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Leonid Chaichenets, Dirk Hundertmark, Peer Kunstmann, Nikolaos Pattakos

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Leonid Chaichenets, Dirk Hundertmark, Peer Kunstmann, Nikolaos Pattakos

Institute for Analysis, Karlsruhe Institute of Technology (KIT), 76128 Karlsruhe, Germany

Abstract

We show the local well-posedness of the Cauchy problem for the cubic nonlinear Schrödinger equation on modulation spaces \( M^s_{p,q}(\mathbb{R}^d) \) for \( d \in \mathbb{N}, 1 \leq p, q \leq \infty \) and \( s > d \left( 1 - \frac{1}{q} \right) \) for \( q > 1 \) or \( s \geq 0 \) for \( q = 1 \). This improves [4, Theorem 1.1] by Bényi and Okoudjou where only the case \( q = 1 \) is considered. Our result is based on the algebra property of modulation spaces with indices as above for which we give an elementary proof via a new Hölder-like inequality for modulation spaces.

1. Introduction

We study the Cauchy problem for the cubic nonlinear Schrödinger equation (NLS)

\[
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) + \Delta u(x,t) \pm |u|^2 u(x,t) &= 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\
u(x,0) &= u_0(x) & x \in \mathbb{R}^d,
\end{aligned}
\]

where the initial data \( u_0 \) is in a modulation space \( M^s_{p,q}(\mathbb{R}^d) \). A definition of \( M^s_{p,q}(\mathbb{R}^d) \) will be given in the next paragraph. As usual, we are interested in mild solutions \( u \) of (1), i.e. \( u \in C([0,T), M^s_{p,q}(\mathbb{R}^d)) \) for a \( T > 0 \) which satisfy the corresponding integral equation

\[
u(\cdot,t) = e^{it\Delta} u_0 \pm i \int_0^t e^{i(t-\tau)\Delta} \left( |u|^2 u(\cdot,\tau) \right) d\tau \quad (\forall t \in [0,T)).\]

Modulation spaces \( M^s_{p,q}(\mathbb{R}^d) \) were introduced by Feichtinger in [6]. Here, we give a short summary of their definition and properties. (We refer to Section 2 and the literature mentioned there for more information, the notation we use is explained at the end of the introduction.) Fix a so-called window function \( g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \). The short-time Fourier transform \( V_g f \) of a tempered distribution \( f \in \mathcal{S}'(\mathbb{R}^d) \) with respect to the window \( g \) is defined by

\[
V_g f (x,\cdot) = \mathcal{F} \left( \overline{g(\xi)} f(x) \right) \in \mathcal{S}'(\mathbb{R}^d) \quad \forall x \in \mathbb{R}^d.
\]

In fact, \( V_g f : \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C} \) can be represented by a continuous function \( \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C} \). Hence, taking a weighted, mixed \( L^p \)-norm is possible and we define

\[
M^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \left| \| f \|_{M^s_{p,q}(\mathbb{R}^d)} < \infty \right. \right\}, \text{where } \| f \|_{M^s_{p,q}(\mathbb{R}^d)} = \left\| \xi \mapsto \langle \xi \rangle^s \| V_g f (\cdot,\xi) \|_p \right\|_q
\]

for \( s \in \mathbb{R}, 1 \leq p, q \leq \infty \). It can be shown, that the \( M^s_{p,q}(\mathbb{R}^d) \) are Banach spaces and that different choices of the window function \( g \) lead to equivalent norms.

Our main result is

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Email addresses: leonid.chaichenets@kit.edu (Leonid Chaichenets), dirk.hundertmark@kit.edu (Dirk Hundertmark), peer.kunstmann@kit.edu (Peer Kunstmann), nikolaos.pattakos@gmail.com (Nikolaos Pattakos)

Theorem 1 (Local well-posedness). Let $d \in \mathbb{N}$ and $1 \leq p, q \leq \infty$. For $q > 1$ let $s > d \left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Assume that $u_0 \in M^s_{p,q}(\mathbb{R}^d)$. Then, there exists a unique maximal mild solution $u \in C\left([0,T^*), M^s_{p,q}(\mathbb{R}^d)\right)$ of (1) and the blow-up alternative

$$T^* < \infty \quad \Rightarrow \quad \limsup_{t \to T^*} \|u(t)\|_{M^s_{p,q}(\mathbb{R}^d)} = \infty$$

holds. Furthermore, for any $0 < T' < T^*$ there exists a neighborhood $V$ of $u_0$ in $M^s_{p,q}$, such that the initial data to solution map

$$V \to C\left([0,T'], M^s_{p,q}(\mathbb{R}^d)\right), \quad v_0 \mapsto v,$$

is Lipschitz continuous.

Let us remark that the only known local well-posedness results in modulation spaces until now are [13, Theorem 1.1] by Wang, Zhao and Guo for $M^0_{2,1}(\mathbb{R}^d)$ and its generalization [4] Theorem 1.1] due to Bényi and Okoudjou for $M^s_{p,1}(\mathbb{R}^d)$ with $1 \leq p \leq \infty$ and $s \geq 0$. Local well-posedness results without persistence (i.e. initial data in a modulation space, but the solution is not a curve on it) include [17, Theorem 1.4] for $u_0 \in M^s_{p,q}(\mathbb{R}^d)$ with $2 \leq q < \infty$.

