

NLS in the modulation space $M_{2,q}(\mathbb{R})$

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NLS IN THE MODULATION SPACE $M_{2,q}(\mathbb{R})$.

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ABSTRACT. We show the existence of weak solutions in the extended sense of the Cauchy problem for the cubic nonlinear Schrödinger equation in the modulation space $M_{2,q}^s(\mathbb{R})$, $1 \leq q \leq 2$ and $s \geq 0$. In addition, for either $s \geq 0$ and $1 \leq q \leq \frac{3}{2}$ or $\frac{3}{2} < q \leq 2$ and $s > \frac{2}{3} - \frac{1}{q}$ we show that the Cauchy problem is unconditionally wellposed in $M_{2,q}^s(\mathbb{R})$. It is done with the use of the differentiation by parts technique which had been previously used in the periodic setting.

1. INTRODUCTION AND MAIN RESULTS

In this paper we study the one dimensional cubic NLS:

$$(1) \quad \begin{cases} iu_t - u_{xx} \pm |u|^2 u = 0 & , (t, x) \in \mathbb{R}^2 \\ u(0, x) = u_0(x) & , x \in \mathbb{R} \end{cases}$$

with initial data u_0 in the modulation space $M_{2,q}^s(\mathbb{R})$. We are interested in existence of solutions and in unconditional wellposedness of the problem. Modulation spaces were introduced by Feichtinger in [6] and have been used extensively in the study of nonlinear dispersive equations. See [13] for many of their properties such as embeddings in other known function spaces and equivalent expressions for their norm. Since their introduction, they have become canonical for both time-frequency and phase-space analysis since they provide an excellent substitute in estimates that are known to fail on Lebesgue spaces.

Let us mention some already known results on local wellposedness of NLS (1) with initial data in a modulation space. From [6] (Proposition 6.9) it is known that for $s > 1/q'$ or $s \geq 0$ and $q = 1$ the modulation space $M_{p,q}^s(\mathbb{R})$ is a Banach algebra and therefore an easy Banach contraction principle argument together with the fact that $e^{it\Delta}$ is a bounded operator from $M_{p,q}^s(\mathbb{R})$ to itself (see [2] and [3]) implies that NLS (1) is locally wellposed for $u_0 \in M_{p,q}^s(\mathbb{R})$ with solution $u \in C([0, T]; M_{p,q}^s(\mathbb{R}))$, $T > 0$. Also in [8] the case $u_0 \in M_{2,q}(\mathbb{R})$, $2 \leq q < \infty$, was considered which is a space that does not belong to the previous family of Banach algebras.

The definition of modulation spaces is the following: Set $Q_0 = [-\frac{1}{2}, \frac{1}{2})$ and $Q_k = Q_0 + k$ for all $k \in \mathbb{Z}$. Consider a family of functions $\{\sigma_k\}_{k \in \mathbb{Z}} \subset C^\infty(\mathbb{R})$ satisfying

- $\exists c > 0 : \forall k \in \mathbb{Z} : \forall \eta \in Q_k : |\sigma_k(\eta)| \geq c$,
- $\forall k \in \mathbb{Z} : \text{supp}(\sigma_k) \subseteq \{\xi \in \mathbb{R} : |\xi - k| \leq 1\}$,
- $\sum_{k \in \mathbb{Z}} \sigma_k = 1$,

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- $\forall m \in \mathbb{N}_0 : \exists C_m > 0 : \forall k \in \mathbb{Z} : \forall \alpha \in \mathbb{N} : \alpha \leq m \Rightarrow \|D^\alpha \sigma_k\|_\infty \leq C_m$

and define the isometric decomposition operators

$$(2) \quad \square_k := \mathcal{F}^{(-1)} \sigma_k \mathcal{F}, \quad (\forall k \in \mathbb{Z}).$$

Then the norm of a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ in the modulation space $M_{p,q}^s(\mathbb{R})$, $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, is

$$(3) \quad \|f\|_{M_{p,q}^s} := \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \|\square_k f\|_p^q \right)^{\frac{1}{q}} = \left(\sum_{k \in \mathbb{Z}} (1 + |k|^2)^{\frac{sq}{2}} \|\square_k f\|_p^q \right)^{\frac{1}{q}},$$

with the usual interpretation when the index q is equal to infinity. Different choices of such sequences of functions $\{\sigma_k\}_{k \in \mathbb{Z}}$ lead to equivalent norms in $M_{p,q}^s(\mathbb{R})$. When $s = 0$ we denote the space $M_{p,q}^0(\mathbb{R})$ by $M_{p,q}(\mathbb{R})$. In the special case where $p = q = 2$ we have $M_{2,2}^s(\mathbb{R}) = H^s(\mathbb{R})$ the usual Sobolev spaces. Here we will use that for $s > 1/q'$ and $1 \leq p, q \leq \infty$, the embedding

$$(4) \quad M_{p,q}^s(\mathbb{R}) \hookrightarrow C_b(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{C} / f \text{ continuous and bounded}\},$$

and for $\left(1 \leq p_1 \leq p_2 \leq \infty, 1 \leq q_1 \leq q_2 \leq \infty, s_1 \geq s_2\right)$ or $\left(1 \leq p_1 \leq p_2 \leq \infty, 1 \leq q_2 < q_1 \leq \infty, s_1 > s_2 + \frac{1}{q_2} - \frac{1}{q_1}\right)$ the embedding

$$(5) \quad M_{p_1, q_1}^{s_1}(\mathbb{R}) \hookrightarrow M_{p_2, q_2}^{s_2}(\mathbb{R}),$$

are both continuous and can be found in [6] (Proposition 6.8 and Proposition 6.5). Also, by [13] it is known that for any $1 < p \leq \infty$ we have the embedding $M_{p,1}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which together with the fact that $M_{2,2}(\mathbb{R}) = L^2(\mathbb{R})$ and interpolation, imply that for any $p \in [2, \infty]$ we have the embedding $M_{p,p'}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$. Later in Subsection 2.4 we will use this fact for $p = 3$, that is

$$(6) \quad M_{3, \frac{3}{2}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R}).$$

In order to give a meaning to solutions of the NLS in $C([0, T], M_{2,q}(\mathbb{R}))$ and to the nonlinearity $\mathcal{N}(u) := u|u|^2$ we need the following definitions which first appeared in [4], [5] where power series solutions to the cubic NLS was studied (see also [7] for similar considerations on the KdV):

Definition 1. *A sequence of Fourier cutoff operators is a sequence of Fourier multiplier operators $\{T_N\}_{N \in \mathbb{N}}$ on $\mathcal{S}'(\mathbb{R})$ with multipliers $m_N : \mathbb{R} \rightarrow \mathbb{C}$ such that*

- m_N has compact support on \mathbb{R} for every $N \in \mathbb{N}$,
- m_N is uniformly bounded,
- $\lim_{N \rightarrow \infty} m_N(x) = 1$, for any $x \in \mathbb{R}$.

Definition 2. *Let $u \in C([0, T], M_{2,q}^s(\mathbb{R}))$. We say that $\mathcal{N}(u)$ exists and is equal to a distribution $w \in \mathcal{S}'((0, T) \times \mathbb{R})$ if for every sequence $\{T_N\}_{N \in \mathbb{N}}$ of Fourier cutoff operators we have*

$$(7) \quad \lim_{N \rightarrow \infty} \mathcal{N}(T_N u) = w,$$

in the sense of distributions on $(0, T) \times \mathbb{R}$.

Definition 3. We say that $u \in C([0, T], M_{2,q}^s(\mathbb{R}))$ is a weak solution in the extended sense of NLS (1) if the following are satisfied

- $u(0, x) = u_0(x)$,
- the nonlinearity $\mathcal{N}(u)$ exists in the sense of Definition 2,
- u satisfies (1) in the sense of distributions on $(0, T) \times \mathbb{R}$, where the nonlinearity $\mathcal{N}(u) = u|u|^2$ is interpreted as above.

Our main result which guarantees existence of weak solutions in the extended sense is the following:

Theorem 4. Let $1 \leq q \leq 2$ and $s \geq 0$. For $u_0 \in M_{2,q}^s(\mathbb{R})$ there exists a weak solution in the extended sense $u \in C([0, T]; M_{2,q}^s(\mathbb{R}))$ of NLS (1) with initial condition u_0 , where the time T of existence depends only on $\|u_0\|_{M_{2,q}^s}$. Moreover, the solution map is Lipschitz continuous.

Remark 5. The restriction on the range of q appears by the construction of the solution of the NLS. That is, we decompose the NLS into countably many parts and at the end we sum all of them together. In order for the summation to make sense in the appropriate space we obtain $1 \leq q < 3$ (see remarks after (78) below). Moreover, when estimating the resonant operator R_2^t in Lemma 10 the restriction $q \leq 2$ appears naturally.

The next theorem is about the unconditional wellposedness of NLS (1) with initial data in a modulation space, that is, uniqueness in $C([0, T], M_{2,q}^s(\mathbb{R}))$ without intersecting with any auxiliary function space (see [11] where this notion first appeared):

Theorem 6. For $u_0 \in M_{2,q}^s(\mathbb{R})$, with either $s \geq 0$ and $1 \leq q \leq \frac{3}{2}$ or $\frac{3}{2} < q \leq 2$ and $s > \frac{2}{3} - \frac{1}{q}$, the solution u with initial condition u_0 constructed in Theorem 4 is unique in $C([0, T], M_{2,q}^s(\mathbb{R}))$.

Remark 7. When $q = 2$ the value $s = \frac{1}{6}$ is also allowed in the previous theorem since then we have the space $M_{2,2}^{\frac{1}{6}}(\mathbb{R}) = H^{\frac{1}{6}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$.

For its proof we are going to use the differentiation by parts technique that was introduced in [1] to attack similar problems for the KdV equation but with periodic initial data. In [9] this technique was used to prove unconditional wellposedness of the periodic cubic NLS. In this paper we use this technique to attack an NLS with a continuous Fourier variable, in the sense that our initial data is far from being periodic. For this reason there are some major differences and some difficulties that do not occur in the periodic setting. We follow very closely the ideas of [9] but we have to replace numbers and estimates for sums of numbers by operators and estimates for sums of suitable operator norms. This will become clearer in the next section where the proofs of Theorem 4 and Theorem 6 will be given. Here let us mention that a similar approach was used in [14] to study the cubic NLS and the mKdV on the real line and obtain unconditional wellposedness results with

initial data in the Sobolev space $H^s(\mathbb{R})$. Finally, similar techniques were used in [12] to study the quadratic nonlinear Klein-Gordon equation.

Since we are interested in the space $M_{2,q}^s(\mathbb{R})$ there is a more convenient expression for its norm which is the one we are going to use in our calculations. Let us denote by $\tilde{\square}_k$ the frequency projection operator $\mathcal{F}^{(-1)}1_{[k,k+1]}\mathcal{F}$, where $1_{[k,k+1]}$ is the characteristic function of the interval $[k, k+1]$, $k \in \mathbb{Z}$. Then it can be proved that

$$(8) \quad \|f\|_{M_{2,q}^s} \approx \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \|\tilde{\square}_k f\|_2^q \right)^{\frac{1}{q}},$$

or in other words, the two norms are equivalent in $M_{2,q}^s(\mathbb{R})$.

To conclude this section, firstly, we need that for $S(t) = e^{it\Delta}$ the Schrödinger semigroup we have the equality:

$$(9) \quad \|S(t)f\|_2 = \|f\|_2,$$

and secondly, we need the multiplier estimate (see [13], Proposition 1.9):

Lemma 8. *Let $1 \leq p \leq \infty$ and $\sigma \in C_c^\infty(\mathbb{R})$. Then the multiplier operator $T_\sigma : S(\mathbb{R}) \rightarrow S'(\mathbb{R})$ defined by*

$$(T_\sigma f) = \mathcal{F}^{-1}(\sigma \cdot \hat{f}), \quad \forall f \in S(\mathbb{R})$$

is bounded on $L^p(\mathbb{R})$ and

$$\|T_\sigma\|_{L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})} \lesssim \|\tilde{\sigma}\|_{L^1(\mathbb{R})}.$$

A useful consequence is that for $1 \leq p_1 \leq p_2 \leq \infty$ the following holds:

$$(10) \quad \|\square_k f\|_{p_2} \lesssim \|\square_k f\|_{p_1},$$

where the implicit constant is independent of k and the function f . This is done by considering a "fattened" function $\tilde{\sigma}_0$ which is identically 1 on the support of σ_0 and then by defining $\tilde{\sigma}_k(\xi) = \tilde{\sigma}_0(\xi - k)$, $\tilde{\square}_k = \mathcal{F}^{-1}\tilde{\sigma}_k\mathcal{F}$, for $k \in \mathbb{Z}$, we have that

$$\|\square_k f\|_{p_2} = \|\tilde{\square}_k \square_k f\|_{p_2} = \|\mathcal{F}^{-1}(\tilde{\sigma}_k) * \square_k f\|_{p_2} \leq \|\mathcal{F}^{-1}(\tilde{\sigma}_k)\|_r \|\square_k f\|_{p_1} = \|\mathcal{F}^{-1}(\tilde{\sigma}_0)\|_r \|\square_k f\|_{p_1},$$

where we applied Young's inequality with indices $1 + \frac{1}{p_2} = \frac{1}{r} + \frac{1}{p_1}$ and we used that all $\tilde{\sigma}_k$ are translations of $\tilde{\sigma}_0$.

Let us also recall the following number theoretic fact (see [10], Theorem 315) which is going to be used throughout the proof of Theorem 4: Given an integer m , let $d(m)$ denote the number of divisors of m . Then we have

$$(11) \quad d(m) \lesssim e^{c \frac{\log m}{\log \log m}} = o(m^\epsilon),$$

for all $\epsilon > 0$.

Lastly, before we proceed into the next section let us fix the notation: For a number $1 \leq p \leq \infty$ we write p' for its dual exponent, that is the number that satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. We denote by $S'(\mathbb{R})$ the space of tempered distributions and by $D'(\mathbb{R})$ the space of distributions. For two quantities A, B (they can be functions or numbers) whenever we write $A \lesssim B$ we mean that there is a universal constant $C > 0$ such that $A \leq CB$.

Next section consists of four subsections. In Subsection 2.1 the first steps of the iteration process are presented and in Subsection 2.2 the tree notation and the induction step finish the infinite iteration procedure. Then, in Subsection 2.3 Theorem 4 is proved where the solution is constructed through an approximation by smooth solutions and in Subsection 2.4 the unconditional uniqueness of Theorem 6 is presented under the extra assumption that the solution lies in the space $C([0, T], L^3(\mathbb{R}))$.

2. PROOF OF THE MAIN THEOREMS

2.1. The first steps of the iteration process. In this subsection we present the first steps of the differentiation by parts technique adapted to the continuous setting, that is NLS (1) with initial data that is not periodic. Since it is the first time that this is done, we try to be detailed for the interested reader. We will also use the same notation as in [9] so that a direct comparison between the two papers can be made and the differences can be emphasised.

