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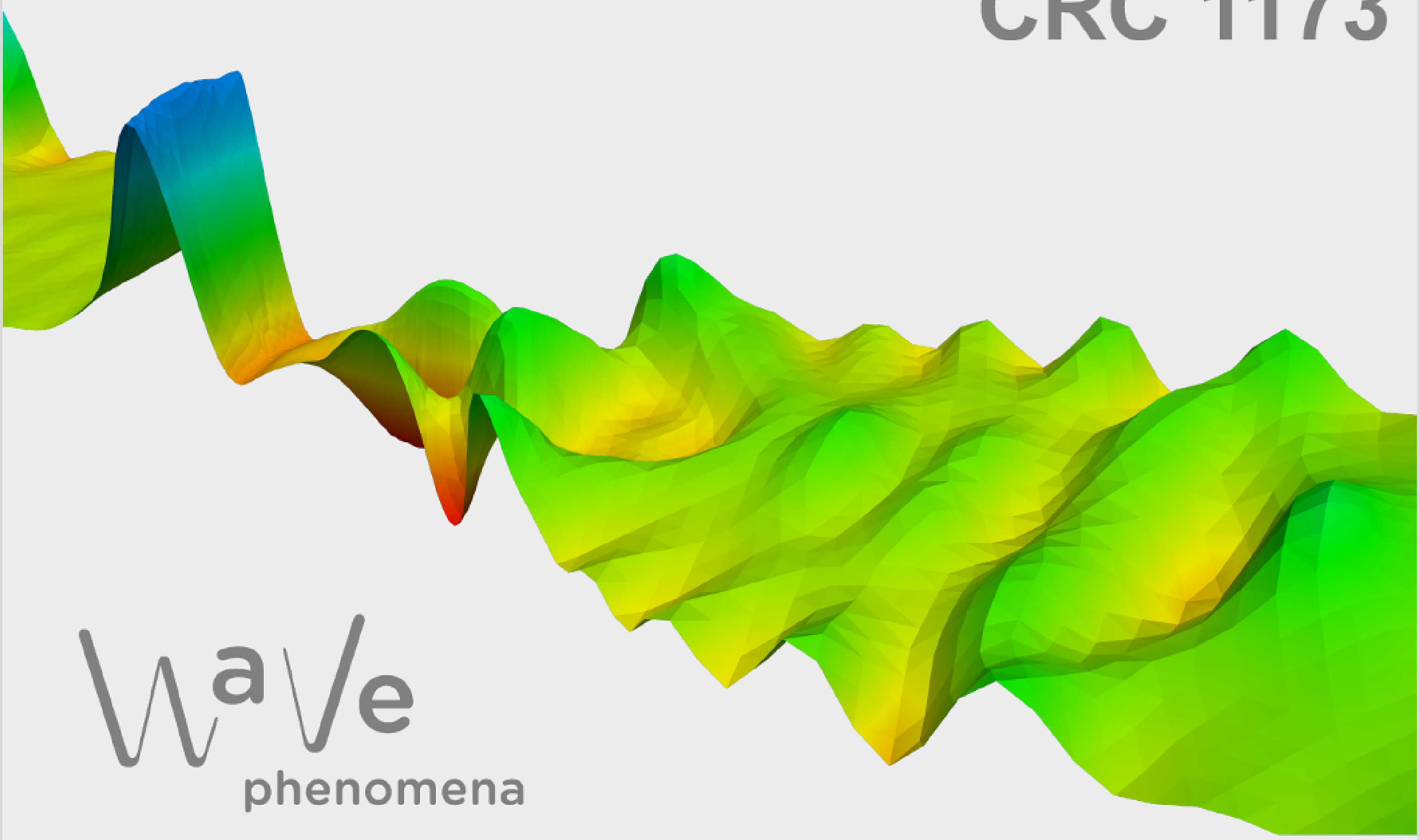

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FROM THE KLEIN-GORDON-ZAKHAROV SYSTEM TO THE KLEIN-GORDON EQUATION

MARKUS DAUB, GUIDO SCHNEIDER, KATHARINA SCHRATZ

Abstract. In a singular limit the Klein-Gordon (KG) equation can be derived from the Klein-Gordon-Zakharov (KGZ) system. We point out that for the original system posed on a d-dimensional torus the solutions of the KG equation do not approximate the solutions of the KGZ system. The KG system has to be modified to make correct predictions about the dynamics of the KGZ system. We explain that this modification is not necessary for the approximation result for the whole space $\mathbb{R}^d$ with $d \geq 3$.

1. Introduction

The Klein-Gordon-Zakharov (KGZ) system occurs as a model in plasma physics where it describes the interaction between so called Langmuir waves and ion sound waves in plasma via some ion density fluctuation n and the electric field z. It is derived from the Euler equation for the electrons and ions, coupled with the Maxwell equation for the electric field, cf. [MN02]. We consider the KGZ system in the form

$$\partial_t^2 z - \Delta z + z = -nz, \qquad \alpha^{-2}\partial_t^2 n - \Delta n = \Delta z^2, \tag{1}$$

with $x \in \mathbb{R}^d$ or $x \in \mathbb{T}^d$, $t \in \mathbb{R}$, $z(x,t) \in \mathbb{R}^d$, and $n(x,t) \in \mathbb{R}$ with $d \in \{1,2,3\}$. For the definition of z^2 see the short section at the end of the introduction about the notation used in this paper.

We are interested in the singular limit $\alpha \to \infty$. In this limit we obtain

$$\partial_t^2 Z - \Delta Z + Z = -NZ, \qquad -\Delta N = \Delta Z^2. \tag{2}$$

The choice $N = -Z^2$ yields the Klein-Gordon (KG) equation

$$\partial_t^2 Z - \Delta Z + Z = |Z|^2 Z. \tag{3}$$

The question occurs whether solutions of the KG equation (3) allow to approximate solutions of the KGZ system (1). The following simple observation makes immediately clear that this is a very delicate question whose answer depends on the

