h_n)_n$, which converges weakly in $H^1(\mathbb{R})$. Since h_n converges strongly to f in $L^2(\mathbb{R})$, an easy argument shows that $f \in H^1(\mathbb{R})$ and h_n converges weakly to f in $H^1(\mathbb{R})$. But then by standard properties of Hilbert spaces, we also have

$$\|f\|_{H^1(\mathbb{R})} \le \liminf_{n\to\infty}\|h_n\|_{H^1(\mathbb{R})} \tag{4.29}$$

and since $\|f\|^2_{H^1(\mathbb{R})} = \|f\|^2 + \|f'\|^2 = \lambda + \|f'\|^2$ and the same for $\|h_n\|^2_{H^1(\mathbb{R})}$, we have

$$\int_{\mathbb{R}}|f'(x)|^2\,dx \le \liminf_{n\to\infty}\int_{\mathbb{R}}|h_n'(x)|^2\,dx. \tag{4.30}$$

Thus, by Lemma 4.5 the map $f \mapsto N(f)$ is continuous in L^2, we see that H is lower semicontinuous when $(h_n)_n$ converges strongly to f and $(h_n')_n$ converges weakly to f' in $L^2(\mathbb{R})$. Hence

$$E_\lambda^{d_{\mathrm{av}}} \le H(f) \le \liminf H(h_n) = E_\lambda^{d_{\mathrm{av}}}, \tag{4.31}$$

that is, f is a minimizer of (1.1).

Now, it remains to show that the existence of a minimizer of (1.1) for $d_{\mathrm{av}} = 0$. Once we get the tightness the remaining proof is analogous to the existence proof in [11]. However, for readers' convenience, we give the proof here. The idea is to use Proposition 4.4 and Lemma A.1 in order to massage an arbitrary minimizing sequence into a strongly convergent sequence.

Let $(f_n)_n \subset L^2(\mathbb{R})$ be an arbitrary minimizing sequence of the variational problem (1.1). Proposition 4.4 guarantees the existence of shifts $y_n \in \mathbb{R}$ and boosts $\xi_n \in \mathbb{R}$ such that (4.19) and (4.20) hold. Define the shifted and boosted sequence $(h_n)_n = (f_{\xi_n, y_n, n})_n$ by

$$h_n(x) = f_{\xi_n, y_n, n}(x) := e^{i\xi_n x} f_n(x - y_n) \quad \text{for } x \in \mathbb{R}.$$

Note that $\|h_n\|_2^2 = \|f_n\|_2^2 = \lambda$ since shifts and boost are unitary operations on $L^2(\mathbb{R})$ and $N(f_n) = N(h_n)$, see Appendix B. Hence $(h_n)_n$ is also a minimizing sequence. Certainly $|h_n(x)| = |f_n(x - y_n)|$ for all $n \in \mathbb{N}$. The Fourier transform of h_n is given by

$$\widehat{h}_n(\eta) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\eta} e^{ix\xi_n} f_n(x - y_n)\, dx = e^{-iy_n\eta} \widehat{f}_n(\eta - \xi_n). \tag{4.32}$$

Thus also $|\widehat{h}_n(\eta)| = |\widehat{f}_n(\eta - \xi_n)|$. In particular, (4.19) and (4.20) show that the minimizing sequence $(h_n)_n$ is tight in the sense of Lemma A.1.

Since $(h_n)_n$ is bounded in $L^2(\mathbb{R})$, the weak compactness of the unit ball, guarantees the existence of a weakly converging subsequence of $(h_n)_n$, denoted again by $(h_n)_n$. Obviously, this subsequence is also tight in the sense of Lemma A.1 and hence converges even strongly in $L^2(\mathbb{R})$. We set

$$f = \lim_{n\to\infty} h_n.$$

By strong convergence $\|f\|^2 = \lim_{n\to\infty} \|h_n\|^2 = \lambda$. To conclude that f is the sought after minimizer we note that by Lemma 4.5 the map $f \mapsto N(f) = \iint_{\mathbb{R}^2} V(|T_r f|^2) dx \mu(dt)$ is continuous on $L^2(\mathbb{R})$. Hence

$$N(f) = \lim_{n\to\infty} N(h_n) = E_\lambda^{d_{\mathrm{av}}}$$

where the last equality follows since $(h_n)_n$ is a minimizing sequence. Thus f is a minimizer for the variational problem (1.1).