From here on, we consider only the case $s = 0$ in Theorem 4 since for $s > 0$ similar considerations apply. See Remark 24 at the end of Subsection 2.2 for a more detailed argument. Also, as we mentioned before we are going to use expression (8) for the norm in $M_{2,q}(\mathbb{R})$ and for convenience we will write \square_n instead of $\tilde{\square}_n$ and σ_k instead of $1_{[k,k+1]}$.

For $n \in \mathbb{Z}$ let us define

$$(12) \quad u_n(t, x) = \square_n u(t, x),$$

$$(13) \quad v(t, x) = e^{it\partial_x^2} u(t, x),$$

$$(14) \quad v_n(t, x) = e^{it\partial_x^2} u_n(t, x) = \square_n [e^{it\partial_x^2} u(t, x)] = \square_n v(t, x).$$

Also for $(\xi, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ we define the function

$$\Phi(\xi, \xi_1, \xi_2, \xi_3) = \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2,$$

which is equal to

$$\Phi(\xi, \xi_1, \xi_2, \xi_3) = 2(\xi - \xi_1)(\xi - \xi_3),$$

if $\xi = \xi_1 - \xi_2 + \xi_3$. Our main equation (1) implies that

$$(15) \quad i\partial_t u_n - (u_n)_{xx} \pm \square_n (|u|^2 u) = 0,$$

and by calculating ($u = \sum_k \square_k u$)

$$\square_n (u\bar{u}u) = \square_n \sum_{n_1, n_2, n_3} u_{n_1} \bar{u}_{n_2} u_{n_3} = \sum_{n_1 - n_2 + n_3 \approx n} \square_n [u_{n_1} \bar{u}_{n_2} u_{n_3}],$$

where by $\approx n$ we mean $= n$ or $= n + 1$ or $= n - 1$. Later, during the calculations we will also write $\xi \approx n$ where ξ is going to be a continuous variable and n an integer. By that we will mean that $\xi \in [n, n + 1)$ or more generally that ξ is in an interval around n .

Next we do the change of variables $u_n(t, x) = e^{-it\partial_x^2} v_n(t, x)$ and arrive at the expression

$$(16) \quad \partial_t v_n = \pm i \sum_{n_1 - n_2 + n_3 \approx n} \square_n \left(e^{it\partial_x^2} [e^{-it\partial_x^2} v_{n_1} \cdot e^{it\partial_x^2} \bar{v}_{n_2} \cdot e^{-it\partial_x^2} v_{n_3}] \right).$$

We continue by presenting the first steps of our splitting procedure. Define the 1st generation operators by

$$(17) \quad Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})(x) = \square_n \left(e^{it\partial_x^2} [e^{-it\partial_x^2} v_{n_1} \cdot e^{it\partial_x^2} \bar{v}_{n_2} \cdot e^{-it\partial_x^2} v_{n_3}] \right),$$

and continue with the splitting

$$(18) \quad \partial_t v_n = \pm i \sum_{n_1 - n_2 + n_3 \approx n} Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) = \sum_{\substack{n_1 \approx n \\ \text{or} \\ n_3 \approx n}} \dots + \sum_{n_1 \not\approx n \not\approx n_3} \dots$$

We define the resonant part

$$(19) \quad R_2^t(v)(n) - R_1^t(v)(n) = \left(\sum_{n_1 \approx n} Q_n^{1,t} + \sum_{n_3 \approx n} Q_n^{1,t} \right) - \sum_{\substack{n_1 \approx n \\ \text{and} \\ n_3 \approx n}} Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}),$$

with R_2^t being equal to the sum of the first two summands and R_1^t being equal with the last summand, and the non-resonant part

$$(20) \quad N_1^t(v)(n) = \sum_{n_1 \not\approx n \not\approx n_3} Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}),$$

which implies the following expression for our NLS (we drop the factor $\pm i$ in front of the sum since they will play no role in our analysis)

$$(21) \quad \partial_t v_n = R_2^t(v)(n) - R_1^t(v)(n) + N_1^t(v)(n).$$

Remark 9. In the following part of the paper a series of lemmata will be presented. Unless stated otherwise we will always assume that $1 \leq q < \infty$.

For the resonant part we have the lemma:

Lemma 10. For $j = 1, 2$

$$\|R_j^t(v)\|_{l^q L^2} \lesssim \|v\|_{M_{2,q}}^3,$$

and

$$\|R_j^t(v) - R_j^t(w)\|_{l^q L^2} \lesssim (\|v\|_{M_{2,q}}^2 + \|w\|_{M_{2,q}}^2) \|v - w\|_{M_{2,q}}.$$

Proof. Let us start with R_1^t . By definition for fixed n , $R_1^t(n)$ consists of finitely many summands, since $|n - n_1|, |n - n_3| \leq 1$ and $|n - n_2| \leq 3$. We will handle $Q_n^{1,t}(v_n, \bar{v}_n, v_n)$ and the remaining summands can be treated similarly. Since,

$$Q_n^{1,t}(v_n, \bar{v}_n, v_n) = \square_n \left(e^{it\partial_x^2} [e^{-it\partial_x^2} v_n \cdot e^{it\partial_x^2} \bar{v}_n \cdot e^{-it\partial_x^2} v_n] \right),$$

and since the Schrödinger operator is an isometry on L^2 our claim follows by Bernstein's inequality (see Lemma 8). For the difference $R_1^t(v) - R_1^t(w)$ we have to estimate terms of the form $|e^{-it\partial_x^2} v_n|^2 |e^{-it\partial_x^2} v_n - e^{-it\partial_x^2} w_n|$ in the $l^q L^2$ norm. For the L^2 norm we apply Hölder's inequality and obtain the upper bound

$$\|e^{-it\partial_x^2} v_n\|_8^2 \|e^{-it\partial_x^2} v_n - e^{-it\partial_x^2} w_n\|_4 \lesssim \|e^{-it\partial_x^2} v_n\|_2^2 \|e^{-it\partial_x^2} v_n - e^{-it\partial_x^2} w_n\|_2 = \|v_n\|_2^2 \|v_n - w_n\|_2,$$

where we used (10) and (9), and then proceed with the l^q norm as

$$\left(\sum_{n \in \mathbb{Z}} \|v_n\|_2^{2q} \|v_n - w_n\|_2^q \right)^{\frac{1}{q}} \leq \left(\sup_{n \in \mathbb{Z}} \|v_n\|_2^2 \right) \left(\sum_{n \in \mathbb{Z}} \|v_n - w_n\|_2^q \right)^{\frac{1}{q}} = \|v\|_{M_{2,\infty}}^2 \|v - w\|_{M_{2,q}}.$$

From (5) we have $\|v\|_{M_{2,\infty}} \leq \|v\|_{M_{2,q}}$ which finishes the proof. Similar considerations apply to all other lemmata of the paper where estimates of the same form appear.

For the R_2^t operator, it suffices to estimate the sum

$$\sum_{\substack{n_1 - n_2 + n_3 \approx n \\ n_1 \approx n}} Q_n^t(v_{n_1}, \bar{v}_{n_2}, v_{n_3})$$

which consists of finitely many sums depending on whether $n_1 = n - 1$, or $n_1 = n$, or $n_1 = n + 1$. Let us only treat

$$\square_n e^{it\partial_x^2} \left(e^{-it\partial_x^2} v_n \sum_{n_2 \in \mathbb{Z}} |e^{-it\partial_x^2} v_{n_2}|^2 \right),$$

since for the remaining sums similar considerations apply. The L^2 norm equals

$$\left\| \square_n \left(u_n \sum_{n_2 \in \mathbb{Z}} |u_{n_2}|^2 \right) \right\|_2 \lesssim \left\| u_n \sum_{n_2 \in \mathbb{Z}} |u_{n_2}|^2 \right\|_2 \leq \sum_{n_2 \in \mathbb{Z}} \left\| u_n |u_{n_2}|^2 \right\|_2 \leq \sum_{n_2 \in \mathbb{Z}} \|u_n\|_4 \|u_{n_2}\|_8^2,$$

where we used that the Schrödinger operator is an isometry in L^2 , Lemma 8 and Hölder's inequality. With the use of (10) this last sum is bounded from above by $\|u_n\|_2 \|u\|_{M_{2,2}}^2$ and since $1 \leq q \leq 2$ we can use the embedding $l^q \hookrightarrow l^2$ to arrive at $\|u_n\|_2 \|u\|_{M_{2,q}}^2$. Then, the l^q norm in the discrete variable implies

$$\|R_2^t(v)\|_{l^q L^2} \lesssim \|v\|_{M_{2,q}}^3.$$

□

For the non-resonant part N_1^t we have to split as

$$(22) \quad N_1^t(v)(n) = N_{11}^t(v)(n) + N_{12}^t(v)(n),$$

where

$$N_{11}^t(v)(n) = \sum_{A_N(n)} Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}),$$

and

$$(23) \quad A_N(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 \approx n, n_1 \not\approx n \not\approx n_3, |\Phi(n, n_1, n_2, n_3)| \leq N\}.$$

We also define the set

$$(24) \quad A_N(n)^c = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 \approx n, n_1 \not\approx n \not\approx n_3, |\Phi(n, n_1, n_2, n_3)| > N\}.$$

The number $N > 0$ is considered to be large and will be fixed at the end of the proof. With the use of inequality (11) we estimate N_{11}^t as follows:

Lemma 11.

$$\|N_{11}^t(v)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}+} \|v\|_{M_{2,q}}^3,$$

and

$$\|N_{11}^t(v) - N_{11}^t(w)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}+} (\|v\|_{M_{2,q}}^2 + \|w\|_{M_{2,q}}^2) \|v - w\|_{M_{2,q}}.$$

Proof. Obviously,

$$\|N_{11}^t(v)\|_{L^2} \leq \sum_{A_N(n)} \|Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_{L^2},$$

which from (9), Lemma 8 and Hölder's inequality is estimated above by

$$\sum_{A_N(n)} \|u_{n_1} \bar{u}_{n_2} u_{n_3}\|_{L^2} \leq \sum_{A_N(n)} \|u_{n_1}\|_{L^6} \|u_{n_2}\|_{L^6} \|u_{n_3}\|_{L^6}.$$

Here we make use of (10) and Hölder's inequality in the discrete variable to obtain the upper bound

$$\sum_{A_N(n)} \|u_{n_1}\|_{L^2} \|u_{n_2}\|_{L^2} \|u_{n_3}\|_{L^2} \leq \left(\sum_{A_N(n)} 1^{q'} \right)^{\frac{1}{q'}} \left(\sum_{A_N(n)} \|u_{n_1}\|_{L^2}^q \|u_{n_2}\|_{L^2}^q \|u_{n_3}\|_{L^2}^q \right)^{\frac{1}{q}}.$$

Fix n and $\mu \in \mathbb{Z}$ such that $|\mu| \leq N$. From (11) there are at most $o(N^{0+})$ many choices for n_1 and n_3 , and so for n_2 from $n \approx n_1 - n_2 + n_3$, satisfying

$$\mu = 2(n - n_1)(n - n_3).$$

Therefore, we arrive at

$$\|N_{11}^t(v)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}+} \left(\sum_{n \in \mathbb{Z}} \sum_{A_N(n)} \|u_{n_1}\|_{L^2}^q \|u_{n_2}\|_{L^2}^q \|u_{n_3}\|_{L^2}^q \right)^{\frac{1}{q}},$$

and this final summation is estimated by Young's inequality providing us with the bound ($\|u\|_{M_{2,q}} = \|v\|_{M_{2,q}}$)

$$\|N_{11}^t(v)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}+} \|v\|_{M_{2,q}}^3.$$

□

In order to continue, we have to look at the N_{12}^t part more closely keeping in mind that we are on $A_N(n)^c$. Our goal is to find a suitable splitting in order to continue our iteration. In the following we perform all formal calculations assuming that v is a sufficiently smooth solution. Later, in Subsection 2.4 we justify these formal computations also for $v \in C([0, T], M_{2,q}^s(\mathbb{R}))$, with $1 \leq q \leq \frac{3}{2}$, $s \geq 0$ or $\frac{3}{2} < q \leq 2$, $s > \frac{2}{3} - \frac{1}{q}$.

From (17) we know that

$$\mathcal{F}(Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi) = \sigma_n(\xi) \int_{\mathbb{R}^2} e^{-2it(\xi - \xi_1)(\xi - \xi_3)} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi - \xi_1 - \xi_3) \hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3,$$

and by the usual product rule for the derivative we can write the previous integral as the sum of the following expressions

$$(25) \quad \partial_t \left(\sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3 \right) - \\ \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \partial_t \left(\hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) \right) d\xi_1 d\xi_3.$$

Therefore, we have the splitting

$$(26) \quad \mathcal{F}(Q_n^{1,t}) = \partial_t \mathcal{F}(\tilde{Q}_n^{1,t}) - \mathcal{F}(T_n^{1,t})$$

or equivalently

$$(27) \quad Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) = \partial_t(\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})) - T_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}),$$

which allows us to write

$$(28) \quad N_{12}^t(v)(n) = \partial_t(N_{21}^t(v)(n)) + N_{22}^t(v)(n),$$

where

$$(29) \quad N_{21}^t(v)(n) = \sum_{A_N(n)^c} \tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}),$$

and

$$(30) \quad N_{22}^t(v)(n) = \sum_{A_N(n)^c} T_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}).$$

Moreover, we have

$$\mathcal{F}(\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{\hat{u}_{n_1}(\xi_1) \hat{u}_{n_2}(\xi-\xi_1-\xi_3) \hat{u}_{n_3}(\xi_3)}{(\xi-\xi_1)(\xi-\xi_3)} d\xi_1 d\xi_3,$$

and we define

$$(31) \quad \mathcal{F}(R_n^{1,t}(u_{n_1}, \bar{u}_{n_2}, u_{n_3}))(\xi) = \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{\hat{u}_{n_1}(\xi_1) \hat{u}_{n_2}(\xi-\xi_1-\xi_3) \hat{u}_{n_3}(\xi_3)}{(\xi-\xi_1)(\xi-\xi_3)} d\xi_1 d\xi_3,$$

which is the same as the operator

$$(32) \quad R_n^{1,t}(w_{n_1}, \bar{w}_{n_2}, w_{n_3})(x) = \int_{\mathbb{R}^3} e^{ix\xi} \sigma_n(\xi) \frac{\hat{w}_{n_1}(\xi_1) \hat{w}_{n_2}(\xi-\xi_1-\xi_3) \hat{w}_{n_3}(\xi_3)}{(\xi-\xi_1)(\xi-\xi_3)} d\xi_1 d\xi_3 d\xi.$$