Theorem 1 generalizes [4, Theorem 1.1] due to Bényi and Okoudjou’s and our proofs being based on the well-known Banach’s contraction principle, an estimate for the norm of the Schrödinger propagator and the fact that the considered modulation spaces $M^s_{p,q}(\mathbb{R}^d)$ are Banach $^*$-algebras with respect to pointwise multiplication. Let us state the two latter ingredients formally and comment on them.

The first is given by

Proposition 2 (Algebra property). Let $d \in \mathbb{N}$ and $1 \leq p, q \leq \infty$. For $q > 1$ let $s > d \left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Then $M^s_{p,q}(\mathbb{R}^d)$ is a Banach $^*$-algebra with respect to pointwise multiplication and complex conjugation. These operations are well-defined due to the following embedding

$$M^s_{p,q}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) \mid f \text{ bounded}\}.$$

Proposition 2 had been observed already in 1983 by Feichtinger in his pioneering work on modulation spaces, cf. [6, Proposition 6.9] where he proves it using a rather abstract approach via Banach convolution triples. This might explain why the algebra property seems to be not well-known in the PDE community. In [4, Corollary 2.6] Proposition 2 for $q = 1$ is stated without referring to Feichtinger and a proof via the theory of pseudodifferential operators is said to be along the lines of [2, Theorem 3.1]. In contrast to these approaches, our proof of the algebra property is elementary. It follows from the new Hölder-like inequality stated in

Theorem 3 (Hölder-like inequality). Let $d \in \mathbb{N}$ and $1 \leq p_1, p_2, q \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For $q > 1$ let $s > d \left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Then there exists a constant $C = C(d, s, q) > 0$ such that

$$\|fg\|_{M^s_{p,q}(\mathbb{R}^d)} \leq C \|f\|_{M^s_{p_1,q}(\mathbb{R}^d)} \|g\|_{M^s_{p_2,q}(\mathbb{R}^d)}.$$  

(5)

\footnote{For us a Banach $^*$-algebra $X$ is a Banach algebra over $\mathbb{C}$ on which a continuous involution $^*$ is defined, i.e. $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda x}^*$, $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$. We neither require $X$ to have a unit nor $C = 1$ in the estimates $\|x \cdot y\| \leq C \|x\| \|y\|$, $\|x^*\| \leq C \|x\|$.}
for all \( f \in M_{p_1,q_1}^*(\mathbb{R}^d) \), \( g \in M_{p_2,q_2}^*(\mathbb{R}^d) \). The pointwise multiplication is well-defined due to the embedding formulated in Proposition 4.

Crucial for the proof of Theorem 3 is the algebra property of the sequence spaces \( l_q^s(\mathbb{Z}^d) \) stated in Lemma 2 (\( s, q \) and \( d \) are as in Theorem 3; \( l_q^s(\mathbb{Z}^d) \) is defined at the end of the introduction).

The second crucial ingredient for the proof of Theorem 3 is the boundedness of the Schrödinger propagator \( e^{it\Delta} \) on all modulation spaces \( M_{p,q}^s(\mathbb{R}^d) \). Let us fix the window function \( x \mapsto e^{-|x|^2} \) in the definition of the modulation space norm. Then we have (notation is explained at the end of the introduction)

**Theorem 4 (Schrödinger propagator bound).** There is a constant \( C > 0 \) such that for any \( d \in \mathbb{N} \), \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \) the inequality

\[
\|e^{it\Delta}\|_{L^p(M_{p,q}^s(\mathbb{R}^d))} \leq C^d(1 + |t|)^{d + \frac{1}{2} - \frac{s}{2} - \frac{1}{2}}
\]

holds for all \( t \in \mathbb{R} \). Furthermore, the exponent of the time dependence is sharp.

The boundedness has been obtained e.g. in [3 Theorem 1] whereas the sharpness was proven in [5 Proposition 4.1]. We sketch a simple proof of Theorem 4 in Section 2.

The remainder of our paper is structured as follows. We start with Section 2 providing an overview over modulation spaces, showing that Proposition 4 follows from Theorem 3 and sketching a simple proof of Theorem 4. In Section 3 we prove an algebra property of the weighted sequence spaces \( l_q^s(\mathbb{Z}^d) \) for sufficiently large \( s \). In the subsequent Section 4 we prove the Hölder-like inequality from Theorem 3. Finally, we prove Theorem 4 on the local well-posedness in Section 5.

