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CRC Preprint 2017/33, December 2017

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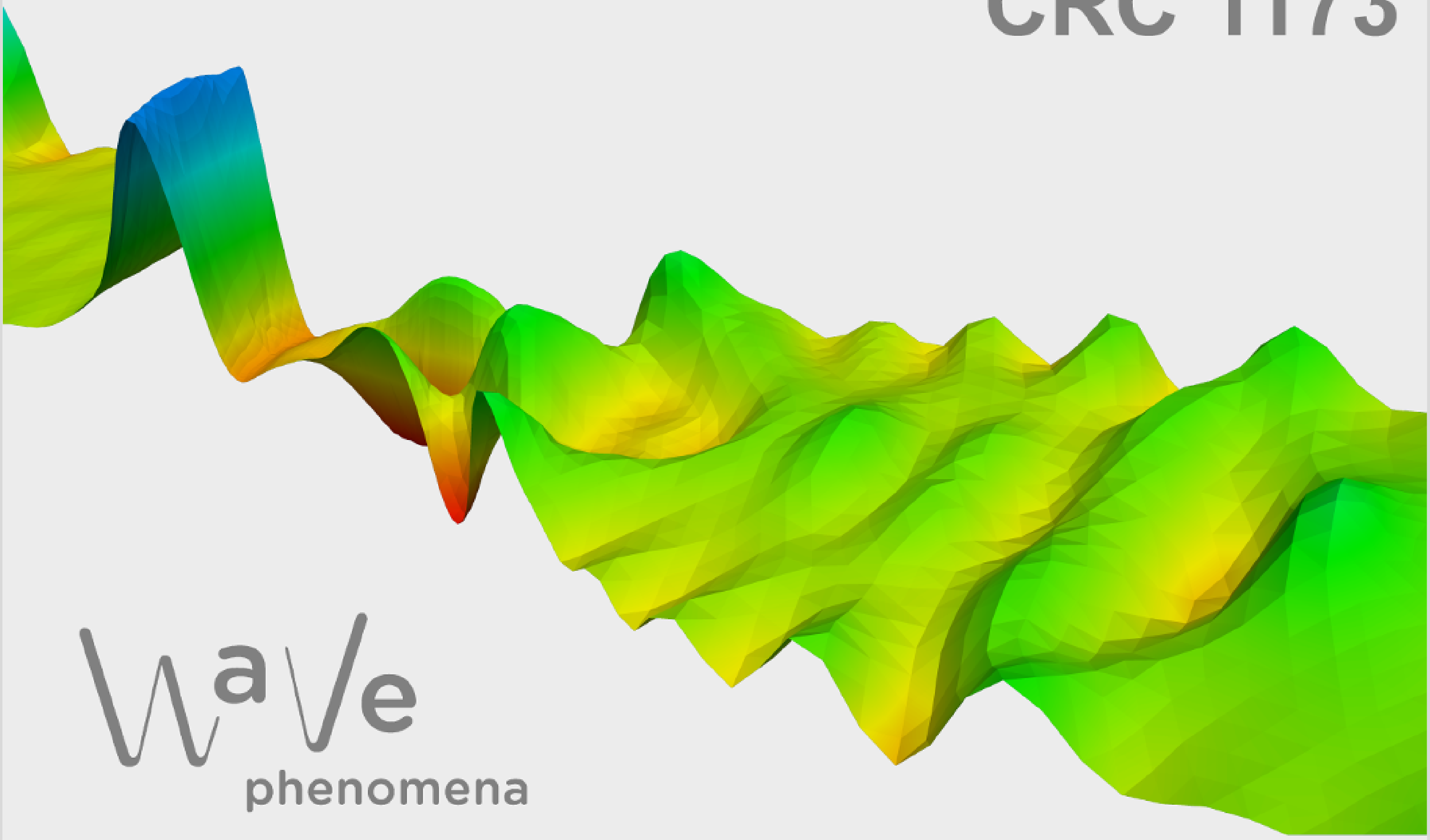
CRC 1173



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DFG

and

Klaus Tschira Stiftung gemeinnützige GmbH

KTS

ISSN 2365-662X

Exponential adiabatic midpoint rule for the dispersion-managed nonlinear Schrödinger equation

Tobias Jahnke* and Marcel Mikl*

December 13, 2017

Abstract

Modeling long-haul data transmission through dispersion-managed optical fiber cables leads to a nonlinear Schrödinger equation where the linear part is multiplied by a large, discontinuous and rapidly changing coefficient function. Typical solutions oscillate with high frequency and have low regularity in time, such that traditional numerical methods suffer from severe step-size restrictions and typically converge only with low order. We construct and analyze a norm-conserving, uniformly convergent time-integrator called the adiabatic exponential midpoint rule by extending techniques developed in [26]. This method is several orders of magnitude more accurate than standard schemes for a relevant set of parameters. In particular, we prove that the accuracy of the method improves considerably if the step-size is chosen in a special way.

Mathematics Subject Classification (2010): 65M12, 65M15, 65M70, 65Z05, 35Q55

Keywords: dispersion management, nonlinear Schrödinger equation, highly oscillatory problem, discontinuous coefficients, adiabatic integrator, error bounds, limit dynamics, norm conservation

Acknowledgment. We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173. The second author also acknowledges the support of the Klaus Tschira Stiftung.

1 Introduction

We consider the *dispersion-managed nonlinear Schrödinger equation* (DMNLS)

$$\begin{aligned} \partial_t u(t,x) &= \tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right)\partial_x^2 u(t,x) + \mathrm{i}\,|u(t,x)|^2\,u(t,x)\,, \qquad x\in\mathbb{T}, t\in(0,T] \\ u(0,x) &= u_0(x)\,, \end{aligned} \tag{1}$$

on the one-dimensional torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ and on a time-interval of length $T>0$. This equation provides a model for the propagation of light pulses through optical fibers with strong dispersion management; cf. [1,5,32]. The DMNLS (1)

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differs from the “classical” semilinear Schrödinger equation with cubic nonlinearity in two characteristic features. First, Eq. (1) involves a small positive parameter $0 < \varepsilon \ll T$. Second, the differential operator is multiplied with the coefficient function $\gamma \colon \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(t) = \chi(t) + \varepsilon\alpha\,, \tag{2}$$

where $\alpha > 0$ is the mean dispersion, and where

$$\chi(t) = \begin{cases} -\delta & \text{if} \quad t \in [n, n+1) \quad \text{for even } n \in \mathbb{N}, \\ \delta & \text{if} \quad t \in [n, n+1) \quad \text{for odd } n \in \mathbb{N} \end{cases} \tag{3}$$

is called the dispersion map. We assume that $\delta > \varepsilon\alpha > 0$ such that $\gamma(t) \neq 0$ for every $t \in [0, T]$.

Approximating the solution of (1) is a considerable challenge. Typical solutions of (1) exhibit a highly oscillatory behavior due to the factor i/ε in the right-hand side. Hence, applying traditional numerical time-integrators (e.g. Runge-Kutta or multistep methods) is inefficient, because these schemes yield a very poor accuracy unless a huge number of time-steps with a very small step-size $\tau \ll \varepsilon$ is made. Additional difficulties are caused by the piecewise constant coefficient function γ inducing discontinuities in the time-derivative $t \mapsto \partial_t u(t, \cdot)$. Hence, higher order time derivatives do not exist which contradicts key assumptions required to prove higher order convergence of many time integrators. Moreover, the nonlinear term $\mathrm{i}\,|u(t,x)|^2\, u(t,x)$ makes implicit methods prohibitively costly and complicates the construction of novel methods.

There is a rich literature on numerical methods for differential equations with oscillatory solutions, and reviews can be found, e.g., in [12,13,19,21,31,33]. Time-integration of oscillatory partial differential equations has been analyzed, e.g., in [2–4,8–11,15–18,20,27] and references therein. However, none of these works considers a nonlinear, singularly perturbed partial differential equation with discontinuous and oscillatory coefficients like the DMNLS.

A tailor-made time-integrator for the DMNLS called the *adiabatic midpoint rule* has been constructed and analyzed in [26]. This method is based on a transformation of (1) to a more suitable equivalent evolution equation – the *transformed dispersion-managed nonlinear Schrödinger equation* (tDMNLS) – and on the fact that certain integrals over highly oscillatory exponential functions in the tDMNLS can be computed analytically. It was proved that the adiabatic midpoint rule converges with order one in time with an error constant that does not depend on ε and without any ε-induced step-size restriction. Moreover, it was shown that surprisingly the accuracy improves for special choices of the step-size τ: The error reduces to $\mathcal{O}(\varepsilon^2 + \tau^2)$ if $\tau = \varepsilon k$ with $k \in \mathbb{N}$, and to $\mathcal{O}(\varepsilon\tau)$ if $\tau = \varepsilon/k$, respectively. In both cases the error constant remains independent of ε, which is typically not the case when traditional methods are used. These features make the adiabatic midpoint rule attractive for solving the tDMNLS. The disadvantage of this method is, however, that numerical approximations at different times have a slightly different L_2 norm in general. This behavior is somewhat unphysical, because it can easily be shown that

$$\|u(t, \cdot)\|_{L^2(\mathbb{T})} = \|u(0, \cdot)\|_{L^2(\mathbb{T})} \quad \text{for all} \quad t \geq 0$$

for every solution of the DMNLS. For this reason, we propose an improved time-integrator for the DMNLS – the *adiabatic exponential midpoint rule*. We prove that this new method has the same favorable error behavior for special step-sizes as the adiabatic midpoint rule, and that it does preserve the norm of the numerical solution in contrast to its non-exponential counterpart. Moreover, we show in Section 5.2 that the adiabatic exponential midpoint rule reproduces the *exact* solution of the tDMNLS in certain special but nontrivial cases. It is expected (and confirmed by our numerical experiments in Section 5.4) that these properties of the new integrator also improve the accuracy of the approximation.

The construction of the new method is again based on the tDMNLS introduced in [26]. The difference is that now the transformed problem is linearized in each time-step by freezing some of the degrees of freedom. This leads to a linear evolution equation with time-dependent coefficients, which can be approximated by the exponential of the first term of the Magnus expansion [6,22,23]. As in [26] we encounter integrals over oscillatory functions which can be computed analytically. The construction of the new integrator is in some sense an extension of the approach from [26]. However, the error analysis is considerably more complicated due to the exponential structure of the new method.

In Section 2, we review the derivation of the tDMNLS. After compiling a suitable analytical framework in Section 3, we discuss the asymptotic limit of the tDMNLS for $\varepsilon \to 0$ in Section 4. The content of Sections 2, 4 and (partially) of Section 3 can already be found in [26], but we briefly revisit these important ingredients in order to keep the presentation self-contained. In Section 5, the adiabatic midpoint rule is constructed and we prove the qualitative properties of the method. Furthermore, we state the main results of our error analysis and illustrate the method by numerical examples. The proofs of the error bounds are given in Section 6.

Throughout the paper, we denote by $C > 0$ and $C(\cdot) > 0$ universal constants, possibly taking different values at various appearances. The notation $C(\cdot)$ means that the constant depends only on the values specified in the brackets.

2 Transformation of the problem

If $u \in C([0,T], H^s(\mathbb{T}))$ with $s \geq 2$ is a solution of the DMNLS, then

$$\|\partial_t u(t,\cdot)\|_{L_2(\mathbb{T})} \sim \tfrac{1}{\varepsilon}\|u(t,\cdot)\|_{H^2(\mathbb{T})},$$

for every $t \in (0,T)$ which implicates that the solution oscillates faster and faster when $\varepsilon \to 0$. This unpleasant scaling was the main motivation in [26] to consider the transformed dispersion-managed nonlinear Schrödinger equation (tDMNLS)

$$y_m'(t) = \mathrm{i}\sum_{I_m} y_j(t)\overline{y}_k(t)y_l(t)\exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right), \qquad m \in \mathbb{Z}. \tag{4}$$

Here and below we use the abbreviations

$$\begin{aligned} \omega_{[jklm]} &= j^2 - k^2 + l^2 - m^2, \\ I_m &= \left\{(j,k,l) \in \mathbb{Z}^3 : j-k+l = m\right\}, \end{aligned} \tag{5}$$

and the short-hand notation

$$\sum_{I_m} a_j b_k c_l = \sum_{\substack{(j,k,l)\in\mathbb{Z}^3 \\ j-k+l=m}} a_j b_k c_l\,.$$

The tDMNLS is obtained by substituting the Fourier series

$$u(t,x) = \sum_{m\in\mathbb{Z}} c_m(t)e^{\mathrm{i}mx}, \qquad c_m(t) = \int_{\mathbb{T}} u(t,x)e^{-\mathrm{i}mx}\,\mathrm{d}x \tag{6}$$

into the DMNLS and introducing the new variables

$$y_m(t) := \exp\left(\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) c_m(t)\,, \qquad m \in \mathbb{Z}\,, \tag{7}$$

where $\widehat{\phi}$ is defined by

$$\widehat{\phi}(z) := \int_0^z \gamma(\sigma)\,\mathrm{d}\sigma = \phi(z) + \alpha\varepsilon z \quad \text{with} \quad \phi(z) := \int_0^z \chi(\sigma)\,\mathrm{d}\sigma\,. \tag{8}$$

The transformation (7) is motivated by the fact that the solution of the linear part

$$\begin{aligned} \partial_t w(t,x) &= \tfrac{\mathrm{i}}{\varepsilon}\gamma\left(\tfrac{t}{\varepsilon}\right)\partial_x^2 w(t,x)\,, \qquad x\in\mathbb{T}, t\in(0,T] \\ w(0,x) &= w_0(x)\,, \end{aligned} \tag{9}$$

is simply obtained by keeping

$$y_m(t) = y_m(0) = c_m(0) = \int_{\mathbb{T}} w_0(x)e^{-\mathrm{i}mx}\,\mathrm{d}x$$

constant in time for every $m\in\mathbb{Z}$, setting $c_m(t) = \exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) y_m(t)$ and then applying the inverse Fourier transform to $(c_m)_m$. Hence, solving the linear part of the DMNLS is trivial in the new variables.

Nevertheless, the transformation (7) does not cure the highly oscillatory behavior of solutions completely. In fact, the rapidly changing exponential terms in (4) still cause fast oscillations of $y(t)$. However, an important advantage of the tDMNLS in contrast to the DMNLS is that the right-hand side of (4) is uniformly bounded in the limit $\varepsilon \to 0$ if $y(t) = (y_m(t))_{m\in\mathbb{Z}} \in \ell^1$, cf. Lemma 3.1 (i) below. A second advantage of the tDMNLS concerns the regularity in time. Since (4) involves $\widehat{\phi}$ instead of γ, the right-hand side is now (weakly) differentiable with respect to t, whereas the right-hand side of (1) is discontinuous. These benefits suggest to approximate the tDMNLS numerically instead of the DMNLS as in [26]. However, both evolution equations are equivalent, and solutions of (4) can be obtained from y via the inverse transformation

$$u(t,x) = \sum_{m\in\mathbb{Z}} y_m(t)\exp\left(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right) + \mathrm{i}mx\right)\,. \tag{10}$$

As already pointed out in [26], the drawback of reformulating the DMNLS in terms of the tDMNLS is the occurring multiple sum in (4). Before the transformation evaluations of the nonlinear part of the DMNLS could be implemented in terms of point-wise multiplications, but now the nested summation makes evaluations more costly from a computational point of view. We will address this aspect further in the numerical experiments in Section 5.4.