Writing out the Fourier transforms of the functions inside the integral it is not difficult to see that

$$(33) \quad R_n^{1,t}(w_{n_1}, \bar{w}_{n_2}, w_{n_3})(x) = \int_{\mathbb{R}^3} K_n^{(1)}(x, x_1, y, x_3) w_{n_1}(x) \bar{w}_{n_2}(y) w_{n_3}(x_3) dx_1 dy dx_3,$$

where

$$K_n^{(1)}(x, x_1, y, x_3) = \int_{\mathbb{R}^3} e^{i\xi_1(x-x_1)+i\eta(x-y)+i\xi_3(x-x_3)} \frac{\sigma_n(\xi_1+\eta+\xi_3)}{(\eta+\xi_1)(\eta+\xi_3)} d\xi_1 d\eta d\xi_3 = \\ \mathcal{F}^{-1} \rho_n^{(1)}(x-x_1, x-y, x-x_3)$$

and

$$\rho_n^{(1)}(\xi_1, \eta, \xi_3) = \frac{\sigma_n(\xi_1 + \eta + \xi_3)}{(\eta + \xi_1)(\eta + \xi_3)}.$$

The important estimate that the operator $\tilde{Q}_n^{1,t}$ satisfies is described in:

Lemma 12.

$$(34) \quad \|\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim \frac{\|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|n - n_1| |n - n_3|}.$$

Proof. Observing that $\mathcal{F}(\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi)$ it suffices to estimate the L^2 norm of the operator $R_n^{1,t}$. By duality, let $g \in L^2$, $\|g\|_2 \neq 0$, and consider the pairing

$$(35) \quad \begin{aligned} |\langle R_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}), g \rangle| &= \left| \int_{\mathbb{R}} \mathcal{F}(R_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi) \mathcal{F}(g)(\xi) d\xi \right| = \\ &= \left| \int_{\mathbb{R}^3} \hat{g}(\xi) \sigma_n(\xi) \frac{\hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi - \xi_1 - \xi_3) \hat{v}_{n_3}(\xi_3)}{(\xi - \xi_1)(\xi - \xi_3)} d\xi d\xi_1 d\xi_3 \right| = \\ &= \left| \int_{\mathbb{R}^3} \hat{g}(\xi_1 + \eta + \xi_3) \frac{\sigma_n(\xi_1 + \eta + \xi_3)}{(\eta + \xi_1)(\eta + \xi_3)} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\eta) \hat{v}_{n_3}(\xi_3) d\eta d\xi_1 d\xi_3 \right| = \\ &= \left| \int_{I_{n_1}} \int_{I_{n_2}} \int_{I_{n_3}} \hat{g}(\xi_1 + \eta + \xi_3) \rho_n^{(1)}(\xi_1, \eta, \xi_3) \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\eta) \hat{v}_{n_3}(\xi_3) d\xi_1 d\eta d\xi_3 \right|, \end{aligned}$$

where these three intervals are the compact supports of the functions $\hat{v}_{n_1}, \hat{v}_{n_2}, \hat{v}_{n_3}$ (see (14)). By Hölder's inequality we obtain the upper bound

$$\|\rho_n^{(1)}\|_\infty \|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2 \left(\int_{I_{n_1}} \int_{I_{n_2}} \int_{I_{n_3}} |\hat{g}(\xi_1 + \eta + \xi_3)|^2 d\xi_1 d\eta d\xi_3 \right)^{\frac{1}{2}},$$

and the last triple integral is easily estimated by

$$\|\hat{g}\|_2 (|I_{n_2}| |I_{n_3}|)^{\frac{1}{2}} = \|g\|_2 (|I_{n_2}| |I_{n_3}|)^{\frac{1}{2}}.$$

Therefore, the following is true

$$\|\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_2 = \|R_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim \|\rho_n^{(1)}\|_\infty \|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2,$$

and since $\xi_1 \approx n_1$, $\eta \approx -n_2$ and $\xi_3 \approx n_3$ we obtain

$$\|\rho_n^{(1)}\|_\infty \lesssim \frac{1}{|n - n_1| |n - n_3|},$$

which finishes the proof. \square

Remark 13. Notice that Lemma 12 (this observation applies to Lemma 21 too) is true for any triple of functions f, g, h that lie in $M_{2,q}(\mathbb{R})$ and the only important property is that they are nicely localised on the Fourier side since we consider their box operators $\square_{n_1} f, \square_{n_2} g$ and $\square_{n_3} h$. Also, the same proof implies that the operator $Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})$ satisfies the estimate

$$(36) \quad \|Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim \|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2.$$

These observations will play an important role in Lemma 25 of Subsection 2.3 and Lemma 28 of Subsection 2.4.

Here is the estimate for the N_{21}^t operator:

Lemma 14.

$$\|N_{21}^t(v)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}-1+} \|v\|_{M_{2,q}}^3,$$

and

$$\|N_{21}^t(v) - N_{21}^t(w)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}-1+} (\|v\|_{M_{2,q}}^2 + \|w\|_{M_{2,q}}^2) \|v - w\|_{M_{2,q}}.$$

Proof. From Lemma 12 we have

$$\|N_{21}^t(v)\|_2 \leq \sum_{A_N(n)^c} \|\tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim \sum_{A_N(n)^c} \frac{\|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|n - n_1| |n - n_3|},$$

and by Hölder's inequality the upper bound

$$\left(\sum_{A_N(n)^c} \frac{1}{(|n - n_1| |n - n_3|)^{q'}} \right)^{\frac{1}{q'}} \left(\sum_{A_N(n)^c} \|v_{n_1}\|_2^q \|v_{n_2}\|_2^q \|v_{n_3}\|_2^q \right)^{\frac{1}{q}}.$$

The first sum (for $\mu = |n - n_1| |n - n_3|$) is estimated from above by (with the use of (11))

$$\left(\sum_{\mu=N+1}^{\infty} \frac{\mu^\epsilon}{\mu^{q'}} \right)^{\frac{1}{q'}} \sim (N^{\epsilon+1-q'})^{\frac{1}{q'}} = N^{\frac{1}{q'}-1+},$$

and then with the use of Young's inequality we arrive at

$$\|N_{21}^t(v)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}-1+} \|v\|_{M_{2,q}}^3$$

as claimed. \square

To the remaining part N_{22}^t we have to make use of equality (21) depending on whether the derivative falls on \hat{v}_{n_1} or \hat{v}_{n_2} or \hat{v}_{n_3} . Let us see how we can proceed from here:

$$N_{22}^t(v)(n) = -2i \sum_{A_N(n)^c} \left[\tilde{Q}_n^{1,t}(R_2^t(v)(n_1) - R_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3}) + \tilde{Q}_n^{1,t}(N_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3}) \right]$$

plus the corresponding term for $\partial_t \hat{v}_{n_2}$ (the number 2 that appears in front of the previous sum is because the expression is symmetric with respect to v_{n_1} and v_{n_3}). Therefore, we can write N_{22}^t as a sum

$$(37) \quad N_{22}^t(v)(n) = N_4^t(v)(n) + N_3^t(v)(n),$$

where $N_4^t(v)(n)$ is the sum with the resonant part $R_2^t - R_1^t$. The following Lemma is true:

Lemma 15.

$$\|N_4^t(v)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}-1+} \|v\|_{M_{2,q}}^5,$$

and

$$\|N_4^t(v) - N_4^t(w)\|_{l^q L^2} \lesssim N^{\frac{1}{q'}-1+} (\|v\|_{M_{2,q}}^4 + \|w\|_{M_{2,q}}^4) \|v - w\|_{M_{2,q}}.$$

Proof. Follows by Lemmata 10 and 14 in the sense that we repeat the proof of Lemma 14 and apply Lemma 10 to the part $R_2^t(v)(n_1) - R_1^t(v)(n_1)$. \square

To continue, we have to decompose N_3^t even further. It consists of 3 sums depending on where the operator N_1^t acts. One of them is the following (similar considerations apply for the remaining sums too)

$$(38) \quad \sum_{A_N(n)^c} \tilde{Q}_n^{1,t}(N_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3}),$$

where

$$N_1^t(v)(n_1) = \sum_{m_1 \not\approx n_1 \not\approx m_3} Q_{n_1}^{1,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}),$$

and $n_1 \approx m_1 - m_2 + m_3$. Here we have to consider new restrictions on the frequencies $(m_1, m_2, m_3, n_2, n_3)$ where the "new" triple of frequencies m_1, m_2, m_3 appears as a "child" of the frequency n_1 . Thus, we define the set $(\mu_1 = \Phi(n, n_1, n_2, n_3))$ and $\mu_2 = \Phi(n_1, m_1, m_2, m_3)$

$$(39) \quad C_1 = \{|\mu_1 + \mu_2| \leq 5^3 |\mu_1|^{1 - \frac{1}{100}}\},$$

and split the sum in (38) as

$$(40) \quad \sum_{A_N(n)^c} \sum_{C_1} \dots + \sum_{A_N(n)^c} \sum_{C_1^c} \dots = N_{31}^t(v)(n) + N_{32}^t(v)(n).$$

The following holds:

Lemma 16.

$$\|N_{31}^t(v)\|_{l^q L^2} \lesssim N^{\frac{2}{q'} - \frac{1}{100q'} - 1+} \|v\|_{M_{2,q}}^5,$$

and

$$\|N_{31}^t(v) - N_{31}^t(w)\|_{l^q L^2} \lesssim N^{\frac{2}{q'} - \frac{1}{100q'} - 1+} (\|v\|_{M_{2,q}}^4 + \|w\|_{M_{2,q}}^4) \|v - w\|_{M_{2,q}}.$$

Proof. From (11) we know that for fixed n and μ_1 , there are at most $o(|\mu_1|^+)$ many choices for n_1 and n_3 and for fixed n_1 and μ_2 there are at most $o(|\mu_2|^+)$ many choices for m_1 and m_3 . From (39) we can control μ_2 in terms of μ_1 , that is $|\mu_2| \sim |\mu_1|$. In addition, for fixed $|\mu_1|$ there are at most $O(|\mu_1|^{1 - \frac{1}{100}})$ many choices for μ_2 . Therefore,

$$\begin{aligned} \|N_{31}^t(v)\|_2 &\leq \sum_{A_N(n)^c} \sum_{C_1} \|\tilde{Q}_n^{1,t}(Q_{n_1}^{1,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}), \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim \\ &\sum_{A_N(n)^c} \sum_{C_1} \frac{\|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|n - n_1| |n - n_3|} \leq \\ &\left(\sum_{\mu=N+1}^{\infty} \frac{\mu^{1 - \frac{1}{100}+}}{\mu^{q'}} \right)^{\frac{1}{q'}} \left(\sum_{A_N(n)^c} \sum_{C_1} \|v_{m_1}\|_2^q \|v_{m_2}\|_2^q \|v_{m_3}\|_2^q \|v_{n_2}\|_2^q \|v_{n_3}\|_2^q \right)^{\frac{1}{q}}, \end{aligned}$$

and then by taking the l^q norm in n and applying Young's inequality we are led to the desired estimate. \square

For the N_{32}^t part we have to do the differentiation by parts technique which will create the 2nd generation operators. Our first 2nd generation operator $Q_n^{2,t}$ consists of three sums

$$\begin{aligned} q_{1,n}^{2,t} &= \sum_{A_N(n)^c} \sum_{C_1^c} \tilde{Q}_n^{1,t}(N_1^t(v)(n_1), \bar{v}_{n_2}, v_{n_3}), \\ q_{2,n}^{2,t} &= \sum_{A_N(n)^c} \sum_{C_1^c} \tilde{Q}_n^{1,t}(v_{n_1}, \overline{N_1^t(v)}(n_2), v_{n_3}), \\ q_{3,n}^{2,t} &= \sum_{A_N(n)^c} \sum_{C_1^c} \tilde{Q}_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, N_1^t(v)(n_3)). \end{aligned}$$

Let us have a look at the first sum $q_{1,n}^{2,t}$ (we treat the other two in a similar manner). Its Fourier transform is equal to

$$\sum_{A_N(n)^c} \sum_{C_1^c} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{(\xi-\xi_1)(\xi-\xi_3)} \mathcal{F}(N_1^t(v)(n_1))(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3,$$

where

$$\mathcal{F}(N_1^t(v)(n_1))(\xi_1)$$

equals

$$\sum_{\substack{n_1 \approx m_1 - m_2 + m_3 \\ m_1 \not\approx n_1 \not\approx m_3}} \sigma_{n_1}(\xi_1) \int_{\mathbb{R}^2} e^{-2it(\xi_1-\xi'_1)(\xi_1-\xi'_3)} \hat{v}_{m_1}(\xi'_1) \hat{v}_{m_2}(\xi_1-\xi'_1-\xi'_3) \hat{v}_{m_3}(\xi'_3) d\xi'_1 d\xi'_3.$$

Putting everything together and applying differentiation by parts we can write the integrals inside the sums as

$$\partial_t \left(\sigma_n(\xi) \int_{\mathbb{R}^4} \sigma_{n_1}(\xi_1) \frac{e^{-it(\mu_1+\mu_2)}}{\mu_1(\mu_1+\mu_2)} \hat{v}_{m_1}(\xi'_1) \hat{v}_{m_2}(\xi_1-\xi'_1-\xi'_3) \hat{v}_{m_3}(\xi'_3) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) d\xi'_1 d\xi'_3 d\xi_1 d\xi_3 \right)$$

minus

$$\sigma_n(\xi) \int_{\mathbb{R}^4} \sigma_{n_1}(\xi_1) \frac{e^{-it(\mu_1+\mu_2)}}{\mu_1(\mu_1+\mu_2)} \partial_t \left(\hat{v}_{m_1}(\xi'_1) \hat{v}_{m_2}(\xi_1-\xi'_1-\xi'_3) \hat{v}_{m_3}(\xi'_3) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) \right) d\xi'_1 d\xi'_3 d\xi_1 d\xi_3,$$

where $\mu_1 = (\xi - \xi_1)(\xi - \xi_3)$ and $\mu_2 = (\xi_1 - \xi'_1)(\xi_1 - \xi'_3)$. Equivalently,

$$(41) \quad \mathcal{F}(q_{1,n}^{2,t}) = \partial_t(\tilde{q}_{1,n}^{2,t}) - \mathcal{F}(\tau_{1,n}^{2,t}).$$

Thus, by doing the same at the remaining two sums of $Q_n^{2,t}$, namely $q_{2,n}^{2,t}, q_{3,n}^{2,t}$, we obtain the splitting

$$(42) \quad \mathcal{F}(Q_n^{2,t}) = \partial_t \mathcal{F}(\tilde{Q}_n^{2,t}) - \mathcal{F}(T_n^{2,t}).$$

These new operators $\tilde{q}_{i,n}^{2,t}$, $i = 1, 2, 3$, act on the following "type" of sequences

$$\tilde{q}_{1,n}^{2,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}),$$

with $m_1 - m_2 + m_3 \approx n_1$ and $n_1 - n_2 + n_3 \approx n$,

$$\tilde{q}_{2,n}^{2,t}(v_{n_1}, \bar{v}_{m_1}, v_{m_2}, \bar{v}_{m_3}, v_{n_3}),$$

with $m_1 - m_2 + m_3 \approx n_2$ and $n_1 - n_2 + n_3 \approx n$, and

$$\tilde{q}_{3,n}^{2,t}(v_{n_1}\bar{v}_{n_2}, v_{m_1}, \bar{v}_{m_2}, v_{m_3}),$$

with $m_1 - m_2 + m_3 \approx n_3$ and $n_1 - n_2 + n_3 \approx n$.