**Notation**

We denote generic constants by \( C \). To emphasize on which quantities a constant depends we write e.g. \( C = C(d) \) or \( C = C(d,s) \). Sometimes we omit a constant from an inequality by writing “\( \lesssim \)”, e.g. \( A \lesssim B \) instead of \( A \leq C(d)B \). Special constants are \( d \in \mathbb{N} \) for the dimension, \( 1 \leq p,q \leq \infty \) for the Lebesgue exponents and \( s \in \mathbb{R} \) for the regularity exponent. By \( p' \) we mean the dual exponent of \( p \), that is the number satisfying \( \frac{1}{p} + \frac{1}{p'} = 1 \). To simplify the subsequent claims we shall call a regularity exponent \( s \) sufficiently large, if

\[
s \begin{cases} > \frac{d}{q} & \text{for } q > 1, \\ \geq 0 & \text{for } q = 1. \end{cases}
\]

We denote by \( \mathcal{S}(\mathbb{R}^d) \) the set of Schwartz functions and by \( \mathcal{S}'(\mathbb{R}^d) \) the space of tempered distributions. Furthermore, we denote the Bessel potential spaces or simply \( L^2 \)-based Sobolev spaces by \( H^s = H^s(\mathbb{R}^d) \) or by \( H^s(\mathbb{T}^d) \), if we are on the \( d \)-dimensional Torus \( \mathbb{T}^d \). For the space of bounded continuous functions we write \( C_{0} \) and for the space of smooth functions with compact support we write \( C_{0}^\infty \). The letters \( f, g, h \) denote either generic functions \( \mathbb{R}^d \to \mathbb{C} \) or generic tempered distributions. Whereas \( (a_k)_{k \in \mathbb{Z}^d}, (b_k)_{k \in \mathbb{Z}^d}, (c_k)_{k \in \mathbb{Z}^d} \) or \( (a_k), (b_k), (c_k) \) denote generic complex-valued sequences. By \( \langle \cdot \rangle := \sqrt{1 + |\cdot|^2} \) we denote the Japanese bracket.

For a Banach space \( X \) we write \( X^* \) for its dual and \( \| \cdot \|_X \) for the norm it is canonically equipped with. By \( \mathcal{L}(X) \) we denote the space of all bounded linear maps on \( X \). By \( [X,Y]_p \) we mean complex interpolation between \( X \) and another Banach space \( Y \). For brevity we write \( \| \cdot \|_p \) for the \( p \)-norm on the Lebesgue space \( L^p(\mathbb{R}^d) \), the sequence space \( l^p = l^p(\mathbb{N}) \) or \( l^p = l^p(\mathbb{N}_0) \), \( \| (a_k) \|_q \) or \( \| (a_k) \|_{q,s} := \| (\langle \cdot \rangle^{s} a_k) \|_q \) for the norm on \( \langle \cdot \rangle^{s} \)-weighted sequence spaces \( l^p = l^p_\mathbb{N}(\mathbb{Z}^d) \). Also, we shorten the notation for modulation spaces: \( M_{p,q}^s(\mathbb{R}^d) \) and even \( M_{p,q} \) for \( M_{p,q}^0(\mathbb{R}^d) \). If the norm is clear from the context, we write \( B_r(x) \) for a ball of radius \( r \) around \( x \in X \) and set \( B_r = B_r(0) \).

Furthermore, we denote the Fourier transform by \( \mathcal{F} \) and the inverse Fourier transform by \( \mathcal{F}^{-1} \), where we use the symmetric choice of constants and write also

\[
\hat{f}(\xi) := (\mathcal{F} f)(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\xi \cdot x} f(x) dx, \quad \hat{g}(x) := (\mathcal{F}^{-1} g)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\xi \cdot x} g(\xi) d\xi.
\]
Finally, we introduce the operations $S_x f(y) = f(y - x)$ of translation by $x \in \mathbb{R}^d$, $(M_k f)(y) = e^{ik \cdot y} f(y)$ of modulation by $k \in \mathbb{R}^d$ and $\bar{f}$ of complex conjugation.

2. Modulation spaces

As already mentioned in the introduction, modulation spaces were introduced by Feichtinger in [6] in the setting of locally compact Abelian groups. The textbook [3] by Gröchenig gives a thorough introduction, although it lacks the characterization of modulation spaces via isometric decomposition operators defined below. A presentation incorporating these operators is contained in the paper [12, Section 2.3] by Wang and Hudzik. A survey on modulation spaces and nonlinear evolution equations is given in [10].

A convenient equivalent norm on modulation spaces which we are going to use is constructed as follows:

Set $Q_0 := [-\frac{1}{2}, \frac{1}{2})^d$ and $Q_k := Q_0 + k$ for all $k \in \mathbb{Z}^d$. Consider a smooth partition of unity $(\sigma_k)_{k \in \mathbb{Z}^d} \in (C^\infty_0(\mathbb{R}^d))^\mathbb{Z}^d$ satisfying

(i) $\exists c > 0 : \forall k \in \mathbb{Z}^d : \forall \eta \in Q_k : |\sigma_k(\eta)| \geq c$,

(ii) $\forall k \in \mathbb{Z}^d : \text{supp}(\sigma_k) \subseteq B_\sqrt{\pi}(k)$,