Before closing this section, we simplify notation: For $\mu = (\mu_m)_{m\in\mathbb{Z}}$, $z = (z_m)_{m\in\mathbb{Z}}$ and $t \in [0,T]$ we denote by $A(t,\mu)z$ the sequence with entries

$$\big(A(t,\mu)z\big)_m = \mathrm{i}\sum_{I_m} \mu_j \bar{\mu}_k z_l \exp\Big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\Big) , \qquad m \in \mathbb{Z}\,. \tag{11}$$

With this notation the tDMNLS reads

$$y'(t) = A\big(t, y(t)\big)y(t)\,. \tag{12}$$

3 Analytic setting

First, we point out that the global well-posedness of the DMNLS (1) in the Sobolev space $H^s(\mathbb{T})$ for arbitrary $s \in \mathbb{N}_0$ follows immediately from Theorem 2.1 in [7] due to the fact that $\gamma(t/\varepsilon)$ is constant on every interval $[n\varepsilon,(n+1)\varepsilon)$. In the following, we provide a suitable analytic setting for investigating the tDMNLS and the related numerical methods.

Every solution $u(t,\cdot) \in H^s(\mathbb{T})$ of the DMNLS is related to a sequence $y(t) = (y_m(t))_{m\in\mathbb{Z}}$ given by (6) and (7). According to (10) we have

$$\partial_x^k u(t,x) = \sum_{m\in\mathbb{Z}} (\mathrm{i}m)^k y_m(t) \exp\Big(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right) + \mathrm{i}mx\Big) ,$$

and hence

$$\|u(t,\cdot)\|^2_{H^s(\mathbb{T})} = \sum_{k=0}^{s} \left\|\partial_x^k u(t,\cdot)\right\|^2_{L_2(\mathbb{T})} = 2\pi \sum_{k=0}^{s}\sum_{m\in\mathbb{Z}} |m|^{2k}|y_m|^2\,. \tag{13}$$

We define the inner product

$$\langle w,z\rangle_{\ell^2_s} := \sum_{m\in\mathbb{Z}} |m|_+^{2s} w_m \overline{z}_m, \qquad |m|_+ := \max\{1,|m|\}$$

($\overline{z}_m$ is the complex conjugate of z_m) and consider the tDMNLS in the Hilbert spaces

$$\ell^2_s := \Big\{(z_m)_{m\in\mathbb{Z}} \text{ in } \mathbb{C} \;\mid\; \|z\|_{\ell^2_s} < \infty\Big\} ,$$

with the induced norm $\|z\|_{\ell^2_s} := \sqrt{\langle z,z\rangle_{\ell^2_s}}$; cf. [14]. Then, (6), (7) and (10) yield an isomorphism $\ell^2_s \cong H^s(\mathbb{T})$ with norm equivalence

$$\sqrt{2\pi}\|y(t)\|_{\ell^2_s} \le \|u(t,\cdot)\|_{H^s(\mathbb{T})} \le \sqrt{2\pi}(s+1)\|y(t)\|_{\ell^2_s}$$

for every $s \in \mathbb{N}$.

It was pointed out in [14] that treating convolution-type sums originating from the Fourier transform of a cubic nonlinearity as in (4) is much more convenient in the sequence spaces ℓ^1_s

$$\ell^1_s = \Big\{(z_m)_{m\in\mathbb{Z}} \text{ in } \mathbb{C} \;\mid\; \|z\|_{\ell^1_s} < \infty\Big\} , \tag{14}$$
$$\|z\|_{\ell^1_s} = \sum_{m\in\mathbb{Z}} |m|_+^s\,|z_m| ,$$

than in ℓ^2_s. These spaces are related to ℓ^2_s by the following embedding: If $r, s \in \mathbb{N}$ with $r > s$, then

$$\ell^2_r \hookrightarrow \ell^1_s \hookrightarrow \ell^2_s\,, \quad \text{i.e.} \quad \|z\|_{\ell^2_s} \le \|z\|_{\ell^1_s} \le C\,\|z\|_{\ell^2_r}\,, \tag{15}$$

see [14, Proposition III.2.]. This allows us to prove error bounds in ℓ^1_0 in order to obtain error bounds in ℓ^2_0. Henceforth, we write ℓ^p instead of ℓ^p_0 for $p \in \{1, 2\}$.

The benefit of the space ℓ^1 is reflected in the following principle: If $a, b, c \in \ell^1$ and $d = \big(d_m\big)_{m\in\mathbb{Z}}$ is given by

$$d_m = \sum_{I_m} a_j b_k c_l\,,$$

then $d \in \ell^1$ and

$$\begin{aligned}\|d\|_{\ell^1} &= \sum_{m\in\mathbb{Z}} \Big| \sum_{I_m} a_j b_k c_l \Big| \\ &\le \Big(\sum_{j\in\mathbb{Z}} |a_j| \Big) \Big(\sum_{k\in\mathbb{Z}} |b_k| \Big) \Big(\sum_{l\in\mathbb{Z}} |c_l| \Big) = \|a\|_{\ell^1}\,\|b\|_{\ell^1}\,\|c\|_{\ell^1}\,. \end{aligned} \tag{16}$$

This general principle will often be used in the proofs below.

Finally, for given $t \in [0, T]$ and μ we consider the linear operator

$$A(t, \mu)\colon z \mapsto A(t, \mu)z$$

with $A(t, \mu)z$ defined by (11). For the construction and analysis of our method the following properties of this operator are crucial.

Lemma 3.1. *If $t \in [0, T]$ and $\mu \in \ell^1$ with $M := \|\mu\|_{\ell^1}$, then the following assertions hold.*

(i) The operator $A(t, \mu)\colon \ell^1 \to \ell^1$ is bounded and

$$\|A(t, \mu)z\|_{\ell^1} \le C(M)\,\|z\|_{\ell^1} \quad \textit{for all } z \in \ell^1\,.$$

(ii) The operator $A(t, \mu)\colon \ell^2 \to \ell^2$ is bounded and

$$\|A(t, \mu)z\|_{\ell^2} \le C(M)\|z\|_{\ell^2} \quad \textit{for all } z \in \ell^2\,.$$

(iii) The operator $A(t, \mu)\colon \ell^2 \to \ell^2$ is skew-adjoint.

Remark.

- With the abbreviation

$$M^y_s := \max_{t\in[0,T]} \|y(t)\|_{\ell^1_s}\,, \tag{17}$$

part (i) yields

$$\|y'(t)\|_{\ell^1} \le C(M^y_0) \tag{18}$$

for the solution y of the tDMNLS (12). We will frequently apply (18) in the proofs in Section 6.

- In (ii) and (iii) we require $\mu \in \ell^1$ although the operator acts on ℓ^2.

Proof. Assertion (i) follows from (11), (16), and the fact that $|\exp(-\mathrm{i}z)| = 1$ for $z \in \mathbb{R}$. In order to prove the Assertions (ii) and (iii), we define

$$a_{m,l}(t,\mu) := \mathrm{i} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=m-l}} \mu_j \overline{\mu}_k \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) \quad \text{for} \quad m,l \in \mathbb{Z} \tag{19}$$

which allows us to write

$$\big(A(t,\mu)z\big)_m = \sum_{l\in\mathbb{Z}} a_{m,l}(t,\mu) z_l \,. \tag{20}$$

Because we have

$$\sum_{l\in\mathbb{Z}} |a_{m,l}(t,\mu)| \le \sum_{l\in\mathbb{Z}} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=m-l}} |\mu_j|\cdot|\mu_k| = \|\mu\|_{\ell^1}^2 = M^2\,, \qquad m \in \mathbb{Z}$$

and

$$\sum_{m\in\mathbb{Z}} |a_{m,l}(t,\mu)| \le \sum_{m\in\mathbb{Z}} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=m-l}} |\mu_j|\cdot|\mu_k| = \|\mu\|_{\ell^1}^2 = M^2\,, \qquad l \in \mathbb{Z}$$

Assertion (ii) follows from the Cauchy-Schwarz inequality via

$$\begin{aligned}
\|A(t,\mu)z\|_{\ell^2}^2 &\le \sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} \sqrt{|a_{m,l}(t,\mu)|}\sqrt{|a_{m,l}(t,\mu)|}\,|z_l| \right)^2 \\
&\le \sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} |a_{m,l}(t,\mu)| \right) \left(\sum_{l\in\mathbb{Z}} |a_{m,l}(t,\mu)|\,|z_l|^2 \right) \\
&\le M^2 \sum_{l\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |a_{m,l}(t,\mu)|\,|z_l|^2 \\
&\le M^4 \|z\|_{\ell^2}^2 \,.
\end{aligned}$$

In order to prove that $A(t,\mu)\colon \ell^2 \to \ell^2$ is skew-adjoint, we note that

$$\begin{aligned}
-\overline{a}_{l,m}(t,\mu) &= \mathrm{i} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ j-k=l-m}} \overline{\mu}_j \mu_k \exp\left(\mathrm{i}(j^2-k^2-m^2+l^2)\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) \\
&= \mathrm{i} \sum_{\substack{(j,k)\in\mathbb{Z}^2 \\ k-j=m-l}} \overline{\mu}_j \mu_k \exp\left(-\mathrm{i}(k^2-j^2+m^2-l^2)\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right)\,,
\end{aligned}$$

and hence interchanging the summation indices j and k shows that

$$-\overline{a}_{l,m}(t,\mu) = a_{m,l}(t,\mu)\,. \tag{21}$$

Now the assertion (iii) follows from

$$\begin{aligned}
\langle A(t,\mu)z, x\rangle &= \sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} a_{m,l}(t,\mu) z_l \overline{x}_m \\
&= -\sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \overline{a}_{l,m}(t,\mu) z_l \overline{x}_m = -\langle z, A(t,\mu)x\rangle\,. \qquad \blacksquare
\end{aligned}$$

4 The limit system

The highly oscillatory behavior of solutions of the tDMNLS (12) originates from the exponentials of the form

$$\exp\left(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{t}{\varepsilon}\right)\right) = \exp(-\mathrm{i}\omega\alpha t)\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{t}{\varepsilon}\right)\right), \qquad \omega \in \mathbb{Z} \tag{22}$$

in (11). Averaging the fast part over one period yields

$$\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{t}{\varepsilon}\right)\right) \approx \frac{1}{2\varepsilon}\int_0^{2\varepsilon} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s = \int_0^1 \exp(\mathrm{i}\omega\delta\xi)\,\mathrm{d}\xi\,. \tag{23}$$

After substituting this approximation into (11), we obtain

$$\left(A^{\lim}(t,\mu)z\right)_m \coloneqq \mathrm{i}\sum_{I_m} \mu_j\bar{\mu}_k z_l \exp(-\mathrm{i}\omega_{[jklm]}\alpha t)\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi \tag{24}$$

for two sequences $\mu = (\mu_m)_{m\in\mathbb{Z}}$ and $z = (z_m)_{m\in\mathbb{Z}}$. Theorem 4.1 below states that the corresponding evolution equation

$$v'(t) = A^{\lim}\big(t, v(t)\big)v(t) \tag{25}$$

is the *limit system* of (12) in the sense that solutions of the tDMNLS converge to solutions of (25) for $\varepsilon \to 0$. Note that the smallness parameter ε does not appear in (24).

Assumption 1. *We suppose that for $s = 0, 1, 2, 3$ the limit system* (25) *with initial value $v_0 \in \ell_s^2$ has a unique solution $v \in C\big([0,T], \ell_s^2\big)$.*

Henceforth, we use the abbreviations

$$M_s^v \coloneqq \max_{t\in[0,T]} \|v(t)\|_{\ell_s^1} \quad \text{and} \quad M_s \coloneqq \max\{M_s^y, M_s^v\} \tag{26}$$

where M_s^y is given in (17).

Theorem 4.1 (cf. Theorem 1 in [26], see also [30,34])**.** *Let y and v be solutions of the tDMNLS* (12) *and the limit system* (25)*, respectively. Under Assumption 1 the following estimates hold.*

(i) If $y(0) = v(0) \in \ell_1^2$, then

$$\|y(t) - v(t)\|_{\ell^1} \le \varepsilon C(t,\alpha,\delta,M_0)\,. \quad t \in [0,T]\,.$$

(ii) If $y(0) = v(0) \in \ell_3^2$ and $t_k = \varepsilon k \in [0,T]$ for some $k \in \mathbb{N}$, then

$$\|y(t_k) - v(t_k)\|_{\ell^1} \le \frac{\varepsilon^2}{\delta} C(t_k,\alpha,M_2)\,.$$

If $\alpha = 0$, then the constant depends only on M_0.

Theorem 4.1 states that the solution of the tDMNLS can be approximated by solving the non-oscillatory limit system (25) numerically with a standard method. The problem of this approach is, however, that the error of this approximation cannot be made arbitrarily small. The accuracy depends on the parameter ε, which has a fixed value in applications. Nevertheless, the limit system will be useful later for analyzing the accuracy of the adiabatic exponential integrator constructed in the next section.

5 Adiabatic exponential midpoint rule

5.1 Construction

We are now ready to construct numerical methods to approximate solutions of the tDMNLS (12) at times $t_n = n\tau$ with a step-size $\tau > 0$. Let $y^{(n)} \approx y(t_n)$ and $y^{(n-1)} \approx y(t_{n-1})$ be available. As a first step, we substitute the tDMNLS $y'(t) = A\big(t, y(t)\big)y(t)$ locally by

$$\tilde{y}'(t) = A\big(t, y^{(n)}\big)\tilde{y}(t) \qquad \text{for} \quad t \in [t_{n-1}, t_{n+1}], \qquad \tilde{y}(t_{n-1}) = y^{(n-1)} \tag{27}$$

such that the second argument of A is the approximation at the *midpoint* of the time interval. Then, Eq. (27) is a *linear* evolution equation with a time-dependent operator. A popular class of integrators for such problems are Magnus methods (cf. [6,22,23]), but applying a standard Magnus method to (27) would be inefficient due to the particular properties of our problem. First, we observe that the multiple sum structure of the operator A makes evaluations of compositions of the form $A\big(t, y^{(n)}\big)A\big(s, y^{(n)}\big)z$ computationally very expensive, which would spoil the efficiency of high-order methods. Moreover, the regularity of $t \mapsto A\big(t, y^{(n)}\big)$ is low such that high-order methods cannot be expected to converge with their classical order. For these reasons we truncate the Magnus expansion already after the first term. This yields the approximation

$$y(t_{n+1}) \approx \tilde{y}(t_{n+1}) \approx \exp\big(2\tau\mathcal{M}_n\big[\tau, y^{(n)}\big]\big)y^{(n-1)} \tag{28}$$

with

$$\mathcal{M}_n[\tau, \mu] := \frac{1}{2}\int_{-1}^{1} A(t_n + \sigma\tau, \mu)\,\mathrm{d}\sigma\,. \tag{29}$$

The operator $\mathcal{M}_n[\tau, \mu] \colon \ell^1 \to \ell^1$ is bounded for every $\mu \in \ell^1$ because Lemma 3.1 (i) yields

$$\|\mathcal{M}_n[\tau, \mu]z\|_{\ell^1} \le \sup_{t \in [t_n, t_{n+1}]} \|A(t, \mu)z\|_{\ell^1} \le C(M)\,\|z\|_{\ell^1} \tag{30}$$

for all $z \in \ell^1$. Thus, the operator exponential in (28) is well-defined in terms of the exponential series if $y^{(n)} \in \ell^1$.