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underlying physical space. For functions which are constant in space the KGZ system (1) degenerates into

$$\partial_t^2 z + z = -nz, \qquad \alpha^{-2}\partial_t^2 n = 0, \tag{4}$$

and the KG equation (3) into

$$\partial_t^2 Z + Z = |Z|^2 Z. \tag{5}$$

Therefore $n(t) = n_0 + n_1 t$, with $n_0, n_1 \in \mathbb{R}$, and so z satisfies

$$\partial_t^2 z(t) + z(t) = -(n_0 + n_1 t)z(t). \tag{6}$$

Obviously, the difference $|Z(t) - z(t)|$ between the solution Z of (5) and the solution z of (6) grows in time independently of the small perturbation parameter α^{-1}, cf. Figure 3 for an illustration. Hence for the KGZ system posed on a d-dimensional torus the solutions of the KGZ system cannot be expected to be approximated by the solutions of the KG equation. It turns out that the KG system has to be modified in order to make correct predictions about the dynamics of the KGZ system. Instead of choosing $N(x,t) = -Z(x,t)^2$ on the torus $\mathbb{T}^d$ we consider

$$N(x,t) = -Z(x,t)^2 + \beta(t) \tag{7}$$

with $\beta(t)$ chosen in such a way that $\int_{\mathbb{T}^d} \partial_t^2 N(x,t)dx = 0$. In detail we choose

$$\beta(t) = (2\pi)^{-d} \int_{\mathbb{T}^d} -Z(x,0)^2 + Z(x,t)^2 dx \tag{8}$$

and Z to satisfy the modified KG equation

$$\partial_t^2 Z - \Delta Z + Z = -\beta Z + |Z|^2 Z. \tag{9}$$

Then our approximation result is as follows.

Theorem 1.1. *Consider the KGZ system* (1) *for* $x \in \mathbb{T}^d$ *with* $d \leq 3$. *Let* $(Z, \beta) \in C([0,T_0], H^3 \times \mathbb{R})$ *be a solution of the modified KG equation* (8)-(9). *Then there exist* C *and* $\alpha_0 > 0$ *such that for all* $\alpha > \alpha_0$ *we have a solution* (z,n) *of the KGZ system* (1) *with*

$$\sup_{t\in[0,T_0]} (\|z(\cdot,t) - Z(\cdot,t)\|_{H^2} + \|n(\cdot,t) + Z(\cdot,t)^2 - \beta(t)\|_{H^1}) \leq C\alpha^{-1}.$$

Interestingly this modification is not necessary for $x \in \mathbb{R}^d$ with $d \geq 3$. There the approximation result is as follows.

Theorem 1.2. *Consider the KGZ system* (1) *for* $x \in \mathbb{R}^3$. *Let* $Z \in C([0,T_0], H^3)$ *be a solution of the KG equation* (3). *Then there exist* C *and* $\alpha_0 > 0$ *such that for all* $\alpha > \alpha_0$ *we have a solution* (z,n) *of the KGZ system* (1) *with*

$$\sup_{t\in[0,T_0]} (\|z(\cdot,t) - Z(\cdot,t)\|_{H^2} + \|n(\cdot,t) + Z(\cdot,t)^2\|_{H^1}) \leq C\alpha^{-1}.$$

Before proving these results we close this introduction with a number of remarks.

Remark 1.1. For the KGZ system various different singular limits have been considered, and a number of approximation results have been established, cf. [MN02, MN05, MN08, MN10]. The only rigorous approximation result where a KG equation has been derived from a KGZ system can be found in [CEGT04]. However, as far as we can see the limit considered in [CEGT04] is different from the one considered here, and $x \in \mathbb{T}^d$ is not discussed in [CEGT04].

Remark 1.2. The corresponding limit has been considered for the Zakharov system

$$i\partial_t z + \Delta z = nz, \qquad \alpha^{-2}\partial_t^2 n - \Delta n = \Delta |z|^2,$$

with $z(x,t) \in \mathbb{C}$, $n(x,t) \in \mathbb{R}$, in various papers, cf. [SW86, AA88]. In the limit $\alpha \to \infty$ with the choice $N = -|Z|^2$ the NLS equation $i\partial_t Z + \Delta Z = -Z|Z|^2$ occurs. Due to the fact that the NLS equation preserves the L^2 norm, without any modification we have $\int_{\mathbb{T}^d} \partial_t^2 N(x,t)dx = 0$, and so the previous problem does not occur for the Zakharov system. However, if higher order approximations are considered corresponding assumptions for these can be found in the existing literature [OT92].

Remark 1.3. The proof of the approximation results is a non-trivial task. We have $\partial_t^2 n = \mathcal{O}(\alpha^2)$, but solutions have to be bounded on a $\mathcal{O}(1)$ time scale. We will use energy estimates to use the oscillatory character of the $\mathcal{O}(\alpha^2)$ terms. In our energy E, terms of the form $\int \alpha^{-2}(\partial_t n)^2$ occur which vanish for $\alpha \to \infty$. Therefore, no $\partial_t n$ terms should occur on the right hand side of our energy estimates "$\partial_t E \leq \ldots$". This problem is solved by expressing all $\partial_t n$ terms as total derivative w.r.t. t such that they can be included in the energy on the right hand side. See Section 2.

Remark 1.4. There are not so many examples of amplitude equations which, although derived in a formally correct way, make wrong predictions about the dynamics of the original system, cf. [Sch95, Sch05, SSZ15]. The KG equation without modification for the KGZ system on the torus can be added to this list of examples.

Remark 1.5. The error can be estimated with the help of the previously mentioned energy estimates by using Gronwall's inequality if the so called residual is sufficiently small. The residual contains the terms which do not cancel after inserting the approximation into the equations. Usually, no difficulties occur from this side. However, here the term $\nabla^{-1}\partial_t^2 N$ occurs in the residual. In order to have this term bounded in some Sobolev space on the torus we need $\int_{\mathbb{T}^d} \partial_t^2 N(x,t)dx = 0$. However the mean value of $N = -Z^2$ w.r.t. x will not be preserved as t evolves. Therefore, we use the correction $\beta = \beta(t)$ to get rid of this problem. For $x \in \mathbb{R}^d$ with $d \geq 3$ this correction is not necessary. Since $N \in L^1$ if $Z \in L^2$ the operator ∇^{-1} can be inverted from $L^1 \cap H^s$ to H^{s+1} if $d \geq 3$. See Section 3.

Remark 1.6. Due to the scaling with α^{-2} in (1) it can be expected that the error made by the approximation is of order $\mathcal{O}(\alpha^{-2})$ for $\alpha \to \infty$. This expectation is confirmed by our numerical experiments presented in Section 4 which indicate that the expected order $\mathcal{O}(\alpha^{-2})$ for the error really occurs. However, due to our method of proof using energy estimates, especially (20) and (22), we were only able to prove an error of order $\mathcal{O}(\alpha^{-1})$. We expect that with sup-norm based estimates the optimal rate $\mathcal{O}(\alpha^{-2})$ can be obtained, but we have to leave this to future research.

Remark 1.7. A main utility of approximating the KGZ system (1) by the modified KG system (8)-(9) is that the latter is easier to simulate for $\alpha \gg 1$ due to the reciprocal coupling of the step size of the temporal discretization of (1) with the parameter α. This is explained in detail in Remark 4.1 at the end of the paper.