(iii) $\sum_{k \in \mathbb{Z}^d} \sigma_k = 1$,

(iv) $\forall m \in \mathbb{N}_0 : \exists C_m > 0 : \forall k \in \mathbb{Z}^d : \forall \alpha \in \mathbb{N}_0^d : |\alpha| \leq m \Rightarrow \|D^\alpha \sigma_k\|_\infty \leq C_m$

and define the isometric decomposition operators $\square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}$. Let us mention the fact that $\square_k f \in C^{\infty}(\mathbb{R}^d)$ for $f \in S'(\mathbb{R}^d)$ by [7, Theorem 2.3.1]. We cite from [12, Proposition 1.9] the following often used

**Lemma 5 (Bernstein multiplier estimate).** Let $d \in \mathbb{N}$, $1 \leq p \leq \infty$, $s > \frac{d}{2}$ and $\sigma \in H^s(\mathbb{R}^d)$. Then the multiplier operator $T_\sigma = \mathcal{F}^{-1} \sigma \mathcal{F} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ corresponding to the symbol $\sigma$ is bounded on $L^p(\mathbb{R}^d)$. More precisely, there is a constant $C = C(s, d) > 0$ such that

$$
\|T_\sigma\|_{\mathcal{L}(L^p(\mathbb{R}^d))} \leq C \|\sigma\|_{H^s(\mathbb{R}^d)}.
$$

By Lemma 3 the family $(\square_k)_{k \in \mathbb{Z}^d}$ is bounded in $\mathcal{F}(L^p(\mathbb{R}^d))$ independently of $p$. The aforementioned equivalent norm for the modulation space $M^{s, p}_{q, q}$ is given by

$$
\|f\|_{M^{s, p}_{q, q}} \equiv \left\| (||\square_k f||_p)_{k \in \mathbb{Z}^d} \right\|_{q, s}.
$$

(8)

Choosing a different partition of unity $(\sigma_k)$ yields yet another equivalent norm.

**Lemma 6 (Continuous embeddings).** Let $s_1 \geq s_2$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Then

(a) $M^{s_1, p_1}_{q_1}(\mathbb{R}^d) \subseteq M^{s_2, p_2}_{q_2}(\mathbb{R}^d)$ and the embedding is continuous,

(b) $M^{s, p}_{q, q}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$.

**Lemma 5** is well-known (cf. [12, Proposition 2.5, 2.7]). For convenience we sketch a

**Proof.** (a) One can change indices one by one. The inclusion for “$q$” is by monotonicity and the inclusion for “$s$” is by the embeddings of the $l^p$ spaces. For the “$p$”-embedding consider $\tau \in C^\infty(\mathbb{R}^d)$ such that $\tau|_{B_{s/\tau}} \equiv 1$ and supp($\tau$) $\subseteq B_d$. Define the shifted $\tau_k = S_k \tau$ and the corresponding multiplier operators $\square_k := \mathcal{F}^{-1} \tau_k \mathcal{F}$. Clearly, $\square_k \square_k = \square_k$ and $\square_k f = \frac{1}{(2\pi)^{d/2}} (M_k \hat{\tau}) \ast f$. Hence

$$
\|\square_k f\|_{p_2} = \|\hat{\square_k} \hat{\square_k} f\|_{p_2} = \frac{1}{(2\pi)^{d/2}} \| (M_k \hat{\tau}) \ast (\square_k f)\|_{p_2} \leq \frac{1}{(2\pi)^{d/2}} \|\hat{\tau}\|_1 \|\square_k f\|_{p_1},
$$

where $\frac{1}{p} = 1 - \frac{1}{p_1} + \frac{1}{p_2}$. Recalling (8) finishes the proof.
(b) By part (a) it is enough to show $M_{\infty,1} \hookrightarrow C_b$. For any $f \in M_{\infty,1}$ we have $\sum_{k \in \mathbb{Z}^d} \|\triangle_k f\|_\infty \leq \sum_{k \in \mathbb{Z}^d} \|\triangle_k f\|_\infty < \infty$.

Thus $\sum_{k \in \mathbb{Z}^d} \|\triangle_k f\|_\infty < \infty$. So $f \in C_b$ and $\sum_{k \leq N} \|\triangle_k f\|_\infty < \infty$.

We are now ready to give a

**Proof of Proposition 2** We have $l^1_s \hookrightarrow l^1$ for sufficiently large $s$, since

$$\sum_{k \in \mathbb{Z}^d} |a_k| = \sum_{k \in \mathbb{Z}^d} \frac{1}{(k)^s} |a_k| \leq \left( \sum_{k \in \mathbb{Z}^d} \frac{1}{(k)^s} \right)^{\frac{1}{s}} \left( \sum_{k \in \mathbb{Z}^d} |a_k|^q \right)^{\frac{1}{q}} \leq \frac{C}{s^{\frac{1}{q}}}. $$

Then (8) yields $M^s_{p,q} \hookrightarrow M^s_{p,1}$ and by Lemma 6 (a) we have $M^s_{p,1} \hookrightarrow M^s_{p,q}$. This proves the claimed embedding.