In order to proceed we need a similar lemma for the operator $\tilde{Q}_n^{2,t}$ as the one we had for $\tilde{Q}_n^{1,t}$ (see Lemma 12). Here we state it only for $\tilde{q}_{1,n}^{2,t}$ (remember that we look only at frequencies on $A_N(n)^c$ and C_1^c):

Lemma 17.

(43)

$$\|\tilde{q}_{1,n}^{2,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim \frac{\|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|n - n_1| |n - n_3| |(n - n_1)(n - n_3) + (n_1 - m_1)(n_1 - m_3)|}.$$

Proof. Writing out the Fourier transforms of the functions inside the integral of $\mathcal{F}(\tilde{q}_{1,n}^{2,t})$ it is not hard to see that

$$\mathcal{F}(\tilde{q}_{1,n}^{2,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_{n,n_1}^{2,t}(u_{m_1}, \bar{u}_{m_2}, u_{m_3}, \bar{u}_{n_2}, u_{n_3}))(\xi),$$

where the operator

(44)

$$R_{n,n_1}^{2,t}(w_{m_1}, \bar{w}_{m_2}, w_{m_3}, \bar{w}_{n_2}, w_{n_3})(x) =$$

$$\int_{\mathbb{R}^5} K_{n,n_1}^{(2)}(x, x'_1, y', x'_3, y, x_3) w_{m_1}(x'_1) \bar{w}_{m_2}(y') w_{m_3}(x'_3) \bar{w}_{n_2}(y) w_{n_3}(x_3) dx'_1 dy' dx'_3 dy dx_3$$

and the Kernel $K_{n,n_1}^{(2)}$ is given by the formula

(45)

$$K_{n,n_1}^{(2)}(x, x'_1, y', x'_3, y, x_3) =$$

$$\frac{\int_{\mathbb{R}^5} [e^{i\xi'_1(x-x'_1)+i\eta'(x-y')+i\xi'_3(x-x'_3)+i\eta(x-y)+i\xi_3(x-x_3)}]}{\sigma_n(\xi'_1 + \eta' + \xi'_3 + \eta + \xi_3) \sigma_{n_1}(\xi'_1 + \eta' + \xi'_3)} \frac{d\xi'_1 d\eta' d\xi'_3 d\eta d\xi_3}{(\eta + \eta' + \xi'_1 + \xi'_3)(\eta + \xi_3)[(\eta + \eta' + \xi'_1 + \xi'_3)(\eta + \xi_3) + (\eta' + \xi'_1)(\eta' + \xi'_3)]} =$$

$$(\mathcal{F}^{-1} \rho_{n,n_1}^{(2)})(x - x'_1, x - y', x - x'_3, x - y, x - x_3),$$

and the function $\rho_{n,n_1}^{(2)}$ equals

$$\rho_{n,n_1}^{(2)}(\xi'_1, \eta', \xi'_3, \eta, \xi_3) = \frac{\sigma_n(\xi'_1 + \eta' + \xi'_3 + \eta + \xi_3) \sigma_{n_1}(\xi'_1 + \eta' + \xi'_3)}{(\eta + \eta' + \xi'_1 + \xi'_3)(\eta + \xi_3)[(\eta + \eta' + \xi'_1 + \xi'_3)(\eta + \xi_3) + (\eta' + \xi'_1)(\eta' + \xi'_3)]}.$$

The operator $R_{n,n_1}^{2,t}$ is estimated in L^2 as in the proof of Lemma 12 and the function $\rho_{n,n_1}^{(2)}$ plays the same role as the function $\rho_n^{(1)}$ did for $R_n^{1,t}$, therefore,

$$\|R_{n,n_1}^{2,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim \|\rho_{n,n_1}^{(2)}\|_\infty \|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2,$$

and since $\xi'_1 \approx m_1, \eta' \approx -m_2, \xi'_3 \approx m_3, \eta \approx -n_2, \xi_3 \approx n_3$ we obtain

$$\|\rho_{n,n_1}^{(2)}\|_\infty \lesssim \frac{1}{|n - n_1| |n - n_3| |(n - n_1)(n - n_3) + (n_1 - m_1)(n_1 - m_3)|},$$

which finishes the proof. \square

Remark 18. The operator $\tilde{q}_{3,n}^{2,t}$ satisfies exactly the same bound as $\tilde{q}_{1,n}^{2,t}$ since the only difference between these operators is a permutation of their variables. On the other hand, the operator $\tilde{q}_{2,n}^{2,t}$ is a bit different, since instead of taking only the permutation we have to conjugate the 2nd variable too. Thus, a similar argument as the one given in Lemma 17 leads to the estimate

$$(46) \quad \|\tilde{q}_{2,n}^{2,t}(v_{n_1}, \bar{v}_{m_1}, v_{m_2}, \bar{v}_{m_3}, v_{n_3})\|_2 \lesssim \frac{\|v_{n_1}\|_2 \|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_3}\|_2}{|(n-n_1)(n-n_3)| |(n-n_1)(n-n_3) - (n_2-m_1)(n_2-m_3)|}$$

which is not exactly the same as the one we had for the operators $\tilde{q}_{1,n}^{2,t}, \tilde{q}_{3,n}^{2,t}$ since in the denominator instead of having $\mu_1 + \mu_2$ we have $\mu_1 - \mu_2$ ($\mu_1 = (n-n_1)(n-n_3)$ and in the first case $\mu_2 = (n_1-m_1)(n_1-m_3)$, m_1, m_3 being the "children" of n_1 , whereas in the second case $\mu_2 = (n_2-m_1)(n_2-m_3)$, m_1, m_3 being the "children" of n_2). It is readily checked that this change in the sign does not really affect the calculations that are to follow.

This Lemma allows us to move forward with our iteration process and show that the operators

$$(47) \quad N_0^{(3)}(v)(n) := \sum_{A_N(n)^c} \sum_{C_1^c} \tilde{Q}_n^{2,t} = \sum_{A_N(n)^c} \sum_{C_1^c} \sum_{i=1}^3 \tilde{q}_{i,n}^{2,t}$$

and

$$(48) \quad N_r^{(3)}(v)(n) := \sum_{A_N(n)^c} \sum_{C_1^c} \left(\tilde{q}_{1,n}^{2,t}(R_2^t(v)(m_1) - R_1^t(v)(m_1), \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}) + \right.$$

$$\left. \tilde{q}_{1,n}^{2,t}(v_{m_1}, \overline{R_2^t(v)(m_2) - R_1^t(v)(m_2)}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}) + \dots + \tilde{q}_{3,n}^{2,t}(v_{n_1}, \bar{v}_{n_2}, v_{m_1}, \bar{v}_{m_2}, R_2^t(v)(m_3) - R_1^t(v)(m_3)) \right),$$

are bounded on $l^q L^2$. The operator $N_r^{(3)}$ appears when we substitute each of the derivatives in the operator $\sum_{i=1}^3 \tau_{i,n}^{2,t}$ by the expression given in (21). Notice that the operator $N_0^{(3)}$ has three summands and the operator $N_r^{(3)}$ has $3 \cdot 5 = 15$ summands. Here is the claim:

Lemma 19.

$$\|N_0^{(3)}(v)\|_{l^q L^2} \lesssim N^{-2+\frac{1}{100}+\frac{2}{q}-\frac{1}{100q^7}+} \|v\|_{M_{2,q}}^5,$$

and

$$\|N_0^{(3)}(v) - N_0^{(3)}(w)\|_{l^q L^2} \lesssim N^{-2+\frac{1}{100}+\frac{2}{q}-\frac{1}{100q^7}+} (\|v\|_{M_{2,q}}^4 + \|w\|_{M_{2,q}}^4) \|v - w\|_{M_{2,q}}.$$

$$\|N_r^{(3)}(v)\|_{l^q L^2} \lesssim N^{-2+\frac{1}{100}+\frac{2}{q}-\frac{1}{100q^7}+} \|v\|_{M_{2,q}}^7,$$

and

$$\|N_r^{(3)}(v) - N_r^{(3)}(w)\|_{l^q L^2} \lesssim N^{-2+\frac{1}{100}+\frac{2}{q}-\frac{1}{100q^7}+} (\|v\|_{M_{2,q}}^6 + \|w\|_{M_{2,q}}^6) \|v - w\|_{M_{2,q}}.$$

Proof. Let us start with the operator $N_0^{(3)}$ and for simplicity of the presentation we will consider only the sum with the term $\tilde{q}_{1,n}^{2,t}$. As in the proof of Lemma 16 we have from (11) that for fixed n and μ_1 there are at most $o(|\mu_1|^+)$ many choices for n_1, n_2, n_3 (such that $(n - n_1)(n - n_3) = \mu_1$) and for fixed n_1 and μ_2 there are at most $o(|\mu_2|^+)$ many choices for m_1, m_2, m_3 (such that $(n_1 - m_1)(n_1 - m_3) = \mu_2$). Thus, from Lemma 17 we obtain

$$\sum_{A_N(n)^c} \sum_{C_1^c} \|\tilde{q}_{1,n}^{2,t}(v_{m_1}, \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3})\|_2 \lesssim$$

$$\sum_{A_N(n)^c} \sum_{C_1^c} \frac{\|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|n - n_1| |n - n_3| |(n - n_1)(n - n_3) + (n_1 - m_1)(n_1 - m_3)|}$$

and the RHS is equal to

$$\sum_{A_N(n)^c} \sum_{C_1^c} \frac{\|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|\mu_1| |\mu_1 + \mu_2|}$$

which by Hölder's inequality is bounded above by

$$\left(\sum_{A_N(n)^c} \sum_{C_1^c} \frac{1}{|\mu_1|^{q'} |\mu_1 + \mu_2|^{q'}} |\mu_1|^+ |\mu_2|^+ \right)^{\frac{1}{q'}} \left(\sum_{A_N(n)^c} \sum_{C_1^c} \|v_{m_1}\|_2^q \|v_{m_2}\|_2^q \|v_{m_3}\|_2^q \|v_{n_2}\|_2^q \|v_{n_3}\|_2^q \right)^{\frac{1}{q}}.$$

By a very crude estimate it is not difficult to see that the first sum behaves like the number $N^{-2 + \frac{1}{100} + \frac{2}{q'} - \frac{1}{100q} +}$. Then, by taking the l^q norm and applying Young's inequality for convolutions we are done. For the operator $N_r^{(3)}$ the proof is the same but in addition we use Lemma 10 for the operator $R_2^t - R_1^t$. \square

The operator that remains to be estimated is defined as

$$(49) \quad N^{(3)}(v)(n) := \sum_{A_N(n)^c} \sum_{C_1^c} \left(\tilde{q}_{1,n}^{2,t}(N_1^t(v)(m_1), \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}) + \right.$$

$$\left. \tilde{q}_{1,n}^{2,t}(v_{m_1}, \overline{N_1^t(v)(m_2)}, v_{m_3}, \bar{v}_{n_2}, v_{n_3}) + \dots + \tilde{q}_{3,n}^{2,t}(v_{n_1} \bar{v}_{n_2}, v_{m_1}, \bar{v}_{m_2}, N_1^t(v)(m_3)) \right),$$

which is the same as $N_r^{(3)}$ but in the place of the operator $R_2^t - R_1^t$ we have N_1^t . As before, we write

$$(50) \quad N^{(3)} = N_1^{(3)} + N_2^{(3)},$$

where $N_1^{(3)}$ is the restriction of $N^{(3)}$ onto the set of frequencies

$$(51) \quad C_2 = \{|\tilde{\mu}_3| \leq 7^3 |\tilde{\mu}_2|^{1 - \frac{1}{100}}\} \cup \{|\tilde{\mu}_3| \leq 7^3 |\mu_1|^{1 - \frac{1}{100}}\},$$

where $\tilde{\mu}_2 = \mu_1 + \mu_2$ and $\tilde{\mu}_3 = \mu_1 + \mu_2 + \mu_3$. The following is true:

Lemma 20.

$$\|N_1^{(3)}(v)\|_{l^q L^2} \lesssim N^{-2 + \frac{1}{100} + \frac{3}{q'} - \frac{2}{100q} +} \|v\|_{M_{2,q}^7},$$

and

$$\|N_1^{(3)}(v) - N_1^{(3)}(w)\|_{l^q L^2} \lesssim N^{-2+\frac{1}{100}+\frac{3}{q}-\frac{2}{100q}+} (\|v\|_{M_{2,q}}^6 + \|w\|_{M_{2,q}}^6) \|v - w\|_{M_{2,q}}.$$

Proof. Let us only consider the very first summand of the operator $N_1^{(3)}$, that is the operator $\tilde{q}_{1,n}^{2,t}$ with N_1^t acting on its first variable, since for the other summands similar considerations apply. For the proof we use again the divisor counting argument. From (11) it follows that for fixed n and μ_1 there are at most $o(|\mu_1|^+)$ many choices for n_1, n_2, n_3 ($\mu_1 = (n - n_1)(n - n_3)$, $n \approx n_1 - n_2 + n_3$). For fixed n_1 and μ_2 there are at most $o(|\mu_2|^+)$ many choices for m_1, m_2, m_3 ($\mu_2 = (n_1 - m_1)(n_1 - m_3)$, $n_1 \approx m_1 - m_2 + m_3$) and for fixed m_1 and μ_3 there are at most $o(|\mu_3|^+)$ many choices for k_1, k_2, k_3 ($\mu_3 = (m_1 - k_1)(m_1 - k_3)$, $m_1 \approx k_1 - k_2 + k_3$).