Remark 1.8. The KGZ system (1) can be written as a semilinear evolutionary system

$$\partial_t U = \Lambda U + N(U), \tag{10}$$

with

$$U = (z, w, n, v), \quad \Lambda U = (w, \Delta z - z, \alpha v, \alpha \Delta n), \quad N(U) = (0, -nz, 0, \alpha \Delta z^2),$$

for which we have the following local existence and uniqueness result. For $U_0 \in \mathcal{X}$ with $\mathcal{X} = H^2 \times H^1 \times H^1 \times L^2$ there exists a $T_0 > 0$ such that (1) possesses solutions $U \in C([-T_0, T_0], \mathcal{X})$ with $U|_{t=0} = U_0$. The proof follows by a standard fixed point argument applied to the variation of constant formula

$$U(t) = e^{\Lambda t} U_0 + \int_0^t e^{\Lambda(t-s)} N(U)(s) ds, \tag{11}$$

related to (10), where $e^{\Lambda t} : \mathcal{X} \to \mathcal{X}$ is the strongly continuous semigroup generated by Λ. The right hand side of (11) is a contraction in a ball of $C([-T_0, T_0], \mathcal{X})$.

Notation. We use the following abbreviations. For $a, b \in \mathbb{R}^d$ we write

$$ab = \sum_{j=1}^{d} a_j b_j, \quad a^2 = \sum_{j=1}^{d} a_j^2 = |a|^2, \quad |a| = (\sum_{j=1}^{d} a_j^2)^{1/2},$$

$$(\nabla a)(\nabla b) = \sum_{j=1}^{d} \sum_{l=1}^{d} (\partial_{x_l} a_j)(\partial_{x_l} b_j), \quad (\nabla a)^2 = \sum_{j=1}^{d} \sum_{l=1}^{d} (\partial_{x_l} a_j)^2,$$

$$(\nabla^2 a)^2 = \sum_{j=1}^{d} \sum_{l=1}^{d} \sum_{m=1}^{d} (\partial_{x_l} \partial_{x_m} a_j)^2.$$

The discrete and the continuous Fourier transform of a function u are both denoted by $\widehat{u}$. Possibly different constants which can be chosen independently of the small perturbation parameter $0 < \alpha^{-1} \ll 1$ are denoted by the same symbol C.

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2. The error estimates

The residuals

$$\mathrm{Res}_z = -\partial_t^2 z + \Delta z - z - nz, \qquad \mathrm{Res}_n = -\alpha^{-2}\partial_t^2 n + \Delta n + \Delta z^2, \tag{12}$$

contain the terms which do not cancel after inserting the approximation into the original equations. For both, the modified and non-modified approximations (Z, N) of (z, n) we obtain

$$\mathrm{Res}_z = 0, \qquad \mathrm{Res}_n = -\alpha^{-2}\partial_t^2 N.$$

Then the errors

$$\alpha^{-1} R_z = z - Z, \qquad \alpha^{-1} R_n = n - N$$

satisfy

$$\partial_t^2 R_z - \Delta R_z + R_z = -N R_z - Z R_n - \alpha^{-1} R_z R_n, \tag{13}$$

$$\alpha^{-2}\partial_t^2 R_n - \Delta R_n = 2\Delta(Z R_z) + \alpha^{-1}\Delta(R_z^2) + \alpha \mathrm{Res}_n. \tag{14}$$

2.1. **Derivation of the energy terms.** We multiply the first equation (13) with $\partial_t R_z$, the second equation (14) with $\partial_t \Delta^{-1} R_n$, and integrate both w.r.t. x. We find

$$\begin{aligned} &\frac{1}{2}\partial_t \int \left((\partial_t R_z)^2 + (\nabla R_z)^2 + R_z^2 \right) dx \\ = & -\int (\partial_t R_z) N R_z dx - \int (\partial_t R_z) Z R_n dx - \alpha^{-1} \int (\partial_t R_z) R_z R_n dx \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\frac{1}{2}\partial_t \int \left(\alpha^{-2} (\partial_t \nabla^{-1} R_n)^2 + R_n^2 \right) dx \\ = & -2\int (\partial_t R_n) Z R_z dx - \alpha^{-1} \int (\partial_t R_n) R_z^2 dx - \alpha \int (\partial_t \Delta^{-1} R_n) \mathrm{Res}_n dx, \end{aligned} \tag{16}$$

where $\widehat{\Delta^{-1} R}(k) = -|k|^{-2}\widehat{R}(k)$ and $\widehat{\nabla^{-1} R}(k) = |k|^{-1}\widehat{R}(k)$. In the following we consider "(15)+γ(16)", whereof we would like to estimate all terms on the right hand side by the energy terms on the left hand side such that finally we can apply Gronwall's inequality to get uniform bounds independently of α. The major difficulty in obtaining these bounds are the $\partial_t R_n$ terms on the right hand side since these occur on the left hand side only with an α^{-2} in front. In order to get rid of this problem we try to rewrite all $\partial_t R_n$ terms on the right hand side as a total time derivative such that they can be included into the energy terms on the left hand side. We find

$$\begin{aligned}
&\frac{1}{2}\partial_t \int \left((\partial_t R_z)^2 + (\nabla R_z)^2 + R_z^2\right) + \gamma\left(\alpha^{-2}(\partial_t \nabla^{-1} R_n)^2 + R_n^2\right) dx \\
= \;& -\int (\partial_t R_z) N R_z dx - \int (\partial_t R_z) Z R_n dx - \alpha^{-1} \int (\partial_t R_z) R_z R_n dx \\
& -2\gamma \int (\partial_t R_n) Z R_z dx - \alpha^{-1}\gamma \int (\partial_t R_n) R_z^2 dx - \alpha\gamma \int (\partial_t \Delta^{-1} R_n) \mathrm{Res}_n dx \\
= \;& -\int (\partial_t R_z) N R_z dx + (2\gamma - 1) \int (\partial_t R_z) Z R_n dx + 2\gamma \int R_n (\partial_t Z) R_z dx \\
& +\alpha^{-1}(2\gamma - 1) \int (\partial_t R_z) R_z R_n dx - 2\gamma \partial_t \int R_n Z R_z dx \\
& -\alpha^{-1}\gamma \partial_t \int R_n R_z^2 dx - \alpha\gamma \int (\partial_t \Delta^{-1} R_n) \mathrm{Res}_n dx.
\end{aligned}$$