Choosing $\sigma_k$ real-valued in (8) shows that complex conjugation does not change the modulation space norm.

Choosing $p_1 = p_2 = 2p$ in Theorem 8 and applying Lemma 6 (a) shows the estimate for the continuity of pointwise multiplication and finishes the proof.

**Lemma 7 (Dual space).** For $s \in \mathbb{R}$, $1 \leq p, q < \infty$ we have

$$(M^s_{p,q})^* = M^{-s}_{p',q'}$$

(see [12, Theorem 3.1]).

**Theorem 8 (Complex interpolation).** For $1 \leq p_1, q_1 < \infty$, $1 \leq p_2, q_2 < \infty$, $s_1, s_2 \in \mathbb{R}$ and $\theta \in (0,1)$ one has

$$[M^s_{p_1,q_1} (\mathbb{R}^d), M^s_{p_2,q_2} (\mathbb{R}^d)]_{\theta} = M^{s \theta}_{p,q} (\mathbb{R}^d),$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{\theta}{q_2}, \quad s = (1-\theta)s_1 + \theta s_2$$

(see [6, Theorem 6.1 (D)]).

Using these results we sketch a

**Proof of Theorem 3** We have $V \varphi(e^{it\Delta} f) = V e^{-it\Delta} g$ by duality, i.e. the Schrödinger time evolution of the initial data can be interpreted as the backwards time evolution of the window function. The price for changing from window $g_0$ to window $g_1$ is $\|V g_0 g_1\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}$ by [8] Proposition 11.3.2 (c)]. For $g(x) = e^{-|x|^2}$ one explicitly calculates

$$\|V e^{-it\Delta} g\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = C d (1 + |t|)^{\frac{d}{2}},$$

which proves the claimed bound for $p \in \{1, \infty\}$. Conservation for $p = 2$ is easily seen from [8]. Complex interpolation between the cases $p = 2$ and $p = \infty$ yields [8] for $2 \leq p \leq \infty$. The remaining case $1 < p < 2$ is covered by duality.

Optimality in the case $1 \leq p \leq 2$ is proven by choosing the window $g$ and the argument $f$ to be a Gaussian and explicitly calculating $\|e^{it\Delta} f\|_{M^s_{p,q}} \approx (1 + |t|)^{d(\frac{1}{d} - \frac{1}{2})}$. This implies the optimality for $2 < p \leq \infty$ by duality.
3. Algebra property of some weighted sequence spaces

Let us recall the definition of the $\langle \cdot \rangle^s$-weighted sequence spaces

$$l^s_q(\mathbb{Z}^d) = \left\{ (a_k) \in \mathbb{C}^{(\mathbb{Z}^d)} \left| \|(a_k)\|_{q,s} < \infty \right\} \right.,$$

where

$$\|(a_k)\|_{q,s} = \begin{cases} (\sum_{k \in \mathbb{Z}^d} (\langle k \rangle^s |a_k|^q)^{\frac{1}{q}})^{\frac{1}{s}} & \text{for } 1 \leq q < \infty, \\ \sup_{k \in \mathbb{Z}^d} (\langle k \rangle^s |a_k|) & \text{for } q = \infty, \end{cases}$$

and $s \in \mathbb{R}, d \in \mathbb{N}$. We have

**Lemma 9 (Algebra property).** Let $1 \leq q \leq \infty$. For $q > 1$ let $s > d \left( 1 - \frac{1}{q} \right)$ and for $q = 1$ let $s \geq 0$. Then $l^s_q(\mathbb{Z}^d)$ is a Banach algebra with respect to convolution

$$(a_l) \ast (b_m) = \left( \sum_{m \in \mathbb{Z}^d} a_{l-m} b_m \right)_{k \in \mathbb{Z}^d},$$

which is well-defined, as the series above always converge absolutely.

This result is most likely not new. For the sake of self-containedness of the presentation, and because we could not come up with any suitable reference, we will give a proof. The inspiration for Lemma 9 comes from the fact that $H^s(\mathbb{R}^d)$ for $s > \frac{d}{2}$ is a Banach algebra with respect to pointwise multiplication and $l^2_2(\mathbb{Z}^d) = \mathcal{F}(H^s(\mathbb{R}^d))$. A proof for the algebra property of $H^s(\mathbb{R}^d)$ can be given using the Littlewood-Paley decomposition, see e.g. [1, Proposition II.A.2.1.1 (ii)]. We were able to adapt that proof to the $l^s_q(\mathbb{Z}^d)$ case, even for $q \neq 2$, by noting that we are already on the Fourier side.