To prove that the above minimizer is a weak solution of the associated Euler-Lagrange equation (1.14) is standard in the calculus of variations. For vanishing average dispersion and a cubic nonlinearity it is done in [11] and the proof given there carries over to our more general setting with the obvious changes in notation. ∎

Appendix A. Strong convergence in L^2 and tightness

A key step in our existence proof of minimizers of the variational problems (1.1) is the following characterization of strong convergence in $L^2(\mathbb{R})$ which is given in [11].

Lemma A.1. *A sequence $(f_n)_n \subset L^2(\mathbb{R})$ is strongly converging to f in $L^2(\mathbb{R})$ if and only if it is weakly convergent to f and*

$$\lim_{L\to\infty} \limsup_{n\to\infty} \int_{|\eta|>L} |\widehat{f}_n(\eta)|^2 \, d\eta = 0, \tag{A.1}$$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |f_n(x)|^2 \, dx = 0, \tag{A.2}$$

where $\widehat{f}$ is the Fourier transform of f.

Appendix B. Galilei transformations and space-time localization properties of Gaussian coherent states

We will only discuss the one-dimensional case which is somewhat easier since we do not have to deal with rotations in one dimension. The unitary operator implementing the shift $S_y : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(S_y f)(x) = f(x-y)$ is given by

$$S_y = e^{-iyP} \tag{B.1}$$

where $P = -i\partial_x$ is the momentum operator. Indeed, since e^{-iyP} corresponds to multiplication by e^{-iyk} in Fourier space, we have

$$(e^{-iyP} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x-y)k} \widehat{f}(k) \, dk = f(x-y).$$

Boosts, i.e., shifts in momentum space are given by $e^{iv\cdot} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, i.e., multiplication by e^{ivx}, since

$$\widehat{e^{iv\cdot} f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix(k-v)} f(x) \, dx = \widehat{f}(k-v). \tag{B.2}$$

Finally, if G is a bounded (measurable) function then $G(P)$ is defined by

$$\widehat{G(P) f}(k) = G(k)\widehat{f}(k).$$

Of course, for any $y \in \mathbb{R}$, the operators $G(P)$ and e^{-iyP} commute, $G(P)e^{-iyP} = e^{-iyP}G(P)$. Moreover, for any $v \in \mathbb{R}$ the commutation relation

$$G(P)e^{iv\cdot} = e^{iv\cdot}G(P+v) \tag{B.3}$$

holds. Indeed, Computing the Fourier transform $\mathcal{F}$ yields

$$\begin{aligned}
\mathcal{F}\big(G(P)e^{iv\cdot} f\big)(k) &= G(k)\widehat{e^{iv\cdot} f}(k) = G(k)\widehat{f}(k-v) \\
&= (G(\cdot+v)\widehat{f})(k-v) = \mathcal{F}\big(G(P+v)f\big)(k-v) \\
&= \mathcal{F}\big(e^{iv\cdot}G(P+v)f\big)(k).
\end{aligned}$$

In particular, choosing $G(P) = e^{-irP^2}$, we arrive at the commutation relation

$$\begin{aligned}
e^{-irP^2} e^{iv\cdot} e^{-iyP} &= e^{iv\cdot} e^{-iyP} e^{-ir(P+v)^2} = e^{iv\cdot} e^{-iyP} e^{-ir(P^2+2vP+v^2)} \\
&= e^{-irv^2} e^{iv\cdot} e^{-i(y+2rv)P} e^{-irP^2}.
\end{aligned} \tag{B.4}$$

Now let $f \in L^2(\mathbb{R})$. Then $u(r) = T_r f = e^{-irP^2} f$ is the solution of the (one-dimensional) Schrödinger equation $-i\partial_r u = P^2 u = -\partial_x^2 u$ with initial condition $u(0) = f$. Using (B.4), the solution of the free Schrödinger equation for the translated and boosted initial condition $f_{y,v} = e^{iv\cdot}e^{-iyP}f$ is given by

$$\begin{aligned} u_{y,v}(r,x) := T_r f_{y,v}(x) &= \big(e^{-irP^2}e^{iv\cdot}e^{-iyP}f\big)(x) \\ &= \big(e^{-irv^2}e^{iv\cdot}e^{-i(y+2rv)P}e^{-irP^2}f\big)(x) \\ &= e^{-irv^2}e^{ivx}\big(e^{-i(y+2rv)P}e^{-irP^2}f\big)(x) \\ &= e^{-irv^2}e^{ivx}\big(e^{-irP^2}f\big)(x-y-2rv) \\ &= e^{-irv^2}e^{ivx}(T_r f)(x-y-2rv), \end{aligned} \tag{B.5}$$

that is, on the level of the solutions of the free time-dependent Schrödinger equation, translations and boosts of the initial condition are implemented by the Galilei transformations $\mathcal{G}_{y,v}$ given by $(\mathcal{G}_{y,v}u)(r,x) := u_{y,v}(r,x) = e^{-irv^2}e^{ivx}u(r,x-y-2rv)$. Except for the time-dependent phase factor e^{-irv^2}, formula (B.5) is exactly what one would have guessed from classical mechanics