In order to turn the approximation (28) into a numerical method, we have to compute the integral

$$\frac{1}{2}\int_{-1}^{1} a_{m,l}(t_n + \sigma\tau, y^{(n)})\,\mathrm{d}\sigma = \frac{\mathrm{i}}{2}\sum_{\substack{(j,k)\in\mathbb{Z}^2\\ j-k=m-l}} y_j^{(n)}\overline{y}_k^{(n)}\int_{-1}^{1}\exp\Big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\big(\tfrac{t_n+\sigma\tau}{\varepsilon}\big)\Big)\,\mathrm{d}\sigma,$$

which is the entry with indices (m, l) of (29) for $\mu = y^{(n)}$ according to (19) and (20). Using quadrature rules for this task (as in interpolatory Magnus methods) would require a tiny step-size $\tau \ll \varepsilon$ because $t \mapsto \widehat{\phi}\big(\frac{t_n+\sigma\tau}{\varepsilon}\big)$ oscillates rapidly, and the discontinuity of the (weak) derivative of $\widehat{\phi}$ would cause additional problems. Fortunately, all integrals of the form

$$\int_{-1}^{1}\exp\Big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\big(\tfrac{t_n+\sigma\tau}{\varepsilon}\big)\Big)\,\mathrm{d}\sigma \tag{31}$$

can be computed analytically because the piecewise linear function $\widehat{\phi}$ is explicitly known from (2), (3), and (8); details can be found in [29, Chapter 4.1]. This is one reason for the favorable approximation properties of adiabatic integrators. All in all, we obtain the *adiabatic exponential midpoint rule*

$$y^{(n+1)} = \exp\Big(2\tau\mathcal{M}_n\big[\tau, y^{(n)}\big]\Big)y^{(n-1)}\,, \qquad n \in \mathbb{N}\,. \tag{32}$$

Clearly, the adiabatic exponential midpoint rule is a two-step scheme since both $y^{(n)}$ and $y^{(n-1)}$ are required to compute $y^{(n+1)}$. The starting step can be done with the corresponding one-step method

$$y^{(n+1)} = \exp\big(\tau\mathcal{E}_n[\tau, y^{(n)}]\big)y^{(n)}\,, \tag{33}$$

with

$$\mathcal{E}_n[\tau, \mu] := \int_0^1 A(t_n + \sigma\tau, \mu)\,\mathrm{d}\sigma\,. \tag{34}$$

This method will be referred to as adiabatic exponential Euler method[1].

An approximation $u^{(n)} \approx u(t_n, \cdot)$ to the solution of the original DMNLS can be obtained via the inverse transformation

$$u^{(n)}(x) = \sum_{m\in\mathbb{Z}} y_m^{(n)} \exp\Big(-\mathrm{i}m^2\widehat{\phi}\left(\tfrac{t_n}{\varepsilon}\right) + \mathrm{i}mx\Big) \tag{35}$$

which is the counterpart of (10).

Remarks.

1. The term "adiabatic" in the names of the above methods refers to the fact that the construction principle is adapted from [24,25] where similar methods for quantum dynamics close to the adiabatic limit have been proposed. We remark, however, that the differential equation considered there is linear and does not involve any discontinuous coefficients. The name "adiabatic integrator" was coined in [19,28].

2. In the construction of the adiabatic integrators (33) and (32) we assume that the approximation $y^{(n)}$ is in ℓ^1. This is necessary in order to apply Lemma 3.1, which ensures that $\mathcal{E}_n\big[\tau, y^{(n)}\big]$ and $\mathcal{M}_n[\tau, y^{(n)}]$ are bounded operators on ℓ^1. However, it can be shown by a classical bootstrapping argument that we have indeed $y^{(n)} \in \ell^1$ for all $n \in \mathbb{N}$ with $n\tau \in [0, T]$ if $y^{(0)} \in \ell^1$ and if the step-size τ is sufficiently small; cf. Appendix B in [29].

3. Because the function $\widehat{\phi}$ in (31) consists of a slowly moving linear α-part and a rapidly changing periodic ϕ-part (see (8)) there are at least two possible ways to deal with this integral. We can either fix the α-part at t_n, i.e. at the midpoint of the time interval, or retain it inside the integral. Fixing the α-part gives us a periodic integral allowing for a more efficient computation. However, this comes at a cost of higher regularity requirements for the initial value and we observe a slightly higher error constant in our numerical experiments. In case of the adiabatic midpoint rule the α-part was fixed, cf. [26], whereas we keep the α-part inside the integral in the adiabatic exponential midpoint rule. For more details on this subject we refer to Chapter 4 in [29].

[1] The $\mathcal{E}_n$ in (33) stands for "Euler", the $\mathcal{M}_n$ in (32) is for "midpoint".

5.2 Qualitative properties

Next, we will show that the two adiabatic exponential integrators (33) and (32) yield numerical approximations with constant norm and provide the *exact* solution in simple but nontrivial situations, as discussed in the introduction. The adiabatic integrators proposed in [26] do *not* have these favourable properties.

Lemma 5.1. *Let $y^{(n)}$ be the approximation of the tDMNLS* (12) *with the adiabatic exponential Euler method* (33)*, or with the adiabatic exponential midpoint rule* (32) *with step-size $\tau > 0$. Let $u^{(n)}$ be defined by* (35)*.*

(i) If $y^{(n)} \in \ell^1$ for all $n \in \mathbb{N}_0$ with $t_n \leq T$, then the norm of the solution is conserved by the numerical methods, i.e.

$$\|y^{(n)}\|_{\ell^2} = \|y^{(0)}\|_{\ell^2} \qquad \textit{and} \qquad \|u^{(n)}\|_{L^2(\mathbb{T})} = \|u(0,\cdot)\|_{L^2(\mathbb{T})} \tag{36}$$

for all $n \in \mathbb{N}_0$ with $t_n \leq T$.

(ii) If $u(0,x) = r\exp(\mathrm{i}\kappa x)$ for some $r > 0$ and $\kappa \in \mathbb{Z}$, and if $y^{(0)} = y(0)$ is related to $u(0,x)$ via the transformation (10)*, then $u^{(n)}$ is exact, i.e.*

$$u^{(n)} = u(t_n, x) = r\exp\Big(\mathrm{i}r^2 t_n - \mathrm{i}\kappa^2 \widehat{\phi}\left(\tfrac{t_n}{\varepsilon}\right) + \mathrm{i}\kappa x\Big), \tag{37}$$

for all $n \in \mathbb{N}_0$ with $t_n \leq T$.

Proof. If $y^{(n)} \in \ell^1$ for all n, then we know from Lemma 3.1(iii) that $A(t, y^{(n)}) \colon \ell^2 \to \ell^2$ is skew-adjoint for all t. Hence, $\mathcal{E}_n[\tau, y^{(n)}]$ and $\mathcal{M}_n[\tau, y^{(n)}]$ are skew-adjoint on ℓ^2 for arbitrary $\tau > 0$, which means that both $\exp\big(2\tau\mathcal{E}_n[\tau, y^{(n)}]\big)$ and $\exp\big(2\tau\mathcal{M}_n[\tau, y^{(n)}]\big)$ are unitary on ℓ^2. With (33) or (32), respectively, this implies that $\|y^{(n)}\|_{\ell^2} = \|y^{(0)}\|_{\ell^2}$ for both methods. The assertion for $u^{(n)}$ in (36) follows from the fact that $\|u^{(n)}\|_{L^2(\mathbb{T})} = \sqrt{2\pi}\|y^{(n)}\|_{\ell^2}$ according to (13).

It can easily be verified that (37) is indeed the exact solution of (1) with initial data $u(0,x) = r\exp(\mathrm{i}\kappa x)$. Now we consider the adiabatic exponential Euler method and prove (ii) by induction. For $n = 0$, (37) is true by assumption because $\widehat{\phi}(0) = 0$ by definition (8). Now suppose that (37) holds for some $n \in \mathbb{N}$. Then, the m-th entry of the transformed variable $y^{(n)}$ is

$$y_m^{(n)} = \begin{cases} r\exp\left(\mathrm{i}r^2 t_n\right) & \text{if } m = \kappa \\ 0 & \text{otherwise} \end{cases}$$

according to (10). With (34) and (11) we see that the m-th entry of $\mathcal{E}_n\big[\tau, y^{(n)}\big]y^{(n)}$ simplifies to

$$\begin{aligned} \Big(\mathcal{E}_n\big[\tau, y^{(n)}\big]y^{(n)}\Big)_m &= \mathrm{i}\sum_{I_m} y_j^{(n)}\overline{y}_k^{(n)}y_l^{(n)} \int_0^1 \exp\Big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\sigma\tau}{\varepsilon}\right)\Big)\,\mathrm{d}\sigma \\ &= \mathrm{i}\delta_{m\kappa}|y_\kappa^{(n)}|^2 y_\kappa^{(n)} \\ &= \mathrm{i}\delta_{m\kappa} r^2 y_\kappa^{(n)} \end{aligned}$$

where $\delta_{m\kappa}$ is the Kronecker symbol. Hence, the next approximation computed with (34) is

$$y_m^{(n+1)} = \begin{cases} \exp(\mathrm{i}r^2\tau)y_m^{(n)} = r\exp\left(\mathrm{i}r^2 t_{n+1}\right) & \text{if } m = \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

and via (10) we obtain

$$u^{(n+1)} = r \exp\left(\mathrm{i}r^2 t_{n+1} - \mathrm{i}\kappa^2 \widehat{\phi}\left(\tfrac{t_{n+1}}{\varepsilon}\right) + \mathrm{i}\kappa x\right) = u(t_{n+1}, x).$$

The same arguments apply *mutatis mutandis* to the adiabatic exponential midpoint rule (32). ■

5.3 Accuracy

In this section we summarize the results of our error analysis for the time-discretization with the adiabatic exponential midpoint rule. The three error bounds stated in Theorems 5.2, 5.3, and 5.4 below are similar to Theorems 2-4 from [26] for the (non-exponential) adiabatic midpoint rule, but the proofs require new techniques in addition to those developed in [26] because the exponential integrator (32) has a completely different structure.

Henceforth, $y(t)$ always denotes the exact solution of the tDMNLS (12) and $y^{(n)}$ is the approximation at time $t_n = n\tau$ computed by the adiabatic exponential midpoint rule (32) with starting step (33).

The initial data $y(0) = y^{(0)}$ are obtained by transforming u_0 of (1) via (6) and (7). We recall that the assumption $u_0 \in H^s(\mathbb{T})$ for some $s \in \mathbb{N}_0$ implies that $u(t, \cdot) \in H^s(\mathbb{T})$ for all $t \in [0, T]$, and hence $y(t) \in \ell^2_s \subset \ell^1_{s-1}$ for all $t \in [0, T]$; cf. Section 3. Conversely, the following error bounds in ℓ^1 for the transformed variables yield error bounds in $L_2(\mathbb{T})$ for the original variables because it follows from (10), (35) and (15), that

$$\|u(t_n, \cdot) - u^{(n)}\|_{L_2(\mathbb{T})} = \sqrt{2\pi}\|y(t_n) - y^{(n)}\|_{\ell^2} \leq \sqrt{2\pi}\left\|y(t_n) - y^{(n)}\right\|_{\ell^1}.$$

Theorem 5.2. *If $u_0 \in H^1(\mathbb{T})$, then the bound*

$$\left\|y(t_n) - y^{(n)}\right\|_{\ell^1} \leq \tau C(T, M_0^y), \quad \tau n \leq T,$$

holds for sufficiently small step-sizes τ.

Theorem 5.3. *If $u_0 \in H^3(\mathbb{T})$ and if we choose step-sizes $\tau = \varepsilon/k$ for some $k \in \mathbb{N}$, then the bound*

$$\left\|y(t_n) - y^{(n)}\right\|_{\ell^1} \leq \varepsilon\tau\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big), \quad \tau n \leq T,$$

holds for sufficiently small step-sizes τ.

Theorem 5.4. *Suppose that Assumption 1 holds. If $u_0 \in H^3(\mathbb{T})$ and if we choose step-sizes $\tau = \varepsilon k$ for some $k \in \mathbb{N}$, then the bound*

$$\left\|y(t_n) - y^{(n)}\right\|_{\ell^1} \leq \big(\tfrac{\varepsilon^2}{\delta} + \tau^2\big) C(T, \alpha, M_2), \quad \tau n \leq T,$$

holds for sufficiently small step-sizes τ. In case of $\alpha = 0$ the constant depends only on T and M_0.

Theorems 5.2-5.4 will be proved in Section 6.

Discussion. For traditional methods of order $p \in \mathbb{N}$ the error constant typically scales like ε^{-q} for some $q \geq p$ such that a reasonable accuracy can only be expected if $\tau \ll \varepsilon^{q/p}$. Theorem 5.2 states that the adiabatic exponential midpoint rule converges at least with order one, and that the error constant does *not* depend on ε. For this reason the method yields higher accuracy for "large" step-sizes than, e.g., the second-order Strang splitting, as we will see in the numerical examples below.

For smooth and non-oscillatory problems (i.e. for $\varepsilon = 1$ and smooth functions ξ and $\widehat{\phi}$) one would expect a global error in $\mathcal{O}(\tau^2)$ for the adiabatic exponential midpoint rule. Unfortunately, second-order convergence is not achieved in the present setting with oscillations and discontinuities. However, Theorems 5.3 and 5.4 state that the accuracy improves significantly if a special step-size is chosen, namely an integer multiple or fraction of ε. This interesting behavior is illustrated by numerical examples in Section 5.4.

The condition "for sufficiently small step-sizes" in Theorems 5.2-5.4 is necessary to ensure the ℓ^1-regularity of the numerical solution; cf. Remark 2 in Section 5.1. This regularity is required for the construction of the scheme (cf. (30)) and to apply the principle (16) if one of the factors is the numerical solution. We point out that this condition does *not* depend on ε, it does *not* impose a severe restriction on the length of the time-step. In all our numerical tests – not only those presented in this paper – we have never observed any problems even when the step-size was much larger than ε.