The derivatives can be controlled in a similar way. We multiply the gradient of the first equation (13) with $\partial_t \nabla R_z$, the second equation (14) with $\partial_t R_n$, and integrate both w.r.t. x. We find

$$\begin{aligned}
&\frac{1}{2}\partial_t \int \left((\partial_t \nabla R_z)^2 + (\nabla^2 R_z)^2 + (\nabla R_z)^2\right) dx \\
= \;& -\int (\partial_t \nabla R_z) \nabla (N R_z) dx - \int (\partial_t \nabla R_z) \nabla (Z R_n) dx - \alpha^{-1} \int (\partial_t \nabla R_z) \nabla (R_z R_n) dx
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\frac{1}{2}\partial_t \int \left(\alpha^{-2}(\partial_t R_n)^2 + (\nabla R_n)^2\right) dx \\
= \;& -2 \int (\partial_t \nabla R_n) \nabla (Z R_z) dx - \alpha^{-1} \int (\partial_t \nabla R_n) \nabla (R_z^2) dx + \alpha \int (\partial_t R_n) \mathrm{Res}_n dx.
\end{aligned} \tag{18}$$

We consider "(17)+γ(18)". The problem again are the $\partial_t R_n$ terms on the right hand side since these occur on the left hand side only with an α^{-2} in front. We proceed as above and find

$$\begin{aligned}
&\frac{1}{2}\partial_t \int \left((\partial_t \nabla R_z)^2 + (\nabla^2 R_z)^2 + (\nabla R_z)^2\right) + \gamma\left(\alpha^{-2}(\partial_t R_n)^2 + (\nabla R_n)^2\right) dx \\
= \;& -\int (\partial_t \nabla R_z) \nabla (N R_z) dx - \int (\partial_t \nabla R_z) \nabla (Z R_n) dx - \alpha^{-1} \int (\partial_t \nabla R_z) \nabla (R_z R_n) dx \\
& -2\gamma \int (\partial_t \nabla R_n) \nabla (Z R_z) dx - \alpha^{-1}\gamma \int (\partial_t \nabla R_n) \nabla (R_z^2) dx + \alpha\gamma \int (\partial_t R_n) \mathrm{Res}_n dx
\end{aligned}$$

$$\begin{aligned} = & -\int (\partial_t \nabla R_z)\nabla(NR_z)dx - \int (\partial_t \nabla R_z)\nabla(ZR_n)dx + 2\gamma \int (\nabla R_n)\partial_t \nabla(ZR_z)dx \\ & -\alpha^{-1}\int (\partial_t \nabla R_z)\nabla(R_z R_n)dx + \alpha^{-1}\gamma \int (\partial_t \nabla(R_z^2))\nabla R_n dx \\ & -2\gamma\partial_t \int (\nabla R_n)\nabla(ZR_z)dx - \alpha^{-1}\gamma\partial_t \int (\nabla R_n)\nabla(R_z^2)dx + \alpha\gamma \int (\partial_t R_n)\mathrm{Res}_n dx. \end{aligned}$$

In a similar fashion the higher order derivatives can be controlled, but for notational simplicity we restricted ourselves to the case $d \leq 3$.

2.2. **The energy estimates.** We collect all total time derivatives in our energy

$$E = E_0 + E_\nabla$$

with $E_0 = \widetilde{E}_0 + \widetilde{E}_{0,r}$ and $E_\nabla = \widetilde{E}_\nabla + \widetilde{E}_{\nabla,r}$, where

$$\begin{aligned} \widetilde{E}_0 &= \int ((\partial_t R_z)^2 + (\nabla R_z)^2 + R_z^2 + \alpha^{-2}\gamma(\partial_t \nabla^{-1} R_n)^2 + \gamma R_n^2)dx, \\ \widetilde{E}_\nabla &= \int ((\partial_t \nabla R_z)^2 + (\nabla^2 R_z)^2 + (\nabla R_z)^2 + \alpha^{-2}\gamma(\partial_t R_n)^2 + \gamma(\nabla R_n)^2)dx, \\ \widetilde{E}_{0,r} &= \int (4\gamma R_n Z R_z + 2\alpha^{-1}\gamma R_n R_z^2)dx, \\ \widetilde{E}_{\nabla,r} &= \int (4\gamma(\nabla R_n)\nabla(ZR_z) + 4\alpha^{-1}\gamma(\nabla R_n)R_z(\nabla R_z))dx. \end{aligned}$$

Obviously the square root of our initial energy terms collected in

$$\widetilde{E} = \widetilde{E}_0 + \widetilde{E}_\nabla$$

allows to estimate the $H^1 \times H^2$ norm for (R_n, R_z). The next lemma shows that this is also true for the square root of our updated energy E.

Lemma 2.1. *Let $d \leq 3$, $C_1 = \|Z\|_{L^\infty} + \|\nabla Z\|_{L^\infty}$, and $E \leq C_2$. Then for given C_1 there exists a $\gamma > 0$, and then for a given C_2 an $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$ we have*

$$\frac{1}{4}\widetilde{E}^{1/2} \leq E^{1/2} \leq 4\widetilde{E}^{1/2}.$$

Proof. We are done with the proof of Lemma 2.1 if we estimate $\widetilde{E}_{0,r} + \widetilde{E}_{\nabla,r}$ by $\widetilde{E}/2$. We start with the α-independent quadratic term in $\widetilde{E}_{0,r}$ and $\widetilde{E}_{\nabla,r}$. We find

$$|\int 4\gamma R_n Z R_z dx| \leq 4\gamma C_1 \|R_n\|_{L^2} \|R_z\|_{L^2}$$

and

$$|\int 4\gamma(\nabla R_n)(\nabla Z)R_z dx + \int 4\gamma(\nabla R_n)Z\nabla R_z dx| \leq 8\gamma C_1\|R_n\|_{H^1}\|R_z\|_{H^1}.$$

Hence, for these terms it is sufficient to estimate $16\gamma C_1 ab$, with $a = \|R_n\|_{H^1}$ and $b = \|R_z\|_{H^1}$, by $\widetilde{E}/4$. Since $\widetilde{E}/4$ obviously allows to estimate $(a^2+\gamma b^2)/4$ we have to guarantee that

$$16\gamma C_1 ab < \frac{1}{4}(a^2+\gamma b^2). \tag{19}$$

Rescaling $b = B/\gamma^{1/2}$ and $A = a$ yields

$$64\gamma^{1/2}C_1 AB < A^2 + B^2$$

which holds if $64\gamma^{1/2}C_1 \in (-2,2)$. This shows the validity of (19) if $\gamma > 0$ is suitably chosen. The terms in $E - \widetilde{E}$ with α^{-1} can be controlled for $\alpha > 0$ sufficiently big. Note that $Z \in H^3$ implies $Z \in W^{1,\infty}$ due to Sobolev's embedding theorem. □