First, let us assume that our frequencies satisfy $|\tilde{\mu}_3| \lesssim |\tilde{\mu}_2|^{1-\frac{1}{100}}$. Since, $\tilde{\mu}_3 = \tilde{\mu}_2 + \mu_3$ we have $|\mu_3| \sim |\tilde{\mu}_2|$. Moreover, for fixed $|\tilde{\mu}_2|$ (equivalently, for fixed μ_1, μ_2) there are at most $O(|\tilde{\mu}_2|^{1-\frac{1}{100}})$ many choices for $\tilde{\mu}_3$ and hence, for $\mu_3 = \tilde{\mu}_3 - \tilde{\mu}_2$. In addition, $|\mu_2| \lesssim \max(|\mu_1|, |\tilde{\mu}_2|)$ and we should recall that since we are on C_1^c we have $|\tilde{\mu}_2| = |\mu_1 + \mu_2| > 5^3 |\mu_1|^{1-\frac{1}{100}} > 5^3 N^{1-\frac{1}{100}}$. Then, the expression

$$\sum_{A_N(n)^c} \sum_{C_1^c} \sum_{C_2} \|\tilde{q}_{1,n}^{2,t}(Q_{m_1}^{1,t}(v_{k_1}, \bar{v}_{k_2}, v_{k_3}), \bar{v}_{m_2}, v_{m_3}, \bar{v}_{n_2}, v_{n_3})\|_2$$

with the use of Lemma 17 and a trivial bound of the operator $Q_{m_1}^{1,t}$ in L^2 (see proof of Lemma 11) we obtain the upper bound

$$\begin{aligned} & \sum_{A_N(n)^c} \sum_{C_1^c} \sum_{C_2} \frac{\|v_{k_1}\|_2 \|v_{k_2}\|_2 \|v_{k_3}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|n - n_1| |n - n_3| |(n - n_1)(n - n_3) + (n_1 - m_1)(n_1 - m_3)|} = \\ & \sum_{A_N(n)^c} \sum_{C_1^c} \sum_{C_2} \frac{\|v_{k_1}\|_2 \|v_{k_2}\|_2 \|v_{k_3}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|\mu_1| |\tilde{\mu}_2|} \end{aligned}$$

and by Hölder's inequality we obtain

$$(52) \quad \left(\sum_{\substack{|\mu_1| > N \\ |\tilde{\mu}_2| > 5^3 N^{1-\frac{1}{100}}}} \frac{|\mu_1|^+ |\mu_2|^+ |\mu_3|^+ |\tilde{\mu}_2|^{1-\frac{1}{100}}}{|\mu_1|^{q'} |\tilde{\mu}_2|^{q'}} \right)^{\frac{1}{q'}} \times \\ \left(\sum_{A_N(n)^c} \sum_{C_1^c} \sum_{C_2} \|v_{k_1}\|_2^q \|v_{k_2}\|_2^q \|v_{k_3}\|_2^q \|v_{m_2}\|_2^q \|v_{m_3}\|_2^q \|v_{n_2}\|_2^q \|v_{n_3}\|_2^q \right)^{\frac{1}{q}}.$$

The first sum is bounded above by

$$(53) \quad \left(\sum_{\substack{|\mu_1| > N \\ |\tilde{\mu}_2| > 5^3 N^{1-\frac{1}{100}}}} \frac{1}{|\mu_1|^{q'-\epsilon} |\tilde{\mu}_2|^{q'-1+\frac{1}{100}-\epsilon}} \right)^{\frac{1}{q'}} \lesssim \left(N^{3(1-\frac{1}{100})-q'(2-\frac{1}{100})+\frac{1}{100^2}+} \right)^{\frac{1}{q'}}$$

and by the use of Young's inequality at the second sum we are done.

On the other hand, if $|\tilde{\mu}_3| \lesssim |\mu_1|^{1-\frac{1}{100}}$, then for fixed μ_1, μ_2 there are at most $O(|\mu_1|^{1-\frac{1}{100}})$ many choices for $\tilde{\mu}_3$ and hence for μ_3 . After this observation, the calculations are exactly the same as before but the first sum of (52) becomes

$$(54) \quad \left(\sum_{\substack{|\mu_1| > N \\ |\tilde{\mu}_2| > 5^3 N^{1-\frac{1}{100}}}} \frac{1}{|\mu_1|^{q'-1+\frac{1}{100}-\epsilon} |\tilde{\mu}_2|^{q'-\epsilon}} \right)^{\frac{1}{q'}} \lesssim \left(N^{3-\frac{2}{100}-q'(2-\frac{1}{100})} \right)^{\frac{1}{q'}}.$$

Between the two exponents of N in (53) and (54) we see that (54) is the dominating one and the proof is complete. \square

To the remaining part, namely $N_2^{(3)}$, we have to apply the differentiation by parts technique again. Note that here we only look at frequencies such that

$$|\tilde{\mu}_3| = |\mu_1 + \mu_2 + \mu_3| > 7^3 |\mu_1|^{1-\frac{1}{100}} > 7^3 N^{1-\frac{1}{100}},$$

or equivalently, frequencies that are on the set C_2^c . Instead, we will present the general J th step of the iteration procedure and prove the required lemmata. To do this, we need to use the tree notation as it was introduced in [9].

2.2. The Tree Notation and the Induction Step. A tree T is a finite, partially ordered set with the following properties:

- For any $a_1, a_2, a_3, a_4 \in T$ if $a_4 \leq a_2 \leq a_1$ and $a_4 \leq a_3 \leq a_1$ then $a_2 \leq a_3$ or $a_3 \leq a_2$.
- There exists a maximum element $r \in T$, that is $a \leq r$ for all $a \in T$ which is called the root.

We call the elements of T the nodes of the tree and in this content we will say that $b \in T$ is a child of $a \in T$ (or equivalently, that a is the parent of b) if $b \leq a, b \neq a$ and for all $c \in T$ such that $b \leq c \leq a$ we have either $b = c$ or $c = a$.

A node $a \in T$ is called terminal if it has no children. A nonterminal node $a \in T$ is a node with exactly 3 children a_1 , the left child, a_2 , the middle child, and a_3 , the right child. We define the sets

$$(55) \quad T^0 = \{\text{all nonterminal nodes}\},$$

and

$$(56) \quad T^\infty = \{\text{all terminal nodes}\}.$$

Obviously, $T = T^0 \cup T^\infty$, $T^0 \cap T^\infty = \emptyset$ and if $|T^0| = j \in \mathbb{Z}_+$ we have $|T| = 3j + 1$ and $|T^\infty| = 2j + 1$. We denote the collection of trees with j parental nodes by

$$(57) \quad T(j) = \{T \text{ is a tree with } |T| = 3j + 1\}.$$

Next, we say that a sequence of trees $\{T_j\}_{j=1}^J$ is a chronicle of J generations if:

- $T_j \in T(j)$ for all $j = 1, 2, \dots, J$.
- T_{j+1} is obtained by changing one of the terminal nodes of T_j into a nonterminal node with exactly 3 children, for all $j = 1, 2, \dots, J - 1$.

Let us also denote by $\mathcal{I}(J)$ the collection of trees of the J th generation. It is easily checked by an induction argument that

$$(58) \quad |\mathcal{I}(J)| = 1 \cdot 3 \cdot 5 \dots (2J - 1) =: (2J - 1)!!.$$

Given a chronicle $\{T_j\}_{j=1}^J$ of J generations we refer to T_J as an ordered tree of the J th generation. We should keep in mind that the notion of ordered trees comes with associated chronicles. It includes not only the shape of the tree but also how it "grew".

Given an ordered tree T we define an index function $n : T \rightarrow \mathbb{Z}$ such that

- $n_a \approx n_{a_1} - n_{a_2} + n_{a_3}$ for all $a \in T^0$, where a_1, a_2, a_3 are the children of a ,
- $n \not\approx n_{a_1}$ and $n \not\approx n_{a_3}$, for all $a \in T^0$,
- $|\mu_1| := 2|n_r - n_{r_1}| |n_r - n_{r_3}| > N$, where r is the root of T ,

and we denote the collection of all such index functions by $\mathcal{R}(T)$.

For the sake of completeness, as it was done in [9], given an ordered tree T with the chronicle $\{T_j\}_{j=1}^J$ and associated index functions $n \in \mathcal{R}(T)$, we need to keep track of the generations of frequencies. Fix an $n \in \mathcal{R}(T)$ and consider the very first tree T_1 . Its nodes are the root r and its children r_1, r_2, r_3 . We define the first generation of frequencies by

$$(n^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}) := (n_r, n_{r_1}, n_{r_2}, n_{r_3}).$$

From the definition of the index function we have

$$n^{(1)} \approx n_1^{(1)} - n_2^{(1)} + n_3^{(1)}, \quad n_1^{(1)} \not\approx n^{(1)} \not\approx n_3^{(1)}.$$

The ordered tree T_2 of the second generation is obtained from T_1 by changing one of its terminal nodes $a = r_k \in T_1^\infty$ for some $k = 1, 2, 3$ into a nonterminal node. Then, the second generation of frequencies is defined by

$$(n^{(2)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

Thus, we have $n^{(2)} = n_k^{(1)}$ for some $k = 1, 2, 3$ and from the definition of the index function we have

$$n^{(2)} \approx n_1^{(2)} - n_2^{(2)} + n_3^{(2)}, \quad n_1^{(2)} \not\approx n^{(2)} \not\approx n_3^{(2)}.$$

This should be compared with what happened in the calculations we presented before when passing from the first step of the iteration process into the second step. Every time we apply the differentiation by parts technique we introduce a new set of frequencies.

After $j - 1$ steps, the ordered tree T_j of the j th generation is obtained from T_{j-1} by changing one of its terminal nodes $a \in T_{j-1}^\infty$ into a nonterminal node. Then, the j th generation frequencies are defined as

$$(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}),$$

and we have $n^{(j)} = n_k^{(m)}$ ($= n_a$) for some $m = 1, 2, \dots, j - 1$ and $k = 1, 2, 3$, since this corresponds to the frequency of some terminal node in T_{j-1} . In addition, from the definition of the index function we have

$$n^{(j)} \approx n_1^{(j)} - n_2^{(j)} + n_3^{(j)}, \quad n_1^{(j)} \not\approx n^{(j)} \not\approx n_3^{(j)}.$$

Finally, we use μ_j to denote the corresponding phase factor introduced at the j th generation. That is,

$$(59) \quad \mu_j = 2(n^{(j)} - n_1^{(j)})(n^{(j)} - n_3^{(j)}),$$

and we also introduce the quantities

$$(60) \quad \tilde{\mu}_J = \sum_{j=1}^J \mu_j, \quad \hat{\mu}_J = \prod_{j=1}^J \tilde{\mu}_j.$$

We should keep in mind that everytime we apply differentiation by parts and split the operators, we need to control the new frequencies that arise from this procedure. For this reason we need to define the sets (see (39) and (51)):

$$(61) \quad C_J := \{|\tilde{\mu}_{J+1}| \leq (2J+3)^3 |\tilde{\mu}_J|^{1-\frac{1}{100}}\} \cup \{|\tilde{\mu}_{J+1}| \leq (2J+3)^3 |\mu_1|^{1-\frac{1}{100}}\}.$$

Let us see how to use this notation and terminology in our calculations. On the very first step, $J = 1$, we have only one tree, the root node r and its three children r_1, r_2, r_3 (sometimes, when it is clear from the context, we will identify the nodes and the frequencies assigned to them, that is, we have the root $n = n_r$ and its three children $n_{r_1} = n_1, n_{r_2} = n_2, n_{r_3} = n_3$) and we have only one operator that needs to be controlled in order to proceed further, namely $\tilde{q}_n^{1,t} := \tilde{Q}_n^{1,t}$.

On the second step, $J = 2$, we have three operators $\tilde{q}_{n,n_1}^{2,t} := \tilde{q}_{1,n}^{2,t}, \tilde{q}_{n,n_2}^{2,t} := \tilde{q}_{2,n}^{2,t}, \tilde{q}_{n,n_3}^{2,t} := \tilde{q}_{3,n}^{2,t}$ that play the same role as $\tilde{q}_n^{1,t}$ did for the first step. Let us observe that for each one of these operators we must have estimates on their L^2 norms in order to be able and continue the iteration. These estimates were provided by Lemmata 12 and 17.

On the general J th step we will have $|\mathcal{I}(J)|$ operators of the $\tilde{q}_{T^0, \mathbf{n}}^{J,t}$ "type" each one corresponding to one of the ordered trees of the J th generation, $T \in T(J)$, where \mathbf{n} is an arbitrary fixed index function on T . We have the subindices T^0 and \mathbf{n} because each one of these operators has Fourier transform supported on the cubes with centers the frequencies assigned to the nodes that belong to T^0 .

Let us denote by T_α all the nodes of the ordered tree T that are descendants of the node $\alpha \in T^0$, i.e. $T_\alpha = \{\beta \in T : \beta \leq \alpha, \beta \neq \alpha\}$.

We also need to define the principal and final "signs" of a node $a \in T$ which are functions from the tree T into the set $\{\pm 1\}$:

$$(62) \quad \text{psgn}(a) = \begin{cases} +1, & a \text{ is not the middle child of his father} \\ +1, & a = r, \text{ the root node} \\ -1, & a \text{ is the middle child of his father} \end{cases}$$

$$(63) \quad \text{fsgn}(a) = \begin{cases} +1, & \text{psgn}(a) = +1 \text{ and } a \text{ has an even number of middle predecessors} \\ -1, & \text{psgn}(a) = +1 \text{ and } a \text{ has an odd number of middle predecessors} \\ -1, & \text{psgn}(a) = -1 \text{ and } a \text{ has an even number of middle predecessors} \\ +1, & \text{psgn}(a) = -1 \text{ and } a \text{ has an odd number of middle predecessors,} \end{cases}$$

where the root node $r \in T$ is not considered a middle father.

The operators $\tilde{q}_{T^0, \mathbf{n}}^{J,t}$ are defined through their Fourier transforms as

$$(64) \quad \mathcal{F}(\tilde{q}_{T^0, \mathbf{n}}^{J,t}(\{w_{n_\beta}\}_{\beta \in T^\infty}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_{T^0, \mathbf{n}}^{J,t}(\{e^{-it\partial_x^2} w_{n_\beta}\}_{\beta \in T^\infty}))(\xi),$$

where the operator $R_{T^0, \mathbf{n}}^{J,t}$ acts on the functions $\{w_{n_\beta}\}_{\beta \in T^\infty}$ as

$$(65) \quad R_{T^0, \mathbf{n}}^{J,t}(\{w_{n_\beta}\}_{\beta \in T^\infty})(x) = \int_{\mathbb{R}^{2J+1}} K_{T^0}^{(J)}(x, \{x_\beta\}_{\beta \in T^\infty}) \left[\otimes_{\beta \in T^\infty} w_{n_\beta}(x_\beta) \right] \prod_{\beta \in T^\infty} dx_\beta,$$

and the Kernel $K_{T^0, \mathbf{n}}^{(J)}$ is defined as

$$(66) \quad K_{T^0, \mathbf{n}}^{(J)}(x, \{x_\beta\}_{\beta \in T^\infty}) = \mathcal{F}^{-1}(\rho_{T^0, \mathbf{n}}^{(J)})(\{x - x_\beta\}_{\beta \in T^\infty}).$$

Here is the formula for the function $\rho_{T^0, \mathbf{n}}^{(J)}$ with $(|T^\infty| = 2J + 1)$ -variables, $\xi_\beta, \beta \in T^\infty$:

$$(67) \quad \rho_{T^0, \mathbf{n}}^{(J)}(\{\xi_\beta\}_{\beta \in T^\infty}) = \left[\prod_{\alpha \in T^0} \sigma_{n_\alpha} \left(\sum_{\beta \in T^\infty \cap T_\alpha} \text{fsgn}(\beta) \xi_\beta \right) \right] \frac{1}{\hat{\mu}_T},$$

where we denote by

$$(68) \quad \hat{\mu}_T = \prod_{\alpha \in T^0} \tilde{\mu}_\alpha, \quad \tilde{\mu}_\alpha = \sum_{\beta \in T^0 \setminus T_\alpha} \mu_\beta,$$

and for $\beta \in T^0$ we have

$$(69) \quad \mu_\beta = 2(\xi_\beta - \xi_{\beta_1})(\xi_\beta - \xi_{\beta_3}),$$

where we impose the relation $\xi_\alpha = \xi_{\alpha_1} - \xi_{\alpha_2} + \xi_{\alpha_3}$ for every $\alpha \in T^0$ that appears in the calculations until we reach the terminal nodes of T^∞ . This is because in the definition of the function $\rho_{T^0}^{J,t}$ we need the variables "ξ" to be assigned only at the terminal nodes of the tree T . We use the notation μ_β in similarity to μ_j of equation (59) because this is the "continuous" version of the discrete case. In addition, the variables $\xi_{\alpha_1}, \xi_{\alpha_2}, \xi_{\alpha_3}$ that appear in the expression (67) are supported in such a way that $\xi_{\alpha_1} \approx n_{\alpha_1}, \xi_{\alpha_2} \approx n_{\alpha_2}, \xi_{\alpha_3} \approx n_{\alpha_3}$. This is because the functions σ_{n_α} are supported in such a way. Therefore, $|\hat{\mu}_T| \sim |\hat{\mu}_J|$.