A simple calculation now shows that any functional of the form

$$f \mapsto N(f) = \iint_{\mathbb{R}^2} V(|T_r f(x)|)\, dx\psi dr$$

is invariant under translations and boosts of f in $L^2(\mathbb{R})$.

Now, we come to one of the major tools for our analysis, the so-called coherent states.

Definition B.1 (Coherent states)**.** Let $h \in L^2$, $\|h\| = 1$, y, $v \in \mathbb{R}$ and $h_{y,v} := e^{iv\cdot}e^{-iyP}h$, i.e.,

$$h_{y,v}(x) = e^{ivx}h(x-y) \tag{B.6}$$

for $x \in \mathbb{R}$ and define the coherent rank-one projection $P_{y,v} := |h_{y,v}\rangle\langle h_{y,v}|$ in Dirac's notation, i.e., given by

$$f \mapsto P_{y,v}f := h_{y,v}\langle h_{y,v}, f\rangle. \tag{B.7}$$

A well-known property of coherent states is their completeness expressed in

Lemma B.2 (Completeness of coherent states)**.** *Let $h \in L^2(\mathbb{R})$ with $\|h\| = 1$ and $h_{y,v}$ the shifted and boosted h as above. Then, in a weak sense,*

$$\frac{1}{2\pi}\iint_{\mathbb{R}^2} dydvP_{y,v} = \frac{1}{2\pi}\iint_{\mathbb{R}^2} dydv|h_{y,v}\rangle\langle h_{y,v}| = \mathbf{1} \tag{B.8}$$

on L^2. Moreover,

$$\frac{1}{2\pi}\int_{\mathbb{R}} dv\, \langle\varphi, P_{y,v}\varphi\rangle = \int |h(x-y)|^2|\varphi(x)|^2\, dx, \tag{B.9}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}} dy\, \langle\varphi, P_{y,v}\varphi\rangle = \int |\hat{h}(\eta-v)|^2|\hat{\varphi}(\eta)|^2\, d\eta, \tag{B.10}$$

Proof. The completeness expressed in (B.8) is well-known, see [7, 16, 17], the other two are less known. We give a short proof for the convenience of the reader: In order to see that the operator A given by its matrix elements

$$\langle\varphi_1, A\varphi_2\rangle := \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} dydv\langle\varphi_1, h_{y,v}\rangle\langle h_{y,v}, \varphi_2\rangle$$

is the identity on L^2 it is enough, by polarization, to take $\varphi_1 = \varphi_2 = \varphi$ and to check $\langle \varphi, A\varphi\rangle = \langle \varphi, \varphi\rangle$ for all $\varphi \in L^2$. Note

$$\langle h_{y,v}, \varphi\rangle = \int_{\mathbb{R}} e^{-ivx}\overline{h(x-y)}\varphi(x)\, dx = (2\pi)^{1/2}(\widehat{\overline{h_{y,0}}\varphi})(v).$$

and thus by Plancherel,

$$\frac{1}{2\pi}\int_{\mathbb{R}} dv\, \langle \varphi, P_{y,v}\varphi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} dv|\langle h_{y,v},\varphi\rangle|^2 = \int_{\mathbb{R}} dx\, |h_{y,0}(x)\varphi(x)|^2 = \int_{\mathbb{R}} dx|h(x-y)\varphi(x)|^2,$$

so (B.9) follows and we also see

$$\langle \varphi, A\varphi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_{\mathbb{R}} dv|\langle h_{y,v},\varphi\rangle|^2 = \int_{\mathbb{R}} dy\int_{\mathbb{R}} dx|h(x-y)\varphi(x)|^2 = \int_{\mathbb{R}} |\varphi(x)|^2\, dx$$

thus, in addition, (B.8) follows. For (B.10) we note that a short calculation reveals

$$\widehat{h_{y,v}}(\eta) = e^{-iy(\eta-v)}\hat{h}(\eta - v) = e^{iyv}\hat{h}_{v,-y}(\eta).$$