Since

$$\|R_n\|_{H^1} + \|R_z\|_{H^2} \leq \widetilde{E}^{1/2} \leq 4E^{1/2},$$

for the proofs of Theorem 1.1 and Theorem 1.2 it is sufficient to find an $\mathcal{O}(1)$ bound for E. In order to do so, we take the time derivative of the energy parts. We find

$$\partial_t E_0 = s_1 + s_2 + s_3 + s_4 + s_5$$

with

$$\begin{aligned}
|s_1| &= |2\int(\partial_t R_z)NR_z dx| \leq C\|N\|_{L^\infty}\|\partial_t R_z\|_{L^2}\|R_z\|_{L^2} \leq CE,\\
|s_2| &= |2(2\gamma-1)\int(\partial_t R_z)ZR_n dx| \leq C\|Z\|_{L^\infty}\|\partial_t R_z\|_{L^2}\|R_n\|_{L^2} \leq CE,\\
|s_3| &= |4\gamma\int(\partial_t Z)R_z R_n dx| \leq C\|\partial_t Z\|_{L^\infty}\|R_z\|_{L^2}\|R_n\|_{L^2} \leq CE,\\
|s_4| &= |2\alpha^{-1}(2\gamma-1)\int(\partial_t R_z)R_z R_n dx|\\
&\leq C\alpha^{-1}\|\partial_t R_z\|_{L^2}\|R_z\|_{L^\infty}\|R_n\|_{L^2} \leq C\alpha^{-1}E^{3/2},
\end{aligned}$$

and

$$\begin{aligned}
|s_5| &= |2\alpha\gamma\int(\partial_t\Delta^{-1}R_n)\mathrm{Res}_n dx| \leq 2\gamma\alpha^{-1}\|\partial_t\nabla^{-1}R_n\|_{L^2}\alpha^2\|\nabla^{-1}\mathrm{Res}_n\|_{L^2}\\
&\leq (\gamma\alpha^{-1}\|\partial_t\nabla^{-1}R_n\|_{L^2})^2 + (\alpha^2\|\nabla^{-1}\mathrm{Res}_n\|_{L^2})^2 \qquad (20)\\
&\leq CE + (\|\nabla^{-1}\partial_t^2 N\|_{L^2})^2.
\end{aligned}$$

The constant C can be chosen independently of the small perturbation parameter $0 < \alpha^{-1} \ll 1$. We used Sobolev's embedding theorem

$$\|R_z\|_{L^\infty} \leq \|R_z\|_{H^s} \leq CE^{1/2} \tag{21}$$

if $s > d/2$ where the last estimate holds since we assumed $d \leq 3$. Next we estimate

$$\partial_t E_\nabla = s_7 + \ldots + s_{19}$$

with

$$\begin{aligned}
|s_7| &= |2\int (\partial_t \nabla R_z) N \nabla R_z dx| \leq C\|N\|_{L^\infty} \|\partial_t \nabla R_z\|_{L^2} \|\nabla R_z\|_{L^2} \leq CE, \\
|s_8| &= |2\int (\partial_t \nabla R_z)(\nabla N) R_z dx| \leq C\|\nabla N\|_{L^\infty} \|\partial_t \nabla R_z\|_{L^2} \|R_z\|_{L^2} \leq CE, \\
|s_9| &= |2\int (\partial_t \nabla R_z)(\nabla Z) R_n dx| \leq C\|\nabla Z\|_{L^\infty} \|\partial_t \nabla R_z\|_{L^2} \|R_n\|_{L^2} \leq CE, \\
|s_{10}| &= |2\int (\partial_t \nabla R_z) Z \nabla R_n dx| \leq C\|Z\|_{L^\infty} \|\partial_t \nabla R_z\|_{L^2} \|\nabla R_n\|_{L^2} \leq CE, \\
|s_{11}| &= |4\gamma \int (\nabla R_n)(\partial_t \nabla Z) R_z dx| \leq C\|\partial_t \nabla Z\|_{L^\infty} \|R_z\|_{L^2} \|\nabla R_n\|_{L^2} \leq CE, \\
|s_{12}| &= |4\gamma \int (\nabla R_n)(\partial_t Z) \nabla R_z dx| \leq C\|\partial_t Z\|_{L^\infty} \|\nabla R_z\|_{L^2} \|\nabla R_n\|_{L^2} \leq CE, \\
|s_{13}| &= |4\gamma \int (\nabla R_n) Z \partial_t \nabla R_z dx| \leq C\|Z\|_{L^\infty} \|\partial_t \nabla R_z\|_{L^2} \|\nabla R_n\|_{L^2} \leq CE, \\
|s_{14}| &= |4\gamma \int (\nabla R_n)(\nabla Z) \partial_t R_z dx| \leq C\|\nabla Z\|_{L^\infty} \|\partial_t R_z\|_{L^2} \|\nabla R_n\|_{L^2} \leq CE, \\
|s_{15}| &= |2\alpha^{-1} \int (\partial_t \nabla R_z)(\nabla R_z) R_n dx| \\
&\leq C\alpha^{-1} \|\partial_t \nabla R_z\|_{L^2} \|\nabla R_z\|_{L^4} \|R_n\|_{L^4} \leq C\alpha^{-1} E^{3/2}, \\
|s_{16}| &= |2\alpha^{-1} \int (\partial_t \nabla R_z) R_z (\nabla R_n) dx| \\
&\leq C\alpha^{-1} \|\partial_t \nabla R_z\|_{L^2} \|R_z\|_{L^\infty} \|\nabla R_n\|_{L^2} \leq C\alpha^{-1} E^{3/2}, \\
|s_{17}| &= |4\alpha^{-1}\gamma \int (\partial_t R_z)(\nabla R_z)(\nabla R_n) dx| \\
&\leq C\alpha^{-1} \|\partial_t R_z\|_{L^4} \|\nabla R_z\|_{L^4} \|\nabla R_n\|_{L^2} \leq C\alpha^{-1} E^{3/2}, \\
|s_{18}| &= |4\alpha^{-1}\gamma \int (\partial_t \nabla R_z) R_z (\nabla R_n) dx| \\
&\leq C\alpha^{-1} \|\partial_t \nabla R_z\|_{L^2} \|R_z\|_{L^\infty} \|\nabla R_n\|_{L^2} \leq C\alpha^{-1} E^{3/2},
\end{aligned}$$

and

$$\begin{aligned} |s_{19}| &= |2\alpha\gamma \int (\partial_t R_n)\mathrm{Res}_n dx| \leq 2\gamma\alpha^{-1}\|\partial_t R_n\|_{L^2}\alpha^2\|\mathrm{Res}_n\|_{L^2} \\ &\leq (\gamma\alpha^{-1}\|\partial_t R_n\|_{L^2})^2 + (\alpha^2\|\mathrm{Res}_n\|_{L^2})^2 \\ &\leq CE + (\|\partial_t^2 N\|_{L^2})^2. \end{aligned} \tag{22}$$