For the induction step of our iteration process it is easy to check that the following Lemma is true, which should be compared with Lemmata 12 and 17:

Lemma 21.

$$(70) \quad \|\tilde{q}_{T^0, \mathbf{n}}^{J,t}(\{v_{n_\beta}\}_{\beta \in T^\infty})\|_2 \lesssim \left(\prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2 \right) \frac{1}{|\hat{\mu}_T|},$$

for every tree $T \in T(J)$ and index function $\mathbf{n} \in \mathcal{R}(T)$.

Given an index function \mathbf{n} and $2J + 1$ functions $\{v_{n_\beta}\}_{\beta \in T^\infty}$ and $\alpha \in T^\infty$ we define the action of the operator N_1^t (see (20)) on the set $\{v_{n_\beta}\}_{\beta \in T^\infty}$ to be the same set as before but with the difference that we have substituted the function v_{n_α} by the new function $N_1^t(v)(n_\alpha)$. We will denote this new set of functions $N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta \in T^\infty})$. Similarly, the

action of the operator $R_2^t - R_1^t$ (see (19)) on the set of functions $\{v_{n_\beta}\}_{\beta \in T^\infty}$ will be denoted by $(R_2^{t,\alpha} - R_1^{t,\alpha})(\{v_{n_\beta}\}_{\beta \in T^\infty})$.

The operator of the J th step, $J \geq 2$, that we want to estimate is given by the formula:

$$(71) \quad N_2^{(J)}(v)(n) := \sum_{T \in T^{(J-1)}} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T^0}^{J-1,t}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta \in T^\infty})).$$

Applying differentiation by parts on the Fourier side (keep in mind that from the splitting procedure we are on the sets $A_N(n)^c, C_1^c, \dots, C_{J-1}^c$) we obtain the expression

$$(72) \quad N_2^{(J)}(v)(n) = \partial_t(N_0^{(J+1)}(v)(n)) + N_r^{(J+1)}(v)(n) + N^{(J+1)}(v)(n),$$

where

$$(73) \quad N_0^{(J+1)}(v)(n) := \sum_{T \in T^{(J)}} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T^0, \mathbf{n}}^{J,t}(\{v_{n_\beta}\}_{\beta \in T^\infty}),$$

and

$$(74) \quad N_r^{(J+1)}(v)(n) := \sum_{T \in T^{(J)}} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T^0, \mathbf{n}}^{J,t}((R_2^{t,\alpha} - R_1^{t,\alpha})(\{v_{n_\beta}\}_{\beta \in T^\infty})),$$

and

$$(75) \quad N^{(J+1)}(v)(n) := \sum_{T \in T^{(J)}} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T^0, \mathbf{n}}^{J,t}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta \in T^\infty})).$$

We also split the operator $N^{(J+1)}$ as the sum

$$(76) \quad N^{(J+1)} = N_1^{(J+1)} + N_2^{(J+1)},$$

where $N_1^{(J+1)}$ is the restriction of $N^{(J+1)}$ onto C_J and $N_2^{(J+1)}$ onto C_J^c . First, we generalise Lemma 19 by estimating the operators $N_0^{(J+1)}$ and $N_r^{(J+1)}$:

Lemma 22.

$$\|N_0^{(J+1)}(v)\|_{l^q L^2} \lesssim N^{-\frac{(q'-1)}{q'}J + \frac{(q'-1)}{100q'}(J-1)+} \|v\|_{M_{2,q}^{2J+1}},$$

and

$$\|N_0^{(J+1)}(v) - N_0^{(J+1)}(w)\|_{l^q L^2} \lesssim N^{-\frac{(q'-1)}{q'}J + \frac{(q'-1)}{100q'}(J-1)+} (\|v\|_{M_{2,q}^{2J}} + \|w\|_{M_{2,q}^{2J}}) \|v - w\|_{M_{2,q}}.$$

$$\|N_r^{(J+1)}(v)\|_{l^q L^2} \lesssim N^{-\frac{(q'-1)}{q'}J + \frac{(q'-1)}{100q'}(J-1)+} \|v\|_{M_{2,q}^{2J+3}},$$

and

$$\|N_r^{(J+1)}(v) - N_r^{(J+1)}(w)\|_{l^q L^2} \lesssim N^{-\frac{(q'-1)}{q'}J + \frac{(q'-1)}{100q'}(J-1)+} (\|v\|_{M_{2,q}^{2J+2}} + \|w\|_{M_{2,q}^{2J+2}}) \|v - w\|_{M_{2,q}}.$$

Proof. As in the proof of Lemma 19 for fixed $n^{(j)}$ and μ_j there are at most $o(|\mu_j|^+)$ many choices for $n_1^{(j)}, n_2^{(j)}, n_3^{(j)}$. In addition, let us observe that μ_j is determined by $\tilde{\mu}_1, \dots, \tilde{\mu}_j$ and $|\mu_j| \lesssim \max(|\tilde{\mu}_{j-1}|, |\tilde{\mu}_j|)$, since $\mu_j = \tilde{\mu}_j - \tilde{\mu}_{j-1}$. Then, for a fixed tree $T \in T(J)$, by Lemma 21 the estimate for the operator $\tilde{q}_{T^0, \mathbf{n}}^{J,t}$ is as follows (remember that $|\hat{\mu}_T| \sim |\hat{\mu}_J| = \prod_{k=1}^J |\tilde{\mu}_k|$):

$$\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = \mathbf{n}}} \|\tilde{q}_{T^0, \mathbf{n}}^{J,t}(\{v_\beta\}_{\beta \in T^\infty})\|_2 \lesssim \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = \mathbf{n}}} \left(\prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2 \right) \left(\prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|} \right),$$

and by Hölder's inequality this is bounded from above by

$$(77) \quad \left(\sum_{\substack{|\mu_1| > N \\ |\tilde{\mu}_j| > (2j+1)^3 N^{1-\frac{1}{100}} \\ j=2, \dots, J}} \prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|^{q'}} |\mu_k|^+ \right)^{\frac{1}{q'}} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = \mathbf{n}}} \prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2^q \right)^{\frac{1}{q}}.$$

The first sum behaves like $N^{-\frac{(q'-1)J + \frac{(q'-1)}{100q'}(J-1)+}{q'}}$ and for the remaining part we take the l^q norm in n and by the use of Young's inequality we are done.

We have to make two observations for this lemma. Note that there is an extra factor $\sim J$ when we estimate the differences $N_0^{(J+1)}(v) - N_0^{(J+1)}(w)$ since $|a^{2J+1} - b^{2J+1}| \lesssim (\sum_{j=1}^{2J+1} a^{2J+1-j} b^{j-1})|a - b|$ has $O(J)$ many terms. Also, we have $c_J = |\mathcal{I}(J)|$ many summands in the operator $N_0^{(J+1)}$ since there are c_J many trees of the J th generation and c_J behaves like a double factorial in J (see (58)). However, these observations do not cause any problem since the constant that we obtain from estimating the first sum of (77) decays like a fractional power of a double factorial in J , or to be more precise we have

$$(78) \quad \frac{c_J}{\prod_{j=2}^J (2j+1)^{3 \frac{q'-1}{q'}}}.$$

This fraction for large values of J behaves like $J^J / J^{(3-\frac{3}{q'})J} = 1/J^{(2-\frac{3}{q'})J}$ and in order to maintain the decay in the denominator we use the assumption of Theorem 4 namely that $1 \leq q \leq 2$. For the operator $N_r^{(J+1)}$ the proof is the same but in addition we use Lemma 10 for the operator $R_2^t - R_1^t$. \square

The estimate for the operator $N_1^{(J+1)}$, which generalises Lemma 20, is the following:

Lemma 23.

$$\|N_1^{(J+1)}(v)\|_{l^q L^2} \lesssim N^{-1 + \frac{2}{q'} - \frac{1}{100q'} + (1 - \frac{1}{100})(\frac{1}{q'} - 1)(J-1)+} \|v\|_{M_{2,q}^{2J+3}},$$

and

$$\|N_1^{(J+1)}(v) - N_1^{(J+1)}(w)\|_{l^q L^2} \lesssim N^{-1 + \frac{2}{q'} - \frac{1}{100q'} + (1 - \frac{1}{100})(\frac{1}{q'} - 1)(J-1)+} (\|v\|_{M_{2,q}^{2J+2}} + \|w\|_{M_{2,q}^{2J+2}}) \|v - w\|_{M_{2,q}}.$$

Proof. As before, for fixed $n^{(j)}$ and μ_j there are at most $o(|\mu_j|^+)$ many choices for $n_1^{(1)}, n_2^{(1)}, n_3^{(1)}$ and note that μ_j is determined by $\tilde{\mu}_1, \dots, \tilde{\mu}_j$.

Let us assume that $|\tilde{\mu}_{J+1}| = |\tilde{\mu}_J + \mu_{J+1}| \lesssim (2J+3)^3 |\tilde{\mu}_J|^{1-\frac{1}{100}}$ holds in (61). Then, $|\mu_{J+1}| \lesssim |\tilde{\mu}_J|$ and for fixed $\tilde{\mu}_J$ there are at most $o(|\tilde{\mu}_J|^{1-\frac{1}{100}})$ many choices for $\tilde{\mu}_{J+1}$ and therefore, for $\mu_{J+1} = \tilde{\mu}_{J+1} - \tilde{\mu}_J$. For a fixed tree $T \in T(J)$ and $\alpha \in T^\infty$, by Lemma 21 and a trivial bound of the operator $Q_{n_\alpha}^{1,t}$ in L^2 (see proof of Lemma 11) the estimate for the operator $\tilde{q}_{T^0, \mathbf{n}}^{J,t}$ is as follows (remember that $|\hat{\mu}_T| \sim |\hat{\mu}_J| = \prod_{k=1}^J |\tilde{\mu}_k|$):

$$\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \|\tilde{q}_{T^0, \mathbf{n}}^{J,t}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta \in T^\infty}))\|_2 \lesssim \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \left(\|v_{n_{\alpha_1}}\|_2 \|v_{n_{\alpha_2}}\|_2 \|v_{n_{\alpha_3}}\|_2 \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|v_{n_\beta}\|_2 \right) \left(\prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|} \right),$$

and by Hölder's inequality we obtain the upper bound

$$(79) \quad \left(\sum_{\substack{|\mu_1| > N \\ |\tilde{\mu}_j| > (2j+1)^3 N^{1-\frac{1}{100}} \\ j=2, \dots, J}} |\tilde{\mu}_J|^{1-\frac{1}{100}} \prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|^{q'}} |\mu_k|^+ \right)^{\frac{1}{q'}} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \|v_{n_{\alpha_1}}\|_2^q \|v_{n_{\alpha_2}}\|_2^q \|v_{n_{\alpha_3}}\|_2^q \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|v_{n_\beta}\|_2^q \right)^{\frac{1}{q}}.$$

An easy calculation shows that the first sum behaves like $N^{-1+\frac{2}{q'}-\frac{1}{100q'}+(1-\frac{1}{100})(\frac{1}{q'}-1)(J-1)+}$ and then by taking the l^q norm by the use of Young's inequality we are done.

If $|\tilde{\mu}_{J+1}| \lesssim (2J+3)^3 |\mu_1|^{1-\frac{1}{100}}$ holds in (61), then for fixed μ_j , $j = 1, \dots, J$, there are at most $O(|\mu_1|^{1-\frac{1}{100}})$ many choices for μ_{J+1} . The same argument as above leads us to exactly the same expressions as in (79) but with the first sum replaced by the following:

$$\left(\sum_{\substack{|\mu_1| > N \\ |\tilde{\mu}_j| > (2j+1)^3 N^{1-\frac{1}{100}} \\ j=2, \dots, J}} |\mu_1|^{1-\frac{1}{100}} \prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|^{q'}} |\mu_k|^+ \right)^{\frac{1}{q'}},$$

which again is bounded from above by $N^{-1+\frac{2}{q'}-\frac{1}{100q'}+(1-\frac{1}{100})(\frac{1}{q'}-1)(J-1)+}$ and the proof is complete. \square

Remark 24. As it was done in [9], for $s > 0$ we have to observe that all previous lemmata hold true if we replace the $l^q L^2$ norm by the $l_s^q L^2$ norm and the $M_{2,q}(\mathbb{R})$ norm by the $M_{2,q}^s(\mathbb{R})$ norm. To see this, consider $n^{(j)}$ large. Then, there exists at least one of $n_1^{(j)}, n_2^{(j)}, n_3^{(j)}$ such that $|n_k^{(j)}| \geq \frac{1}{3}|n^{(j)}|$, $k \in \{1, 2, 3\}$, since we have the relation $n^{(j)} \approx n_1^{(j)} - n_2^{(j)} + n_3^{(j)}$.

Therefore, in the estimates of the J th generation, there exists at least one frequency $n_k^{(j)}$ for some $j \in \{1, \dots, J\}$ with the property

$$\langle n \rangle^s \leq 3^{js} \langle n_k^{(j)} \rangle^s \leq 3^{Js} \langle n_k^{(j)} \rangle^s.$$

This exponential growth does not affect our calculations due to the double factorial decay in the denominator of (78).

2.3. Existence of Weak Solutions in the Extended Sense. In this subsection we prove Theorem 4. The calculations are the same as in [9] where we just need to replace their L^2 norm by the $M_{2,q}(\mathbb{R})$ norm. We will present them for the sake of completion.

Let us start by defining the partial sum operator $\Gamma_{v_0}^{(J)}$ as

$$(80) \quad \begin{aligned} \Gamma_{v_0}^{(J)} v(t) &= v_0 + \sum_{j=2}^J N_0^{(j)}(v)(n) - \sum_{j=2}^J N_0^{(j)}(v_0)(n) \\ &+ \int_0^t R_1^\tau(v)(n) + R_2^\tau(v)(n) + \sum_{j=2}^J N_r^{(j)}(v)(n) + \sum_{j=1}^J N_1^{(j)}(v)(n) \, d\tau, \end{aligned}$$

where we have $N_1^{(1)} := N_{11}^t$ from (22), $N_0^{(2)} := N_{21}^t$ from (28), $N_1^{(2)} := N_{31}^t$ from (40) and $N_r^{(2)} := N_4^t$ from (37) and $v_0 \in M_{2,q}(\mathbb{R})$ is a fixed function.