5.4 Numerical examples

In the following, we illustrate Theorems 5.2-5.4 by numerical examples. We consider the tDMNLS with $T = 1$, $\varepsilon \in \{0.01, 0.002\}$, $\alpha = 0.1$ and $\delta = 1$. Moreover, we choose the initial value $u_0(x) = e^{-3x^2} e^{3\mathrm{i}x}$ and 64 equidistant grid points in the interval $[-\pi, \pi]$. To this setting, we apply the adiabatic exponential midpoint rule (32) as well as the adiabatic exponential Euler method (33). These methods are compared with the adiabatic midpoint rule proposed in [26] and the classical Strang splitting (for the DMNLS). The reference solution is computed by the Strang splitting with a very large number of steps ($> 10^6$).

The left panels of Figure 1 show the accuracy of the Strang splitting, the adiabatic exponential Euler method (33), and the adiabatic exponential midpoint rule (32) for different step-sizes τ and $\varepsilon = 0.01$ (top) and $\varepsilon = 0.002$ (bottom). The behavior of the Strang splitting and the adiabatic exponential midpoint rule appears to be volatile, i.e. small changes of the step-size may change the error by a factor of 10 or even 100. Moreover, we observe that the adiabatic Euler – a first-order method – yields a significantly higher accuracy than Strang splitting for large step-sizes. However, the highest accuracy is obtained with the adiabatic exponential midpoint rule. Apparently, the method is "better than order one for many step-sizes", however, several outliers reveal first order convergence as stated in Theorem 5.2. The right panels of Figure 1 display again the error of the adiabatic exponential midpoint rule but now only for step-sizes chosen as integer multiples and fractions of ε, again for $\varepsilon = 0.01$ (top) and $\varepsilon = 0.002$ (bottom). Moreover, the accuracy of the (non-exponential) adiabatic midpoint rule is shown. We observe second-order convergence for $\tau > \varepsilon$ and convergence in $\mathcal{O}(\tau\varepsilon)$ for $\tau < \varepsilon$ as stated in Theorem 5.3 and Theorem 5.4, respectively. In

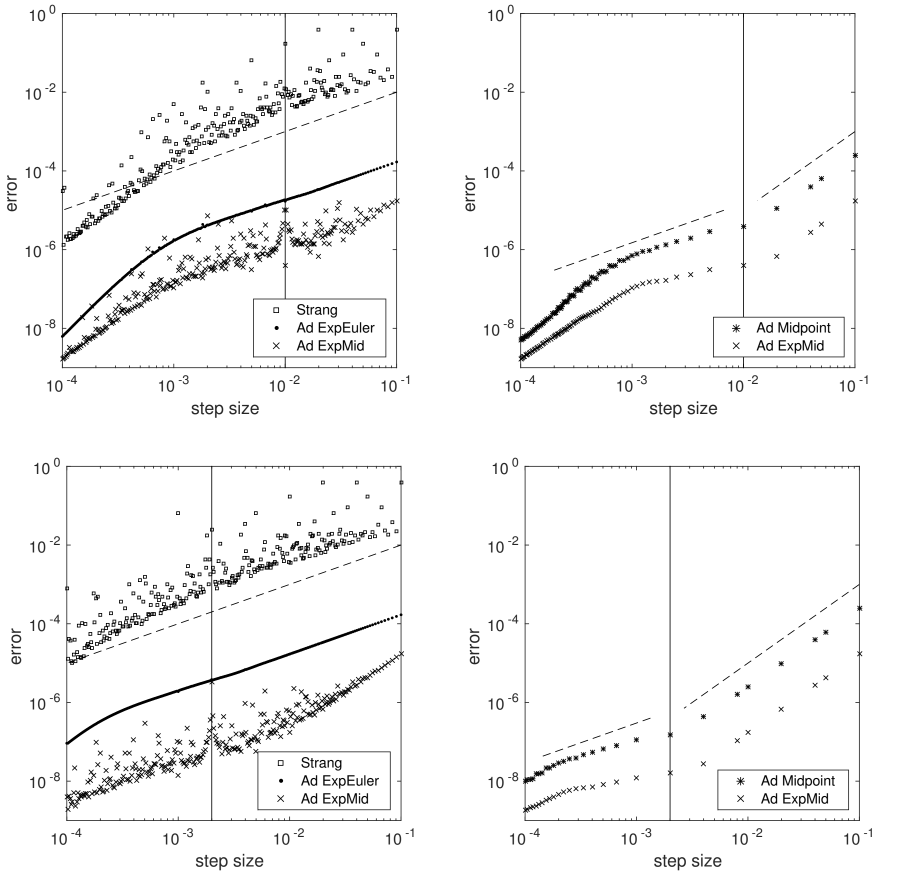

Figure 1: Maximal ℓ^2-error over time of the adiabatic exponential midpoint rule (32) for $\varepsilon = 0.01$ (top) and $\varepsilon = 0.002$ (bottom). In the left panels the accuracy of the adiabatic exponential Euler rule (33) and the Strang splitting is shown for comparison. In the two panels on the right the accuracy of the adiabatic exponential midpoint rule is compared with the (non-exponential) adiabatic midpoint rule from [26], and the step-sizes are chosen according to Theorem 5.3 and Theorem 5.4, respectively.

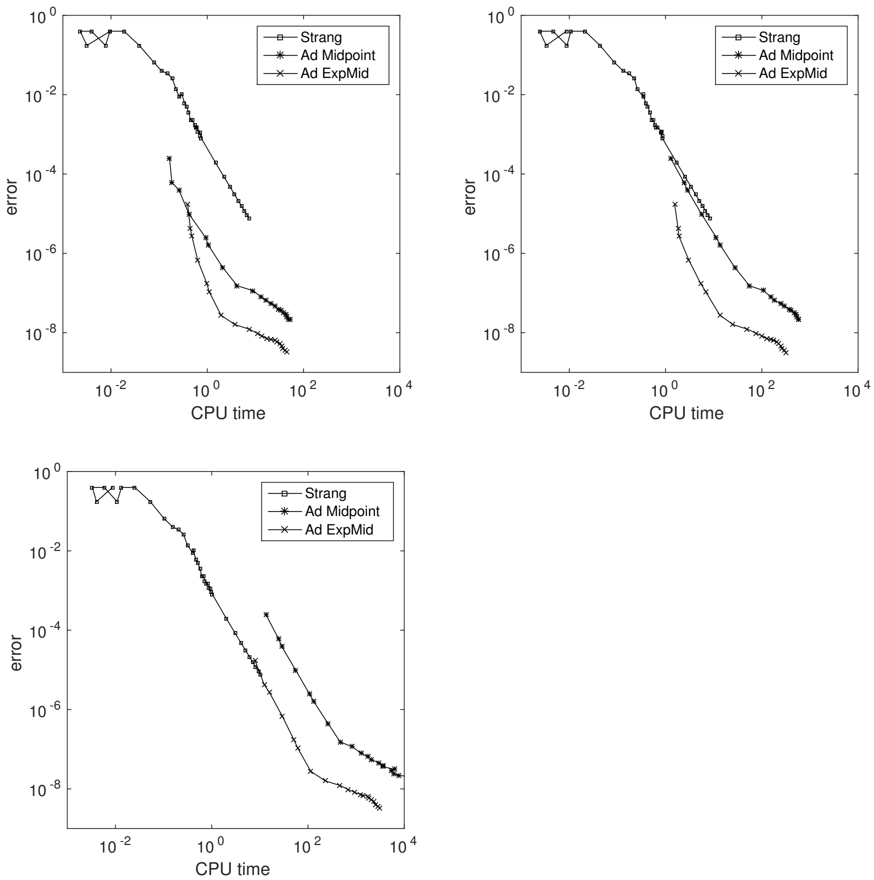

Figure 2: CPU-time in seconds versus maximal ℓ^2-error over time of the Strang splitting, the (non-exponential) adiabatic midpoint rule, and the adiabatic exponential midpoint rule for $\varepsilon = 0.002$ and 64 (top left), 128 (top right) and 256 (bottom left) equidistant grid points in space. All computations have been conducted in Matlab (version R2015a) on a laptop with an Intel i7-4710MQ CPU (4 cores at 2.50 GHz) and 16 GB of RAM.

addition, the experiment confirms our conjecture that the adiabatic exponential method has a smaller error constant than its non-exponential counterpart.

In the next numerical experiment, we investigate the trade-off between the tiny, cheap time steps of the Strang splitting method and the larger, more expensive time steps of the adiabatic exponential midpoint rule. For this purpose, we consider 64, 128, and 256 equidistant grid points in the interval $[-\pi, \pi]$ in the above experiment and fix $\varepsilon = 0.002$.

The panels of Figure 2 show the computational times in relation to the accuracy of adiabatic exponential midpoint rule (32) for these different space discretizations. In addition, the performance of the Strang splitting method and the (non-exponential) adiabatic midpoint rule from [26] is shown. The time step-sizes[2] are chosen as integer multiples and fractions of ε. We compute up to $5 \cdot 10^3$

[2]A better performance of the Strang splitting might be obtained with a "lucky guess" for

time-steps with the adiabatic methods and up to 10^5 time-steps with the Strang splitting method. In the top left panel, we observe that the adiabatic methods clearly outperform the Strang splitting method in terms of computational costs versus accuracy for 64 grid points in space. The adiabatic exponential midpoint rule also outperforms the Strang splitting for 128 grid points in space (top right), whereas the (non-exponential) adiabatic method is only on a par with the Strang splitting in this setting. Although the adiabatic exponential midpoint rule is still equal to the Strang splitting for 256 grid points in space (bottom left), the Strang splitting becomes more efficient for finer discretizations because the computational work required for computing the nested summation in the adiabatic methods increases cubically with the number of grid points in space.

However, we remark that typical solutions considered in mathematical physics are somewhat smooth in space, such that a moderate number of Fourier modes provides a reasonable accuracy of the space discretization; cf. Figures 3 and 6 in [32]. Moreover, increasing the number of time-steps for the Strang splitting yields only more accuracy to a certain extend because at some point rounding errors prevent a higher accuracy. We report a decreasing accuracy for our implementation of the Strang splitting method for step-sizes $\tau < 10^{-7}$.

6 Error analysis

6.1 Preparations

Before we start the proofs of Theorems 5.2-5.4, we make a few preparations. If $\mu \in \ell^1$, then the operator $\mathcal{M}_n[\tau, \mu] : \ell^1 \to \ell^1$ defined in (29) is bounded and thus generates a uniformly continuous semigroup of bounded linear operators in ℓ^1. Hence, we have for $\mu \in \ell^1$ and $M := \|\mu\|_{\ell^1}$ the basic estimate

$$\left\|\exp\big(t\mathcal{M}_n[\tau, \mu]\big)z\right\|_{\ell^1} \le e^{tC(M)} \|z\|_{\ell^1} . \tag{38}$$

Moreover, we introduce the (possibly) operator-valued functions

$$\varphi_k(z) = \int_0^1 e^{(1-\theta)z} \frac{\theta^{k-1}}{(k-1)!} \,\mathrm{d}\theta \quad \text{for} \quad k \ge 1 , \tag{39}$$

cf. [21]. These φ-functions allow us to expand

$$\exp\big(t\mathcal{M}_n[\tau, \mu]\big) = \sum_{k=0}^{m-1} \frac{t^k}{k!} \mathcal{M}_n^k[\tau, \mu] + (t\mathcal{M}_n[\tau, \mu])^m \varphi_m\big(t\mathcal{M}_n[\tau, \mu]\big) , \tag{40}$$

for $m \ge 1$. Here, the operator $\varphi_m\big(t\mathcal{M}_n[\tau, \mu]\big) : \ell_1 \to \ell_1$ in the remainder term is bounded by

$$\left\|\varphi_m\big(t\mathcal{M}_n[\tau, \mu]\big)z\right\|_{\ell^1} \le C(M) \|z\|_{\ell^1} . \tag{41}$$

Henceforth, we use the abbreviations

$$\mathcal{M}_n := \mathcal{M}_n[\tau, y^{(n)}] \quad \text{and} \quad \mathcal{M}_n^{\mathrm{ex}} := \mathcal{M}_n[\tau, y(t_n)] \tag{42}$$

a better step-size.

to simplify notation. As a first step, we reformulate the adiabatic exponential midpoint rule (32) as a one-step method: If we define

$$\mathbf{y}_{n+1} = \begin{pmatrix} y^{(n+1)} \\ y^{(n)} \end{pmatrix}, \qquad \mathbf{y}(t_{n+1}) = \begin{pmatrix} y(t_{n+1}) \\ y(t_n) \end{pmatrix},$$

then method (32) is given by

$$\mathbf{y}_{n+1} = \mathbf{M}_n \mathbf{y}_n \qquad \text{with} \qquad \mathbf{M}_n = \begin{pmatrix} 0 & \exp\left(2\tau\mathcal{M}_n\right) \\ I & 0 \end{pmatrix}. \tag{43}$$

In particular, one can use the one-step formulation (43) to show that

$$\left\| y^{(n)} \right\|_{\ell^1} \leq C(M_0^y) \quad \text{for all} \quad n\tau \leq T, \tag{44}$$

for sufficiently small step-sizes τ using a standard bootstrapping argument; cf. Remark 2 in Section 5.1. Moreover, the global error $\mathbf{e}_N = \mathbf{y}_N - \mathbf{y}(t_N)$ propagates according to

$$\mathbf{e}_{N+1} = \mathbf{M}_N \mathbf{e}_N + \mathbf{d}_{N+1}, \qquad \mathbf{e}_0 = 0,$$

where

$$\mathbf{d}_1 = \mathbf{e}_1 \quad \text{and} \quad \mathbf{d}_{n+1} = \mathbf{M}_n \mathbf{y}(t_n) - \mathbf{y}(t_{n+1}), \quad n \geq 1. \tag{45}$$

Solving this recursion yields

$$\mathbf{e}_N = \mathbb{M}_1 \mathbf{d}_1 + \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \tag{46}$$

with $\mathbb{M}_N = I$ and $\mathbb{M}_{n+1} = \mathbf{M}_{N-1}\mathbf{M}_{N-2}\dots\mathbf{M}_{n+1}$ for $n+1 \leq N-1$. It follows from the basic estimate (38) that

$$\|\mathbb{M}_n z\|_{\ell^1} \leq e^{TC(M_0^y)} \|z\|_{\ell^1} \tag{47}$$

for $z \in \ell^1$.

Finally, we recall that the starting step $y^{(1)}$ is obtained by the adiabatic exponential Euler method (33). For arbitrary $y(0) = y^{(0)} \in \ell_1^2$ it can be shown with (47) and straightforward computation that

$$\|\mathbb{M}_1 \mathbf{d}_1\|_{\ell^1} \leq e^{TC(M_0^y)} \left\| y^{(1)} - y(t_1) \right\|_{\ell^1} \leq \tau^2 C(T, M_0^y). \tag{48}$$

Remark. One can show that the adiabatic exponential Euler method (33) is a first-order scheme, and that its error constant is independent of ε. The proof is a rather straightforward application of the principle "stability and consistency yield convergence" and is therefore omitted in this paper. The details are given in [29, Section 7.3].