By Plancherel

$$\langle h_{y,v},\varphi\rangle = \langle \widehat{h_{y,v}}, \widehat{\varphi}\rangle = \int_{\mathbb{R}} e^{iy(\eta-v)}\overline{\hat{h}(\eta-v)}\widehat{\varphi}(\eta)\, d\eta = (2\pi)^{1/2}e^{-iyv}\mathcal{F}^{-1}\left[\overline{\widehat{h}_{v,0}}\widehat{\varphi}\right](y).$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Again by Plancherel, we thus have

$$\frac{1}{2\pi}\int_{\mathbb{R}} dy\, \langle \varphi, P_{y,v}\varphi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} dy\, |\langle \widehat{h_{y,v}}, \widehat{\varphi}\rangle|^2 = \int_{\mathbb{R}} d\eta\, \left|\overline{\widehat{h}_{v,0}(\eta)}\widehat{\varphi}(\eta)\right|^2 = \int_{\mathbb{R}} d\eta\, \left|\overline{\widehat{h}(\eta-v)}\widehat{\varphi}(\eta)\right|^2$$

and (B.10) follows. ∎

We use coherent states in order to localize a wave function simultaneously in real and Fourier spaces and since Gaussians have nice localization properties simultaneously in real and Fourier spaces, it is natural to use Gaussian coherent states for this.

Lemma B.3 (Space-time localization properties of Gaussian coherent states)**.** *Let* $g(x) = \pi^{-1/4}e^{-x^2/2}$ *be the standard* L^2 *normalized Gaussian and*

$$g_{y,v}(x) := e^{ivx}g(x-y) \tag{B.11}$$

its shifted and boosted version. Let

$$P_L^{\leq} := \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_{|v|\leq L} dv|g_{y,v}\rangle\langle g_{y,v}| \tag{B.12}$$

and

$$P_L^{>} := \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_{|v|> L} dv|g_{y,v}\rangle\langle g_{y,v}|. \tag{B.13}$$

Then $P_L^{\leq} + P_L^{>} = \mathbf{1}$, $0 \leq P_L^{\leq} \leq \mathbf{1}$, *and* $0 \leq P_L^{>} \leq \mathbf{1}$ *as operators. Moreover* $P_L^{>}$ *localizes a wave function in the region of large frequencies* $|\eta| \gtrsim L$ *in the sense that for any* $f \in H^\alpha$ *we have*

$$\|P_L^{>} f\| \lesssim L^{-\alpha}\|f\|_{H^\alpha}. \tag{B.14}$$

where the implicit constant does not depend on f *nor* L.

Moreover, the time-evolution of the shifted and boosted Gaussian $g_{y,v}$ *is given by*

$$(T_r g_{y,v})(x) = \frac{1}{\pi^{1/4}\sqrt{1+2ir}}e^{-irv^2}e^{ivx}e^{-\frac{(x-y-2rv)^2}{2(1+2ir)}} \tag{B.15}$$

and for any $f_1, f_2 \in L^2$ *which have separated supports we have the bilinear estimate*

$$\sup_{|r|\leq R} \|T_r P_L^{\leq} f_1 T_r P_L^{\leq} f_2\|_{L_x^p} \lesssim A_R L^2 e^{L^2/p - B_{p,R}s^2}\|f_1\|\|f_2\|,\ 1 \leq p < \infty, \tag{B.16}$$

where $A_R := \sqrt{1+4R^2}$, $B_{p,R} := 2^{-4}(\sqrt{p(1+4R^2)}+1)^{-2}$, *and* $s := \operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2)$.

Proof. The first assertions are clear, since by Lemma B.2 we have $P_L^{\le} + P_L^{>} = \mathbf{1}$ and certainly $P_L^{\le}$ and $P_L^{>} \ge 0$ in the sense of operators. So also $P_L^{\le} = \mathbf{1} - P_L^{>} \le \mathbf{1}$ and similarly $P_L^{>} \le \mathbf{1}$.

