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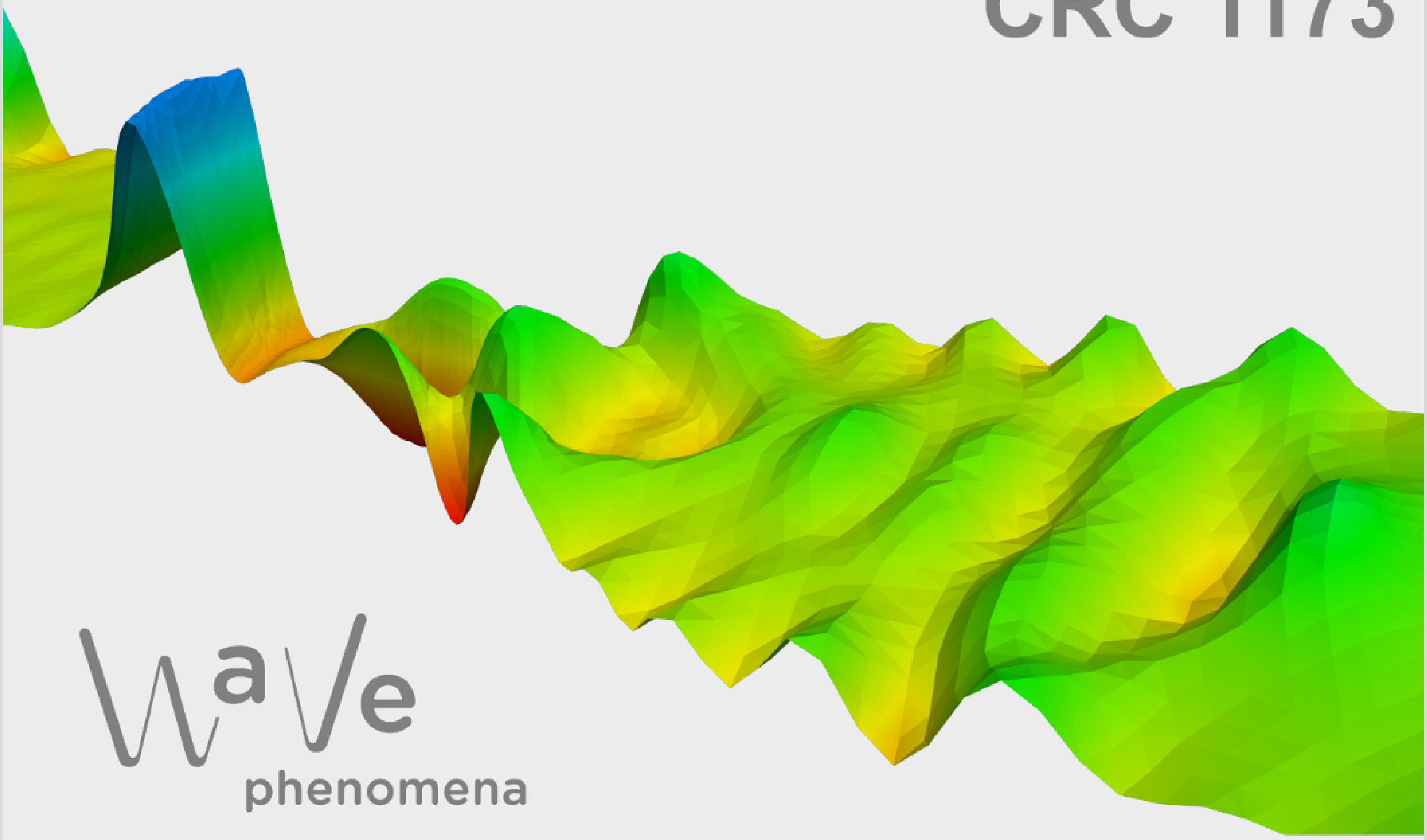

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Effective slow dynamics models for a class of dispersive systems

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Abstract

We consider dispersive systems of the form

$$\partial_t U = \Lambda_U U + B_U(U,V), \qquad \varepsilon \partial_t V = \Lambda_V V + B_V(U,U)$$

in the singular limit $\varepsilon \to 0$, where Λ_U, Λ_V are linear and B_U, B_V bilinear mappings. We are interested in deriving error estimates for the approximation obtained through the regular limit system

$$\partial_t \psi_U = \Lambda_U \psi_U - B_U(\psi_U, \Lambda_V^{-1} B_V(\psi_U, \psi_U))$$

from a more general point of view. Our abstract approximation theorem applies to a number of semilinear systems, such as the Dirac-Klein-Gordon system, the Klein-Gordon-Zakharov system, and a mean field polaron model. It extracts the common features of scattered results in the literature, but also gains an approximation result for the Dirac-Klein-Gordon system which has not been documented in the literature before. We explain that our abstract approximation theorem is sharp in the sense that there exists a quasilinear system of the same structure where the regular limit system makes wrong predictions.

1 Introduction

What have the following systems, where $0 < \varepsilon \ll 1$ is a small parameter, in common? Written as first order systems,

- the Dirac-Klein-Gordon (DKG) system, cf. Section 3.1,

$$-i\gamma^0\partial_t u - i\sum_{\mu=1}^{3}\gamma^\mu\partial_{x_\mu}u + m_u u = vu, \qquad \varepsilon^2\partial_t^2 v = \Delta v - m_v^2 v + \overline{u}^\tau\gamma^0 u, \tag{1}$$

- the Klein-Gordon-Zakharov (KGZ) system, cf. Section 3.2,

$$\partial_t^2 u = \Delta u - u - \gamma uv, \qquad \varepsilon^2\partial_t^2 v = \Delta v + \Delta(|u|^2), \tag{2}$$

- the mean field polaron model, cf. Section 3.3,

$$i\partial_t u = \Delta u - \gamma uv, \qquad \varepsilon^2\partial_t^2 v = -v + \Delta^{-1}(|u|^2), \tag{3}$$

- and the Zakharov system, cf. Section 4.2,

$$i\partial_t u = -\Delta u - \gamma vu, \qquad \varepsilon^2\partial_t^2 v = \Delta v + \Delta|u|^2, \tag{4}$$

are all of the form

$$\partial_t U = \Lambda_U U + B_U(U,V), \qquad \varepsilon\partial_t V = \Lambda_V V + B_V(U,U), \tag{5}$$

where Λ_U, Λ_V are linear and B_U, B_V bilinear mappings. Moreover, in the singular limit $\varepsilon \to 0$ effective equations for the slow dynamics occur, namely

- the Dirac-Hartree equation

$$-i\gamma^0\partial_t u - i\sum_{\mu=1}^{3}\gamma^\mu\partial_{x_\mu}u + m_u u = ((-\Delta + m_v^2)^{-1}(\overline{u}^\tau\gamma^0 u))u \tag{6}$$

  for the DKG system,

- the Klein-Gordon equation

$$\partial_t^2 u = \Delta u - u - \gamma u|u|^2 \tag{7}$$

  for the KGZ system,

- the Hartree equation

$$i\partial_t u = \Delta u - \gamma u\Delta^{-1}(|u|^2) \tag{8}$$

  for the mean field polaron model,

- the NLS equation

$$i\partial_t u = -\Delta u + \gamma|u|^2 u \tag{9}$$

  for the Zakharov system,

and

$$\partial_t \psi_U = \Lambda_U \psi_U - B_U(\psi_U, \psi_V), \qquad \text{with} \qquad \psi_V = \Lambda_V^{-1} B_V(\psi_U, \psi_U) \tag{10}$$

for the abstract system (5).

It is the goal of this paper to discuss the validity of the following approximation result (formulated for (5)) for this class of dispersive systems from a more general point of view.

Theorem 1.1. *Let X_ψ, X_U, and X_V be suitably chosen Banach spaces, and let $\psi_U \in C([0,T_0], X_\psi)$ be a solution of (10). Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (U,V) of (5) with*

$$\sup_{t\in[0,T_0]} (\|U - \psi_U\|_{X_U} + \|V - \psi_V\|_{X_V}) \leq C\varepsilon.$$

At a first view such error estimates are a non-trivial task since in the equivalent formulation

$$\partial_t U = \Lambda_U U + B_U(U,V), \qquad \partial_t V = \varepsilon^{-1}\Lambda_V V + \varepsilon^{-1} B_V(U,U), \tag{11}$$

in the V-equation there are terms of order $\mathcal{O}(\varepsilon^{-1})$ on the right hand side which can lead to growth rates of order $\mathcal{O}(e^{\varepsilon^{-1}t})$ and which would make it impossible to prove the bounds for the error on an $\mathcal{O}(1)$ time scale.

At a second view one finds such approximation results scattered in the literature, for the KGZ system in [DSS16], for the mean field polaron model in [GSS17], and for the Zakharov system for instance in [SW86, AA88]. In fact the underlying idea to prove such approximation results is rather simple. We eliminate the dangerous nonlinear term $\varepsilon^{-1}B_V(U,U)$ by the change of coordinates

$$W = V + M(U,U), \quad \text{with} \quad M(U,U) = \Lambda_V^{-1} B_V(U,U). \tag{12}$$

We find

$$\begin{aligned} \partial_t U &= \Lambda_U U + B_U(U, W - \Lambda_V^{-1} B_V(U,U)), \\ \partial_t W &= \varepsilon^{-1}\Lambda_V W + 2M(\Lambda_U U + B_U(U, W - \Lambda_V^{-1} B_V(U,U)), U). \end{aligned} \tag{13}$$

If Λ_V is the generator of a uniformly bounded semigroup, for the transformed system (13), by a simple application of Gronwall's inequality, the required estimates can be obtained. Hence, it is the first goal of this paper to show that the scattered results in the literature can be handled all with the same abstract approximation theorem and that it also applies to the DKG system (1) to Dirac-Hartree limit (6) which has not been handled in the literature before.

At a third view, however, it turns out that the problem is more subtle. Although the Zakharov system (4) formally falls into this class of systems of the form (5), it cannot be handled with our abstract approximation result. In fact, it is not a technical problem. Using the resonances of the system we really construct a counter example which shows that, in case of a 'wrong' sign in the nonlinearity of (4), the NLS approximation fails to make correct predictions. Therefore, it is the second goal of this paper to extract the reasons why for the first three systems (1)-(3) the limit systems (6)-(8) make correct predictions independently of the sign of the nonlinearity and why for the fourth system (4) the sign of the nonlinearity plays an essential role.

Remark 1.2. Systems with the same interaction structure in the nonlinear terms, i.e.,

$$\partial_t U = \ldots + B_U(U,V), \qquad \varepsilon\partial_t V = \ldots + B_V(U,U)$$

occur in various situations. The first class of examples are coupled quantum mechanical systems such as (1)-(3). Other examples in this class are the Klein-Gordon-Schödinger system with Yukawa coupling, cf. [FT75] or the perturbed Zakharov system considered in [BBC96]. The argument with the change of coordinates (12) still works if the bilinear mapping $B_U(U,V)$ is replaced by a general nonlinearity $F_U(U,V)$ and if the bilinear mapping $B_V(U,U)$ is replaced by a general nonlinearity $F_V(U)$. Essential is that the nonlinear terms in the V-equation only depend on U. This generalized interaction structure also occurs in coupled amplitude systems describing long-short wave interactions or slow-fast oscillations. Examples can be found for instance in [MN05, SZ13].