The constant C can be chosen independently of the small perturbation parameter $0 < \alpha^{-1} \ll 1$. Beside (21) we used Sobolev's embedding theorem $\|\nabla R_z\|_{L^4} \leq \|\nabla R_z\|_{H^1}$ for $-d/4 < 1 - d/2$ to obtain

$$\|\nabla R_z\|_{L^4} + \|\partial_t R_z\|_{L^4} \leq CE^{1/2}.$$

Remember that we restricted ourselves for notational simplicity to $d \leq 3$. For $d \geq 4$ more and more higher derivatives have to be considered. It is obvious that they can be handled in exactly the same manner.

2.3. **Application of Gronwall's inequality.** Assume for the moment that we have a bound C_{res} for $\|\nabla^{-1}\partial_t N\|_{L^2}$ and $\|\partial_t N\|_{L^2}$, cf. Lemma 3.1 and Lemma 3.2. Summing up all estimates gives then the inequality

$$\partial_t E \leq C_1 E + C_2 \alpha^{-1} E^{3/2} + C_{res}^2,$$

with constants C_1, C_2, and C_{res} independent of the small perturbation parameter $0 < \alpha^{-1} \ll 1$ and the energy E. In order to show that the energy E can be bounded independently of α we set $M = C_{res}^2 T_0 \mathrm{e}^{(C_1+C_2)T_0}$ and choose $\alpha_0^{-1} > 0$ such that $\alpha_0^{-1} M^{1/2} \leq 1$. Then define $\widetilde{T} = \inf\{t \geq 0 : E(t) > M\}$. Assume that $\widetilde{T} < T_0$. For all $t < \widetilde{T}$ and $\alpha^{-1} \in (0, \alpha_0^{-1}]$ we have

$$\partial_t E \leq C_1 E + C_2 \alpha^{-1} E^{3/2} + C_{res}^2 \leq (C_1 + C_2)E + C_{res}^2.$$

Hence, Gronwall's inequality yields

$$E(t) \leq C_{res}^2 t \mathrm{e}^{(C_1+C_2)t} \leq C_{res}^2 T_0 \mathrm{e}^{(C_1+C_2)T_0} = M \tag{23}$$

for all $t \in [0, T_0]$. This contradicts $\widetilde{T} < T_0$. Therefore, we established an $\mathcal{O}(1)$ bound for the energy.

Remark 2.1. a) We assumed in (23) that $R_z|_{t=0} = 0$, $R_n|_{t=0} = 0$, $\partial_t R_z|_{t=0} = 0$, and $\partial_t R_n|_{t=0} = 0$. It is obvious that we can choose an arbitrary initial condition of order $\mathcal{O}(1)$ for the energy $E(0)$. Note that $E(0) = \mathcal{O}(1)$ allows for $\partial_t R_n|_{t=0} = \mathcal{O}(\alpha)$ due to the coefficient α^{-2} in front of $\|\partial_t \nabla^{-1} R_n\|_{H^1}^2$ in the definition of E.
b) By making the ansatz $\alpha^{-\theta} R_z = z - Z$ and $\alpha^{-\theta} R_n = n - N$ with $\theta \in (0,1)$ the initial difference between the approximation and the correct solution even can be made larger and a modified approximation theorem with an error $\mathcal{O}(\alpha^{-\theta})$ can be established. Note that this allows for $\partial_t n|_{t=0} = \mathcal{O}(\alpha^{1-\theta})$.

3. Estimates for the residual

It remains to bound the residual terms. Usually no difficulties occur from this side. However, here we need to bound $\|\nabla^{-1}\partial_t^2 N\|_{L^2}$. This turns out to be a nontrivial task since the operator ∇^{-1} is unbounded from L^2 to L^2.

3.1. **On the torus.**

As explained in the introduction, in order to have the term $\nabla^{-1}\partial_t^2 N$ bounded in some Sobolev space on the torus we need $\int_{\mathbb{T}^d} \partial_t^2 N(x,t)dx = 0$. We achieved this by introducing the correction $\beta = \beta(t)$. So we have

Lemma 3.1. *Consider the KGZ system* (1) *for $x \in \mathbb{T}^d$. Let $(Z,\beta) \in C([0,T_0], H^2 \times \mathbb{R})$ be a solution of the modified KG equation* (8)-(9). *Then there exists a $C_{res} > 0$ such that*

$$\sup_{t\in[0,T_0]} (\|\nabla^{-1}\partial_t^2 N\|_{L^2}^2 + \|\partial_t^2 N\|_{L^2}^2) \leq C_{res}^2$$

where N is defined in (7)*.*

Proof. The operator ∇^{-1} can be applied to $\partial_t^2 N$ and estimated via Parseval's identity via

$$\begin{aligned}
\|\nabla^{-1}\partial_t^2 N\|_{L^2}^2 &= \|x \mapsto \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{-1}\partial_t^2 \widehat{N}_k(t)\mathrm{e}^{ikx}\|_{L^2}^2 \\
&= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} ||k|^{-1}\partial_t^2 \widehat{N}_k(t)|^2 \leq \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\partial_t^2 \widehat{N}_k(t)|^2 = \|\partial_t^2 N\|_{L^2}^2.
\end{aligned}$$

By definition $\|\partial_t^2 N\|_{L^2}$ can be estimated by $C\cdot(\|Z\|_{L^\infty}\|\partial_t^2 Z\|_{L^2} + \|\partial_t Z\|_{L^4}^2)$. Since Z satisfies the modified KG equation this can be estimated by a multiple of

$$\|Z\|_{L^\infty}(\|Z\|_{H^2} + \||Z|^2 Z\|_{L^2}) + \|\partial_t Z\|_{L^4}^2.$$

Using Sobolev's inequality for $d \leq 3$ all terms can be estimated in terms of $\|Z\|_{H^2}$. □

3.2. **On $\mathbb{R}^d$ for $d \geq 3$.**

For $x \in \mathbb{R}^d$ with $d \geq 3$ the correction with $\beta = \beta(t)$ is not necessary. There we have