To prove (B.14), we first note that because of $0 \le P_L^{>} \le \mathbf{1}$, one has

$$\|P_L^{>} f\|^2 = \langle P_L^{>1/2} f, P_L^{>} P_L^{>1/2} f\rangle \le \langle f, P_L^{>} f\rangle.$$

Let $P_{y,v} := |g_{y,v}\rangle\langle g_{y,v}|$, then

$$\begin{aligned}\langle f, P_L^{>} f\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_{|v|>L} dv\, \langle f, P_{y,v} f\rangle = \int_{|v|>L}\int_{\mathbb{R}} |\hat{g}(\eta - v)|^2 |\hat{f}(\eta)|^2\, d\eta dv \\ &= \frac{1}{\sqrt{\pi}} \int_{|v|>L}\int_{\mathbb{R}} e^{-(\eta-v)^2} |\hat{f}(\eta)|^2\, d\eta dv = \int_{\mathbb{R}} H_L(\eta) |\hat{f}(\eta)|^2\, d\eta \end{aligned} \tag{B.17}$$

due to (B.10) and $\hat{g} = g$ where we set

$$H_L(\eta) := \frac{1}{\sqrt{\pi}} \int_{|v|>L} e^{-(\eta-v)^2}\, dv.$$

Note that H_L is even, $0 < H_L \le 1$, increasing on $[0,\infty)$, and $\lim_{\eta\to\infty} H_L(\eta) = 1$. A short calculation reveals

$$H_L(L) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{2L}^{\infty} e^{-v^2}\, dv$$

so $H_L(L)$ is extremely close to $1/2$ for large L. For $|\eta| \le L/2$ and $|v| \ge L$, one has $|v - \eta| \ge |v| - |\eta| \ge |v| - L/2 \ge L/2$, hence

$$H_L(\eta) \le \frac{2}{\sqrt{\pi}} \int_L^{\infty} e^{-\frac{L}{2}(v - \frac{L}{2})}\, dv = \frac{4}{\sqrt{\pi} L} e^{-\frac{L^2}{4}} \quad \text{for all } |\eta| \le \frac{L}{2}.$$

So

$$\begin{aligned}\int_{\mathbb{R}} H_L(\eta)|\hat{f}(\eta)|^2\, d\eta &= \int_{|\eta|\le L/2} H_L(\eta)|\hat{f}(\eta)|^2\, d\eta + \int_{|\eta|>L/2} H_L(\eta)|\hat{f}(\eta)|^2\, d\eta \\ &\le \frac{4}{\sqrt{\pi}L} e^{-\frac{L^2}{4}} \|f\|^2 + \int_{|\eta|>L/2} |\hat{f}(\eta)|^2\, d\eta.\end{aligned}$$

Using

$$\int_{|\eta|>L/2} |\hat{f}(\eta)|^2\, d\eta \le (L/2)^{-2\alpha} \int_{|\eta|>L/2} |\eta|^{2\alpha} |\hat{f}(\eta)|^2\, d\eta \le (L/2)^{-2\alpha} \|f\|^2_{H^\alpha}$$

completes the proof of (B.14).

To prove formula (B.15) first note that for a centered Gaussian $g(x) = A_0 e^{-x^2/\sigma_0}$, $\operatorname{Re}\sigma_0 > 0$, the time evolution $T_r g$ can be found by making the ansatz

$$(T_r g)(x) = A(r) e^{-x^2/\sigma(r)} =: u(r,x).$$

A short calculation, using that $u(r,x)$ solves $i\partial_r u = -\partial_x^2 u$, reveals that a and σ solve

$$iA' = \frac{2A}{\sigma} \quad \text{and } \sigma' = 4i,$$

thus $A(r)$ and $\sigma(r)$ are given by

$$A(r) = A_0 \frac{\sqrt{\sigma_0}}{\sqrt{\sigma(r)}} \quad \text{and} \quad \sigma(r) = \sigma_0 + 4ir. \tag{B.18}$$

Taking $\sigma_0 = 2$ and $A_0 = \pi^{-1/4}$ we get

$$(T_r g_{0,0})(x) = \pi^{-1/4}\frac{1}{\sqrt{1+2ir}}e^{-\frac{x^2}{2(1+2ir)}}. \tag{B.19}$$

Now we use the Galilei transformation formula (B.5) to arrive at

$$(T_r g_{y,v})(x) = \pi^{-1/4}\frac{e^{-irv^2}e^{ivx}}{\sqrt{1+2ir}}e^{-\frac{(x-y-2rv)^2}{2(1+2ir)}}$$

which is (B.15).