Let us recall that the Littlewood-Paley decomposition of a tempered distribution is a series essentially such that the Fourier transform of $l$-th summand has its support in the annulus with radii comparable to $2^l$. In the same spirit we formulate

**Lemma 10 (Discrete Littlewood-Paley characterization).** Let $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Define $C(s) = 2^{s|l|}$.

$$A_0 := \{0\} \subseteq \mathbb{Z}^d, \quad \text{and} \quad A_l := \left\{ k \in \mathbb{Z}^d \left| 2^{(l-1)} \leq |k| < 2^l \right. \right\} \quad \forall l \in \mathbb{N}.$$

(a) (Necessary condition) For any $(a_k) \in l^s_q(\mathbb{Z}^d)$ there is a sequence $(C_l) \in l^q(\mathbb{N}_0)$ such that $\|C_l\|_q = 1$ and

$$\left\|(\mathbb{1}_{A_l}(k)a_k)\|_q \leq C(s)2^{-ls}C_l \| (a_k)\|_{q,s} \quad \forall l \in \mathbb{N}_0.$$  

(b) (Sufficient condition) Conversely, if for some $N \geq 0$ and $(C_l) \in l^q(\mathbb{N}_0)$ with $\|C_l\|_q \leq 1$ the estimate

$$\left\| (\mathbb{1}_{A_l}(k)a_k) \right\|_q \leq \frac{1}{C(s)}2^{-ls}C_l N \quad \forall l \in \mathbb{N}_0$$

holds, then $(a_k) \in l^s_q(\mathbb{Z}^d)$ and $\| (a_k)\|_{q,s} \leq N$.

**Proof.** Observe that $2^{|l-1|} \leq \langle k \rangle < 2^{|l+1|}$ so $(k)^t \leq 2^{|l|/2}t$ for each $l \in \mathbb{N}_0$, $k \in A_l$ and $t \in \mathbb{R}$.

(a) For $(a_k) = 0$ there is nothing to show, so assume $\| (a_k)\|_{q,s} > 0$. Then for any $l \in \mathbb{N}_0$

$$\left\| (\mathbb{1}_{A_l}(k)a_k) \right\|_q = \left\| (\mathbb{1}_{A_l}(k)\langle k \rangle^s a_k) \right\|_q \leq \frac{C(s)}{2^{|s|}} \left\| (\mathbb{1}_{A_l}(k)a_k) \right\|_{q,s} = C(s)2^{-ls}C_l \| (a_k)\|_{q,s},$$

where $C_l := \frac{\|(\mathbb{1}_{A_l}(k)a_k)\|_{q,s}}{\| (a_k)\|_{q,s}}$. 

6
(b) We have \( (a_k) = \left( \sum_{l=0}^{\infty} \mathbb{1}_{A_l}(k)a_k \right) \). Thus, for \( q < \infty \),

\[
\| (a_k) \|_q, s = \sum_{l=0}^{\infty} \| (k)^s \mathbb{1}_{A_l}(k)a_k \|_q \leq C(s)^q \sum_{l=0}^{\infty} 2^{ls} \| (\mathbb{1}_{A_l}(k)a_k) \|_q \leq N^q \sum_{l=0}^{\infty} C^q_l \leq N^q.
\]

Similarly, for \( q = \infty \), we have

\[
\| (a_k) \|_{\infty, s} = \sup_{l \in \mathbb{N}_0} \| (k)^s \mathbb{1}_{A_l}(k)a_k \|_\infty \leq \sup_{l \in \mathbb{N}_0} C(s) 2^{ls} \| (\mathbb{1}_{A_l}(k)a_k) \|_\infty \leq N \sup_{l \in \mathbb{N}_0} C_l \leq N.
\]

For the proof of Lemma 9 we will require yet another sufficient condition. The discrete Littlewood-Paley decomposition in Lemma 10 consisted of sequences having their supports in disjoint dyadic annuli. We now consider non-disjoint dyadic balls \( B_m \).

**Lemma 11 (Sufficient condition for balls).** Let \( 1 \leq q \leq \infty \) and \( s > 0 \). Define \( C(s) = \frac{2^s}{s-2} \) and

\[
B_m := \{ k \in \mathbb{Z}^d \mid |k| < 2^m \} \quad \forall m \in \mathbb{N}_0.
\]

For each \( m \in \mathbb{N}_0 \) let \( (a_{k,m})_{k \in \mathbb{Z}^d} \) be such that \( \text{supp} \ (a_{k,m}) \subseteq B_m \). If for some \( N \geq 0 \) and \( (C_m) \in l^q(\mathbb{N}_0) \) with \( \| (C_m) \|_q \leq 1 \) the estimate

\[
\| (a_{k,m})_{k \in \mathbb{Z}^d} \|_q \leq \frac{1}{C(s)} 2^{-ms} C_m N \quad \forall m \in \mathbb{N}_0
\]

holds, then

\[
(a_k) := \left( \sum_{m=0}^{\infty} a_{k,m} \right) \in l^q_s(\mathbb{Z}^d) \quad \text{and} \quad \| (a_k) \|_{q, s} \leq N.
\]