Remark 1.3. General slow-fast systems

$$\partial_t u = f(u,v), \qquad \varepsilon\partial_t v = g(u,v)$$

play a big role in applications. In the ODE case, with the geometric singular perturbation theory, cf. [Fen79, JK94], there exist powerful tools to analyze such systems. For a recent overview see [Kue15].

The plan of this paper is as follows. In Section 2 we present an abstract approximation result for dispersive systems of the form (5). The approximation result is used in Section 3.1 to prove the validity of the Dirac-Hartree approximation (6) for the DKG system (1), in Section 3.2 to prove the validity of the Klein-Gordon approximation (7) for the KGZ system (2), and in Section 3.3 to prove the validity of the Hartree approximation (8) for the high frequency limit of a mean field polaron model (3). In Section 4.2 we explain that although the NLS approximation limit for the (quasilinear) Zakharov system is of the above abstract form the validity analysis is different. In Section 4.3 we use the resonances of the original systems to construct solutions which grow in time. We explain why these growing solutions for the systems (1)-(3) do not contradict the previous approximation results, but why on the other hand these growing solutions allow to construct the above mentioned counter example for the system of Section 4.2. This discussion is illustrated by numerical experiments in Section 4.4.

Notation. Constants which can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$ are denoted with the same symbol C. We write $\int$ for $\int_{\mathbb{R}^d}$. The Fourier transform of a function u is denoted with $\widehat{u}$. We introduce the norm $\|\cdot\|_{L^2_s}$ by

$$\|\widehat{u}\|_{L^2_s}^2 = \int |\widehat{u}(k)|^2(1+|k|^2)^s \mathrm{d}k$$

and define the Sobolev norm $\|u\|_{H^s} = \|\widehat{u}\|_{L^2_s}$, but use equivalent versions, too. Sometimes we use the short-hand notation $\|f(k)\|_{L^p(\mathrm{d}k)}$ for $\|k \mapsto f(k)\|_{L^p}$.

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2 The abstract approximation theorem

For the system

$$\partial_t U = \Lambda_U U + B_U(U,V), \qquad \varepsilon \partial_t V = \Lambda_V V + B_V(U,U), \tag{14}$$

with $0 \leq \varepsilon \ll 1$ a small perturbation parameter, we pose a number of assumptions.

(S1) The operator Λ_U is the generator of a strongly continuous group $(e^{\Lambda_U t})_{t\in\mathbb{R}}$ in some Banach space X_U. The group is uniformly bounded, i.e., there exists a constant $C_\Lambda > 0$ such that $\|e^{\Lambda_U t}\|_{X_U \to X_U} \leq C_\Lambda$ for all $t \in \mathbb{R}$.

(S2) Similarly, Λ_V is the generator of a strongly continuous group $(e^{\Lambda_V t})_{t\in\mathbb{R}}$ in some Banach space X_V satisfying $\|e^{\Lambda_V t}\|_{X_V \to X_V} \leq C_\Lambda$ for all $t \in \mathbb{R}$.

(B1) For the bilinear mapping $B_U : X_U \times X_V \to X_U$ there exists a C_B such that

$$\|B_U(U,V)\|_{X_U} \leq C_B \|U\|_{X_U} \|V\|_{X_V}$$

for all $U \in X_U$ and $V \in X_V$.

(B2) For the bilinear mapping $B_V : X_U \times X_U \to X_V$ there exists a C_B such that

$$\|B_V(U,\widetilde{U})\|_{X_V} \leq C_B \|U\|_{X_U} \|\widetilde{U}\|_{X_U}$$

for all $U, \widetilde{U} \in X_U$.

In the singular limit $\varepsilon \to 0$ we have the regular system

$$\partial_t \psi_U = \Lambda_U \psi_U + B_U(\psi_U, \psi_V), \qquad 0 = \Lambda_V \psi_V + B_V(\psi_U, \psi_U). \tag{15}$$

We assume that

(I) The bilinear mapping $\Lambda_V^{-1} B_V(\cdot,\cdot) : X_U \times X_U \to X_V$ exists and there exists a C_I such that

$$\|\Lambda_V^{-1} B_V(U,\widetilde{U})\|_{X_V} \leq C_I \|U\|_{X_U} \|\widetilde{U}\|_{X_U}$$

for all $U, \widetilde{U} \in X_U$.

Inserting $\psi_V = -\Lambda_V^{-1} B_V(\psi_U, \psi_U)$ into the equation for ψ_U gives the regular limit system

$$\partial_t \psi_U = \Lambda_U \psi_U - B_U(\psi_U, \Lambda_V^{-1} B_V(\psi_U, \psi_U)). \tag{16}$$

In order to prove that ψ_U makes correct predictions about the dynamics of (14) for $\varepsilon > 0$ small, we follow the ideas explained in the introduction. We eliminate the dangerous nonlinear term $\varepsilon^{-1} B_V(U,U)$ by a change of coordinates

$$W = V + M(U,U), \quad \text{with} \quad M(U,U) = \Lambda_V^{-1} B_V(U,U).$$

By assumption **(I)** the bilinear mapping $M(\cdot,\cdot) : X_U \times X_U \to X_V$ exists and there exists a C_I such that

$$\|M(U,\widetilde{U})\|_{X_V} \leq C_I \|U\|_{X_U} \|\widetilde{U}\|_{X_U} \tag{17}$$

for all $U, \widetilde{U} \in X_U$. We find

$$\begin{aligned} \partial_t W &= \partial_t V + M(\partial_t U, U) + M(U, \partial_t U) \\ &= \varepsilon^{-1}\Lambda_V V + \varepsilon^{-1} B_V(U,U) + 2M(\Lambda_U U + B_U(U,V), U) \\ &= \varepsilon^{-1}\Lambda_V W + 2M(\Lambda_U U + B_U(U,V), U) \end{aligned}$$

such that after the transformation

$$\begin{aligned} \partial_t U &= \Lambda_U U + B_U(U, W - \Lambda_V^{-1} B_V(U,U)), \\ \partial_t W &= \varepsilon^{-1}\Lambda_V W + 2M(\Lambda_U U + B_U(U, W - \Lambda_V^{-1} B_V(U,U)), U). \end{aligned} \tag{18}$$

For obtaining local existence and uniqueness of this system we additionally assume that

(M) The bilinear mapping $M(\Lambda_U \cdot, \cdot) : X_U \times X_U \to X_V$ exists and there exists a C_I such that

$$\|M(\Lambda_U U, \widetilde{U})\|_{X_V} \leq C_I \|U\|_{X_U} \|\widetilde{U}\|_{X_U}$$

for all $U, \widetilde{U} \in X_U$.

As before the limit system is given by

$$\partial_t \psi_U = \Lambda_U \psi_U + B_U(\psi_U, -\Lambda_V^{-1} B_V(\psi_U, \psi_U)), \qquad \psi_W = 0. \tag{19}$$

The formal error made by inserting the approximation ψ_U and ψ_W into (18) can be measured by the residuals

$$\begin{aligned} \mathrm{Res}_U(U,W) &= -\partial_t U + \Lambda_U U + B_U(U, W - \Lambda_V^{-1} B_V(U,U)), \\ \mathrm{Res}_W(U,W) &= -\partial_t W + \varepsilon^{-1}\Lambda_V W \\ &\quad + 2M(\Lambda_U U + B_U(U, W - \Lambda_V^{-1} B_V(U,U)), U). \end{aligned} \tag{20}$$

For the approximations ψ_U and ψ_W we find

$$\mathrm{Res}_U(\psi_U, 0) = 0, \qquad \mathrm{Res}_W(\psi_U, 0) = 2M(\Lambda_U \psi_U + B_U(\psi_U, -\Lambda_V^{-1} B_V(\psi_U, \psi_U)), \psi_U).$$

By posing additional assumptions on Λ_U, Λ_V, B_U, and B_V it can be proven that $\mathrm{Res}_W(\psi_U, 0)$ can be bounded independently of $0 < \varepsilon \ll 1$. However, in order to keep the number of assumptions on a reasonable level and to be more flexible for the subsequent applications we assume the following for the residual terms.

(Res) Let $\psi_U \in C([0,T_0], X_\psi)$ be a solution of (16) with $X_\psi \subset X_U$ another suitably chosen Banach space. Then there exists a $C_{res} > 0$ such that for all $\varepsilon \in (0,1]$ we have

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_W(\psi_U, 0)\|_{X_V} \le C_{res}, \qquad \sup_{t\in[0,T_0]} \|\Lambda_V^{-1}\mathrm{Res}_W(\psi_U, 0)\|_{X_V} \le C_{res},$$

and

$$\sup_{t\in[0,T_0]} \|\Lambda_V^{-1}\partial_t \mathrm{Res}_W(\psi_U, 0)\|_{X_V} \le C_{res}.$$

Our abstract approximation result is as follows.

Theorem 2.1. *Assume the validity of* **(S1)**, **(S2)**, **(B1)**, **(B2)**, **(I)**, **(M)**, *and* **(Res)**. *Let $\psi_U \in C([0,T_0], X_\psi)$ be a solution of (16). Then there exist $\varepsilon_0 > 0$ and $C_U > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (U, W) of (18) with*

$$\sup_{t\in[0,T_0]} (\|U - \psi_U\|_{X_U} + \|W\|_{X_V}) \le C\varepsilon,$$

respectively, (U, V) of (14) with

$$\sup_{t\in[0,T_0]} (\|U - \psi_U\|_{X_U} + \|V - \psi_V\|_{X_V}) \le C\varepsilon.$$

Proof. We introduce

$$\begin{aligned} N_U(Z) &= B_U(U, W - \Lambda_V^{-1} B_V(U,U)), \\ N_W(Z) &= 2M(\Lambda_U U + B_U(U, W - \Lambda_V^{-1} B_V(U,U)), U), \end{aligned}$$

where $Z = (U, W)$, such that (18) can be written as

$$\partial_t U = \Lambda_U U + N_U(Z), \qquad \partial_t W = \varepsilon^{-1}\Lambda_V W + N_W(Z). \tag{21}$$

The error $\varepsilon R = \varepsilon(R_U, R_V) = Z - \psi_Z$ made by the approximation $\psi_Z = (\psi_U, \psi_W)$ satisfies

$$\begin{aligned} \partial_t R_U &= \Lambda_U R_U + \varepsilon^{-1}(N_U(\psi_Z + \varepsilon R) - N_U(\psi_Z)), \\ \partial_t R_W &= \varepsilon^{-1}\Lambda_V R_W + \varepsilon^{-1}(N_W(\psi_Z + \varepsilon R) - N_W(\psi_Z)) + \varepsilon^{-1}\mathrm{Res}_W(\psi_Z). \end{aligned} \tag{22}$$

In order to estimate the error R we consider the variation of constant formula

$$\begin{aligned} R_U(t) &= \int_0^t e^{\Lambda_U(t-\tau)}\varepsilon^{-1}(N_U(\psi_Z + \varepsilon R) - N_U(\psi_Z))(\tau) d\tau, \\ R_W(t) &= \int_0^t e^{\varepsilon^{-1}\Lambda_V(t-\tau)}\varepsilon^{-1}(N_W(\psi_Z + \varepsilon R) - N_W(\psi_Z))(\tau) d\tau + s_1, \end{aligned} \tag{23}$$

where

$$s_1 = \varepsilon^{-1}\int_0^t e^{\varepsilon^{-1}\Lambda_V(t-\tau)}\mathrm{Res}_W(\psi_U, 0)(\tau) d\tau$$

The factor ε^{-1} in s_1 can be removed by integration by parts, namely

$$\begin{aligned} s_1 &= e^{\varepsilon^{-1}\Lambda_V(t-\tau)}\Lambda_V^{-1}\mathrm{Res}_W(\psi_U,0)(\tau)\big|_{\tau=0}^{t} \\ &\quad + \int_0^t e^{\varepsilon^{-1}\Lambda_V(t-\tau)}\Lambda_V^{-1}\partial_\tau \mathrm{Res}_W(\psi_U,0)(\tau)d\tau. \end{aligned}$$

The terms occurring in (23) can be estimated by using the assumptions **(S1)**, **(S2)**, **(Res)**, and the following lemma.