To prove (B.16), fix $|r| \le R$ and note that

$$(T_r P_L^{\le} f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} dy \int_{|v|\le L} dv\, (T_r g_{y,v})(x)\langle g_{y,v}, f\rangle.$$

Thus using (B.15) and the triangle inequality

$$|(T_r P_L^{\le} f)(x)| \le \frac{1}{2\pi(\pi(1+4r^2))^{1/4}}\int_{\mathbb{R}} dy \int_{|v|\le L} dv\, e^{-\frac{(x-y-2rv)^2}{2(1+4r^2)}}|\langle g_{y,v}, f\rangle|$$

together with

$$A(r,L) := \int_{\mathbb{R}} dy \int_{|v|\le L} dv\, e^{-\frac{(x-y-2rv)^2}{2(1+4r^2)}} = 2L(2\pi(1+4r^2))^{1/2},$$

which is independent of x, by translation invariance of Lebesgue measure we can thus bound

$$|(T_r P_L^{\le} f)(x)| \le \frac{A(r,L)}{2\pi(\pi(1+4r^2))^{1/4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \nu_x(dy,dv)|\langle g_{y,v}, f\rangle|$$

with the probability measure $\nu_x(dy,dv) := \frac{1}{A(r,L)}e^{-\frac{(x-y-2rv)^2}{2(1+4r^2)}}\mathbf{1}_{|v|\le L}\, dydv$. Hence Jensen's inequality [14] for the convex function $r \to |r|^p$, $1 \le p < \infty$, shows

$$\begin{aligned}\big|(T_r P_L^{\le} f)(x)\big|^p &\le \frac{A(r,L)^p}{(2\pi)^p(\pi(1+4r^2))^{p/4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \nu_x(dy,dv)|\langle g_{y,v}, f\rangle|^p \\ &\lesssim L^{p-1}(1+4r^2)^{\frac{p-2}{4}}\int_{\mathbb{R}} dy \int_{|v|\le L} dv\, e^{-\frac{(x-y-2rv)^2}{2(1+4r^2)}}|\langle g_{y,v}, f\rangle|^p.\end{aligned}$$

Therefore,

$$\begin{aligned}\|(T_r P_L^{\le} f_1)(T_r P_L^{\le} f_2)\|_{L_x^p}^p &\lesssim L^{2(p-1)}(1+4r^2)^{\frac{p-2}{2}}\int_{\mathbb{R}} dy_1 \int_{|v_1|\le L} dv_1 \int_{\mathbb{R}} dy_2 \int_{|v_2|\le L} dv_2 \\ &\qquad |\langle g_{y_1,v_1}, f_1\rangle|^p|\langle g_{y_2,v_2}, f_2\rangle|^p \int_{\mathbb{R}} dx\, e^{-\frac{(x-y_1-2rv_1)^2+(x-y_2-2rv_2)^2}{2(1+4r^2)}} \\ &\lesssim L^{2(p-1)}(1+4r^2)^{\frac{p-1}{2}}\int_{\mathbb{R}} dy_1 \int_{|v_1|\le L} dv_1 \int_{\mathbb{R}} dy_2 \int_{|v_2|\le L} dv_2\, |\langle g_{y_1,v_1}, f_1\rangle|^p|\langle g_{y_2,v_2}, f_2\rangle|^p\; e^{-\frac{[(y_1-y_2)+2r(v_1-v_2)]^2}{4(1+4r^2)}}\end{aligned} \tag{B.20}$$

where we used

$$\int_{\mathbb{R}} dx\, e^{-\frac{(x-y_1-2rv_1)^2+(x-y_2-2rv_2)^2}{2(1+4r^2)}} = (\pi(1+4r^2))^{1/2}e^{-\frac{((y_1-y_2)+2r(v_1-v_2))^2}{4(1+4r^2)}}$$

by a simple convolution of Gaussians. Since $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$ for any $a, b \in \mathbb{R}$, the lower bound

$$[(y_1-y_2)+2r(v_1-v_2)]^2 \ge \frac{1}{2}(y_1-y_2)^2 - 16r^2L^2$$

holds for all y_1, y_2, and $|v_1|, |v_2| \le L$. Moreover,

$$|\langle g_{y,v}, f\rangle| \le \int_{\mathbb{R}} |g_{y,v}(x)||f(x)|\,dx = \pi^{-1/4}\int_{\mathbb{R}} e^{-\frac{1}{2}(x-y)^2}|f(x)|\,dx = (g_{0,0} * |f|)(y),$$

and thus (B.20) gives the upper bound

$$\|T_r P_L^{\le} f_1 T_r P_L^{\le} f_2\|_{L_x^p}^p \lesssim L^{2p}\, e^{L^2}(1+4r^2)^{\frac{p-1}{2}} \int_{\mathbb{R}} dy_1 \int_{\mathbb{R}} dy_2\, e^{-\frac{(y_1-y_2)^2}{8(1+4r^2)}} [g_{0,0} * |f_1|(y_1)]^p [g_{0,0} * |f_2|(y_2)]^p. \tag{B.21}$$