In the following we will denote by $X_T = C([0, T], M_{2,q}(\mathbb{R}))$. Our goal is to show that the series appearing on the RHS of (80) converge absolutely in X_T for sufficiently small $T > 0$, if $v \in X_T$, even for $J = \infty$. Indeed, by Lemmata 10, 11, 22, and 23 we obtain

$$(81) \quad \begin{aligned} \|\Gamma_{v_0}^{(J)} v\|_{X_T} &\leq \|v_0\|_{M_{2,q}} + C \sum_{j=2}^J N^{-(1-\frac{1}{q'})(j-1) + \frac{q'-1}{100q'}(j-2)+} (\|v\|_{X_T}^{2j-1} + \|v_0\|_{M_{2,q}}^{2j-1}) \\ &+ CT \left[\|v\|_{X_T}^3 + \sum_{j=2}^J N^{-(1-\frac{1}{q'})(j-1) + \frac{q'-1}{100q'}(j-2)+} \|v\|_{X_T}^{2j+1} \right. \\ &\left. + N^{\frac{1}{q'}+} \|v\|_{X_T}^3 + \sum_{j=2}^J N^{-1 + \frac{2}{q'} - \frac{1}{100q'} + (1-\frac{1}{100})(\frac{1}{q'}-1)(J-2)+} \|v\|_{X_T}^{2j+1} \right]. \end{aligned}$$

Let us assume that $\|v_0\|_{M_{2,q}} \leq R$ and $\|v\|_{X_T} \leq \tilde{R}$, with $\tilde{R} \geq R \geq 1$. From (81) we have

$$(82) \quad \begin{aligned} \|\Gamma_{v_0}^{(J)} v\|_{X_T} &\leq R + CN^{\frac{1}{q'}-1+} R^3 \sum_{j=0}^{J-2} (N^{\frac{1}{q'}-1+ + \frac{q'-1}{100q'}} R^2)^j + CN^{\frac{1}{q'}-1+} \tilde{R}^3 \sum_{j=0}^{J-2} (N^{\frac{1}{q'}-1+ + \frac{q'-1}{100q'}} \tilde{R}^2)^j \\ &+ CT \left[(1 + N^{\frac{1}{q'}+}) \tilde{R}^3 + CN^{\frac{1}{q'}-1+} \tilde{R}^5 \sum_{j=0}^{J-2} (N^{\frac{1}{q'}-1+ + \frac{q'-1}{100q'}} \tilde{R}^2)^j \right] \end{aligned}$$

$$+N^{\frac{2}{q'}-1-\frac{1}{100q'}}+\tilde{R}^5\sum_{j=0}^{J-2}(N^{\frac{1}{q'}-1+\frac{q'-1}{100q'}}\tilde{R}^2)^j].$$

We choose $N = N(\tilde{R})$ large enough, such that $N^{\frac{1}{q'}-1+\frac{q'-1}{100q'}}\tilde{R}^2 = N^{99\frac{1-q'}{100q'}}\tilde{R}^2 \leq \frac{1}{2}$, or equivalently,

$$(83) \quad N \geq (2\tilde{R}^2)^{\frac{100q'}{99(q'-1)}},$$

so that the geometric series on the RHS of (82) converge and are bounded by 2. Therefore, we arrive at

$$(84) \quad \begin{aligned} \|\Gamma_{v_0}^{(J)}v\|_{X_T} &\leq R + 2CN^{\frac{1}{q'}-1+}R^3 + 2CN^{\frac{1}{q'}-1+}\tilde{R}^3 \\ &\quad + CT\left[(1+N^{\frac{1}{q'}+})\tilde{R}^2 + 2N^{\frac{1}{q'}-1+}\tilde{R}^4 + 2N^{\frac{199-100q'}{100q'}+}\tilde{R}^4\right]\tilde{R}, \end{aligned}$$

and we choose $T > 0$ sufficiently small such that

$$(85) \quad CT\left[(1+N^{\frac{1}{q'}+})\tilde{R}^2 + 2N^{\frac{1}{q'}-1+}\tilde{R}^4 + 2N^{\frac{199-100q'}{100q'}+}\tilde{R}^4\right] < \frac{1}{10}.$$

With the use of (83) we see that $2CN^{\frac{1}{q'}-1+}\tilde{R}^3 \leq CN^{\frac{1-q'}{100q'}+}\tilde{R}$ and by further imposing N to be sufficiently large such that

$$(86) \quad CN^{\frac{1-q'}{100q'}+} < \frac{1}{10},$$

we have

$$(87) \quad \|\Gamma_{v_0}^{(J)}v\|_{X_T} \leq R + \frac{R}{10} + \frac{\tilde{R}}{5} = \frac{11}{10}R + \frac{1}{5}\tilde{R}.$$

Thus, for sufficiently large N and sufficiently small $T > 0$ the partial sum operators $\Gamma_{v_0}^{(J)}$ are well defined in X_T , for every $J \in \mathbb{N} \cup \{\infty\}$. We will write Γ_{v_0} for $\Gamma_{v_0}^{(\infty)}$.

Our next step is given an initial data $v_0 \in M_{2,q}(\mathbb{R})$ to construct a solution $v \in X_T$ in the sense of Definition 3. To this direction, let $s > \max\{\frac{1}{q'}, \frac{1}{2} + \frac{1}{q}\}$ (so that $M_{2,q}^s(\mathbb{R})$ is a Banach Algebra that embeds in $L^2(\mathbb{R})$) and consider a sequence $\{v_0^{(m)}\}_{m \in \mathbb{N}} \in M_{2,q}^s(\mathbb{R}) \subset M_{2,q}(\mathbb{R})$ whose Fourier transforms are all compactly supported (thus, all $v_0^{(m)}$ are smooth functions) and such that $v_0^{(m)} \rightarrow v_0$ in $M_{2,q}(\mathbb{R})$ as $m \rightarrow \infty$. Let $R = \|v_0\|_{M_{2,q}} + 1$ and we can assume that $\|v_0^{(m)}\|_{M_{2,q}} \leq R$, for all $m \in \mathbb{N}$. Denote by $v^{(m)}$ the local in time solution of NLS (1) in $M_{2,q}^s(\mathbb{R})$ with initial condition $v_0^{(m)}$. It satisfies the Duhamel formulation:

$$(88) \quad \begin{aligned} v^{(m)}(t) &= v_0^{(m)} + i \int_0^t N_1^\tau(v^{(m)}) - R_1^\tau(v^{(m)}) + R_2^\tau(v^{(m)}) \, d\tau = \\ &= v_0^{(m)} + \sum_{j=2}^{\infty} N_0^{(j)}(v^{(m)})(n) - \sum_{j=2}^{\infty} N_0^{(j)}(v_0^{(m)})(n) \end{aligned}$$

$$+ \int_0^t R_1^\tau(v^{(m)})(n) + R_2^\tau(v^{(m)})(n) + \sum_{j=2}^{\infty} N_r^{(j)}(v^{(m)})(n) + \sum_{j=1}^{\infty} N_1^{(j)}(v^{(m)})(n) d\tau = \Gamma_{v_0^{(m)}} v^{(m)}.$$

To see this it suffices to prove that the remainder term $N_2^{(J+1)}(v)$ given by (71) goes to zero in the $l^q L^2$ norm as J goes to infinity for the smooth solutions $v^{(m)}$. Indeed, we have the following lemma:

Lemma 25. *Let w be one of the smooth solutions $v^{(m)}$. Then*

$$\lim_{J \rightarrow \infty} \|N_2^{(J+1)}(w)\|_{l^q L^2} = 0.$$

Proof. Obviously,

$$\|N_2^{(J+1)}(w)\|_2 \leq \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \|\tilde{q}_{T^0}^{J,t}(N_1^{t,\alpha}(\{w_{n_\beta}\}_{\beta \in T^\infty}))\|_2,$$

which by Lemma 21 is bounded by

$$\sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|w_{n_\beta}\|_2 \frac{\|N_1^t(w)(n_\alpha)\|_2}{\prod_{k=1}^J |\tilde{\mu}_k|}.$$

By the definition of the operator $N_1^t(w)$ (see (20)) and Remark 13 we arrive at the upper bound

$$\sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \left[\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|w_{n_\beta}\|_2 \left(\sum_{\substack{n_\alpha \approx n_1 - n_2 + n_3 \\ n_1 \neq n_\alpha \neq n_3}} \|w_{n_1}\|_2 \|w_{n_2}\|_2 \|w_{n_3}\|_2 \right) \frac{1}{\prod_{k=1}^J |\tilde{\mu}_k|} \right].$$

Hölder's inequality for the sum inside the brackets with indices $1/q + 1/q' = 1$ implies the estimate (which is basically the same as in the proof of Lemma 22)

$$\frac{1}{J^{(3-\frac{3}{q'})J}} \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|w_{n_\beta}\|_2^q \left(\sum_{\substack{n_\alpha \approx n_1 - n_2 + n_3 \\ n_1 \neq n_\alpha \neq n_3}} \|w_{n_1}\|_2 \|w_{n_2}\|_2 \|w_{n_3}\|_2 \right)^q \right)^{\frac{1}{q}}.$$

Now we take the l^q norm to bound $\|N_2^{(J+1)}(w)\|_{l^q L^2}$ by

$$\frac{1}{J^{(3-\frac{3}{q'})J}} \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \left(\sum_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|w_{n_\beta}\|_2^q (\{\|w_{n_1}\|_2\} * \{\|w_{n_2}\|_2\} * \{\|w_{n_3}\|_2\})^q (n_\alpha) \right)^{\frac{1}{q}},$$

and by applying Young's inequality in l^1 for $2J + 1$ sequences we see that

$$\|N_2^{(J+1)}(w)\|_{l^q L^2} \lesssim \frac{1}{J^{(2-\frac{3}{q'})J}} \|w\|_{M_{2,q}}^{2J} \|\{\|w_{n_1}\|_2\} * \{\|w_{n_2}\|_2\} * \{\|w_{n_3}\|_2\}\|_{l^q}.$$

In general, we do not know if the l^q norm of this convolution is finite but since w is sufficiently smooth we may assume that $w \in M_{2,1}$ which is a space (actually a Banach

algebra) with bigger norm than $M_{2,q}$ and we obtain

$$\|N_2^{(J+1)}(w)\|_{l^q L^2} \lesssim \frac{1}{J^{(2-\frac{3}{q})J}} \|w\|_{M_{2,1}}^{2J+3},$$

from which the claim follows and the proof is complete. \square

Next we will show that (88) holds in X_T for the same time $T = T(R) > 0$ independent of $m \in \mathbb{N}$. Indeed, fix $m \in \mathbb{N}$ and observe that the norm $\|v^{(m)}\|_{X_t} = \|v^{(m)}\|_{C([0,t],M_{2,q})}$ is continuous in t . Since $\|v_0^{(m)}\|_{M_{2,q}} \leq R$ there is a time $T_1 > 0$ such that $\|v^{(m)}\|_{X_{T_1}} \leq 4R$. Then, by repeating the previous calculations with $\tilde{R} = 4R$ and keeping one of the factors as $\|v^{(m)}\|_{X_{T_1}}$ we get

$$(89) \quad \|v^{(m)}\|_{X_{T_1}} = \|\Gamma_{v_0^{(m)}} v^{(m)}\|_{X_{T_1}} \leq \frac{11}{10}R + \frac{1}{5}\|v^{(m)}\|_{X_{T_1}},$$

if N and T_1 satisfy (83), (85) and (86). Therefore, we have

$$(90) \quad \|v^{(m)}\|_{X_{T_1}} \leq \frac{19}{10}R < 2R.$$

Thus, from the continuity of $t \rightarrow \|v^{(m)}\|_{X_t}$, there is $\epsilon > 0$ such that $\|v^{(m)}\|_{X_{T_1+\epsilon}} \leq 4R$. Then again, from (89) and (90) with $T_1 + \epsilon$ in place of T_1 we derive that $\|v^{(m)}\|_{X_{T_1+\epsilon}} \leq 2R$ as long as N and $T_1 + \epsilon$ satisfy (83), (85) and (86). By observing that these conditions are independent of $m \in \mathbb{N}$ we obtain a time interval $[0, T]$ such that $\|v^{(m)}\|_{X_T} \leq 2R$ for all $m \in \mathbb{N}$.

A similar computation on the difference, by possibly taking larger N and smaller T leads to the estimate

$$(91) \quad \begin{aligned} \|v^{(m_1)} - v^{(m_2)}\|_{X_T} &= \|\Gamma_{v_0^{(m_1)}} v^{(m_1)} - \Gamma_{v_0^{(m_2)}} v^{(m_2)}\|_{X_T} \leq \\ &(1 + \frac{1}{10})\|v_0^{(m_1)} - v_0^{(m_2)}\|_{M_{2,q}} + \frac{1}{5}\|v^{(m_1)} - v^{(m_2)}\|_{X_T}, \end{aligned}$$

which implies

$$(92) \quad \|v^{(m_1)} - v^{(m_2)}\|_{X_T} \leq c \|v_0^{(m_1)} - v_0^{(m_2)}\|_{M_{2,q}},$$

for some $c > 0$ and therefore, the sequence $\{v^{(m)}\}_{m \in \mathbb{N}}$ is Cauchy in the Banach space X_T . Let us denote by v^∞ its limit in X_T and by $u^\infty = S(t)v^\infty$. We will show that u^∞ satisfies NLS (1) in the interval $[0, T]$ in the sense of Definition 3. For convenience, we drop the superscript ∞ and write u, v . In addition, let $u^{(m)} := S(t)v^{(m)}$, where $v^{(m)}$ is the smooth solution to (21) with smooth initial data $v_0^{(m)}$ as described above and note that $u^{(m)}$ is the smooth solution to (1) with smooth initial data $u_0^{(m)} := v_0^{(m)}$. Furthermore, $u^{(m)} \rightarrow u$ in X_T because $v^{(m)} \rightarrow v$ in X_T and since convergence in the modulation space $M_{2,q}(\mathbb{R})$ implies convergence in the sense of distributions we conclude that $\partial_x u^{(m)} \rightarrow \partial_x u$ and $\partial_t u^{(m)} \rightarrow \partial_t u$ in $\mathcal{D}'((0, T) \times \mathbb{R})$. Since $u^{(m)}$ satisfies NLS (1) for every $m \in \mathbb{N}$ we have that

$$\mathcal{N}(u^{(m)}) = u^{(m)}|u^{(m)}|^2 = -i\partial_t u^{(m)} + \partial_x^2 u^{(m)},$$

also converges to some distribution $w \in \mathcal{S}'((0, T) \times \mathbb{R})$. Our claim is the following:

Proposition 26. Let w be the limit of $\mathcal{N}(u^{(m)})$ in the sense of distributions as $m \rightarrow \infty$. Then, $w = \mathcal{N}(u)$, where $\mathcal{N}(u)$ is to be interpreted in the sense of Definition 2.