Lemma 2.2. *For all $C_\psi > 0$ there exist constants $C_1, \ldots, C_4$ such that if $\|\psi_Z\|_{X_Z} \leq C_\psi$, then*

$$\begin{aligned} &\|\varepsilon^{-1}(N_U(\psi_Z+\varepsilon R) - N_U(\psi_Z)\|_{X_U} + \|\varepsilon^{-1}(N_W(\psi_Z+\varepsilon R) - N_W(\psi_Z)\|_{X_V} \\ &\leq C_1\|R\|_{X_Z} + C_2\varepsilon\|R\|_{X_Z}^2 + C_3\varepsilon^2\|R\|_{X_Z}^3 + C_4\varepsilon^3\|R\|_{X_Z}^4, \end{aligned}$$

where $\|(U,V)\|_{X_Z} = \|U\|_{X_U} + \|V\|_{X_V}$.

Proof. The nonlinearities N_U and N_W are compositions of bilinear mappings satisfying the estimates from **(B1)**, **(B2)**, **(I)**, **(M)**, and (17). Expanding the corresponding expressions immediately gives the desired estimates. □

Introducing

$$\mathcal{S}(t) = \sup_{\tau\in[0,t]} \|R(\tau)\|_{X_Z}$$

yields the estimate

$$\mathcal{S}(t) \leq \int_0^t C_\Lambda(C_1\mathcal{S}(\tau) + C_2\varepsilon\mathcal{S}(\tau)^2 + C_3\varepsilon^2\mathcal{S}(\tau)^3 + C_4\varepsilon^3\mathcal{S}(\tau)^4 + 2C_{res})d\tau.$$

For

$$C_2\varepsilon\mathcal{S}(\tau)^2 + C_3\varepsilon^2\mathcal{S}(\tau)^3 + C_4\varepsilon^3\mathcal{S}(\tau)^4 \leq 1$$

we find

$$\mathcal{S}(t) \leq \int_0^t C_\Lambda(C_1\mathcal{S}(\tau) + 1 + 2C_{res})d\tau.$$

Gronwall's inequality then shows

$$\mathcal{S}(t) \leq (1+2C_{res})te^{C_\Lambda C_1 t} \leq (1+2C_{res})T_0e^{C_\Lambda C_1 T_0} =: M.$$

Choosing finally $\varepsilon_0 > 0$ so small that

$$C_2\varepsilon_0 M^2 + C_3\varepsilon_0^2 M^3 + C_4\varepsilon_0^3 M^4 \leq 1,$$

we are done. □

Remark 2.3. It is obvious that the previous arguments still work for systems, with general nonlinearities F_U and F_V, of the form

$$\partial_t U = \Lambda_U U + F_U(U,V), \qquad \varepsilon \partial_t V = \Lambda_V V + F_V(U),$$

with associated limit system

$$\partial_t \psi_U = \Lambda_U \psi_U - F_U(\psi_U, \Lambda_V^{-1} F_V(\psi_U)),$$

if we assume a certain smoothness of $F_U : X_U \times X_V \to X_U$, $F_V : X_U \to X_V$, and adapt the assumptions **(B1)**, **(B2)**, **(I)**, and **(M)**.

3 Applications

From a functional-analytical point of view the proof of the above approximation theorem is not difficult. The difficulties are transferred to the application of the theorem and are discussed below for the first three examples of the introduction.

3.1 The Dirac-Hartree approximation for the DKG system

The Dirac-Klein-Gordon system is used as a model for proton-proton interactions where one proton is scattered in a meson field, cf. [BH17]. It is given by

$$-i\gamma^0 \partial_t u - i \sum_{\mu=1}^{3} \gamma^\mu \partial_{x_\mu} u + m_u u = vu, \qquad \varepsilon^2 \partial_t^2 v = \Delta v - m_v^2 v + \overline{u}^\tau \gamma^0 u, \tag{24}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, and describes a vector field $u(x,t) \in \mathbb{C}^4$ and a scalar field $v(x,t) \in \mathbb{R}$. We have the small perturbation parameter $0 \leq \varepsilon \ll 1$, the masses $m_u, m_v > 0$, the Dirac matrices $\gamma^\mu \in \mathbb{C}^{4\times 4}$ with

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

for $j = 1,2,3$, and the Pauli matrices $\sigma^j \in \mathbb{C}^{2\times 2}$ with

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

u is called a spinor field and u^τ is the transposed vector associated to u.

It is the goal of this section to explain that the singular limit $\varepsilon \to 0$ of the DKG system, where we obtain the Dirac-Hartree equation

$$-i\gamma^0 \partial_t u - i \sum_{\mu=1}^{3} \gamma^\mu \partial_{x_\mu} u + m_u u = ((-\Delta + m_v^2)^{-1} (\overline{u}^\tau \gamma^0 u)) u \tag{25}$$

as limit system, is covered by Theorem 2.1. In order to do so, we write (24) as a first order system

$$\begin{aligned}
\partial_t u &= -\sum_{\mu=1}^{3} \gamma^0\gamma^\mu \partial_\mu u - i m_u \gamma^0 u + i\gamma^0 v u, \\
\partial_t v &= \varepsilon^{-1} i\omega_v \widetilde{v}, \\
\partial_t \widetilde{v} &= \varepsilon^{-1} i\omega_v v - \varepsilon^{-1}\frac{1}{i\omega_v}\overline{u}^\tau \gamma^0 u,
\end{aligned}$$

where ω_v and $1/\omega_v$ are defined via their symbols in Fourier space

$$\widehat{\omega}_v(k) = \sqrt{|k|^2 + m_v^2} \qquad \text{and} \qquad 1/\widehat{\omega}_v(k).$$

In order to apply our abstract approximation result from Section 2 we set $U = u$, $V = (v, \widetilde{v})$, and

$$\Lambda_U = -\sum_{\mu=1}^{3} \gamma^0\gamma^\mu \partial_\mu - i m_u \gamma^0, \qquad \Lambda_V = \begin{pmatrix} 0 & i\omega_v \\ i\omega_v & 0 \end{pmatrix}$$

for the linear operators. For the nonlinear terms we choose

$$B_U(U,V) = i\gamma^0 v u, \qquad B_V(U,U) = \begin{pmatrix} 0 \\ -\frac{1}{i\omega_v}\overline{u}^\tau \gamma^0 u \end{pmatrix}.$$

Finally, the Banach spaces X_U and X_V are given by the Sobolev spaces $X_U = (H^s)^4$ and $X_V = H^s$ which are closed under multiplication for $s > 3/2$ due to Sobolev's embedding theorem.

We are now going to check the assumptions of Theorem 2.1. The validity of the assumptions **(S1)**, **(S2)**, **(B1)**, and **(B2)** is obvious. We have for instance

$$\|e^{\Lambda_U t} U\|_{H^s} = \|e^{\widehat{\Lambda}_U t}\widehat{U}\|_{L^2_s} \leq \|e^{\widehat{\Lambda}_U t}\|_{L^\infty} \|\widehat{U}\|_{L^2_s} \leq \|U\|_{H^s}$$

or

$$\begin{aligned}
\|B_V(U,U)\|_{H^s} &= \left\| \frac{1}{i\omega_v}\overline{u}^\tau \gamma^0 u \right\|_{L^2_s} \leq \left\| \frac{1}{\widehat{\omega}_v(k)} \right\|_{L^\infty(\mathrm{d}k)} \|\widehat{\overline{u}^\tau \gamma^0 u}\|_{L^2_s} \\
&\leq C\|\overline{u}^\tau \gamma^0 u\|_{H^s} \leq C\|U\|_{H^s}^2.
\end{aligned}$$

For checking the assumption **(I)** we first note that

$$\Lambda_V^{-1} B_V(U_1,U_2) = -\frac{1}{2}\begin{pmatrix} 0 & i\omega_v \\ i\omega_v & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1}{i\omega_v}(\overline{u}_1^t \gamma^0 u_2 + \overline{u}_2^t \gamma^0 u_1) \end{pmatrix}.$$

such that

$$\|M(U_1,U_2)\|_{H^{s+2}} = \|\Lambda_V^{-1} B_V(U_1,U_2)\|_{H^{s+2}} \;\; \leq \;\; C\|U_1\|_{H^s}\|U_2\|_{H^s}$$

which is better than what is actually needed for **(I)**. The validity of the assumption **(M)** follows from

$$\|M(\Lambda_U U_1, U_2)\|_{H^{s+1}} \leq C\|\Lambda_U U_1\|_{H^{s-1}}\|U_2\|_{H^{s-1}} \leq C\|U_1\|_{H^s}\|U_2\|_{H^{s-1}}.$$

We obtain for the residual, first

$$\begin{aligned}
\|\mathrm{Res}_W(\psi_U,0)\|_{H^s} &= 2\|M(\Lambda_U\psi_U + B_U(\psi_U, -\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^s} \\
&\leq C(\|M(\Lambda_U\psi_U,\psi_U)\|_{H^s} + \|M(B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^s}) \\
&\leq C\|\Lambda_U\psi_U\|_{H^{s-2}}\|\psi_U\|_{H^{s-2}} + C\|B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U))\|_{H^{s-2}}\|\psi_U\|_{H^{s-2}} \\
&\leq C\|\psi_U\|_{H^{s-1}}\|\psi_U\|_{H^{s-2}} + C\|\psi_U\|_{H^{s-2}}\|-\Lambda_V^{-1}B_V(\psi_U,\psi_U)\|_{H^{s-2}}\|\psi_U\|_{H^{s-2}} \\
&\leq C\|\psi_U\|_{H^{s-1}}\|\psi_U\|_{H^{s-2}} + C\|\psi_U\|_{H^{s-2}}\|\psi_U\|^2_{H^{s-4}}\|\psi_U\|_{H^{s-2}}.
\end{aligned}$$

Since Λ_V^{-1} gains one derivative we have next

$$\|\Lambda_V^{-1}\mathrm{Res}_W(\psi_U,0)\|_{H^s} \leq C\|\psi_U\|_{H^{s-2}}\|\psi_U\|_{H^{s-3}} + C\|\psi_U\|_{H^{s-3}}\|\psi_U\|^2_{H^{s-5}}\|\psi_U\|_{H^{s-3}}. \tag{26}$$

Finally, for the same reason we find

$$\begin{aligned}
&\|\Lambda_V^{-1}\partial_t\mathrm{Res}_W(\psi_U,0)\|_{H^s} \\
\leq\;& C\|\partial_t M(\Lambda_U\psi_U + B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^{s-1}} \\
\leq\;& C(\|M(\Lambda_U\partial_t\psi_U,\psi_U)\|_{H^{s-1}} + \|M(\Lambda_U\psi_U,\partial_t\psi_U)\|_{H^{s-1}} \\
&+\|M(B_U(\partial_t\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^{s-1}} \\
&+2\|M(B_U(\psi_U,-\Lambda_V^{-1}B_V(\partial_t\psi_U,\psi_U)),\psi_U)\|_{H^{s-1}} \\
&+\|M(B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\partial_t\psi_U)\|_{H^{s-1}}).
\end{aligned}$$

Replacing then $\partial_t\psi_U$ by the right-hand side of (19) allows to estimate all terms in terms of H^s-norms of ψ_U. The term losing most derivatives – namely two – is $\Lambda_U\partial_t\psi_U$. Since we gained two derivatives in (26) also the third estimate of **(Res)** can be obtained by choosing $\psi_U \in H^s = X_\psi = X_U$.