Lemma 3.2. *Consider the KGZ system* (1) *for $x \in \mathbb{R}^d$ and $d \geq 3$. Let $Z \in C([0,T_0], H^2)$ be a solution of the KG equation* (3) *and $N = -Z^2$. Then there exists a $C_{res} > 0$ such that*

$$\sup_{t\in[0,T_0]} (\|\nabla^{-1}\partial_t^2 N\|_{L^2}^2 + \|\partial_t^2 N\|_{L^2}^2) \leq C_{res}^2.$$

Proof. Since

$$\begin{aligned}
\|\nabla^{-1}u\|_{L^2} &\leq C\||k|^{-1}\widehat{u}(k)\|_{L^2(dk)} \\
&\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-1}\widehat{u}(k)\|_{L^2(dk)} + \|\chi_{|k|>1}(k)|k|^{-1}\widehat{u}(k)\|_{L^2(dk)})
\end{aligned}$$

$$
\begin{aligned}
&\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-1}\|_{L^2(dk)}\|\widehat{u}(k)\|_{L^\infty(dk)} \\
&\qquad +\|\chi_{|k|>1}(k)|k|^{-1}\|_{L^\infty(dk)}\|\widehat{u}(k)\|_{L^2(dk)}) \\
&\leq C(\|\chi_{|k|\leq 1}(k)|k|^{-1}\|_{L^2(dk)}\|u\|_{L^1}+\|u\|_{L^2})
\end{aligned}
$$

and since

$$\|\chi_{|k|\leq 1}(k)|k|^{-1}\|_{L^2(dk)} = \int_{|k|\leq 1}|k|^{-2}dk = C_d\int_0^1 r^{d-3}dr < \infty$$

for $d \geq 3$ the operator ∇^{-1} can be inverted from $L^1 \cap L^2$ to L^2 if $d \geq 3$.

The L^2 norm of $\partial_t^2 N$ can be estimated exactly as in the proof of Lemma 3.1. The L^1 norm of $\partial_t^2 N$ is estimated as

$$\|\partial_t^2 N\|_{L^1} = \|\partial_t^2(Z^2)\|_{L^1} \leq 2(\|Z\|_{L^2}\|\partial_t^2 Z\|_{L^2} + \|\partial_t Z\|_{L^2}^2)$$

where all terms on the right hand side can be estimated again in terms of $\|Z\|_{H^2}$ via the KG equation. □

4. Some numerical illustrations

In order to illustrate the approximation of the KGZ system (1) through the modified KG equation (9) in the limit $\alpha \to \infty$, we implemented first-order centered Finite Difference schemes in space and time for both equations. The numerical stability of the Finite Difference method requires that the step size Δt of the temporal discretization depends on the value of α and the step size of the spatial discretization Δx, cf. the CFL condition $\alpha\Delta t/\Delta x < 1$, respectively $\Delta t < \Delta x/\alpha$. Moreover, the evaluation of the integral on the right hand side of (8) requires a small spatial step size Δx to reduce numerical inaccuracies in the approximation result due to the numerical integration. In summary, a small Δx and a large value of α lead to a tiny step size Δt. This numerical challenge finally limits the values of α for the numerical illustrations.

We test the approximation behavior in the one-dimensional case for the initial condition

$$z(x,0) = A\sin(x), \quad \partial_t z(x,0) = 0, \tag{24}$$

$$n(x,0) = -z(x,0)^2 + \underbrace{\beta(0)}_{=0}, \quad \partial_t n(x,0) = -2z(x,0)\cdot\partial_t z(x,0) + \underbrace{\partial_t\beta(0)}_{=0}, \tag{25}$$

$$Z(x,0) = z(x,0), \quad \partial_t Z(x,0) = \partial_t z(x,0), \tag{26}$$

and various amplitudes A. We consider 2^{14} discretization nodes on the interval $[0, 2\pi)$, hence $\Delta x = 2\pi/2^{14}$, and periodic boundary conditions. The step size of the temporal discretization actually should be chosen dependent on α, namely $\Delta t =$

$\Delta x/(2\alpha)$ for instance. However, we set $\Delta t = \Delta x/(2\alpha_{\max})$, where $\alpha_{\max}$ is the largest value under consideration for α. This choice ascertain that the determination of the approximation error is not influenced by the numerical error of the Finite Difference schemes.

The numerical solutions illustrate that the solution of the KGZ system (1) is well approximated by the solution of the modified KG equation (9), cf. Figure 1. In particular, we observe that the approximation is already well for $\alpha = 10$. Solving the KGZ system (1) and the modified KG equation (9) for other values of α additionally illustrates the behavior of the approximation error. Choosing a general ansatz of the form $f(x) = c \cdot x^m$ for a nonlinear regression gives an almost x^{-2} decrease of the approximation error in the L^2 norm. More precisely, we consider the logarithmic error data and perform a linear regression, cf. Figure 2. Hence, the numerical approximation error is better than the x^{-1} decrease which we proved in Theorem 1.1, cf. Remark 1.

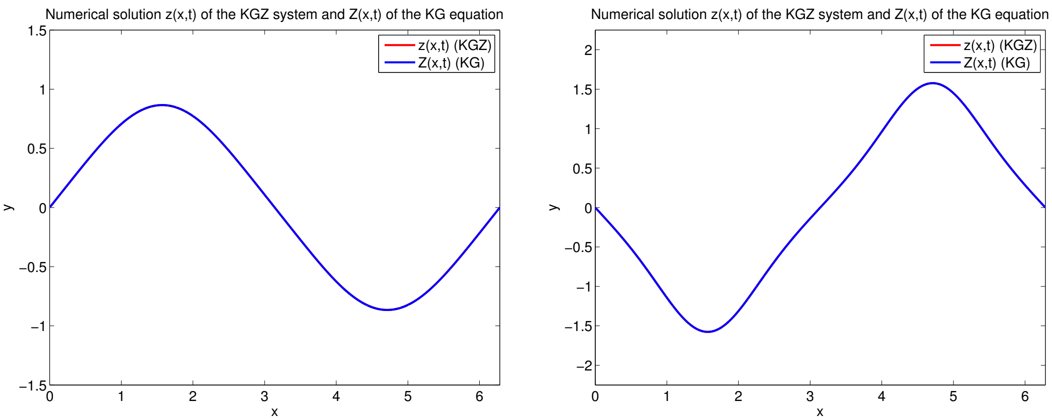

FIGURE 1. The numerical solution of the KGZ system (1) (red graph) and the modified KG equation (9) (blue graph) for the initial condition $z(x, 0) = \sin(x)$ (left figure) and $z(x, 0) = 1.5 \cdot \sin(x)$ (right figure) after $T_0 = 5$ in case of $\alpha = 10$. The red and blue graphs almost coincide. The same statements are true for the n-variable.