Combining (48) with (46) yields

$$\|\mathbf{e}_N\|_{\ell^1} \leq \tau^2 C(T, M_0^y) + \left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1}. \tag{49}$$

This estimate is the starting point for each of the proofs of Theorems 5.2 and 5.3, where we derive suitable ℓ^1-estimates for the remaining sum in the right-hand side.

6.2 Proof of Theorem 5.2

For arbitrary step-sizes $\tau > 0$, we aim for the bound

$$\left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \leq \tau C(T, M_0^y) \sum_{n=1}^{N-1} \|\mathbf{e}_n\|_{\ell^1} + \tau C(T, M_0^y) \,. \tag{50}$$

Then, substituting into (49) and applying the discrete Gronwall lemma completes the proof. Thanks to (47), we immediately obtain

$$\left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \leq C(T, M_0^y) \sum_{n=1}^{N-1} \|\mathbf{d}_{n+1}\|_{\ell^1} \,. \tag{51}$$

If we can prove that

$$\|\mathbf{d}_{n+1}\|_{\ell^1} \leq \tau C(M_0^y) \|\mathbf{e}_n\|_{\ell^1} + \tau^2 C(T, M_0^y) \,, \tag{52}$$

then the bound (50) follows. As a first step, we observe that by (45) and (43)

$$\mathbf{d}_{n+1} = \begin{pmatrix} d_{n+1} \\ 0 \end{pmatrix} \quad \text{with} \quad d_{n+1} = \exp\big(2\tau \mathcal{M}_n\big) y(t_{n-1}) - y(t_{n+1}) \,, \tag{53}$$

and hence it is sufficient to derive an estimate for d_{n+1}. According to (12) we have

$$y'(t) = \mathcal{M}_n y(t) + \Big(A\big(t, y(t)\big) - \mathcal{M}_n \Big) y(t) \,.$$

Thus, applying the variation of constants formula gives

$$\begin{aligned} y(t_{n+1}) = {}& \exp\big(2\tau \mathcal{M}_n\big) y(t_{n-1}) \\ &+ \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big) \big(A\big(s, y(s)\big) - \mathcal{M}_n \big) y(s) \, \mathrm{d}s \,. \end{aligned} \tag{54}$$

Now, inserting (54) into (53) results in

$$d_{n+1} = - \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big) \big(A\big(s, y(s)\big) - \mathcal{M}_n \big) y(s) \, \mathrm{d}s \,. \tag{55}$$

Using (40) we get the partition

$$d_{n+1} = -\big(d_{n+1}^{(1)} + d_{n+1}^{(2)} + R_{n+1}^{(1)} \big) \,, \tag{56}$$

where

$$d_{n+1}^{(1)} = \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau - s)\mathcal{M}_n\big) \big(\mathcal{M}_n^{\mathrm{ex}} - \mathcal{M}_n \big) y(s) \, \mathrm{d}s \,, \tag{57}$$

$$d_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}} \big(A\big(s, y(s)\big) - \mathcal{M}_n^{\mathrm{ex}} \big) y(t_n) \, \mathrm{d}s \,, \tag{58}$$

$$\begin{aligned} R_{n+1}^{(1)} = {}& \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \big(A\big(s, y(s)\big) - \mathcal{M}_n^{\mathrm{ex}} \big) y'(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}s \\ &+ 2\mathcal{M}_n \int_{t_{n-1}}^{t_{n+1}} (\tau - s) \varphi_1\big(2(\tau - s)\mathcal{M}_n\big) \Big(A\big(s, y(s)\big) - \mathcal{M}_n^{\mathrm{ex}} \Big) y(s) \, \mathrm{d}s \,. \end{aligned}$$

Thanks to Lemma 3.1, (30), (41), and (18) the bound

$$\left\| R_{n+1}^{(1)} \right\|_{\ell^1} \leq \tau^2 C(M_0^y) \tag{59}$$

follows. Because

$$\left| y_j(t_n)\overline{y}_k(t_n) - y_j^{(n)}\overline{y}_k^{(n)} \right| \leq \left| y_j(t_n) - y_j^{(n)} \right| \cdot |\overline{y}_k(t_n)| + |y_j(t_n)| \cdot \left| \overline{y}_k(t_n) - \overline{y}_k^{(n)} \right| ,$$

we obtain for $z \in \ell^1$

$$\left\| \big(\mathcal{M}_n^{\text{ex}} - \mathcal{M}_n\big) z \right\|_{\ell^1} \leq C(M_0^y) \left\| y(t_n) - y^{(n)} \right\|_{\ell^1} \|z\|_{\ell^1} \leq C(M_0^y) \left\| \mathbf{e}_n \right\|_{\ell^1} \|z\|_{\ell^1} , \tag{60}$$

and hence

$$\left\| d_{n+1}^{(1)} \right\|_{\ell^1} \leq \tau C(M_0^y) \left\| \mathbf{e}_n \right\|_{\ell^1} , \tag{61}$$

by (38) and (44). Moreover, a small computation shows that

$$d_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}} \Big(A\big(s, y(s)\big) - A\big(s, y(t_n)\big) \Big) y(t_n) \, \mathrm{d}s \, .$$

Now, let $[d_{n+1}^{(2)}]_m$ be the m-th entry of $d_{n+1}^{(2)}$. If we partition

$$[d_{n+1}^{(2)}]_m = [S_{n+1}^{(1)}]_m + [S_{n+1}^{(2)}]_m \tag{62}$$

with

$$[S_{n+1}^{(1)}]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} y_j(t_n) \overline{y}_k'(\sigma) y_l(t_n) \exp\Big(-\mathrm{i}\omega_{[jklm]} \widehat{\phi}\left(\tfrac{s}{\varepsilon}\right) \Big) \, \mathrm{d}\sigma \, \mathrm{d}s \, , \tag{63}$$

$$[S_{n+1}^{(2)}]_m = \mathrm{i} \sum_{I_m} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} y_j'(\sigma) \overline{y}_k(s) y_l(t_n) \exp\Big(-\mathrm{i}\omega_{[jklm]} \widehat{\phi}\left(\tfrac{s}{\varepsilon}\right) \Big) \, \mathrm{d}\sigma \, \mathrm{d}s \, , \tag{64}$$

then (16) and (18) imply the estimate

$$\left\| d^{(2)} \right\|_{\ell^1} \leq \tau^2 C(M_0^y) \, . \tag{65}$$

Finally, combining (59), (61) and (65) yields the desired bound (52).

6.3 Proof of Theorem 5.3

In the setting of Theorem 5.3, i.e. $\tau = \varepsilon / k$ for some $k \in \mathbb{N}$, we can improve the bound (50) for the remaining sum in (49). Here, we aim for the estimate

$$\left\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \mathbf{d}_{n+1} \right\|_{\ell^1} \leq \tau C(M_0^y) \sum_{n=1}^{N-1} \left\| \mathbf{e}_n \right\|_{\ell^1} + \tau\varepsilon \big(C(T, M_0^y) + \alpha C(T, M_2^y) \big) \, . \tag{66}$$

Again, substituting into (49) and applying the discrete Gronwall lemma then completes the proof.

The key idea to prove (66) is to exploit cancellation effects in the summation of the error terms; cf. Lemma 6.2. It turns out that these cancellations occur over time intervals of the length 2ε. Since the endpoint of the time interval $[0,T]$ is not necessarily an integer multiple of 2ε, we have to take into account potential extra summands without cancellation. Therefore, we decompose $N-1 = 2kL + n^*$ with $L \in \mathbb{N}_0$, $n^* \in \{0,\dots,2k-1\}$ and partition

$$\Big\| \sum_{n=1}^{N-1} \mathbb{M}_{n+1}\mathbf{d}_{n+1} \Big\|_{\ell^1} \le \Big\| \sum_{n=1}^{2kL-1} \mathbb{M}_{n+1}\mathbf{d}_{n+1} \Big\|_{\ell^1} + \Big\| \sum_{n=2kL}^{2kL+n^*} \mathbb{M}_{n+1}\mathbf{d}_{n+1} \Big\|_{\ell^1} . \tag{67}$$

Because $n^*\tau^2 < 2k\tau^2 = 2\tau\varepsilon$, we immediately conclude from (47) and (52) that

$$\Big\| \sum_{n=2kL}^{2kL+n^*} \mathbb{M}_{n+1}\mathbf{d}_{n+1} \Big\|_{\ell^1} \le \tau C(T, M_0^y) \sum_{n=2kL}^{2kL+n^*} \|\mathbf{e}_n\|_{\ell^1} + \tau\varepsilon C(T, M_0^y)\,. \tag{68}$$

In order make use of the cancellation effects for estimating the other sum in (67), we must avoid the triangle inequality. Hence, we cannot employ the bound (47) in order to estimate the operators $\mathbb{M}_{n+1}$. The following lemma provides an alternative.

Lemma 6.1. *Let $k, L \in \mathbb{N}$. Then, we have*

$$(i) \quad \Big\| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} \mathbb{M}_{n+1}\mathbf{d}_{n+1} \Big\|_{\ell^1} \le C(T, M_0^y) \Big\| \sum_{n=1}^{kL} \mathbf{d}_{2n} \Big\|_{\ell^1} + \tau C(M_0^y) \sum_{n=1}^{kL-1} \Big\| \sum_{j=1}^{n} \mathbf{d}_{2j} \Big\|_{\ell^1}$$

and

$$(ii) \quad \Big\| \sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} \mathbb{M}_{n+1}\mathbf{d}_{n+1} \Big\|_{\ell^1} \le C(T, M_0^y) \Big\| \sum_{n=1}^{kL-1} \mathbf{d}_{2n+1} \Big\|_{\ell^1} + \tau C(M_0^y) \sum_{n=1}^{kL-2} \Big\| \sum_{j=1}^{n} \mathbf{d}_{2j+1} \Big\|_{\ell^1} .$$

Proof. First, we apply the summation by parts formula and obtain

$$\begin{aligned} \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} \mathbb{M}_{n+1}\mathbf{d}_{n+1} &= \sum_{n=1}^{kL} \mathbb{M}_{2n}\mathbf{d}_{2n} \\ &= \mathbb{M}_{2kL} \sum_{n=1}^{kL} \mathbf{d}_{2n} - \sum_{n=1}^{kL-1} \big(\mathbb{M}_{2n+2} - \mathbb{M}_{2n}\big)\Big(\sum_{j=1}^{n} \mathbf{d}_{2j} \Big) . \end{aligned}$$

With the factorization

$$\begin{aligned} \mathbb{M}_{2n+2} - \mathbb{M}_{2n} &= \mathbb{M}_{2n+2} - \mathbb{M}_{2n+2}\mathbf{M}_{2n+1}\mathbf{M}_{2n} \\ &= \mathbb{M}_{2n+2} \left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} \exp\big(2\tau\mathcal{M}_{2n+1}\big) & 0 \\ 0 & \exp\big(2\tau\mathcal{M}_{2n}\big) \end{pmatrix} \right) \end{aligned}$$

and (40), we get

$$\mathbb{M}_{2n+2} - \mathbb{M}_{2n} = 2\tau\mathbb{M}_{2n+2} \begin{pmatrix} \mathcal{M}_{2n+1}\varphi_1\big(2\tau\mathcal{M}_{2n+1}\big) & 0 \\ 0 & \mathcal{M}_{2n}\varphi_1\big(2\tau\mathcal{M}_{2n}\big) \end{pmatrix} ,$$

and hence (30), (47) and (41) yield the bound

$$\big\| \big(\mathbb{M}_{2n+2} - \mathbb{M}_{2n}\big) z \big\|_{\ell^1} \le \tau C(M_0^y)\, \|z\|_{\ell^1} \quad \text{for} \quad z \in \ell^1$$

which implies the first estimate. The second estimate follows analogously. ■

According to Lemma 6.1 it suffices to derive estimates for

$$\left\| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} \mathbf{d}_{n+1} \right\|_{\ell^1} \quad \text{and} \quad \left\| \sum_{\substack{n=1 \\ n \text{ odd}}}^{2kL-1} \mathbf{d}_{n+1} \right\|_{\ell^1} \quad \text{with} \quad k, L \in \mathbb{N}\,. \tag{69}$$

This is because these estimates can also be employed to bound the remaining double sums in Lemma 6.1. Here, we partition $n = (lk + n^*)$ with $l \in \mathbb{N}_0$, $n^* \in \{0, \ldots, k-1\}$ to subdivide the inner sum as in (67), but with $\mathbb{M}_{n+1}$ replaced by identity. Then, the first sum can be bounded by the (yet to be derived) estimates for (69), whereas the second can be treated analogously to (68).

In the following two lemmas, we specify the cancellation effects, which allow us to obtain suitable bounds for the sums (69). The crucial terms for these cancellations are double integrals of the form

$$\mathcal{I}_n = \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right) \mathrm{d}\sigma \exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s. \tag{70}$$

Lemma 6.2. *Let $k, L \in \mathbb{N}$ and suppose that $\tau = \varepsilon/k$. Further, we consider the double integral $\mathcal{I}_n$ given in* (70)*, a sequence $(a_n)_{n\in\mathbb{N}}$, and a sequence $(b_n)_{n\in\mathbb{N}}$ with $|b_n| \le M$ for all $n \in \mathbb{N}$ and with the property*

$$b_{2n} = b_{2(k-n)}, \quad \textit{for} \quad n = 1, \ldots, k/2 - 1$$

and

$$b_{2n-1} = b_{2(k-n)+1}, \quad \textit{for} \quad n = 1, \ldots, k/2\,.$$

Then, we have the estimates

$$(i) \quad \left| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n \right| \le \varepsilon\tau C(M) \sum_{n=1}^{kL-2} \left|a_{2(n+1)} - a_{2n}\right|$$

and

$$(ii) \quad \left| \sum_{\substack{n=1 \\ n \text{ odd}}}^{2kL-1} a_n b_n \mathcal{I}_n \right| \le \varepsilon\tau C(M) \sum_{n=1}^{kL-2} |a_{2n+1} - a_{2n-1}|$$

Remark. Lemma 6.2 is the foundation to improve the error estimate from $\mathcal{O}(\tau)$ in Theorem 5.2 to $\mathcal{O}(\varepsilon\tau)$ in Theorem 5.3. Suppose that $|a_n b_n| \le C_{ab}$ for all $n = 1, \ldots, 2kL-1$ for some constant $C_{ab} > 0$ independent of τ or ε. Because $|\mathcal{I}_n| \le \tau^2$ and $(2kL-1)\tau \le T$ a straightforward estimate via the triangle inequality yields

$$\left| \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n \right| \le C_{ab} \sum_{\substack{n=1 \\ n \text{ even}}}^{2kL-1} |\mathcal{I}_n| \le C_{ab} T\tau = \mathcal{O}(\tau)$$

for the left-hand side of (i). In the proof of Theorem 5.3, however, we have $a_n = \widehat{F}(t_n)$ where $\widehat{F}$ is a differentiable function with bounded derivative $\widehat{F}'$; cf. (94) below. In this case, Lemma 6.2 yields the stronger estimate

$$\begin{aligned}\left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n\right| &\le \varepsilon\tau C(M) \sum_{n=1}^{kL-2} \left|a_{2(n+1)} - a_{2n}\right| \\ &\le \varepsilon\tau C(M)(kL-2)2\tau \max_{t\in[0,T]} \left|\widehat{F}'(t)\right| \\ &\le C(M, \widehat{F}', T)\varepsilon\tau = \mathcal{O}(\varepsilon\tau).\end{aligned}$$

Of course, the same consideration holds for part (ii) of Lemma 6.2.