Thus we checked all assumptions of Theorem 2.1 and so we have

Theorem 3.1. *Let $\psi_u \in C([0,T_0],H^s)$ be a solution of (25) for an $s > 3/2$. Then there exist $\varepsilon_0 > 0$ and $C_U > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions (u,v) of (24) with*

$$\sup_{t\in[0,T_0]}(\|u-\psi_u\|_{H^s} + \|v-\psi_v\|_{H^s}) \leq C\varepsilon,$$

where $\psi_v = (-\Delta + m_v^2)^{-1}(\overline{\psi}_u^T\gamma^0\psi_u)$.

3.2 The Klein-Gordon approximation for the KGZ system

We consider the Klein-Gordon (KG) limit of the Klein-Gordon-Zakharov (KGZ) system, i.e., we consider

$$\partial_t^2 u = \Delta u - u - \gamma uv, \qquad \varepsilon^2\partial_t^2 v = \Delta v + \Delta(|u|^2), \tag{27}$$

with $\gamma = \pm 1$, $t \geq 0$, $x, u(x,t) \in \mathbb{R}^3$, and $v(x,t) \in \mathbb{R}$, for small $0 \leq \varepsilon \ll 1$. In the limit $\varepsilon \to 0$ we obtain the regular limit system, namely the KG equation

$$\partial_t^2 u = \Delta u - u + \gamma \left|u\right|^2 u. \tag{28}$$

The KGZ system occurs as a model in plasma physics where it describes the interaction between so called Langmuir waves and ion sound waves via some ion density fluctuation v and the electric field u, cf. [MN05]. Error estimates that the KG equation or a slightly modified KG equation make correct predictions about the dynamics of the KGZ system for small $\varepsilon > 0$ can be found in [DSS16] in case of periodic boundary conditions. It is the goal of this section to explain that this singular limit of the KGZ system is covered by Theorem 2.1.

In order to do so we write (27) as a first order system

$$\begin{aligned}
\partial_t u &= i\omega_u \widetilde{u}, \\
\partial_t \widetilde{u} &= i\omega_u u - \frac{\gamma}{i\omega_u} uv, \\
\partial_t v &= \varepsilon^{-1} i\omega_v \widetilde{v}, \\
\partial_t \widetilde{v} &= \varepsilon^{-1} i\omega_v v + \varepsilon^{-1} i\omega_v \left|u\right|^2,
\end{aligned}$$

where ω_u, $1/\omega_u$ and ω_v are defined via their symbols in Fourier space

$$\widehat{\omega}_u(k) = \sqrt{|k|^2 + 1} \quad \text{and} \quad \widehat{\omega}_v(k) = |k|.$$

In order to apply our abstract approximation result from Section 2 we set $U = (u, \widetilde{u})$, $V = (v, \widetilde{v})$, and

$$\Lambda_U = \begin{pmatrix} 0 & i\omega_u \\ i\omega_u & 0 \end{pmatrix}, \quad \Lambda_V = \begin{pmatrix} 0 & i\omega_v \\ i\omega_v & 0 \end{pmatrix}$$

for the linear operators. For the nonlinear terms we choose

$$B_U(U,V) = \begin{pmatrix} 0 \\ -\frac{\gamma}{i\omega_u} uv \end{pmatrix}, \quad B_V(U,U) = \begin{pmatrix} 0 \\ i\omega_v(|u|^2) \end{pmatrix}.$$

Finally, the Banach spaces X_U and X_V are given by the Sobolev spaces $X_U = H^{s+1}$ and $X_V = H^s$ with $s \geq 1$. We are now going to check the assumptions of Theorem 2.1. As is Section 3.1 the validity of the assumptions **(S1)**, **(S2)**, **(B1)**, and **(B2)** is obvious. We have for instance

$$\begin{aligned}
\|B_U(U,V)\|_{H^{s+1}} &= \left\| \frac{\gamma}{i\omega_u} \widehat{uv} \right\|_{L^2_{s+1}} \leq \left\| \frac{(1+k^2)^{1/2}}{\widehat{\omega}_u(k)} \right\|_{L^\infty(\mathrm{d}k)} \|\widehat{uv}\|_{L^2_s} \\
&\leq C\|uv\|_{H^s} \leq C\|U\|_{H^s}\|V\|_{H^s} \leq C\|U\|_{H^{s+1}}\|V\|_{H^s}.
\end{aligned}$$

For checking the assumption **(I)** we first note that

$$\Lambda_V^{-1} B_V(U_1, U_2) = \begin{pmatrix} u_1 \cdot u_2 \\ 0 \end{pmatrix}.$$

such that

$$\|\Lambda_V^{-1} B_V(U_1,U_2)\|_{H^s} \quad \leq \quad C\|U_1\|_{H^s}\|U_2\|_{H^s}$$

which is better than what is actually needed for **(I)**. The validity of the assumption **(M)** follows from

$$\|M(\Lambda_U U_1,U_2)\|_{H^s} \quad \leq \quad C\|\Lambda_U U_1\|_{H^s}\|U_2\|_{H^s} \leq C\|U_1\|_{H^{s+1}}\|U_2\|_{H^s}.$$

Checking the assumption **(Res)** is less trivial and depends on whether we consider the KGZ system in $\mathbb{R}^3$ or with periodic boundary conditions, cf. [DSS16]. This distinction plays no role for

$$\begin{aligned}
\|\mathrm{Res}_W(\psi_U,0)\|_{H^s} &= 2\|M(\Lambda_U\psi_U + B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^s} \\
&\leq C(\|M(\Lambda_U\psi_U,\psi_U)\|_{H^s} + \|M(B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^s}) \\
&\leq C\|\Lambda_U\psi_U\|_{H^s}\|\psi_U\|_{H^s} + C\|B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U))\|_{H^s}\|\psi_U\|_{H^s} \\
&\leq C\|\psi_U\|_{H^{s+1}}\|\psi_U\|_{H^s} + C\|\psi_U\|_{H^{s-1}}\|-\Lambda_V^{-1}B_V(\psi_U,\psi_U)\|_{H^{s-1}}\|\psi_U\|_{H^s} \\
&\leq C\|\psi_U\|_{H^{s+1}}\|\psi_U\|_{H^s} + C\|\psi_U\|_{H^{s-1}}\|\psi_U\|^2_{H^{s-1}}\|\psi_U\|_{H^s}.
\end{aligned}$$

However, for the time derivative both cases have to be treated differently. For the periodic case we refer to [DSS16] and restrict ourselves here to $x \in \mathbb{R}^3$. We use

Lemma 3.2. *For $x \in \mathbb{R}^3$ the operator Λ_V^{-1} is invertible from $H^{s-1} \cap L^1$ to H^s for $s \geq 1$, i.e., there exists a $C > 0$ such that*

$$\|\Lambda_V^{-1}V\|_{H^s} \leq C\|V\|_{H^{s-1}\cap L^1}.$$

Proof. Since

$$\begin{aligned}
\|\Lambda_V^{-1}V\|_{L^2} &\leq C\||k|^{-1}\widehat{V}(k)\|_{L^2(\mathrm{d}k)} \\
&\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-1}\widehat{V}(k)\|_{L^2(\mathrm{d}k)} + \|\chi_{|k|>1}(k)|k|^{-1}\widehat{V}(k)\|_{L^2(\mathrm{d}k)}) \\
&\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-1}\|_{L^2(\mathrm{d}k)}\|\widehat{V}(k)\|_{L^\infty(\mathrm{d}k)} \\
&\qquad + \|\chi_{|k|>1}(k)|k|^{-1}\|_{L^\infty(\mathrm{d}k)}\|\widehat{V}(k)\|_{L^2(\mathrm{d}k)}) \\
&\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-1}\|_{L^2(\mathrm{d}k)}\|V\|_{L^1} + \|V\|_{L^2})
\end{aligned}$$

and since

$$\|\chi_{|k|\leq 1}(k)|k|^{-1}\|^2_{L^2(\mathrm{d}k)} = \int_{|k|\leq 1}|k|^{-2}\mathrm{d}k = C_d\int_0^1 r^{d-3}\mathrm{d}r < \infty$$

for $d \geq 3$, the operator Λ_V can be inverted from $L^1 \cap L^2$ to L^2 if $d \geq 3$. For the derivatives no singularity at $k = 0$ occurs and the estimates follow in a trivial way. □

Since $\mathrm{Res}_W(\psi_U,0)$ contains only terms which are at least quadratic, by the Cauchy-Schwarz inequality we not only have that $\|\mathrm{Res}_W(\psi_U,0)\|_{H^s}$ is bounded for ψ_U in H^{s+1}, but also that $\|\mathrm{Res}_W(\psi_U,0)\|_{H^s\cap L^1}$ is bounded. Hence, applying Lemma 3.2 shows

$$\|\Lambda_V^{-1}\mathrm{Res}_W(\psi_U,0)\|_{H^s} \leq C\|\psi_U\|_{H^{s+1}}\|\psi_U\|_{H^s} + C\|\psi_U\|_{H^{s-1}}\|\psi_U\|^2_{H^{s-1}}\|\psi_U\|_{H^s}.$$

Finally, we find

$$
\begin{aligned}
& \|\Lambda_V^{-1}\partial_t \mathrm{Res}_W(\psi_U,0)\|_{H^s} \\
\leq\ & C\|\partial_t M(\Lambda_U\psi_U + B_U(\psi_U, -\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^{s-1}\cap L^1} \\
\leq\ & C(\|M(\Lambda_U\partial_t\psi_U,\psi_U)\|_{H^{s-1}\cap L^1} + \|M(\Lambda_U\psi_U,\partial_t\psi_U)\|_{H^{s-1}\cap L^1} \\
& +\|M(B_U(\partial_t\psi_U, -\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{H^{s-1}\cap L^1} \\
& +2\|M(B_U(\psi_U, -\Lambda_V^{-1}B_V(\partial_t\psi_U,\psi_U)),\psi_U)\|_{H^{s-1}\cap L^1} \\
& +\|M(B_U(\psi_U, -\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\partial_t\psi_U)\|_{H^{s-1}\cap L^1})
\end{aligned}
$$

Replacing then $\partial_t\psi_U$ by the right-hand side of (19) allows to estimate all terms in terms of H^s-norms of ψ_U. The term losing most derivatives – namely two – is $\Lambda_U\partial_t\psi_U$ such that also the third estimate of **(Res)** can be obtained by choosing $\psi_U \in H^{s+1} = X_\psi = X_U$. The L^1-estimate follows again from the Cauchy-Schwarz inequality. Thus we checked all assumptions of Theorem 2.1 and so we have

Theorem 3.3. *Let $\psi_u \in C([0,T_0],H^{s+1})$ be a solution of (28) for an $s > 3/2$. Then there exist $\varepsilon_0 > 0$ and $C_U > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions (u,v) of (27) with*

$$
\sup_{t\in[0,T_0]} (\|u-\psi_u\|_{H^{s+1}} + \|v+|\psi_u|^2\|_{H^s}) \leq C\varepsilon.
$$

3.3 The high frequency limit of a mean field polaron model

We consider the high frequency limit of a mean field polaron model [BNAS00], i.e., we consider

$$
i\partial_t u = \Delta u - \gamma uv, \qquad \varepsilon^2\partial_t^2 v = -v + \Delta^{-1}(|u|^2), \tag{29}
$$

with $\gamma = \pm 1$, $t \geq 0$, $x \in \mathbb{R}^3$, $v(x,t) \in \mathbb{R}$, and $u(x,t) \in \mathbb{C}$ for small $0 \leq \varepsilon \ll 1$. This model describes a free electron interacting with a dielectric polarizable continuum under a number of assumptions. The evolution of the lattice polarization is modelled as a harmonic oscillator, described by the electrostatic potential $v = v(x,t) \in \mathbb{R}$, subject to an external force coming from the single electron, which is described by its wave function $u = u(x,t) \in \mathbb{C}$. In the limit $\varepsilon \to 0$ we obtain the Hartree equation

$$
i\partial_t u = \Delta u - \gamma u(\Delta^{-1}(|u|^2) \tag{30}
$$

as regular limit system. It is the goal of this section to explain that this singular limit of the mean field polaron model is covered by Theorem 2.1.