**Proof.** We want to apply the sufficient condition for annuli. Observe, that \( A_l \cap B_m = \emptyset \) if \( l > m \). Hence

\[
\| (\mathbb{1}_{A_l}(k)a_k) \|_q = \left\| \left( \sum_{m=0}^{\infty} \mathbb{1}_{A_l \cap B_m}(k)a_{k,m} \right) \right\|_q \leq \sum_{m=0}^{\infty} \| (a_{k,m}) \|_q \leq \frac{1}{C(s)} N 2^{-ls} \sum_{m=0}^{\infty} 2^{-(m-l)s} C_m =: \tilde{C}_l
\]

for all \( l \in \mathbb{N}_0 \). It remains to show that \( (\tilde{C}_l) \in l^q(\mathbb{N}_0) \) and \( \| (\tilde{C}_l) \|_q \leq \frac{1}{1-\frac{1}{q}} \). We can assume \( 1 < q < \infty \), as the proof for the other cases is easier and follows the same lines. We have

\[
\tilde{C}_l = \sum_{m=0}^{\infty} 2^{-(m-l)s} \frac{1}{2^s} \times 2^{-(m-l)s} C_m \leq \left( \sum_{m=0}^{\infty} 2^{-ms} \right)^{q-1} \times \left( \sum_{m=0}^{\infty} 2^{-(m-l)s} C_m \right)^{\frac{1}{q}}
\]

for all \( l \in \mathbb{N}_0 \). Using the geometric series formula we recognize \( \sum_{m=0}^{\infty} 2^{-ms} = \frac{1}{1-\frac{1}{2}} \) and

\[
\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} 2^{-(m-l)s} C_m = \sum_{l=0}^{\infty} C_l 2^{-ms} \sum_{l=0}^{\infty} 2^{ls} = \sum_{m=0}^{\infty} C_m 2^{-ms} \left( \frac{2^{(m+1)s} - 1}{2^s - 1} \right) \leq \frac{1}{1-\frac{1}{2}} \sum_{m=0}^{\infty} C_m.
\]

Recalling \( \| (C_m) \|_q \leq 1 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \) finishes the proof.

We are now ready to give a
11. By the preceding remark, all of the occurring series are absolutely convergent and hence the following is obvious. Consider now the case $s > 0$.

To that end, let us study what happens to the parts of the Littlewood-Paley decompositions of $(a_i)$ and $(b_m)$ under convolution. Let the annuli $A_i$ and the balls $B_j$ ($i, j \in \mathbb{N}_0$) be defined as in the Lemmas 10 and 11. By the preceding remark, all of the occurring series are absolutely convergent and hence the following manipulations are justified:

$$
(a_i) \ast (b_m) = \left( \sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l)a_i \right) \ast \left( \sum_{j=0}^{\infty} \mathbb{1}_{A_j}(m)b_m \right)
= \sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l)a_i \ast \left( \sum_{j=0}^{\infty} \mathbb{1}_{A_j}(m)b_m \right) + \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \left( \mathbb{1}_{A_i}(l)a_i \ast \mathbb{1}_{A_j}(m)b_m \right)
= \sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l)a_i \ast \left( \mathbb{1}_{B_i}(m)b_m \right) + \sum_{j=1}^{\infty} \left( \sum_{i=0}^{j-1} \mathbb{1}_{A_i}(l)a_i \right) \ast \left( \mathbb{1}_{A_j}(m)b_m \right)
= \sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l)a_i \ast \left( \mathbb{1}_{B_i}(m)b_m \right) + \sum_{j=0}^{\infty} \left( \mathbb{1}_{B_j}(l)a_j \right) \ast \left( \mathbb{1}_{A_j+1}(m)b_m \right)
$$

Observe that $\text{supp} ((a_{k,i})_k) \subseteq B_{i+1}$ and $\text{supp} ((b_{k,j})_k) \subseteq B_{j+2}$ by the properties of convolution and so the sufficient condition for balls could be applied. Indeed we have

$$
\left\| (a_{k,i})_k \right\|_q \leq \left\| (\mathbb{1}_{B_i}(m)b_m)_m \right\|_1 \left\| (\mathbb{1}_{A_i}(l)a_i)_l \right\|_q \lesssim 2^{-iq}C_i \left\| (b_m) \right\|_{q,s} \left\| (a_i) \right\|_{q,s},
$$

where we used Young’s inequality, the embedding $l^q_s \hookrightarrow l^q$, and the necessary condition for $(a_i) \in l^q_s$ from Lemma 10 ($C_i$ was called $C_i$ there). Hence, $\sum_{i=0}^{\infty} (a_{k,i})_k \in l^q_s$ with $\left\| \sum_{i=0}^{\infty} (a_{k,i})_k \right\|_{q,s} \lesssim \left\| (a_i) \right\|_{q,s} \left\| (b_m) \right\|_{q,s}$ by Lemma 11. The same argument applies to $\sum_{j=0}^{\infty} (b_{k,j})_k$ and finishes the proof.