Proof of Lemma 6.2. A short computation using the symmetry and periodicity of ϕ, i.e.

$$\phi(1+s) = \phi(1-s)\,, \qquad \phi(2+s) = \phi(2-s) \tag{71}$$

and

$$\phi(s) = \phi(2+s)\,, \tag{72}$$

shows that $\mathcal{I}_{2k} = 0$. Moreover, one can verify that

$$\mathcal{I}_k = 0\,, \quad \mathcal{I}_{2n} + \mathcal{I}_{2(k-n)} = 0\,, \quad \text{for} \quad n = 1, \ldots, k/2 - 1$$

and

$$\mathcal{I}_{2n-1} + \mathcal{I}_{2(k-n)+1} = 0\,, \quad \text{for} \quad n = 1, \ldots, k/2\,.$$

For more details of these computations we refer to [26] or [29, Lemma 13]. These symmetric behavior results in

$$\sum_{n=1}^{lk-1} b_{2n}\mathcal{I}_{2n} = 0 \quad \text{for} \quad l \in \mathbb{N}\,. \tag{73}$$

Applying the summation by parts formula gives

$$\begin{aligned}\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n &= \sum_{n=1}^{kL-1} a_{2n} b_{2n} \mathcal{I}_{2n} \\ &= \left(\sum_{n=1}^{kL-1} b_{2n}\mathcal{I}_{2n}\right) a_{2(kL-1)} - \sum_{n=1}^{kL-2}\left(\sum_{j=1}^{n} b_{2j}\mathcal{I}_{2j}\right)\left(a_{2(n+1)} - a_{2n}\right).\end{aligned} \tag{74}$$

The first part vanishes immediately with (73). For the second part, we partition $n = (kl-1) + n^*$ for $l \in \mathbb{N}$, $n^* \in \{0, \ldots, k-1\}$ and subdivide

$$\sum_{j=1}^{n} b_{2j}\mathcal{I}_{2j} = \sum_{j=1}^{lk-1} b_{2j}\mathcal{I}_{2j} + \sum_{j=lk}^{lk+n^*} b_{2j}\mathcal{I}_{2j}\,. \tag{75}$$

Again, the first sum vanishes with (73). Because $|\mathcal{I}_n| \le 2\tau^2$ and $n^*\tau^2 \le \varepsilon\tau$, we obtain

$$\left|\sum_{j=lk}^{lk+n^*} b_{2j}\mathcal{I}_{2j}\right| \le \tau^2 n^* C(M) \le \tau\varepsilon C(M)\,,$$

and hence the estimate

$$\left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n b_n \mathcal{I}_n\right| \le \varepsilon\tau C(M) \sum_{n=1}^{kL-2} \left|a_{2(n+1)} - a_{2n}\right| \tag{76}$$

follows. The estimate (ii) follows analogously. ■

For technical reasons we also need the following variant of Lemma 6.2.

Lemma 6.3. *Let $k, L \in \mathbb{N}$ and suppose that $\tau = \varepsilon/k$ and $\tau k L \le T$. Further, we consider the double integral*

$$\widehat{\mathcal{I}}_n = \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \exp\left(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{\sigma}{\varepsilon}\right)\right) \mathrm{d}\sigma \exp\left(-\mathrm{i}\tilde{\omega}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right) \mathrm{d}s\,. \tag{77}$$

and a sequence $(a_n)_{n\in\mathbb{N}}$. Then, with the sequence $(\hat{a}_n)_{n\in\mathbb{N}}$ given by

$$\hat{a}_n = \exp\left(-\mathrm{i}(\omega+\tilde{\omega})\alpha t_n\right) a_n\,,$$

we have the estimates

$$(i) \quad \left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n \widehat{\mathcal{I}}_n\right| \le \varepsilon\tau C \sum_{n=1}^{kL-2} \left|\hat{a}_{2(n+1)} - \hat{a}_{2n}\right| + \alpha\tau^2 C(T) \max_{n\le 2kL} \left\{ |\omega a_n| + |\tilde{\omega} a_n| \right\}$$

and

$$(ii) \quad \left|\sum_{\substack{n=1\\ n \text{ odd}}}^{2kL-1} a_n \widehat{\mathcal{I}}_n\right| \le \varepsilon\tau C \sum_{n=1}^{kL-2} \left|\hat{a}_{2n+1} - \hat{a}_{2n-1}\right| + \alpha\tau^2 C(T) \max_{n\le 2kL} \left\{ |\omega a_n| + |\tilde{\omega} a_n| \right\}.$$

Proof. By the definition (8), we have

$$\exp\left(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\right) = \left(\exp\left(-\mathrm{i}\omega\alpha t_n\right) - \mathrm{i}\omega\alpha \int_{t_n}^{s} \exp\left(-\mathrm{i}\omega\alpha\xi\right) \mathrm{d}\xi\right) \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right)\,. \tag{78}$$

This allows us to partition (77) into

$$\widehat{\mathcal{I}}_n = \exp\left(-\mathrm{i}(\omega+\tilde{\omega})\alpha t_n\right) \mathcal{I}_n - \mathrm{i}\alpha(\omega R^{(1)} + \tilde{\omega} R^{(2)})\,,$$

with

$$\left|R^{(1)}\right| \le \tau^3 C \quad \text{and} \quad \left|R^{(2)}\right| \le \tau^3 C\,,$$

Now we obtain inequality (i) by estimating

$$\left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} a_n \widehat{\mathcal{I}}_n\right| \le \left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} \hat{a}_n \mathcal{I}_n\right| + \alpha\tau^2 C(T) \max_{n\le 2kL} \left\{ |\omega a_n| + |\tilde{\omega} a_n| \right\},$$

and then applying Lemma 6.2 to the first sum. Inequality (ii) follows analogously. ■

We are now in a position to derive estimates for the sums in (69). However, we consider only the sum over even n, because a corresponding bound for the sum over odd n follows analogously. It suffices to estimate the non-zero part d_{n+1} of $\mathbf{d}_{n+1}$, cf. (53). First, we partition

$$d_{n+1} = -\big(d_{n+1}^{(1)} + d_{n+1}^{(2)} + d_{n+1}^{(3)} + d_{n+1}^{(4)} + R_{n+1}^{(2)}\big)\,, \tag{79}$$

with $d_{n+1}^{(1)}$ and $d_{n+1}^{(2)}$ defined in (57) and (58), respectively, and

$$\begin{aligned}
d_{n+1}^{(3)} &= \int_{t_{n+1}}^{t_{n+1}} \int_{t_n}^{s} \Big(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\Big) y'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\,,\\
d_{n+1}^{(4)} &= \int_{t_{n-1}}^{t_{n+1}} 2(\tau - s)\mathcal{M}_n \Big(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\Big) y(t_n)\,\mathrm{d}s\,,\\
R_{n+1}^{(2)} &= \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} 2(\tau - s)\mathcal{M}_n \Big(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\Big) y'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\quad + \int_{t_{n-1}}^{t_{n+1}} \big(2(\tau - s)\mathcal{M}_n\big)^2 \varphi_2\big(2(\tau - s)\mathcal{M}_n\big) \Big(A\big(s,y(s)\big) - \mathcal{M}_n^{\mathrm{ex}}\Big) y(s)\,\mathrm{d}s\,.
\end{aligned}$$

By Lemma 3.1, (30), (41), and (18) the bound

$$\Big\|R_{n+1}^{(2)}\Big\|_{\ell^1} \le \tau^3 C(M_0^y) \tag{80}$$

follows immediately. Moreover, we reuse the estimate (61) for the term $d_{n+1}^{(1)}$. It remains to derive suitable estimates for $d_{n+1}^{(2)}$, $d_{n+1}^{(3)}$, and $d_{n+1}^{(4)}$.

Step 1. We start at the partition (62) and solely consider the term (64) because an estimate for $[S_{n+1}^{(1)}]_m$ follows analogously. Replacing $y_j'(\sigma)$ by the tDMNLS gives

$$\begin{aligned}
[S_{n+1}^{(2)}]_m = -\sum_{I_m}\sum_{I_j} \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} & y_p(\sigma)\overline{y}_q(\sigma)y_r(\sigma)\overline{y}_k(\sigma)y_l(\sigma)\\
&\exp\Big(-\mathrm{i}\omega_{[pqrj]}\widehat{\phi}\left(\tfrac{\sigma}{\varepsilon}\right)\Big)\,\mathrm{d}\sigma \exp\Big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\Big)\,\mathrm{d}s\,.
\end{aligned} \tag{81}$$

Now, we fix $m \in \mathbb{Z}$, $(j,k,l) \in I_m$ and $(p,q,r) \in I_j$ and write $\omega = \omega_{[pqrj]}$, $\tilde{\omega} = \omega_{[jklm]}$ and $Y(\sigma) = y_p(\sigma)\overline{y}_q(\sigma)y_r(\sigma)\overline{y}_k(\sigma)y_l(\sigma)$ for short. Then, any summand of (81) can be expanded via

$$\int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} Y(\sigma) \exp\Big(-\mathrm{i}\omega\widehat{\phi}\left(\tfrac{\sigma}{\varepsilon}\right)\Big)\,\mathrm{d}\sigma \exp\Big(-\mathrm{i}\tilde{\omega}\widehat{\phi}\left(\tfrac{s}{\varepsilon}\right)\Big)\,\mathrm{d}s = Y(t_n)\widehat{\mathcal{I}}_n + \widehat{R}_n\,,$$

with $\widehat{\mathcal{I}}_n$ given in (77) and

$$\Big|\widehat{R}_n\Big| \le \tau^3 \max_{\sigma\in[0,T]} |Y'(\sigma)|\,. \tag{82}$$

Moreover, using the abbreviation

$$\widehat{F}(\sigma) = \exp\big(-\mathrm{i}(\omega + \tilde{\omega})\alpha\sigma\big) Y(\sigma)\,,$$

Lemma 6.3 implies

$$\left|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} Y(t_n)\widehat{\mathcal{I}}_n\right| \leq C(T)\Big(\varepsilon\tau \max_{\sigma\in[0,T]} \left|\widehat{F}'(\sigma)\right| + \alpha\tau^2 \max_{\sigma\in[0,T]} \big\{\, |\omega Y(\sigma)| + |\tilde{\omega} Y(\sigma)|\, \big\}\Big)\,.$$

Ultimately, we obtain with the principle (16), (18), and Lemma A.1 (in Appendix) the final estimate

$$\left\|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} d_{n+1}^{(2)}\right\|_{\ell^1} \leq \varepsilon\tau\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big)\,. \tag{83}$$

Step 2. We partition $d_{n+1}^{(3)} = S_{n+1}^{(3)} - S_{n+1}^{(4)}$ with

$$\begin{aligned} S_{n+1}^{(3)} &= \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} A\big(s, y(s)\big) y'(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}s\,, \\ S_{n+1}^{(4)} &= \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \mathcal{M}_n^{\mathrm{ex}}\, y'(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}s\,. \end{aligned} \tag{84}$$

Because $S_{n+1}^{(3)}$ has the same structure as the terms (63) and (64), we obtain as in the previous step

$$\left\|\sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} S_{n+1}^{(3)}\right\|_{\ell^1} \leq \varepsilon\tau\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big)\,. \tag{85}$$

Moreover, substituting the tDMNLS for $y'(\sigma)$ the m-th entry of $S_{n+1}^{(4)}$ reads

$$\begin{aligned} [S_{n+1}^{(4)}]_m = -\frac{1}{2}\sum_{I_m}\sum_{I_l} y_j(t_n)\overline{y}_k(t_n) \int_{-1}^{1} \exp\left(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right) \mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \widehat{Y}_{pqrl}(\sigma) \exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right) \mathrm{d}\sigma\, \mathrm{d}s\,, \end{aligned} \tag{86}$$

with

$$\widehat{Y}_{pqrl}(\sigma) = y_p(\sigma)\overline{y}_q(\sigma)y_r(\sigma)\exp(-\mathrm{i}\omega_{[pqrl]}\sigma\alpha)\,.$$

For fixed $m \in \mathbb{Z}$, $(j,k,l) \in I_m$, and $(p,q,r) \in I_l$, we write $\omega = \omega_{[pqrl]}$, $\tilde{\omega} = \omega_{[jklm]}$ and $\widehat{Y}(s) = \widehat{Y}_{pqrl}(s)$ for short. In addition, we abbreviate

$$f(s) = y_j(s)\overline{y}_k(s) \quad \text{and} \quad \widehat{K}_n = \int_{-1}^{1} \exp\left(-\mathrm{i}\tilde{\omega}\widehat{\phi}\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right) \mathrm{d}\xi\,. \tag{87}$$

Now, we decompose any summand of (86) into

$$f(t_n)\widehat{K}_n \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \widehat{Y}(\sigma) \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma}{\varepsilon}\right)\right) \mathrm{d}\sigma\, \mathrm{d}s = f(t_n)\widehat{Y}(t_n)\widehat{K}_n\mathcal{I}_n + \mathcal{R}_n^{(1)}\,,$$

where $\mathcal{I}_n$ is given by (70) with $\tilde{\omega} = 0$ and

$$\mathcal{R}_n^{(1)} = f(t_n)\widehat{K}_n \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{\sigma_1} \widehat{Y}'(\sigma_2)\,\mathrm{d}\sigma_2 \exp\left(-\mathrm{i}\omega\phi\left(\tfrac{\sigma_1}{\varepsilon}\right)\right) \mathrm{d}\sigma_1\,\mathrm{d}s\,. \tag{88}$$

Moreover, we observe that

$$\widehat{K}_n = K_n \exp\left(-\mathrm{i}\tilde{\omega}\alpha t_n\right) - \tau \mathrm{i}\tilde{\omega}\alpha \int_{-1}^{1} \exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right) \int_0^{\xi} \exp\left(-\mathrm{i}\tilde{\omega}\alpha(t_n+\tau\theta)\right) \mathrm{d}\theta\,\mathrm{d}\xi\,,$$

with

$$K_n := \int_{-1}^{1} \exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right) \mathrm{d}\xi\,. \tag{89}$$

Hence, if we define

$$\widehat{F}(s) = f(s)\exp\left(-\mathrm{i}\tilde{\omega}\alpha s\right)\widehat{Y}(s)\,, \tag{90}$$

we can write any summand of (86) in terms of

$$f(t_n)\widehat{Y}(t_n)\widehat{K}_n\mathcal{I}_n + \mathcal{R}_n^{(1)} = \widehat{F}(t_n)K_n\mathcal{I}_n + \mathcal{R}_n^{(1)} - \mathcal{R}_n^{(2)}\,,$$

where $\mathcal{R}_n^{(1)}$ is given by (88) and

$$\mathcal{R}_n^{(2)} = \mathrm{i}\tau\alpha\tilde{\omega} f(t_n)\widehat{Y}(t_n)\mathcal{I}_n \int_{-1}^{1} \exp\left(-\mathrm{i}\tilde{\omega}\phi\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\right) \int_0^{\xi} \exp\left(-\mathrm{i}\tilde{\omega}\alpha(t_n+\tau\theta)\right) \mathrm{d}\theta\,\mathrm{d}\xi\,. \tag{91}$$