In order to do so, we write (29) as a first order system

$$
\begin{aligned}
& i\partial_t u = \Delta u - \gamma uv, \\
& \partial_t v = i\varepsilon^{-1}\widetilde{v}, \\
& \partial_t \widetilde{v} = i\varepsilon^{-1}v + i\varepsilon^{-1}\Delta^{-1}(|u|^2).
\end{aligned}
$$

In order to apply our abstract approximation result from Section 2 we set $U = u$, $V = (v, \widetilde{v})$, and

$$\Lambda_U = -i\Delta, \quad \Lambda_V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

for the linear operators. For the nonlinear terms we choose

$$B_U(U,V) = -\gamma uv, \quad B_V(U,U) = \begin{pmatrix} 0 \\ i\Delta^{-1}(|u|^2) \end{pmatrix}.$$

Finally, the Banach spaces X_U and X_V are given by the Sobolev spaces

$$X_U = H^s, \quad X_V = L^\infty \cap H^s_{\geq 1}$$

with $s \geq 1$ where

$$\|V\|_{X_V} = \|V\|_{L^\infty} + \sum_{|k|=1}^{s} \|\nabla^k V\|_{L^2}.$$

We are now going to check the previous assumptions. The validity of the assumptions **(S1)**, **(S2)**, and **(B1)** is obvious. We have for instance

$$\|e^{\Lambda_U t} U\|_{H^s} = \|e^{\widehat{\Lambda}_U t} \widehat{U}\|_{L^2_s} \leq \|e^{\widehat{\Lambda}_U t}\|_{L^\infty} \|\widehat{U}\|_{L^2_s} \leq \|U\|_{H^s}$$

and

$$\|B_U(U,V)\|_{H^s} = \|\gamma uv\|_{H^s} \leq C\|U\|_{H^s}\|V\|_{H^s}$$

for $s > 3/2$ due to Sobolev's embedding theorem. The assumption **(B2)** follows by

$$\begin{aligned} \|B_V(U,\widetilde{U})\|_{L^\infty \cap H^s_{\geq 1}} &\leq \||k|^{-2}\widehat{(|u\widetilde{u}|)}\|_{L^1 \cap L^2_s} \leq C\|\widehat{(|u\widetilde{u}|)}\|_{L^2_r \cap L^\infty} \\ &\leq C\|u\widetilde{u}\|_{H^r \cap L^1} \leq C\|U\|_{H^r}\|\widetilde{U}\|_{H^r} \end{aligned}$$

for $r > \max\{s-2, 3/2\}$, where the second estimate holds for $d = 3$, cf. [GSS17, Lemma 2.3]. It is based on

$$\begin{aligned} \|\Delta_V^{-1} v\|_{L^\infty} &\leq C\||k|^{-2}\widehat{v}(k)\|_{L^1(\mathrm{d}k)} \\ &\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-2}\widehat{v}(k)\|_{L^1(\mathrm{d}k)} + \|\chi_{|k|>1}(k)|k|^{-2}\widehat{v}(k)\|_{L^1(\mathrm{d}k)}) \\ &\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-2}\|_{L^1(\mathrm{d}k)}\|\widehat{v}(k)\|_{L^\infty(\mathrm{d}k)} \\ &\qquad + \|\chi_{|k|>1}(k)|k|^{-2}\|_{L^2(\mathrm{d}k)}\|\widehat{v}(k)\|_{L^2(\mathrm{d}k)}) \\ &\leq C(\|v\|_{L^1} + \|v\|_{L^2}) \end{aligned}$$

where we used

$$\|\chi_{|k|\leq 1}(k)|k|^{-2}\|_{L^1(\mathrm{d}k)} = \int_{|k|\leq 1} |k|^{-2}\,\mathrm{d}k = C_d \int_0^1 r^{d-3}\,\mathrm{d}r < \infty$$

for $d \geq 3$ and

$$\|\chi_{|k|\geq 1}(k)|k|^{-2}\|^2_{L^2(\mathrm{d}k)} = \int_{|k|\geq 1} |k|^{-4}\,\mathrm{d}k = C_d \int_1^\infty r^{d-5}\,\mathrm{d}r < \infty$$

for $d \leq 3$. Since Λ_V^{-1} is a bounded operator the validity of assumption **(I)** is a direct consequence of **(B2)**, i.e., we have the estimate

$$\|\Lambda_V^{-1}B_V(U,\widetilde{U})\|_{L^\infty\cap H^s_{\geq 1}} \leq C\|U\|_{H^r}\|\widetilde{U}\|_{H^r}$$

for $r > \max\{s-2, 3/2\}$. The estimate is better than necessary for **(I)**. The validity of the assumption **(M)** follows from

$$\|M(\Lambda_U U,\widetilde{U})\|_{L^\infty\cap H^s_{\geq 1}} \leq C\|\Lambda_U U_1\|_{H^r}\|U_2\|_{H^r} \leq C\|U_1\|_{H^{r+2}}\|U_2\|_{H^r}.$$

We find next

$$\begin{aligned}
&\|\mathrm{Res}_W(\psi_U,0)\|_{L^\infty\cap H^s_{\geq 1}} = 2\|M(\Lambda_U\psi_U + B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{L^\infty\cap H^s_{\geq 1}}\\
\leq\ & C(\|M(\Lambda_U\psi_U,\psi_U)\|_{L^\infty\cap H^s_{\geq 1}} + \|M(B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{L^\infty\cap H^s_{\geq 1}})\\
\leq\ & C\|\Lambda_U\psi_U\|_{H^r}\|\psi_U\|_{H^r} + C\|B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U))\|_{H^r}\|\psi_U\|_{H^r}\\
\leq\ & C\|\psi_U\|_{H^{r+2}}\|\psi_U\|_{H^r} + C\|\psi_U\|_{H^r}\|-\Lambda_V^{-1}B_V(\psi_U,\psi_U)\|_{L^\infty\cap H^r_{\geq 1}}\|\psi_U\|_{H^r}\\
\leq\ & C\|\psi_U\|_{H^{r+2}}\|\psi_U\|_{H^r} + C\|\psi_U\|_{H^r}\|\psi_U\|^2_{H^{\tilde{r}}}\|\psi_U\|_{H^r}.
\end{aligned}$$

for $r > \max\{s-2, 3/2\}$ and $\widetilde{r} > \max\{r-2, 3/2\}$. Since Λ_V^{-1} is a bounded operator we also have

$$\|\Lambda_V^{-1}\mathrm{Res}_W(\psi_U,0)\|_{L^\infty\cap H^s_{\geq 1}} \leq C\|\psi_U\|_{H^{r+2}}\|\psi_U\|_{H^r} + C\|\psi_U\|_{H^r}\|\psi_U\|^2_{H^{\tilde{r}}}\|\psi_U\|_{H^r}$$

for the same values of r and $\widetilde{r}$. Finally, we have

$$\begin{aligned}
&\|\Lambda_V^{-1}\partial_t\mathrm{Res}_W(\psi_U,0)\|_{L^\infty\cap H^s_{\geq 1}}\\
\leq\ & C\|\partial_t M(\Lambda_U\psi_U + B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{L^\infty\cap H^s_{\geq 1}}\\
\leq\ & C(\|M(\Lambda_U\partial_t\psi_U,\psi_U)\|_{L^\infty\cap H^s_{\geq 1}} + \|M(\Lambda_U\psi_U,\partial_t\psi_U)\|_{L^\infty\cap H^s_{\geq 1}}\\
&+\|M(B_U(\partial_t\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\psi_U)\|_{L^\infty\cap H^s_{\geq 1}}\\
&+2\|M(B_U(\psi_U,-\Lambda_V^{-1}B_V(\partial_t\psi_U,\psi_U)),\psi_U)\|_{L^\infty\cap H^s_{\geq 1}}\\
&+\|M(B_U(\psi_U,-\Lambda_V^{-1}B_V(\psi_U,\psi_U)),\partial_t\psi_U)\|_{L^\infty\cap H^s_{\geq 1}})
\end{aligned}$$

Replacing then $\partial_t\psi_U$ by the right-hand side of (19) allows to express all terms in terms of H^s-norms of ψ_U. The term losing most derivatives – namely four – is $\Lambda_U\partial_t\psi_U$. Since M gains two derivatives the third estimate of **(Res)** can be obtained by choosing $\psi_U \in H^{s+2} = X_\psi \subset X_U$. Thus we checked all assumptions of Theorem 2.1 and so we have

Theorem 3.4. *Let $\psi_u \in C([0,T_0], H^{s+2})$ be a solution of (30) for an $s > 3/2$. Then there exist $\varepsilon_0 > 0$ and $C_U > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions (u,v) of (29) with*

$$\sup_{t\in[0,T_0]} (\|u-\psi_u\|_{H^s} + \|v - \Delta^{-1}(|\psi_u|^2)\|_{L^\infty\cap H^s_{\geq 1}}) \leq C\varepsilon.$$

4 Discussion

Our abstract approximation theorem applies to all semilinear systems of the structure (14) which additionally satisfy the assumption **(M)**. Here, we show that the assumptions of the abstract theorem are sharp in the sense that there exists a quasilinear system of the same structure where the regular limit system makes wrong predictions. Hence, it turns out that this limit for non-semilinear systems in general is a rather subtle problem.

4.1 The singular limit and normal form transformations

In order to explain the difference between the systems (1)-(3) on the one hand and (4) on the other hand we have to make a small detour to the theory of normal form transformations.