Proof. Consider a sequence of Fourier cutoff multipliers $\{T_N\}_{N \in \mathbb{N}}$ as in Definition 1. We will prove that

$$\lim_{N \rightarrow \infty} \mathcal{N}(T_N u) = w,$$

in the sense of distributions. Let ϕ be a test function and $\epsilon > 0$ a fixed given number. Our goal is to find $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have

$$(93) \quad |\langle w - \mathcal{N}(T_N u), \phi \rangle| < \epsilon.$$

The LHS can be estimated as

$$\begin{aligned} |\langle w - \mathcal{N}(T_N u), \phi \rangle| &\leq |\langle w - \mathcal{N}(u^{(m)}), \phi \rangle| + |\langle \mathcal{N}(u^{(m)}) - \mathcal{N}(T_N u^{(m)}), \phi \rangle| \\ &\quad + |\langle \mathcal{N}(T_N u^{(m)}) - \mathcal{N}(T_N u), \phi \rangle|. \end{aligned}$$

The first term is estimated very easily since by the definition of w we have that

$$(94) \quad |\langle w - \mathcal{N}(u^{(m)}), \phi \rangle| < \frac{1}{3} \epsilon,$$

for sufficiently large $m \in \mathbb{N}$.

To continue, let us consider the second summand for fixed m . By writing the difference $\mathcal{N}(u^{(m)}) - \mathcal{N}(T_N u^{(m)})$ as a telescoping sum we have to estimate terms of the form

$$\left| \int \int (I - T_N) u^{(m)} |u^{(m)}|^2 \phi \, dx \, dt \right|,$$

where I denotes the identity operator. By Hölder's inequality and (4) we obtain that this integral is bounded by

$$\begin{aligned} \|\phi\|_{L_{T,x}^2} \|u^{(m)}\|_{L_{T,x}^\infty}^2 \|(I - T_N)u^{(m)}\|_{L_{T,x}^2} &\lesssim C_\phi \|u^{(m)}\|_{C((0,T), M_{2,q}^s)}^2 \|(I - T_N)u^{(m)}\|_{L_{T,x}^2} \\ &\leq C_{\phi,m} \|(I - T_N)u^{(m)}\|_{L_{T,x}^2}, \end{aligned}$$

where $L_{T,x}^2 = L^2((0, T) \times \mathbb{R})$. By definition of the Fourier cutoff operators, the function $\mathcal{F}\left((I - T_N)u^{(m)}(\cdot, t)\right)(\xi)$ converges pointwise in t and ξ and by an application of the Dominated Convergence Theorem, there is $N_0 = N_0(m)$ with the property

$$(95) \quad C_{\phi,m} \|(I - T_N)u^{(m)}\|_{L_{T,x}^2} < \frac{1}{3} \epsilon,$$

for all $N \geq N_0$.

For the last term, we need to observe two things. Firstly, let us consider the sequence $\{\mathcal{N}(T_N u^{(m)})\}_{m \in \mathbb{N}}$, for each fixed N . By applying the iteration process that we described in the previous subsection to $\{S(-t)\mathcal{N}(T_N u^{(m)})\}_{m \in \mathbb{N}}$, which is basically the nonlinearity in equation (21) up to the operator T_N , we see that $\{\mathcal{N}(T_N u^{(m)})\}_{m \in \mathbb{N}}$ is Cauchy in $\mathcal{S}'((0, T) \times \mathbb{R})$, as $m \rightarrow \infty$ for each fixed $N \in \mathbb{N}$ since the sequence $u^{(m)}$ is Cauchy in

$C((0, T), M_{2,q}(\mathbb{R}))$. Since the multipliers m_N of T_N are uniformly bounded in N we conclude that this convergence is uniform in N .

Secondly, let us observe that for fixed N , $T_N u$ is in $C((0, T), H^\infty(\mathbb{R}))$ since $u \in M_{2,q}(\mathbb{R})$ and the multiplier m_N of T_N is compactly supported. Hence, $\mathcal{N}(T_N u) = T_N u |T_N u|^2$ makes sense as a function. Therefore, for fixed N by Hölder's inequality we get

$$\begin{aligned} |\langle \mathcal{N}(T_N u^{(m)}) - \mathcal{N}(T_N u), \phi \rangle| &\leq \|\phi\|_{L^4_{T,x}} (\|T_N u^{(m)}\|_{L^4_{T,x}}^2 + \|T_N u\|_{L^4_{T,x}}^2) \|T_N u^{(m)} - T_N u\|_{L^4_{T,x}} \\ &\leq C_{\phi, \|u\|_{X_T}} M^{\frac{3}{4}} T^{\frac{3}{4}} \|u^{(m)} - u\|_{X_T} < \frac{1}{3} \epsilon, \end{aligned}$$

where the number $M = M(N) > 0$ is chosen so that $\text{supp}(m_N) \subset [-M, M]$. Here we used Hölder's inequality in the interval $(0, T)$ to pass from the L^4 norm to the L^∞ norm and in the space variable an application of Parseval's identity together with the fact that the multiplier operators T_N have compactly supported symbols m_N . Hence, $\mathcal{N}(T_N u^{(m)})$ converges to $\mathcal{N}(T_N u)$ in $\mathcal{S}'((0, T) \times \mathbb{R})$ as $m \rightarrow \infty$ for each fixed N .

From these two observations we derive that $\mathcal{N}(T_N u^{(m)}) \rightarrow \mathcal{N}(T_N u)$ in $\mathcal{S}'((0, T) \times \mathbb{R})$ as $m \rightarrow \infty$ uniformly in N . Equivalently,

$$(96) \quad |\langle \mathcal{N}(T_N u^{(m)}) - \mathcal{N}(T_N u), \phi \rangle| < \frac{1}{3} \epsilon,$$

for all large m , uniformly in N . Therefore, (93) follows by choosing m sufficiently large so that (94) and (96) hold, and then choosing $N_0 = N_0(m)$ such that (95) holds. \square

Finally, we have shown that the function $u = u^\infty$ is a solution to the NLS (1) in the sense of Definition 3.

2.4. Unconditional Uniqueness. In Subsections 2.1 and 2.2 we switched the order of space integration with time differentiation and summation in the discrete variable with time differentiation too. In the following we justify these formal computations and obtain the unconditional wellposedness of Theorem 6.

In this subsection we assume that $u_0 \in M_{2,q}^s(\mathbb{R})$ with either $s \geq 0$ and $1 \leq q \leq \frac{3}{2}$ or $\frac{3}{2} < q \leq 2$ and $s > \frac{2}{3} - \frac{1}{q}$ (see also Remark 7 for the case $q = 2$ and $s = 1/6$) which by (5) and (6) implies that

$$(97) \quad M_{2,q}^s(\mathbb{R}) \hookrightarrow M_{3,\frac{3}{2}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R}).$$

By (97) we know that if u is a solution of NLS (1) in the space $C([0, T], M_{2,q}^s(\mathbb{R}))$ then u and hence $v = e^{it\partial_x^2} u$ are elements of $X_T \hookrightarrow C([0, T], L^3(\mathbb{R}))$. Thus, the nonlinearity of NLS (1) makes sense as an element of $C([0, T], L^1(\mathbb{R}))$ and by (16) we obtain that $\partial_t v_n \in C([0, T], L^1(\mathbb{R}))$. Next, let us state a lemma:

Lemma 27. *Let $f(t, x), \partial_t f(t, x) \in C([0, T], L^1(\mathbb{R}^d))$ and define the distribution $\int_{\mathbb{R}^d} f(\cdot, x) dx$ by*

$$\left\langle \int_{\mathbb{R}^d} f(\cdot, x) dx, \phi \right\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(t, x) \phi(t) dx dt,$$

with $\phi \in C_c^\infty(\mathbb{R})$. Then, $\partial_t \int_{\mathbb{R}^d} f(\cdot, x) dx = \int_{\mathbb{R}^d} \partial_t f(\cdot, x) dx$.

Proof. By definition

$$\left\langle \partial_t \int_{\mathbb{R}^d} f(\cdot, x) dx, \phi \right\rangle = - \left\langle \int_{\mathbb{R}^d} f(\cdot, x) dx, \phi' \right\rangle = - \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(t, x) \phi'(t) dx dt,$$

and since $f \in C([0, T], L^1(\mathbb{R}^d))$ we can change the order of integration by Fubini's Theorem and obtain

$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}} f(t, x) \phi'(t) dt dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \partial_t f(t, x) \phi(t) dt dx = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \partial_t f(t, x) \phi(t) dx dt,$$

where in the first equality we used the definition of the weak derivative of f and in the second equality Fubini's Theorem with the fact that $\partial_t f \in C([0, T], L^1(\mathbb{R}^d))$. The last integral is equal to

$$\left\langle \int_{\mathbb{R}^d} \partial_t f(\cdot, x) dx, \phi \right\rangle,$$

and the proof is complete. \square

Consider now (25) for fixed n and ξ . We want to apply Lemma 27 to the function

$$f(t, \xi_1, \xi_3) = \sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3),$$

where $\xi \approx n, \xi_1 \approx n_1, \xi_3 \approx n_3, \xi - \xi_1 - \xi_3 \approx -n_2$ and $(n, n_1, n_2, n_3) \in A_N(n)^c$ given by (24). Notice that $f, \partial_t f \in C([0, T], L^1(\mathbb{R}^2))$ since $v \in C([0, T], M_{2,q}^s(\mathbb{R}))$ and $\partial_t v_n \in C([0, T], L^1(\mathbb{R}))$ for all integers n . Thus,

$$\begin{aligned} & \partial_t \left[\int_{\mathbb{R}^2} \sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3 \right] = \\ & \int_{\mathbb{R}^2} \sigma_n(\xi) \partial_t \left[\sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) \right] d\xi_1 d\xi_3 = \\ & \int_{\mathbb{R}^2} \sigma_n(\xi) \partial_t \left[\frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \right] \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3 + \\ & \int_{\mathbb{R}^2} \sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \partial_t \left[\hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi-\xi_1-\xi_3) \hat{v}_{n_3}(\xi_3) \right] d\xi_1 d\xi_3. \end{aligned}$$

In the second equality we used the product rule which is applicable since $v \in C([0, T], L^3(\mathbb{R}))$ implies that $\partial_t v_n \in C([0, T], L^1(\mathbb{R}))$.

Finally it remains to justify the interchange of differentiation in time and summation in the discrete variable but this is done in exactly the same way as in [9] (Lemma 5.1). Similar arguments justify the interchange on the J th step of the infinite iteration procedure.

Thus, for $v \in C([0, T], M_{2,q}^s(\mathbb{R}))$ with $M_{2,q}^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ we can repeat the calculations of Subsections 2.1 and 2.2 and 2.3 to obtain the following expression in X_T for the solution u of NLS (1) with initial data u_0

$$(98) \quad u = \Gamma_{u_0} u + \lim_{J \rightarrow \infty} \int_0^t N_2^{(J+1)}(u) d\tau,$$

where the limit is an element of X_T . Its existence follows from the fact that the operators $\Gamma_{u_0}^{(J)} u$ converge to $\Gamma_{u_0} u$ in the norm of X_T as $J \rightarrow \infty$. The important estimate about the remainder operator $N_2^{(J)}$ is the following:

Lemma 28.

$$\lim_{J \rightarrow \infty} \|N_2^{(J)}(v)\|_{l^\infty L^2} = 0.$$

Proof. By (72) we can write the remainder operator as the following sum

$$(99) \quad N_2^{(J)}(v)(n) = \partial_t(N_0^{(J+1)}(v)(n)) + \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T^0, \mathbf{n}}^{J, t}(\partial_t^{(\alpha)}(\{v_{n_\beta}\}_{\beta \in T^\infty})),$$

where we define the action of $\partial_t^{(\alpha)}$ onto the set of functions $\{v_{n_\beta}\}_{\beta \in T^\infty}$ to be the same set of functions except for the α node where we replace v_{n_α} by the function $\partial_t v_{n_\alpha}$.

We control the first summand $\partial_t(N_0^{(J+1)}(v)(n))$ by Lemma 22. For the last summand of the RHS of (99) we estimate its L^2 norm in exactly the same way as in the proof of Lemma (25) and arrive at the upper bound

$$\sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|v_{n_\beta}\|_2 \frac{\|\partial_t v_{n_\alpha}\|_2}{\prod_{k=1}^J |\tilde{\mu}_k|},$$

which by Hölder's inequality with exponents $\frac{1}{q} + \frac{1}{q'} = 1$ implies

$$\frac{1}{J^{(3-\frac{3}{q'})J}} \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|v_{n_\beta}\|_2^q \|\partial_t v_{n_\alpha}\|_2^q \right)^{\frac{1}{q}}.$$

Then for the sum inside the parenthesis we apply Young's inequality in the discrete variable where for the first $2J$ functions we take the l^1 norm and for the last the l^∞ norm we arrive at the estimate

$$\|v\|_{M_{2,q}^{2J}}^2 \sup_{n \in \mathbb{Z}} \|\partial_t v_n\|_2 = \|v\|_{M_{2,q}^{2J}}^2 \|\partial_t v_n\|_{l^\infty L^2}.$$

Since by (16) we have $\partial_t v_n = e^{it\partial_x^2} \square_n(|u|^2 u)$ it is straightforward to obtain

$$\|\partial_t v_n\|_{l^\infty L^2} \lesssim \|v\|_{M_{2,q}^3}^3.$$

Indeed, from (9) and since $\square_n(|u|^2 u)$ is nicely localised it suffices to estimate

$$\|\square_n(|u|^2 u)\|_2 \lesssim \|\square_n(|u|^2 u)\|_1 \lesssim \| |u|^2 u \|_1 = \|u\|_3^3 \lesssim \|u\|_{M_{2,q}^3}^3 = \|v\|_{M_{2,q}^3}^3,$$

where we used (10) and (8). Therefore, putting everything together we have

$$\|N_2^{(J)}(v)\|_{l^\infty L^2} \lesssim \frac{1}{J^{(2-\frac{3}{q'})J}} \|v\|_{M_{2,q}^{2J+3}},$$

which finishes the proof. \square

This lemma implies that $\lim_{J \rightarrow \infty} \int_0^t N_2^{(J+1)}(u) d\tau$ is equal to 0 in X_T . From this we obtain the unconditional uniqueness of NLS (1) since if there are two solutions u_1 and u_2 with the same initial datum u_0 we obtain by (92)

$$\|u_1 - u_2\|_{X_T} = \|\Gamma_{u_0} u_1 - \Gamma_{u_0} u_2\|_{X_T} \lesssim \|u_0 - u_0\|_{M_{2,q}^s} = 0.$$

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