Finally, we illustrate the situation with the original KG equation (3). We observe that the solution of the KG equation does not approximate the solution of the KGZ system neither for $\alpha = 10$ nor for a larger value of α, for instance $\alpha = 100$, see Figure 3.

Remark 4.1. As explained above the numerical time-integration of highly-oscillatory problems is very delicate: In order to resolve the oscillations severe time-step restrictions need to be imposed which leads to huge computational costs and often do not

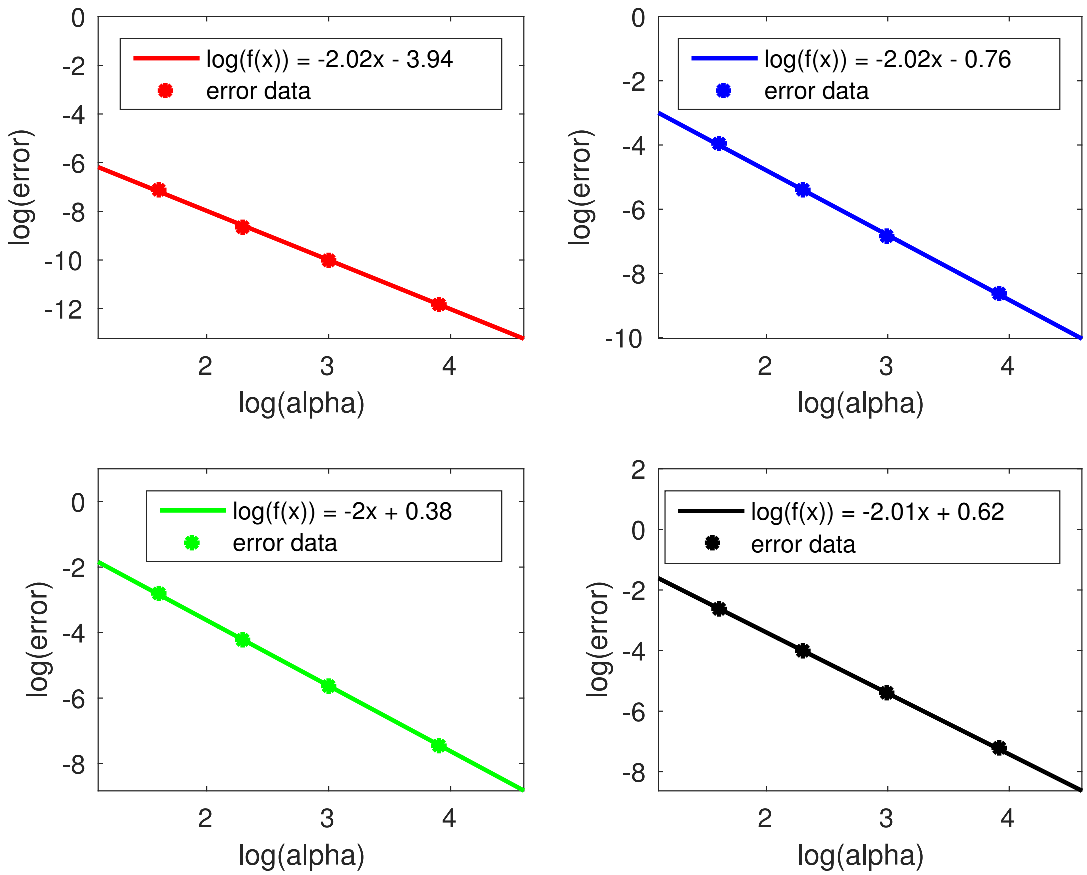

FIGURE 2. The numerical approximation error $\|R_z\|_{L^2} + \|R_n\|_{L^2}$ for $A = 0.1$ (left top), $A = 0.5$ (right top), $A = 1.0$ (left bottom) and $A = 1.5$ (right bottom). The solid lines illustrate the linear regression of the logarithmic error data.

yield a good approximation. The derived modified KG equation (9) allows us to construct efficient numerical time integrators for the KGZ system (1) in the subsonic regime $\alpha \gg 1$ without imposing any α-dependent CFL condition. More precisely, let Z^n denote the numerical approximation to the non-oscillatory modified KG solution $Z(t_n)$ at time $t_n = n\Delta t$ obtained with, for instance, a Gautschi-type integrator (cf. [HL99]). Then, for sufficiently smooth solutions and all $\alpha \geq 1$ the following error bound holds

$$\|z(t_n) - Z^n\|_{H^2} \leq C\|z(t_n) - Z(t_n)\|_{H^2} + \|Z(t_n) - Z^n\|_{H^2} \leq C\left(\alpha^{-1} + \Delta t^2\right). \tag{27}$$

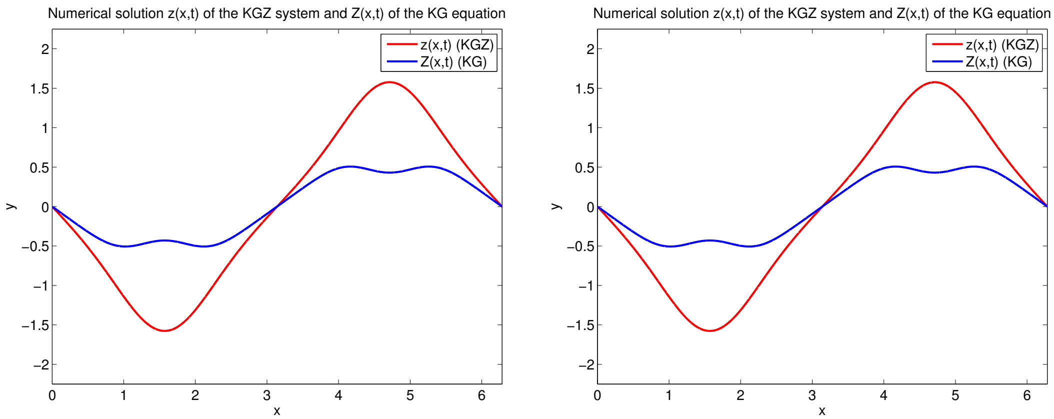

FIGURE 3. The numerical solution of the KGZ system (1) (red graph) and the original KG equation (3) (blue graph) for the initial condition $z(x, 0) = 1.5 \cdot \sin(x)$ after $T_0 = 5$ for $\alpha = 10$ (left figure) and $\alpha = 100$ (right figure). The correct solution (red graph) and the formal approximation (blue graph) behave differently as has been explained in (4)-(6).

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