In order to do so we consider the simple ODE system

$$\partial_t u = iu + iuv, \qquad \varepsilon \partial_t v = iv - iu^2, \tag{31}$$

with $t \in \mathbb{R}$, $u(t), v(t) \in \mathbb{C}$, and $0 \leq \varepsilon \ll 1$ a small perturbation parameter. In the singular limit $\varepsilon \to 0$ the regular limit system

$$\partial_t \psi_U = i\psi_U + i\psi_U \psi_V, \qquad 0 = i\psi_V - i\psi_U^2,$$

and finally

$$\partial_t \psi_U = i\psi_U + i\psi_U^3 \tag{32}$$

is obtained. Although (31) falls into the class of systems (5) and although checking the assumptions of Theorem 2.1 is rather trivial we redo the calculations from above, but this time we collect the terms differently.

We make the transform

$$w = v + \alpha u^2, \tag{33}$$

with the new variable w and a coefficient $\alpha \in \mathbb{C}$ which has to be determined in such a way that the dangerous term $-i\varepsilon^{-1}u^2$ is eliminated. We find

$$\begin{aligned} \partial_t w &= \partial_t v + \alpha(\partial_t u)u + \alpha u(\partial_t u) \\ &= i\varepsilon^{-1} v - i\varepsilon^{-1} u^2 + \alpha(iu + iuv)u + \alpha u(iu + iuv) \\ &= i\varepsilon^{-1} w + ((-\lambda_v + \lambda_u + \lambda_u)\alpha - i\varepsilon^{-1})u^2 + \mathcal{O}(1) + \mathcal{O}(\alpha), \end{aligned}$$

where $\lambda_u = i$ is the eigenvalue of the linearization of the u-equation, and $\lambda_v = i\varepsilon^{-1}$ is the eigenvalue of the linearization of the v-equation. Hence, in order to eliminate the nonlinear term $-i\varepsilon^{-1}u^2$, i.e., in order to compute α, we need the validity of the non-resonance condition

$$-\lambda_v + \lambda_u + \lambda_u \neq 0. \tag{34}$$

If (34) is satisfied we can choose

$$\alpha = \frac{i\varepsilon^{-1}}{-\lambda_v + \lambda_u + \lambda_u} = \frac{i\varepsilon^{-1}}{2i - i\varepsilon^{-1}} = \frac{\varepsilon^{-1}}{2 - \varepsilon^{-1}}$$

which is $\mathcal{O}(1)$ for $\varepsilon \to 0$. After this transformation we have a system

$$\partial_t u = \mathcal{O}(1), \qquad \partial_t w = i\varepsilon^{-1} w + \mathcal{O}(1) \tag{35}$$

for which easily an $\mathcal{O}(1)$ bound for the error on the $\mathcal{O}(1)$ time scale can be obtained with the help of the variation of constant formula and Gronwall's inequality. The right-hand side of (35) can be expressed solely in terms of u and w since the normal form (33) is invertible.

Obviously, as we have done in Section 2 it is not necessary to eliminate all the terms proportional to u^2 in order to obtain a system of the same form as (35). It is sufficient to eliminate the $\mathcal{O}(\varepsilon^{-1})$ terms only by choosing $\alpha = -i\varepsilon^{-1}/\lambda_v = -1$. This is exactly what we used above and as a consequence for problems on the real axis we do not need the validity of the non-resonance condition

$$-\lambda_v(k) + \lambda_u(k-l) + \lambda_u(l) \neq 0 \tag{36}$$

for all Fourier wave numbers $k, l \in \mathbb{R}$, but only the much simpler condition

$$-\lambda_v(k) \neq 0 \tag{37}$$

for all Fourier wave numbers $k \in \mathbb{R}$, which corresponds to the existence of Λ_V^{-1}, respectively the weaker assumption **(I)**.

This observation is restricted to systems with this special nonlinear interaction structure. We will use this correspondence to normal form transformations to construct the counter example. An introduction to normal form transformations can be found in [SVM07]. In the last years they have been used extensively in proving error estimates for the NLS approximation, cf. [DLP$^+$11, Sch16] for an overview.

Remark 4.1. We buy the simpler non-resonance condition (37) or the weaker assumption **(I)** with the additional assumption **(M)** which is necessary to control the terms which are $\mathcal{O}(1)$ and which are not eliminated by the simpler approach. As we have seen for the relevant systems from Section 3 the assumption **(M)** is satisfied.

4.2 The NLS approximation for the Zakharov system

Another system which can be reformulated to have the structure of (14) is the Zakharov system

$$i\partial_t u + \Delta u = -\gamma v u, \qquad \varepsilon^2 \partial_t^2 v - \Delta v = \Delta |u|^2, \tag{38}$$

with $\gamma = \pm 1$, $t, v(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $u(x,t) \in \mathbb{C}$, and $0 < \varepsilon \ll 1$ a small perturbation parameter. In the limit $\varepsilon \to 0$ we obtain the regular limit system

$$i\partial_t u + \Delta u = -\gamma v u, \qquad -v = |u|^2, \quad \text{resp.} \quad i\partial_t u + \Delta u = \gamma |u|^2 u, \tag{39}$$

which is a NLS equation. We proceed in the same way as in Section 3.2 for the KGZ system and write (38) as a first order system

$$\begin{aligned}
\partial_t u &= i\Delta u + i\gamma u v, \\
\partial_t v &= i\varepsilon^{-1}\omega_v \widetilde{v}, \\
\partial_t \widetilde{v} &= i\varepsilon^{-1}\omega_v v + \varepsilon^{-1}\omega_v(|u^2|),
\end{aligned}$$

with ω_v defined via its symbol in Fourier space $\widehat{\omega}_v(k) = |k|$.

Remark 4.2. The Zakharov system is a reduced model obtained from the Klein-Gordon-Zakharov (KGZ) system, considered in Section 3.2, cf. [CEGT04]. It can also be obtained directly from Maxwell's equations coupled with Euler's equations [Tex07]. The Zakharov system is a quasilinear system in the following sense. With the choice $X_U = H^{s+1}$, $X_V = H^s$ assumption **(B1)** would be violated. On the other hand, with the choice $X_U = H^s$ and $X_V = H^s$ assumption **(B2)** would be violated, and so (38) cannot be solved with the variation of constant formula in the above sense. Therefore, the assumptions of our abstract approximation theorem, Theorem 2.1, cannot be satisfied.

In contrast to the semilinear systems of Section 3, where the sign of γ was irrelevant, for the Zakharov system (38) an approximation property only holds if $\gamma = 1$. In fact in Section 4.3 we construct a counter example which shows that in case of $\gamma = -1$ the NLS approximation fails to make correct predictions. For numerical illustrations of this fact see Section 4.4. Before we construct this counter example we make two additional remarks.

Remark 4.3. The Zakharov system (38) can be made semilinear [OT92] by introducing the new variable $w = \partial_t u - i\theta u$ with $\theta > 0$ suitably chosen. Then u can be reconstructed via

$$\Delta u - \theta u = -iw - \gamma v u. \tag{40}$$

For a given v there exists a θ sufficiently big such that for (40) there exists a unique solution $u = u^*(w, v)$ where u is two times more regular than w and v. The second equation of (38) is written as first order system

$$\partial_t v = \varepsilon^{-1}\nabla q, \qquad \partial_t q = \varepsilon^{-1}\nabla v + \varepsilon^{-1}\nabla|u|^2 \tag{41}$$

Since $w = (\partial_t - i\theta)u$ we can apply $(\partial_t - i\theta)$ to the first equation of (38) and find

$$i(\partial_t(\partial_t - i\theta)u + \Delta(\partial_t - i\theta)u = \gamma\partial_t(vu) - i\theta\gamma v u$$

such that

$$i\partial_t w + \Delta w = \gamma w v + \varepsilon^{-1}\gamma u \nabla q.$$

The evolutionary system

$$\begin{aligned}
\partial_t w &= i\Delta w - i\gamma w v - i\varepsilon^{-1}\gamma u^*(w,v)\nabla q, && (42)\\
\partial_t v &= \varepsilon^{-1}\nabla q, && (43)\\
\partial_t q &= \varepsilon^{-1}\nabla v + \varepsilon^{-1}\nabla|u^*(w,v)|^2, && (44)
\end{aligned}$$

for w, q, and v, is semilinear, if we choose $w \in H^{s-1}$ and $v, q \in H^s$ which yields $u^* \in H^{s+1}$. However, the new semilinear system (42)-(44) is no longer of the form of our abstract system (14).

Remark 4.4. For instance in [SW86, AA88] it has been shown that the NLS equation makes correct predictions about the dynamics of the Zakharov system. We follow [AA88] and sketch how the sign of γ enters the proof of an approximation result for the NLS limit for the Zakharov system. In detail, we explain how to obtain an $\mathcal{O}(1)$ bound on an $\mathcal{O}(1)$ time scale for the solutions u, v of (38) in case $\gamma = 1$. We rewrite the Zakharov system as a first order system

$$i\partial_t u + \Delta u = -\gamma v u, \qquad \partial_t v = \nabla \cdot q, \qquad \varepsilon^2 \partial_t q = \nabla v + \nabla |u|^2.$$

We multiply the first equation with $-i\overline{u}$ and integrate this equation w.r.t. x. Adding the complex conjugate gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 = 0. \tag{45}$$

We multiply the first equation with $\partial_t \overline{u}$ and integrate this equation w.r.t. x. Adding the complex conjugate gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 = -\int \gamma (v u \partial_t \overline{u} + v \overline{u} \partial_t u)\,\mathrm{d}x. \tag{46}$$

Multiplying the second equation with v, the third equation with q, and integrating both equations w.r.t. x yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2 + \varepsilon^2 \frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{L^2}^2 = \int q \nabla |u|^2\,\mathrm{d}x = -\int |u|^2 \nabla \cdot q\,\mathrm{d}x = -\int |u|^2 \partial_t v\,\mathrm{d}x. \tag{47}$$

Adding (45)-(47) yields in case $\gamma = 1$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \varepsilon^2\|q\|_{L^2}^2 + \int v|u|^2\,\mathrm{d}x) = 0. \tag{48}$$

For u, v sufficiently small, but $\mathcal{O}(1)$, the square root of the energy on the right-hand side is equivalent to the $H^1 \times L^2 \times L^2$-norm.

This idea has been used in [AA88] to justify the validity of the NLS approximation u_{app}. The equations for the error $R_u = u - u_{app}$, $R_v = v - |u_{app}|^2$ can be handled in the same way if $\gamma = 1$, since q, which is scaled with ε in the energy, does not appear in the nonlinear terms.

In case $\gamma = -1$ the term $\int (v u \partial_t \overline{u} + v \overline{u} \partial_t u - |u|^2 \partial_t v)\,\mathrm{d}x$ remains on the right-hand side of (48). It cannot be estimated in terms of $\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \varepsilon^2\|q\|_{L^2}^2$. In Section 4.3 we explain that in case $\gamma = -1$ the NLS approximation fails to make correct predictions.

4.3 No counter examples and counter examples

In all models there are resonant wave numbers violating the non-resonance condition (36). Depending on the sign of γ these resonances can be stable or unstable. In the unstable case, i.e. $\gamma = -1$ in the models (1)-(4), exponentially growing modes occur. In this section we explain why the unstable resonances for the semilinear systems from Sections 3.1-3.3 only lead to growth rates of $\mathcal{O}(1)$ in coincidence with our approximation theorem. These growing modes cannot be used to construct a counter example, showing that the limit system makes wrong predictions about the original system. Hence, the sign of the coefficient γ in the nonlinear terms is irrelevant in accordance with our abstract approximation theorem, Theorem 2.1.