It is clear that

$$\left|\sum_{n=1}^{2kL-1} \mathcal{R}_n^{(1)}\right| \leq \tau^2 C(T) \max_{\sigma\in[0,T]} \left|f(\sigma)\widehat{Y}'(\sigma)\right| \tag{92}$$

and since $\mathcal{I}_n = \mathcal{O}(\tau^2)$, we have

$$\left|\sum_{n=1}^{2kL-1} \mathcal{R}_n^{(2)}\right| \leq \tau^2 \alpha C(T) \max_{\sigma\in[0,T]} \left|\tilde{\omega} f(\sigma)\widehat{Y}(\sigma)\right|\,. \tag{93}$$

Furthermore, we observe that

$$K_{2n} = K_{2(k-n)}, \quad \text{for} \quad n = 1,\dots,k/2-1$$

and

$$K_{2n-1} = K_{2(k-n)+1}, \quad \text{for} \quad n = 1,\dots,k/2\,,$$

and hence Lemma 6.2 gives the estimate

$$\left|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} \widehat{F}(t_n)K_n\mathcal{I}_n\right| \leq \varepsilon\tau \max_{\sigma\in[0,T]} \left|\widehat{F}'(\sigma)\right|\,. \tag{94}$$

Because the principle (16), (18), and Lemma A.1 (see Appendix) imply suitable bounds for the terms

$$\max_{\sigma\in[0,T]} \left|f(\sigma)\widehat{Y}'(\sigma)\right|\,, \quad \max_{\sigma\in[0,T]} \left|\tilde{\omega} f(\sigma)\widehat{Y}(\sigma)\right| \quad \text{and} \quad \max_{\sigma\in[0,T]} \left|\widehat{F}'(\sigma)\right|\,,$$

we can combine (92), (93) and (94) to obtain

$$\Bigg\| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} S_{n+1}^{(4)} \Bigg\|_{\ell^1} \le \tau\varepsilon\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big)\,. \tag{95}$$

Finally, it follows from (85) and (95) that

$$\Bigg\| \sum_{\substack{n=1\\ n \text{ even}}}^{2kL-1} d_{n+1}^{(3)} \Bigg\|_{\ell^1} \le \tau\varepsilon\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big)\,. \tag{96}$$

Step 3. A short computation gives the partition $d_{n+1}^{(4)} = S_{n+1}^{(5)} + S_{n+1}^{(6)}$, with

$$S_{n+1}^{(5)} = \int_{t_{n-1}}^{t_{n+1}} 2(t_n - s)\mathcal{M}_n A\big(s, y(s)\big)y(t_n)\,\mathrm{d}s\,, \tag{97}$$

$$S_{n+1}^{(6)} = \int_{t_{n-1}}^{t_{n+1}} 2(\tau - t_n)\mathcal{M}_n \Big(A\big(s, y(s)\big) - A\big(s, y(t_n)\big)\Big)y(t_n)\,\mathrm{d}s\,. \tag{98}$$

Because of the relation

$$y_j(s)\overline{y}_k(s) - y_j(t_n)\overline{y}_k(t_n) = \overline{y}_k(s)\int_{t_n}^{s} y_j'(\sigma)\,\mathrm{d}\sigma + y_j(t_n)\int_{t_n}^{s} \overline{y}_k'(\sigma)\,\mathrm{d}\sigma\,,$$

the estimate

$$\Big\|\Big(A\big(s, y(s)\big) - A\big(s, y(t_n)\big)\Big)y(t_n)\Big\|_{\ell^1} \le \tau C(M_0^y)$$

follows from (18), and hence we obtain with (30) the bound

$$\Bigg\| \sum_{n=1}^{2kL-1} S_{n+1}^{(6)} \Bigg\|_{\ell^1} \le \tau^2 C(T, M_0^y)\,. \tag{99}$$

The term (97) requires more attention. First, we expand $S_{n+1}^{(5)} = T_{n+1}^{(1)} + T_{n+1}^{(2)}$ with

$$T_{n+1}^{(1)} = \int_{t_{n-1}}^{t_{n+1}} 2(t_n - s)\Big(\mathcal{M}_n - \mathcal{M}_n^{\mathrm{ex}}\Big)A\big(s, y(s)\big)y(t_n)\,\mathrm{d}s\,,$$

$$T_{n+1}^{(2)} = \mathcal{M}_n^{\mathrm{ex}} \int_{t_{n-1}}^{t_{n+1}} 2(t_n - s)A\big(s, y(s)\big)y(t_n)\,\mathrm{d}s\,.$$

By (60) and Lemma 3.1, we obtain

$$\Bigg\| \sum_{n=1}^{2kL-1} T_{n+1}^{(1)} \Bigg\|_{\ell^1} \le \tau^2 C(M_0^y) \sum_{n=1}^{2kL-1} \|\mathbf{e}_n\|_{\ell^1}\,. \tag{100}$$

Moreover, let $[T_{n+1}^{(2)}]_m$ denote the m-th entry of $T_{n+1}^{(2)}$. Then, we have

$$[T_{n+1}^{(2)}]_m = \sum_{I_m}\sum_{I_l} y_j(t_n)\overline{y}_k(t_n)\int_{-1}^{1} \exp\Big(-\mathrm{i}\omega_{[jklm]}\widehat{\phi}\left(\tfrac{t_n+\tau\xi}{\varepsilon}\right)\Big)\,\mathrm{d}\xi$$
$$\int_{t_{n-1}}^{t_{n+1}} (s - t_n)\widehat{Y}_{pqrl}(s)\exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s\,. \tag{101}$$

With the abbreviations (87) and (90) any fixed summand of (101) reads

$$f(t_n)\widehat{K}_n \int_{t_{n-1}}^{t_{n+1}} (s-t_n)\widehat{Y}(s)\exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}s = \widehat{F}(t_n)K_n\mathcal{I}_n + \tilde{\mathcal{R}}_n^{(1)} - \mathcal{R}_n^{(2)}\,,$$

where $\mathcal{I}_n$ is given by (70) with $\tilde{\omega} = 0$, $\mathcal{R}_n^{(2)}$ is given by (91) and

$$\tilde{\mathcal{R}}_n^{(1)} = f(t_n)\widehat{K}_n \int_{t_{n-1}}^{t_{n+1}} \int_{t_n}^{s} (s-t_n)\widehat{Y}'(\sigma)\exp\left(-\mathrm{i}\omega_{[pqrl]}\phi\left(\tfrac{s}{\varepsilon}\right)\right)\mathrm{d}\sigma\,\mathrm{d}s\,.$$

Because we have

$$\left|\sum_{n=1}^{2kL-1} \tilde{\mathcal{R}}_n^{(1)}\right| \le \tau^2 C(T, M_0^y) \max_{\sigma\in[0,T]} \left|\widehat{Y}'(\sigma)\right|\,,$$

we obtain analogously to (95) with Lemma 6.2 the estimate

$$\left|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} T_{n+1}^{(2)}\right| \le \varepsilon\tau\big(C(T, M_0^y + \alpha C(T, M_2^y)\big)\,. \tag{102}$$

Now, we recall that

$$d_{n+1}^{(4)} = S_{n+1}^{(5)} + S_{n+1}^{(6)} = T_{n+1}^{(1)} + T_{n+1}^{(2)} + S_{n+1}^{(6)}\,,$$

and hence combining (99), (100), and (102) results in

$$\left\|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} d_{n+1}^{(4)}\right\|_{\ell^1} \le \tau^2 C(M_0^y) \sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} \|\mathbf{e}_n\|_{\ell^1} + \tau\varepsilon\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big)\,. \tag{103}$$

Finally, substituting the estimates (61), (80), (83), (96), and (103) into (79) gives the bound

$$\left\|\sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} \mathbf{d}_{n+1}\right\|_{\ell^1} \le \tau C(M_0^y) \sum_{\substack{n=1\\ n\text{ even}}}^{2kL-1} \|\mathbf{e}_n\|_{\ell^1} + \tau\varepsilon\big(C(T, M_0^y) + \alpha C(T, M_2^y)\big)\,. \tag{104}$$

Analogously, one can show a similar bound for the sum over the odd indices resulting in the desired estimate (66).

6.4 Proof of Theorem 5.4

In the setting of Theorem 5.4, i.e. $\tau = \varepsilon k$ for $k \in \mathbb{N}$, we consider approximations of the tDMNLS by the adiabatic exponential midpoint rule (32) as approximations of the limit system (25). According to Theorem 4.1, we have

$$\begin{aligned}\left\|y(t_n) - y^{(n)}\right\|_{\ell^1} &\le \|y(t_n) - v(t_n)\|_{\ell^1} + \left\|v(t_n) - y^{(n)}\right\|_{\ell^1}\\ &\le \frac{\varepsilon^2}{\delta} C(T, \alpha, M_2) + \left\|v(t_n) - y^{(n)}\right\|_{\ell^1}\,.\end{aligned} \tag{105}$$

Now, it remains to show that the numerical solution $y^{(n)}$ yields sufficiently accurate approximations of the exact solution $v(t_n)$ of the limit system. For this purpose, we require a technical lemma from [26] concerning estimates for integrals over products of the function

$$g_\omega(\sigma) = \exp\big(-\mathrm{i}\omega\phi(\sigma)\big) - \frac{\exp(\mathrm{i}\omega\delta) - 1}{\mathrm{i}\omega\delta}\,, \qquad \omega \neq 0\,, \tag{106}$$

with a sufficiently smooth function.

Lemma 6.4 (cf. Lemma 1 in [26]). *Let $\varepsilon > 0$, $\omega \neq 0$, $f \in C^2(\mathbb{R})$, and let g_ω be as in* (106)*. Then we have*

$$(i) \qquad \left| \int_0^2 f(\varepsilon\sigma) g_\omega(\sigma)\,\mathrm{d}\sigma \right| \leq \frac{\varepsilon^2}{\delta} C \max_{\sigma\in[0,2]} \left|\omega^{-1} f''(\varepsilon\sigma)\right|$$

and

$$(ii) \qquad \left| \int_1^3 f(\varepsilon\sigma) g_\omega(\sigma)\,\mathrm{d}\sigma \right| \leq \frac{\varepsilon^2}{\delta} C \max_{\sigma\in[1,3]} \left|\omega^{-1} f''(\varepsilon\sigma)\right| .$$

Next, we define

$$\begin{aligned} \mathbf{v}(t_{n+1}) &= \begin{pmatrix} v(t_{n+1}) \\ v(t_n) \end{pmatrix}, & \tilde{\mathbf{e}}_N &= \mathbf{y}_N - \mathbf{v}(t_N), \\ \tilde{\mathbf{d}}_1 &= \tilde{\mathbf{e}}_1\,, & \tilde{\mathbf{d}}_{n+1} &= \mathbf{M}_n \mathbf{v}(t_n) - \mathbf{v}(t_{n+1}) \quad \text{for } n \geq 1. \end{aligned} \tag{107}$$

As in the proof of Theorem 5.3, this allows us to express the global error $\tilde{\mathbf{e}}_N$ by the recursion formula

$$\tilde{\mathbf{e}}_N = \mathbb{M}_1 \tilde{\mathbf{d}}_1 + \sum_{n=1}^{N-1} \mathbb{M}_{n+1} \tilde{\mathbf{d}}_{n+1}$$

similar to (46). With (48) and (47), we obtain

$$\|\tilde{\mathbf{e}}_N\|_{\ell^1} \leq \tau^2 C(T, M_0^v) + C(T, M_0^v) \sum_{n=1}^{N-1} \left\|\tilde{\mathbf{d}}_{n+1}\right\|_{\ell^1} . \tag{108}$$

In the following, we aim for the bound

$$\sum_{n=1}^{N-1} \left\|\tilde{\mathbf{d}}_{n+1}\right\|_{\ell^1} \leq \left(\tfrac{\varepsilon^2}{\delta} + \tau^2\right)\left(C(T, M_0^v) + (\alpha + \alpha^2) C(T, M_2^v)\right). \tag{109}$$

If (109) is shown, the desired result follows from the discrete Gronwall lemma as in the previous proofs. In particular, we observe that the constant for the global error bound improves as specified if $\alpha = 0$. The key difference to the proof of Theorem 5.2 is that higher order time derivatives of the solution v of the limit equation exist. Hence, higher order Taylor expansions of v are available, whereas we were restricted to the first-order time derivative y' before. As in (53), we have

$$\tilde{\mathbf{d}}_{n+1} = \begin{pmatrix} \tilde{d}_{n+1} \\ 0 \end{pmatrix}, \quad \text{with} \quad \tilde{d}_{n+1} := \exp\big(2\tau\mathcal{M}_n\big) v(t_{n-1}) - v(t_{n+1})\,, \tag{110}$$

where $\mathcal{M}_n$ is given by (42). Thus it remains to derive an estimate for the non-zero part $\tilde{d}_{n+1}$ of $\tilde{\mathbf{d}}_{n+1}$. Thanks to the variation of constant formula we have

$$\begin{aligned} v(t_{n+1}) &= \exp\big(2\tau\mathcal{M}_n\big)v(t_{n-1}) \\ &\quad + \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau-s)\mathcal{M}_n\big)\,\big(A^{\mathrm{lim}}\big(s,v(s)\big)-\mathcal{M}_n\big)\,v(s)\,\mathrm{d}s\,, \end{aligned} \tag{111}$$

cf. (55). Henceforth, we abbreviate

$$\mathcal{M}_n^{\mathrm{lim}} := \mathcal{M}_n[\tau, v(t_n)]$$

in the spirit of (42). Substituting (111) into (110) and using the expansion (40) gives

$$\tilde{d}_{n+1} = -\big(\tilde{d}_{n+1}^{(1)} + \tilde{d}_{n+1}^{(2)} + \tilde{d}_{n+1}^{(3)} + \tilde{d}_{n+1}^{(4)} + \tilde{R}_{n+1}\big)\,, \tag{112}$$

where

$$\tilde{d}_{n+1}^{(1)} = \int_{t_{n-1}}^{t_{n+1}} \exp\big(2(\tau-s)\mathcal{M}_n\big)\big(\mathcal{M}_n^{\mathrm{lim}}-\mathcal{M}_n\big)v(s)\,\mathrm{d}s\,, \tag{113}$$

$$\tilde{d}_{n+1}^{(2)} = \int_{t_{n-1}}^{t_{n+1}} \Big(A^{\mathrm{lim}}\big(s,v(s)\big)-\mathcal{M}_n^{\mathrm{lim}}\Big)v(t_n)\,\mathrm{d}s\,, \tag{114}$$

$$\tilde{d}_{n+1}^{(3)} = \int_{t_{n-1}}^{t_{n+1}} (s-t_n)\Big(A^{\mathrm{lim}}\big(s,v(s)\big)-\mathcal{M}_n^{\mathrm{lim}}\Big)v'(t_n)\,\mathrm{d}s\,, \tag{115}$$

$$\tilde{d}_{n+1}^{(4)} = \int_{t_{n-1}}^{t_{n+1}} 2(\tau-s)\mathcal{M}_n\Big(A^{\mathrm{lim}}\big(s,v(s)\big)-\mathcal{M}_n^{\mathrm{lim}}\Big)v(t_n)\,\mathrm{d}s\,, \tag{116}$$

$$\begin{aligned} \tilde{R}_{n+1} &= \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{\sigma_1} \Big(A^{\mathrm{lim}}\big(s,v(s)\big)-\mathcal{M}_n^{\mathrm{lim}}\Big)v''(\sigma_2)\,\mathrm{d}\sigma_2\,\mathrm{d}\sigma_1\,\mathrm{d}s \\ &\quad + \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} 2(\tau-s)\mathcal{M}_n\Big(A^{\mathrm{lim}}\big(s,v(s)\big)-\mathcal{M}_n^{\mathrm{lim}}\Big)v'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \\ &\quad + \int_{t_{n-1}}^{t_{n+1}} \big(2(\tau-s)\mathcal{M}_n\big)^2\varphi_2\big(2(\tau-s)\mathcal{M}_n\big)\Big(A^{\mathrm{lim}}\big(s,v(s)\big)-\mathcal{M}_n^{\mathrm{lim}}\Big)v(s)\,\mathrm{d}s\,, \end{aligned}$$