Let $K_j := \operatorname{supp} f_j$, $j = 1, 2$ be the support of f_j. Recall that we assume $s := \operatorname{dist}(K_1, K_2) > 0$. Given $0 < \tilde{s} < s/2$, we will enlarge K_j a little bit,

$$\widetilde{K}_j := \{y \in \mathbb{R} |\, \operatorname{dist}(y, K_j) \le \tilde{s}\}.$$

Note that $\operatorname{dist}(\widetilde{K}_1, \widetilde{K}_2) = s - 2\tilde{s} > 0$ and we will split the integral in (B.21) according to the splitting $\mathbb{R}\times\mathbb{R} = (\widetilde{K}_1^c \times \mathbb{R}) \cup (\widetilde{K}_1 \times \mathbb{R}) = (\widetilde{K}_1^c \times \mathbb{R}) \cup (\widetilde{K}_1 \times \widetilde{K}_2^c) \cup (\widetilde{K}_1 \times \widetilde{K}_2)$. As a further preparation, note that the Cauchy-Schwartz inequality implies

$$\begin{aligned} &\left|\iint_{\mathbb{R}^2} e^{-\frac{1}{c}(y_1-y_2)^2} h_1(y_1)h_2(y_2)\,dy_1 dy_2\right| \\ &\le \left[\iint_{\mathbb{R}^2} e^{-\frac{1}{c}(y_1-y_2)^2}|h_1(y_1)|^2 dy_1 dy_2\right]^{1/2} \left[\iint_{\mathbb{R}^2} e^{-\frac{1}{c}(y_1-y_2)^2}|h_2(y_2)|^2 dy_1 dy_2\right]^{1/2} \\ &= \sqrt{c\pi}\|h_1\|\|h_2\|. \end{aligned} \tag{B.22}$$

for any $h_1, h_2 \in L^2(\mathbb{R})$ and $c > 0$. Using this, we can bound

$$\begin{aligned} I_1 &:= \int_{\widetilde{K_1}^c} dy_1 \int_{\mathbb{R}} dy_2\; e^{-\frac{(y_1-y_2)^2}{8(1+4r^2)}} \left[(g_{0,0} * |f_1|)(y_1)\right]^p \left[(g_{0,0} * |f_2|)(y_2)\right]^p \\ &\lesssim (1+4r^2)^{1/2} \left[\int_{\widetilde{K_1}^c} \left[(g_{0,0} * |f_1|)(y_1)\right]^{2p} dy_1\right]^{1/2} \left[\int_{\mathbb{R}} \left[(g_{0,0} * |f_2|)(y_2)\right]^{2p} dy_2\right]^{1/2}. \end{aligned} \tag{B.23}$$

Moreover, by Young's inequality,

$$\int_{\mathbb{R}} \left[(g_{0,0} * |f_2|)(y_2)\right]^{2p} dy_2 \lesssim \|f_2\|^{2p} \tag{B.24}$$

and, on the other hand,

$$\begin{aligned} &\int_{\widetilde{K_1}^c} \left[(g_{0,0} * |f_1|)(y)\right]^{2p} dy = \frac{1}{(2\pi)^p} \int_{\widetilde{K_1}^c} dy \left[\int_{K_1} e^{-\frac{1}{2}(y-z)^2}|f_1(z)|\,dz\right]^{2p} \\ &\lesssim e^{-\frac{p}{2}[\operatorname{dist}(K_1, \widetilde{K_1}^c)]^2} \|e^{-\frac{1}{4}|\cdot|^2} * |f_1|\|_{L^{2p}}^{2p} \\ &\lesssim e^{-\frac{p}{2}[\operatorname{dist}(K_1, \widetilde{K_1}^c)]^2} \|f_1\|^{2p}, \end{aligned} \tag{B.25}$$

where again Young's inequality, similar as for (B.24), has been used in the last inequality. Plugging (B.24) and (B.25) into (B.23), we obtain

$$I_1 \lesssim (1+4r^2)^{1/2} e^{-\frac{p}{4}[\operatorname{dist}(K_1, \widetilde{K_1}^c)]^2} \|f_1\|^p \|f_2\|^p. \tag{B.26}$$