However, we compute the growth rates of the resonant modes for the Zakharov system to be of order $\mathcal{O}(e^{1/\varepsilon})$ in the unstable case. This growth allows us to construct a counter example which shows that in case $\gamma = -1$ the NLS approximation fails to make correct predictions.

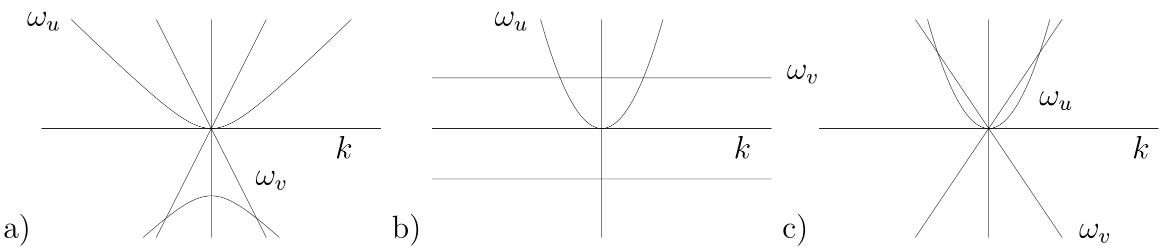

Figure 1: Graphical computation of the resonant wave numbers by intersection of $k \mapsto \omega_v(k)$ and $k \mapsto \omega_u(0) + \omega_u(k)$. a) shows these curves for the system from Section 3.2, b) these curves for the system from Section 3.3, and c) these curves for the system from Section 4.2. The intersection points in a) are at $(\mathcal{O}(\varepsilon), \mathcal{O}(1))$, in b) at $(\mathcal{O}(1/\sqrt{\varepsilon}), \mathcal{O}(1/\varepsilon))$, and in c) at $(\mathcal{O}(1/\varepsilon), \mathcal{O}(1/\varepsilon^2))$.

4.3.1 The KGZ-KG limit

As an illustrative example for the semilinear case we consider the KGZ-KG limit from Section 3.2. The non-resonance condition to eliminate the dangerous nonlinear terms $\varepsilon^{-2}\Delta(|u|^2)$ in (27) is given by

$$\widehat{\omega}_v(k) - \widehat{\omega}_u(k-l) - \widehat{\omega}_u(l) \neq 0 \tag{49}$$

for all $k, l \in \mathbb{R}$, where in the one-dimensional case we set $\widehat{\omega}_v(k) = k$. There are many resonances. Since we already know from our previous analysis that the unstable resonances can lead at most to $\mathcal{O}(1)$-growth rates we refrain from a complete discussion of all resonances, and for expository reasons we restrict ourselves to the special situation $l = 0$. Then the non-resonance condition is given by

$$\omega_v(k) - \omega_u(0) - \omega_u(k) = \pm\frac{k}{\varepsilon} - 1 - \sqrt{1+k^2} \neq 0 \tag{50}$$

It is easy to see that the resonant wave numbers are given by $k_1 = \pm\frac{2\varepsilon}{1-\varepsilon^2} = \mathcal{O}(\varepsilon)$. See Figure 1a). In order to compute the dynamics of the resonant modes we make the ansatz

$$\begin{aligned} u(x,t) &= A_0(t)e^{i\omega_u(0)t} + \varepsilon^n A_1(t)e^{i(k_1x+\omega_u(k_1)t)} + c.c., \\ v(x,t) &= B_0(t) + \varepsilon^n B_1(t)e^{i(k_1x+\omega_v(k_1)t)} + c.c., \end{aligned}$$

for a $n > 0$ fixed, i.e., a small perturbation of the x-independent situation. The linear terms cancel and so we find in lowest order

$$\begin{aligned}
2i\omega_u(0)\partial_t A_0 &= -\gamma A_0 B_0, \\
2i\omega_u(k_1)\partial_t A_1 &= -\gamma(A_1 B_0 + A_0 B_1), \\
\varepsilon^2 \partial_t^2 B_0 &= 0, \\
2\varepsilon^2 i\omega_v(k_1)\partial_t B_1 &= -k_1^2(A_1 \overline{A_0}).
\end{aligned}$$

For simplicity we restrict ourselves to the case $B_0|_{t=0} = 0$. Then we find $A_0 = const.$ and $B_0 = 0$. We compute

$$2i\omega_u(k_1)\partial_t^2 A_1 = -\gamma A_0 \partial_t B_1 = -\gamma A_0 \frac{1}{2\varepsilon^2 i\omega_v(k_1)}(-k_1^2(A_1\overline{A_0})),$$

i.e.

$$\partial_t^2 A_1 = \Gamma A_1 \qquad \text{and} \qquad \partial_t^2 B_1 = \Gamma B_1,$$

where

$$\Gamma = -\gamma \frac{|A_0|^2 k_1^2}{4\varepsilon^2 \omega_u(k_1)\omega_v(k_1)}.$$

We have that Γ has the opposite sign of γ, due to $\omega_u(k_1)\omega_v(k_1) > 0$, and so for $\gamma = -1$ exponential growth occurs. Since $k_1 = \mathcal{O}(\varepsilon)$, $A_0 = \mathcal{O}(1)$, $\omega_u(k_1) = \mathcal{O}(1)$, and $\omega_v(k_1) = \mathcal{O}(1)$ we have $\Gamma = \mathcal{O}(1)$. Thus, in accordance with Theorem 3.3, even in case $\gamma = -1$ the resonant modes B_1 stay $\mathcal{O}(1)$ bounded on the natural $\mathcal{O}(1)$-time scale of (27).

4.3.2 The Zakharov-NLS limit

The non-resonance condition to eliminate the dangerous nonlinear terms $\varepsilon^{-2}\Delta(|u|^2)$ in (38) is given by (49), where again we restrict to the case $l = 0$, i.e., to the non-resonance condition (50). It is easy to see that the resonant wave numbers k_1 are of order $\mathcal{O}(1/\varepsilon)$. See Figure 1c). In order to compute the dynamics of the resonant modes we make the ansatz

$$\begin{aligned}
u(x,t) &= A_0(t)e^{i\omega_u(0)t} + \varepsilon^n A_1(t)e^{i(k_1 x + \omega_u(k_1)t)} + c.c., \\
v(x,t) &= B_0(t) + \varepsilon^n B_1(t)e^{i(k_1 x + \omega_v(k_1)t)} + c.c.,
\end{aligned}$$

for a $n > 0$. The linear terms cancel and so we find in lowest order

$$\begin{aligned}
i\partial_t A_0 &= -\gamma A_0 B_0, \\
i\partial_t A_1 &= -\gamma(A_1 B_0 + A_0 B_1), \\
\varepsilon^2 \partial_t^2 B_0 &= 0, \\
2\varepsilon^2 i\omega_v(k_1)\partial_t B_1 &= -k_1^2(A_1 \overline{A_0}).
\end{aligned}$$

For simplicity we restrict ourselves to the case $B_0|_{t=0} = 0$. Then we find $A_0 = const.$ and $B_0 = 0$. A necessary condition that the NLS equation makes correct predictions is that B_1 stays $\mathcal{O}(1)$-bounded on an $\mathcal{O}(1)$-time scale. We compute

$$i\partial_t^2 A_1 = -\gamma A_0 \partial_t B_1 = -\gamma A_0 \frac{1}{2\varepsilon^2 i\omega_v(k_1)}(-k_1^2(A_1\overline{A_0})),$$

i.e.

$$\partial_t^2 A_1 = \Gamma A_1 \qquad \text{and} \qquad \partial_t^2 B_1 = \Gamma B_1,$$

where

$$\Gamma = -\gamma \frac{|A_0|^2 k_1^2}{2\varepsilon^2 \omega_v(k_1)}.$$

We have that Γ and γ have opposite signs, due to $\omega_v(k_1) > 0$, and so for $\gamma = -1$ exponential growth occurs. Since $k_1 = \mathcal{O}(1/\varepsilon)$, $A_0 = \mathcal{O}(1)$, and $\omega_v(k_1) = \mathcal{O}(1/\varepsilon^2)$ we have $\Gamma = \mathcal{O}(1/\varepsilon^2)$. Hence in case $\gamma = -1$ the resonant modes grow as $\mathcal{O}(e^{\varepsilon^{-1}t})$ which is not $\mathcal{O}(1)$ bounded on the natural $\mathcal{O}(1)$-time scale of (38). Thus, in case of $\gamma = -1$ the associated NLS equation makes wrong predictions about the dynamics of the Zakharov system, cf Section 4.4.

4.4 Numerical illustrations

By numerical experiments we illustrate the differences between the approximation properties occuring for the semilinear system from Section 3.2 and the ones for the quasilinear system from Section 4.2.

4.4.1 The KGZ-KG limit

We choose $2\pi/k_1$-spatially periodic boundary conditions with k_1 the resonant wave number of order $\mathcal{O}(\varepsilon)$. Hence, the spatial domain grows as $\varepsilon \to 0$. Moreover, due to the fact that we consider a bounded interval with periodic boundary conditions the limit system has to be modified. Instead of choosing $v(x,t) = -u(x,t)^2$ we consider

$$v(x,t) = -u(x,t)^2 + \beta(t) \tag{51}$$

with $\beta(t)$ chosen in such a way that $\int_0^{2\pi/k_1} \partial_t^2 v(x,t)\,\mathrm{d}x = 0$. In detail we choose

$$\beta(t) = \frac{k_1}{2\pi} \int_0^{2\pi/k_1} (-u(x,0)^2 + u(x,t)^2)\,\mathrm{d}x \tag{52}$$

and u to satisfy the modified KG equation

$$\partial_t^2 u - \Delta u + u = -\beta u + \gamma |u|^2 u. \tag{53}$$

We refer to [DSS16] for details. As predicted by our analysis, the numerical experiments show a comparable error for $\gamma = 1$ and $\gamma = -1$. See Figure 2.

4.4.2 The Zakharov-NLS limit

We choose $2\pi/k_1$-spatially periodic boundary conditions with $k_1 = 1/\varepsilon$ the resonant wave number. Hence, the spatial domain shrinks as $\varepsilon \to 0$. As predicted by our analysis, the numerical experiments show that the NLS equation provides a good approximation for $\gamma = 1$. However, in case $\gamma = -1$ the Zakharov system (38) behaves much faster in a different way as predicted by the NLS approximation. See Figure 3.