From (38), (30) and (44) it follows that

$$\sum_{n=1}^{N-1}\Big\|\tilde{d}_{n+1}^{(1)}\Big\|_{\ell^1} \le \tau C(M_0)\sum_{n=1}^{N-1}\|\tilde{\mathrm{e}}_n\|_{\ell^1}\ , \tag{117}$$

with M_0 given in (26), cf. (61). Moreover, we obtain

$$\sum_{n=1}^{N-1}\Big\|\tilde{R}_{n+1}\Big\|_{\ell^1} \le \tau^2\big(C(T,M_0)+\alpha C(T,M_2^v)\big)\,, \tag{118}$$

with (30), Lemma 3.1, (41), and Lemma A.1 (see Appendix). In the next three steps, we derive bounds for $\tilde{d}_{n+1}^{(2)}$, $\tilde{d}_{n+1}^{(3)}$ and $\tilde{d}_{n+1}^{(4)}$.

Step 1. A short computation yields

$$\tilde{d}^{(2)}_{n+1} = \int_{t_{n-1}}^{t_{n+1}} \Big(A^{\lim}\big(s, v(s)\big) - A\big(s, v(t_n)\big)\Big) v(t_n)\,\mathrm{d}s\,,$$

and hence the m-th entry of $\tilde{d}^{(2)}_{n+1}$ can be split into $[\tilde{d}^{(2)}_{n+1}]_m = [\tilde{S}^{(1)}_{n+1}]_m - [\tilde{S}^{(2)}_{n+1}]_m$, with

$$[\tilde{S}^{(1)}_{n+1}]_m = \mathrm{i}\sum_{I_m}\int_{t_{n-1}}^{t_{n+1}} \big(v_j(s)\overline{v}_k(s) - v_j(t_n)\overline{v}_k(t_n)\big) v_l(t_n) \\ \exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\,\mathrm{d}s \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi\,, \tag{119}$$

$$[\tilde{S}^{(2)}_{n+1}]_m = \mathrm{i}\sum_{I_m} v_j(t_n)\overline{v}_k(t_n)v_l(t_n)\int_{t_{n-1}}^{t_{n+1}} \exp(-\mathrm{i}\omega_{[jklm]}\alpha s) \\ \left(\exp\left(-\mathrm{i}\omega_{[jklm]}\phi\left(\tfrac{s}{\varepsilon}\right)\right) - \int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi\right)\mathrm{d}s\,. \tag{120}$$

Further, we decompose $[\tilde{S}^{(1)}_{n+1}]_m = [\tilde{T}^{(1)}_{n+1}]_m + [\tilde{T}^{(2)}_{n+1}]_m$ with

$$[\tilde{T}^{(1)}_{n+1}]_m = \mathrm{i}\sum_{I_m}\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} v_j(t_n)\overline{v}'_k(\sigma)v_l(t_n)\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\,\mathrm{d}\sigma\,\mathrm{d}s \tag{121}$$

and

$$[\tilde{T}^{(2)}_{n+1}]_m = \mathrm{i}\sum_{I_m}\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s} v'_j(\sigma)\overline{v}_k(s)v_l(t_n)\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\,\mathrm{d}\sigma\,\mathrm{d}s\,.$$

Then, fixing $\overline{v}'_k(\sigma)$ at $\sigma = t_n$ followed by fixing $\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)$ at $s = t_n$ yields $[\tilde{T}^{(1)}_{n+1}]_m = [\tilde{R}^{(1)}_{n+1}]_m + [\tilde{R}^{(2)}_{n+1}]_m$, where

$$[\tilde{R}^{(1)}_{n+1}]_m = \mathrm{i}\sum_{I_m}\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}}\int_{t_n}^{s}\int_{t_n}^{\sigma_1} v_j(t_n)\overline{v}''_k(\sigma_2)v_l(t_n)\exp(-\mathrm{i}\omega_{[jklm]}\alpha s)\,\mathrm{d}\sigma_2\,\mathrm{d}\sigma_1\,\mathrm{d}s$$

and

$$[\tilde{R}^{(2)}_{n+1}]_m = \alpha\sum_{I_m}\int_0^1 \exp(\mathrm{i}\omega_{[jklm]}\delta\xi)\,\mathrm{d}\xi \\ \int_{t_{n-1}}^{t_{n+1}}(s - t_n)\int_{t_n}^{s} \omega_{[jklm]}v_j(t_n)\overline{v}'_k(t_n)v_l(t_n)\exp(-\mathrm{i}\omega_{[jklm]}\alpha\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\,.$$

Because estimates for

$$\big|v_j(t)\overline{v}''_k(t)v_l(t)\exp(-\mathrm{i}\omega_{[jklm]}\alpha t)\big| = |v_j(t)\overline{v}''_k(t)v_l(t)|$$

and

$$\left|\omega_{[jklm]}v_j(t)\overline{v}_k'(t)v_l(t)\exp(-\mathrm{i}\omega_{[jklm]}\alpha t)\right| = \left|\omega_{[jklm]}v_j(t)\overline{v}_k'(t)v_l(t)\right|$$

follow from Lemma A.1 (see Appendix), we infer

$$\left\|\tilde{R}^{(1)}_{n+1}\right\|_{\ell^1} \le \tau^3\big(C(M_0^v) + \alpha C(M_2^v)\big) \quad \text{and} \quad \left\|\tilde{R}^{(2)}_{n+1}\right\|_{\ell^1} \le \tau^3\alpha C(M_2^v)\,.$$

Hence, we obtain

$$\left\|\tilde{T}^{(1)}_{n+1}\right\|_{\ell^1} \le \tau^3\big(C(M_0^v) + \alpha C(M_2^v)\big)\,.$$

Since an estimate for $\tilde{T}^{(2)}_{n+1}$ follows analogously, we get

$$\sum_{n=1}^{N-1}\left\|\tilde{S}^{(1)}_{n+1}\right\|_{\ell^1} \le \tau^2\big(C(T,M_0^v) + \alpha C(T,M_2^v)\big)\,. \tag{122}$$

In order to bound the sum over the terms $\tilde{S}^{(2)}_{n+1}$, we aim to apply Lemma 6.4. For fixed $m \in \mathbb{Z}$ and $(j,k,l) \in I_m$ we write $\omega = \omega_{[jklm]}$ and $V(s) = v_j(s)\overline{v}_k(s)v_l(s)$. Moreover, we define $f_\omega(s) \coloneqq \exp(-\mathrm{i}\omega\alpha s)$. Then, any fixed summand of $\tilde{S}^{(2)}_{n+1}$ reads

$$\begin{aligned}
V(t_n)\int_{t_{n-1}}^{t_{n+1}} f_\omega(s)&\left(\exp\left(-\mathrm{i}\omega\phi\left(\tfrac{s}{\varepsilon}\right)\right) - \int_0^1 \exp(\mathrm{i}\omega\delta\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \\
&= V(t_n)\int_{t_{n-1}}^{t_{n+1}} f_\omega(s) g_\omega\left(\tfrac{s}{\varepsilon}\right)\mathrm{d}s \\
&= \varepsilon V(t_n)\int_0^{2k} f_\omega(\varepsilon\sigma + t_{n-1}) g_\omega(\sigma)\,\mathrm{d}\sigma \\
&= \varepsilon V(t_n)\sum_{\kappa=1}^{k}\int_0^2 f_\omega\big(\varepsilon(\sigma + 2(\kappa-1)) + t_{n-1}\big) g_\omega(\sigma)\,\mathrm{d}\sigma\,,
\end{aligned}$$

where g_ω is the 2-periodic function given in (106). Because $\left|\omega^{-1}f_\omega''(s)\right| = \alpha^2\left|\omega\right|$, Lemma 6.4 implies

$$\left|\varepsilon V(t_n)\sum_{\kappa=1}^{k}\int_0^2 f_\omega\big(\varepsilon(\sigma + 2(\kappa-1)) + t_{n-1}\big) g_\omega(\sigma)\,\mathrm{d}\sigma\right| \le \tau\alpha^2\frac{\varepsilon^2}{\delta}C\left|\omega V(t_n)\right|\,,$$

and hence we obtain with the principle (16) and Lemma A.1 (see Appendix)

$$\sum_{n=1}^{N-1}\left\|\tilde{S}^{(2)}_{n+1}\right\|_{\ell^1} \le \alpha^2\frac{\varepsilon^2}{\delta}C(T,M_2^v)\,. \tag{123}$$

Combining (123) and (122) yields

$$\sum_{n=1}^{N-1}\left\|\tilde{d}^{(2)}_{n+1}\right\|_{\ell^1} \le \left(\tfrac{\varepsilon^2}{\delta} + \tau^2\right)\big(C(T,M_0^v) + (\alpha+\alpha^2)C(T,M_2^v)\big)\,. \tag{124}$$

Step 2. Since $\int_{t_{n-1}}^{t_{n+1}} (s - t_n) \mathcal{M}_n^{\lim} v'(t_n) \, \mathrm{d}s = 0$, we only have to estimate

$$\tilde{d}_{n+1}^{(3)} = \int_{t_{n-1}}^{t_{n+1}} (s - t_n) A^{\lim}\big(s, v(s)\big) v'(t_n) \, \mathrm{d}s \,.$$

Fixing $A^{\lim}\big(s, v(s)\big)$ at $s = t_n$ and bounding the remainder terms with Lemma A.1 (see Appendix) yields the estimate

$$\sum_{n=1}^{N-1} \Big\| \tilde{d}_{n+1}^{(3)} \Big\|_{\ell^1} \leq \tau^2 \big(C(T, M_0^v) + \alpha C(T, M_2^v) \big) \,. \tag{125}$$

Step 3. A short computation gives

$$\begin{aligned} \tilde{d}_{n+1}^{(4)} = & \int_{t_{n-1}}^{t_{n+1}} 2(\tau - s) \mathcal{M}_n \Big(A^{\lim}\big(s, v(s)\big) - A^{\lim}\big(s, v(t_n)\big) \Big) v(t_n) \, \mathrm{d}s \\ & - \int_{t_{n-1}}^{t_{n+1}} 2(s - t_n) \mathcal{M}_n A^{\lim}\big(s, v(t_n)\big) v(t_n) \, \mathrm{d}s \,. \end{aligned}$$

One can estimate the first term in (126) analogously to the term (98). Moreover, one can bound the second term by fixing $A^{\lim}\big(s, v(t_n)\big)$ at $s = t_n$. Then, the leading order term vanishes due to the symmetry of the integral and the remainder terms can be dealt with the principle (16) and Lemma A.1 (see Appendix). Ultimately, we obtain the estimate

$$\sum_{n=1}^{N-1} \Big\| \tilde{d}_{n+1}^{(4)} \Big\|_{\ell^1} \leq \tau^2 \big(C(T, M_0^v) + \alpha C(T, M_2^v) \big) \,. \tag{126}$$

Finally, combining (117), (124), (125), (126) and (118) yields the desired bound (109).

■

A Appendix

Let y and v be the solutions of the tDMNLS (12) and the limit system (25), respectively, and let

$$Y_{jkl}(t) = y_j(t) \overline{y}_k(t) y_l(t) \quad \text{and} \quad V_{jkl}(t) = V_j(t) \overline{V}_k(t) V_l(t) \,,$$

Lemma A.1. *If $y_0 \in \ell_3^2$, then*

$$(i) \quad \sum_{m \in \mathbb{Z}} \sum_{I_m} \big| \omega_{[jklm]} Y_{jkl}(t) \big| \leq C(M_2^y) \qquad \textit{for all} \quad t \in [0, T] \,,$$

Suppose that Assumption 1 holds. Let $v_0 \in \ell_3^2$, then

$$\begin{aligned} &(ii) \quad \|v'(t)\|_{\ell^1} \leq C(M_0^v) && \textit{for all} \quad t \in [0, T] \,, \\ &(iii) \quad \sum_{m \in \mathbb{Z}} \sum_{I_m} \big| \omega_{[jklm]} V_{jkl}(t) \big| \leq C(M_2^v) && \textit{for all} \quad t \in [0, T] \,, \\ &(iv) \quad \|v''(t)\|_{\ell^1} \leq C(M_0^v) + \alpha C(M_2^v) && \textit{for all} \quad t \in [0, T] \,, \end{aligned}$$

Proof. (i) Because

$$\omega_{[jklm]} = -2(k^2 + jk - jl + kl) \qquad \text{for} \quad (j,k,l) \in I_m\,,$$

we obtain with the principle (16)

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \sum_{I_m} \left|\omega_{[jklm]} Y_{jkl}(t)\right| &= 2 \sum_{m\in\mathbb{Z}} \sum_{I_m} \left|(k^2 + jk - jl + kl) Y_{jkl}(t)\right| \\ &\le 2\big(\|y(t)\|_{\ell^1_0}^2 \|y(t)\|_{\ell^1_2} + 3\,\|y(t)\|_{\ell^1_0} \|y(t)\|_{\ell^1_1}^2 \big) \\ &\le C(M_2^y)\,. \end{aligned} \tag{127}$$

(ii) follows like (18) and (iii) is the same as (i).
(iv) Differentiating (25) yields

$$\|v''(t)\|_{\ell^1} \le \sum_{m\in\mathbb{Z}} \sum_{I_m} \left|V'_{jkl}(t) - \mathrm{i}\omega_{[jklm]} \alpha V_{jkl}(t)\right|\,, \tag{128}$$

and hence (ii) and (iii) yield the desired estimate.

■

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