Furthermore, the bound

$$\begin{aligned} I_2 &:= \int_{\widetilde{K_1}} dy_1 \int_{\widetilde{K_2}^c} dy_2\; e^{-\frac{(y_1-y_2)^2}{8(1+4r^2)}} \left[(g_{0,0} * |f_1|)(y_1)\right]^p \left[(g_{0,0} * |f_2|)(y_2)\right]^p \\ &\lesssim (1+4r^2)^{1/2} e^{-\frac{p}{4}[\operatorname{dist}(K_2, \widetilde{K_2}^c)]^2} \|f_1\|^p \|f_2\|^p \end{aligned} \tag{B.27}$$

follows as the one for I_1, by symmetry.

It remains to get a bound on

$$I_3 := \int_{\widetilde{K_1}} dy_1 \int_{\widetilde{K_2}} dy_2 \; e^{-\frac{(y_1-y_2)^2}{8(1+4r^2)}} \Big[(g_{0,0} * |f_1|)(y_1)\Big]^p \Big[(g_{0,0} * |f_2|)(y_2)\Big]^p \tag{B.28}$$

Since $(y_1 - y_2)^2 \geq (y_1 - y_2)^2/2 + [\mathrm{dist}(\widetilde{K}_1, \widetilde{K}_2)]^2/2$ in the integral in (B.28), we get

$$\begin{aligned} I_3 &\leq e^{-\frac{1}{16(1+4r^2)}[\mathrm{dist}(\widetilde{K_1},\widetilde{K_2})]^2} \int_{\widetilde{K_1}} dy_1 \int_{\widetilde{K_2}} dy_2 \; e^{-\frac{(y_1-y_2)^2}{16(1+4r^2)}} \Big[(g_{0,0} * |f_1|)(y_1)\Big]^p \Big[(g_{0,0} * |f_2|)(y_2)\Big]^p \\ &\lesssim (1+4r^2)^{1/2}\, e^{-\frac{1}{16(1+4r^2)}[\mathrm{dist}(\widetilde{K_1},\widetilde{K_2})]^2} \|g_{0,0} * |f_1|\,\|^p_{L^{2p}} \|g_{0,0} * |f_2|\,\|^p_{L^{2p}} \\ &\lesssim (1+4r^2)^{1/2}\, e^{-\frac{1}{16(1+4r^2)}[\mathrm{dist}(\widetilde{K_1},\widetilde{K_2})]^2} \|f_1\|^p \|f_2\|^p \end{aligned} \tag{B.29}$$

using again (B.24). Combining

$$\|T_r P_L^{\leq} f_1 T_r P_L^{\leq} f_2\|^p_{L^p_x} \lesssim L^{2p}\; e^{L^2} (1+4r^2)^{\frac{p-1}{2}} \Big(I_1 + I_2 + I_3\Big)$$

with(B.26), (B.27), (B.29), $\mathrm{dist}(K_j, \widetilde{K}_j^c) = \tilde{s}$ for $j = 1, 2$, and $\mathrm{dist}(\widetilde{K}_1, \widetilde{K}_2) = s - 2\tilde{s}$, we obtain

$$\|T_r P_L^{\leq} f_1 T_r P_L^{\leq} f_2\|^p_{L^p_x} \lesssim L^{2p}\; e^{L^2} (1+4r^2)^{\frac{p}{2}} \Big[e^{-\frac{p\tilde{s}^2}{4}} + e^{-\frac{(s-2\tilde{s})^2}{16(1+4r^2)}}\Big] \|f_1\|^p \|f_2\|^p$$

choosing $\tilde{s} = s/(2\sqrt{p(1+4r^2)}+2)$, which makes $p\tilde{s}^2/4 = (s-2\tilde{s})^2/(16(1+4r^2))$, gives the upper bound

$$\begin{aligned} \|T_r P_L^{\leq} f_1 T_r P_L^{\leq} f_2\|_{L^p_x} &\lesssim (1+4r^2)^{1/2} L^2 e^{L^2/p} e^{-\frac{s^2}{16(\sqrt{p(1+4r^2)}+1)^2}} \|f_1\|\|f_2\| \\ &\leq (1+4R^2)^{1/2} L^2 e^{L^2/p} e^{-\frac{s^2}{16(\sqrt{p(1+4R^2)}+1)^2}} \|f_1\|\|f_2\| \end{aligned}$$

for all $|r| \leq R$, which proves (B.16). ∎

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