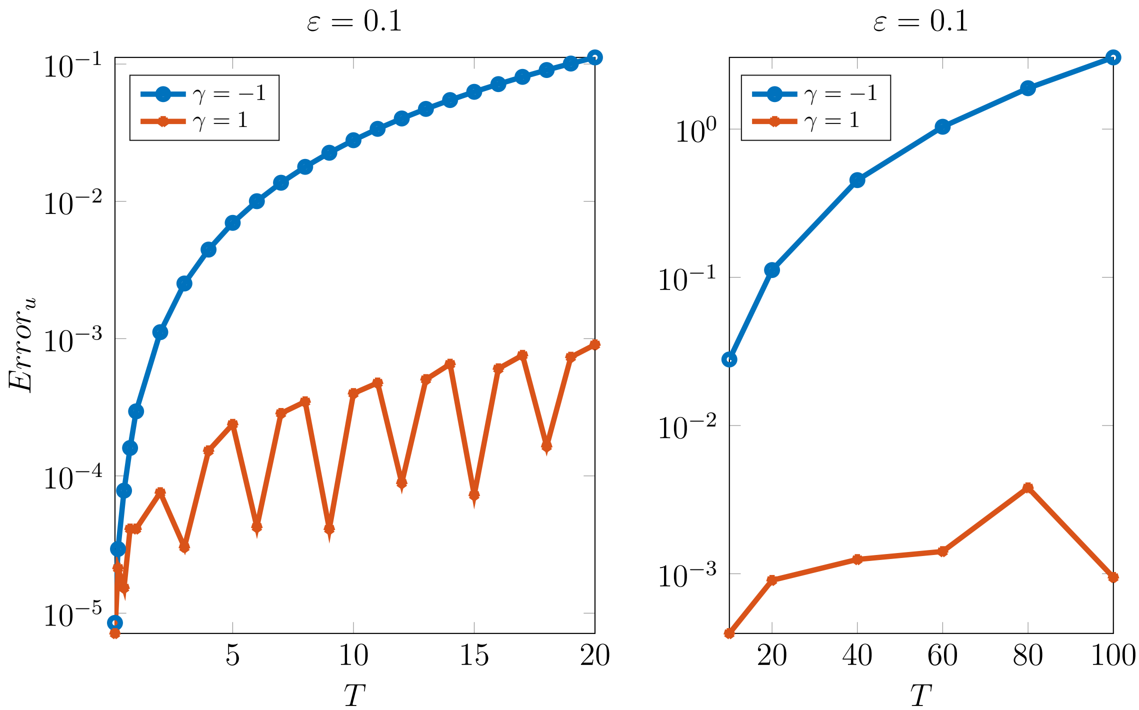

Figure 2: The L^2-norm of the approximation error for u in case $\gamma = 1$ (red lower curve) and in case $\gamma = -1$ (blue upper curve). The computations for the initial conditions $u|_{t=0} = 1 + \varepsilon^2 \cos(k_1 x)$, $\partial_t u|_{t=0} = 0$, $v|_{t=0} = -u|_{t=0}^2$, and $\partial_t v|_{t=0} = 0$ confirm our analysis. The error in case $\gamma = -1$ grows faster than in case $\gamma = 1$, but stays bounded and the unstable resonance does not destroy the approximation property. The computations for $\varepsilon = 0.1$ were made with a split step method for 1024 Fourier modes. The time step was $\tau_{ref} = 4.77 \times 10^{-7}$ for the KGZ system and $\tau_{lim} = 4.88 \times 10^{-4}$ for the KG equation. The lower curve shows the oscillatory character for $\gamma = 1$. The right panel shows that beyond the natural approximation interval the error in case $\gamma = -1$ finally will be of the same size as the solution.

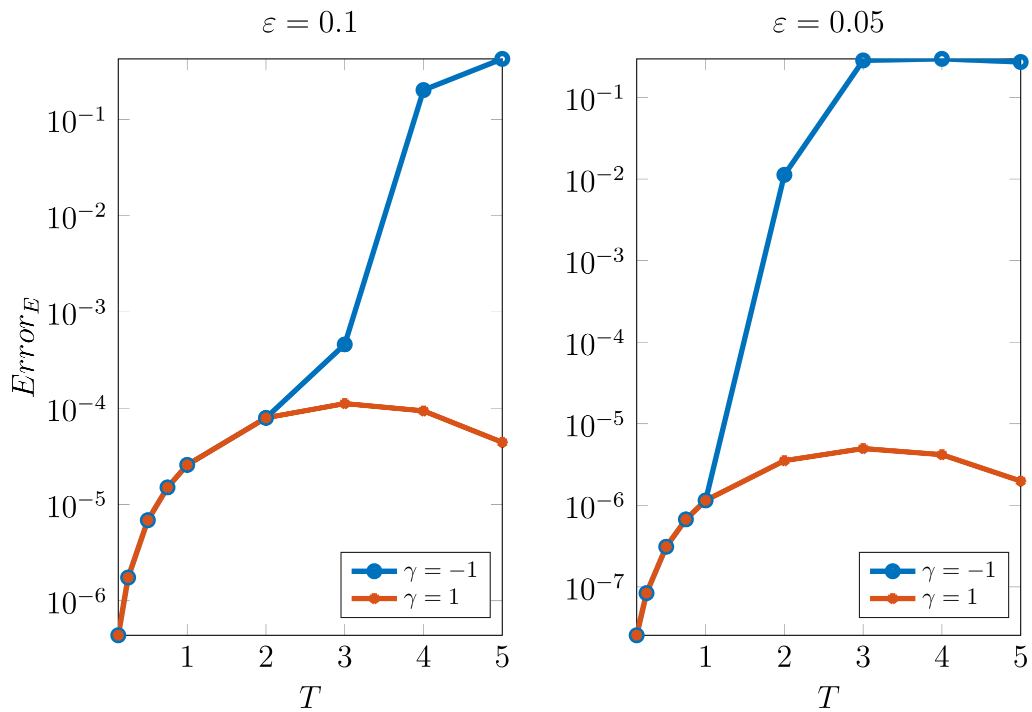

Figure 3: The L^2-norm of the approximation error for u in case $\gamma = 1$ (red lower curve) and in case $\gamma = -1$ (blue upper curve). The computations for the initial conditions $E|_{t=0} = 1$, $n|_{t=0} = -1 + \varepsilon^2 \cos(k_1 x)$, and $\partial_t n|_{t=0} = 0$ confirm our analysis. The error in case $\gamma = -1$ grows faster than in case $\gamma = 1$. The unstable resonance destroys the approximation property since the error in case $\gamma = -1$ (blue upper curve) grows faster for smaller ε. The left panel shows the case $\varepsilon = 0.1$ and the right panel the case $\varepsilon = 0.05$. The computations were made with a split step method for 1024 Fourier modes. The time step was $\tau_{ref} = 4.77 \times 10^{-7}$ for the Zakharov system and $\tau_{lim} = 4.88 \times 10^{-4}$ for the NLS equation.

A Local existence and uniqueness of the limit systems

For completeness we add the following local existence and uniqueness result for (16).

Theorem A.1. *For all $C_0 > 0$ there exists a $T_0 > 0$ such that for all $U_0 \in X_U$ with $\|U_0\|_{X_U} \leq C_0$ there exists a unique mild solution $\psi_U \in C([0,T_0], X_U)$ of* (16) *with $\psi_U|_{t=0} = U_0$.*

Proof. We consider the variation of constant formula

$$\psi_U(t) = e^{\Lambda_U t} U_0 - \int_0^t e^{\Lambda_U (t-\tau)} B_U(\psi_U, \Lambda_V^{-1} B_V(\psi_U, \psi_U))(\tau) d\tau \tag{54}$$

associated to (16). Due to the assumptions **(S1)**, **(B1)**, and **(I)**, for all fixed $C_1 > 0$ the right-hand side of (54) is a contraction in a ball

$$\{\psi_U \in C([0,T_0], X_U) : \|\psi_U - e^{\Lambda_U t} U_0\|_{X_U} \leq C_1\}$$

for a $T_0 > 0$ sufficiently small. By the contraction mapping principle there is a unique fixed point of the right-hand side of (54) which by definition is the unique mild solution $\psi_U \in C([0,T_0], X_U)$ of (16) with $\psi_U|_{t=0} = U_0$. □

Remark A.2. In one of the previous applications, for estimating the residual terms, we used solutions $\psi_U \in C([0,T_0], X_\psi)$ with $X_\psi \subset X_U$ another suitably chosen Banach space. The local existence and uniqueness proof in X_ψ will work exactly the same since X_U and X_ψ have been chosen as Sobolev spaces H^{s_U} and H^{s_ψ} with $s_U < s_\psi$.

References

[AA88] Hélène Added and Stéphane Added. Equations of Langmuir turbulence and nonlinear Schrödinger equation: smoothness and approximation. *J. Funct. Anal.*, 79(1):183–210, 1988.

[BBC96] Luc Bergé, Brigitte Bidégaray, and Thierry Colin. A perturbative analysis of the time-envelope approximation in strong Langmuir turbulence. *Phys. D*, 95(3-4):351–379, 1996.

[BH17] Ioan Bejenaru and Sebastian Herr. On global well-posedness and scattering for the massive Dirac-Klein-Gordon system. *J. Eur. Math. Soc. (JEMS)*, 19(8):2445–2467, 2017.

[BNAS00] P. Bechouche, J. Nieto, E. Ruiz Arriola, and J. Soler. On the time evolution of the mean-field polaron. *Journal of Mathematical Physics*, 41(7):4293–4312, 2000.

[CEGT04] T. Colin, G. Ebrard, G. Gallice, and B. Texier. Justification of the Zakharov model from Klein-Gordon–wave systems. *Comm. Partial Differential Equations*, 29(9-10):1365–1401, 2004.

[DLP+11] Willy Dörfler, Armin Lechleiter, Michael Plum, Guido Schneider, and Christian Wieners. *Photonic crystals. Mathematical analysis and numerical approximation.* Berlin: Springer, 2011.

[DSS16] Markus Daub, Guido Schneider, and Katharina Schratz. From the Klein-Gordon-Zakharov system to the Klein-Gordon equation. *Math. Methods Appl. Sci.*, 39(18):5371–5380, 2016.

[Fen79] Neil Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differ. Equations*, 31:53–98, 1979.

[FT75] Isamu Fukuda and Masayoshi Tsutsumi. On the Yukawa-coupled Klein-Gordon-Schrödinger equations in three space dimensions. *Proc. Japan Acad.*, 51(6):402–405, 1975.

[GSS17] Marcel Griesemer, Jochen Schmid, and Guido Schneider. On the dynamics of the mean-field polaron in the high-frequency limit. *Lett. Math. Phys.*, 107(10):1809–1821, 2017.

[JK94] C.K.R.T. Jones and N. Kopell. Tracking invariant manifolds with differential forms in singularly perturbed systems. *J. Differ. Equations*, 108(1):64–88, 1994.

[Kue15] Christian Kuehn. *Multiple time scale dynamics.* Cham: Springer, 2015.

[MN05] Nader Masmoudi and Kenji Nakanishi. From the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation. *J. Hyperbolic Differ. Equ.*, 2(4):975–1008, 2005.

[OT92] Tohru Ozawa and Yoshio Tsutsumi. Existence and smoothing effect of solutions for the Zakharov equations. *Publ. Res. Inst. Math. Sci.*, 28(3):329–361, 1992.

[Sch16] Guido Schneider. Validity and non-validity of the nonlinear Schrödinger equation as a model for water waves. In *Lectures on the theory of water waves. Papers from the talks given at the Isaac Newton Institute for Mathematical Sciences, Cambridge, UK, July – August, 2014*, pages 121–139. Cambridge: Cambridge University Press, 2016.

[SVM07] J.A. Sanders, F. Verhulst, and James Murdock. *Averaging methods in nonlinear dynamical systems. 2nd ed.* New York, NY: Springer, 2nd ed. edition, 2007.

[SW86] Steven H. Schochet and Michael I. Weinstein. The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence. *Comm. Math. Phys.*, 106(4):569–580, 1986.

[SZ13] Guido Schneider and Dominik Zimmermann. Justification of the Ginzburg-Landau approximation for an instability as it appears for Marangoni convection. *Math. Methods Appl. Sci.*, 36(9):1003–1013, 2013.

[Tex07] Benjamin Texier. Derivation of the Zakharov equations. *Arch. Ration. Mech. Anal.*, 184(1):121–183, 2007.