4. Proof of the Hölder-like inequality, Theorem 3

We have already shown $M^s_{q,q} \hookrightarrow C_0$ in the proof of Proposition 2 in Section 2, so it remains to prove 5. To that end, we shall use 2. Fix a $k \in \mathbb{Z}^d$. By the definition of the operator $\Box_k$ we have

$$
\Box_k (f \ast g) = \frac{1}{(2\pi)^d} \mathcal{F}^{-1} \left( \sigma_k (f \ast \hat{g}) \right) = \frac{1}{(2\pi)^d} \sum_{l,m \in \mathbb{Z}^d} \mathcal{F}^{-1} \left( \sigma_k \left( (\sigma_l f) \ast (\sigma_m \hat{g}) \right) \right).
$$

As the supports of the partition of unity are compact, many summands disappear. Indeed, for any $k, l, m \in \mathbb{Z}^d$

$$
\text{supp} \left( \sigma_k \left( (\sigma_l f) \ast (\sigma_m \hat{g}) \right) \right) \subseteq \text{supp}(\sigma_k) \cap (\text{supp}(\sigma_l) + \text{supp}(\sigma_m)) \subseteq B_{\sqrt{d}}(k) \cap B_{2\sqrt{d}}(l + m)
$$
and so $\sigma_k \left( (\sigma_l f) \ast (\sigma_m \hat{g}) \right) \equiv 0$ if $|k - l| - |m| > 3 \sqrt{d}$. Hence, the double series over $l, m \in \mathbb{Z}^d$ boils down to a finite sum of discrete convolutions

$$
\Box_k (f \ast g) = \frac{1}{(2\pi)^d} \mathcal{F}^{-1} \left( \sigma_k \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} (\sigma_l f) \ast (\sigma_{k-l+m} \hat{g}) \right) = \Box_k \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} (\sigma_l f) \cdot (\Box_{k+m-l}g),
$$

where $M = \left\{ m \in \mathbb{Z}^d \left| \right. |m| \leq 3 \sqrt{d} \right\}$ and $\# M \leq \left( 6 \sqrt{d} + 1 \right)^d < \infty$. That was the job of $\Box_k$ and we now get rid of it,

$$
\left\| \Box_k (f \ast g) \right\|_p \lesssim \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} \left\| (\sigma_l f) \cdot (\Box_{k+m-l}g) \right\|_p,
$$
using the Bernstein multiplier estimate from Lemma 5.

Invoking Hölder’s inequality we further estimate
\[
\left( \| \Box_k (fg) \|_p \right)_k \lesssim \sum_{m \in M} \left( \| \Box_l (f) \|_{p_1} \right)_l * \left( \| \Box_{n+m} (g) \|_{p_2} \right)_n
\]

pointwise in \( k \) and hence
\[
\| fg \|_{M_{p,q}} \lesssim \left( \| \Box_l (f) \|_{p_1} \right)_l * \left( \sum_{m \in M} \left( \| \Box_{n+m} (g) \|_{p_2} \right)_n \right)
\]

by the algebra property of \( l^q \) from Lemma 9. Finally, we remove the sum over \( m \)

\[
\sum_{m \in M} \left( \| \Box_{n+m} (g) \|_{p_2} \right)_n \lesssim \| g \|_{M_{p,q}}
\]

applying Peetre’s inequality \((k + l)^s \leq 2|s| \langle k \rangle^s \langle l \rangle^s\). See e.g. [11 Proposition 3.3.31].

Let us finish the proof remarking that the only estimate involving “\( p \)” we used was Hölder’s inequality and thus indeed \( C = C(d, s, q) \).

\[\square\]

5. Proof of the local well-posedness, Theorem 1

For \( T > 0 \) let \( X(T) = C([0, T], M_{p,q}^s(\mathbb{R}^d)) \). Proposition 2 immediately implies that \( X \) is a Banach *-algebra, i.e.,
\[
\| uv \|_X = \sup_{0 \leq t \leq T} \| uv(\cdot, t) \|_{M_{p,q}^s} \lesssim \left( \sup_{0 \leq t \leq T} \| u(\cdot, t) \|_{M_{p,q}^s} \right) \left( \sup_{0 \leq t \leq T} \| v(\cdot, t) \|_{M_{p,q}^s} \right) = \| u \|_X \| v \|_X.
\]

For \( R > 0 \) we denote by \( M(R, T) = \{ u \in X \| u \|_{X(T)} \leq R \} \) the closed ball of radius \( R \) in \( X(T) \) centered at the origin. We show that for some \( T, R > 0 \) the right-hand side of (2),
\[
(Tu)(\cdot, t) := e^{i\Delta}u_0 + i \int_0^t e^{i(t-\tau)\Delta} \left( |u|^2 u(\cdot, \tau) \right) d\tau \quad (\forall t \in [0, T],)
\]

defines a contractive self-mapping \( T = T(u_0) : M(R, T) \rightarrow M(R, T) \).

To that end let us observe that Theorem 4 implies the homogeneous estimate
\[
\| t \mapsto e^{i\Delta}v \|_X \lesssim (1 + T)^{\frac{d}{2}} \| v \|_{M_{p,q}^s} \quad (\forall v \in M_{p,q}^s),
\]

which, together with the algebra property of \( X(T) \), proves the inhomogeneous estimate
\[
\left\| \int_0^t e^{i(t-\tau)\Delta} \left( |u|^2 u(\cdot, \tau) \right) d\tau \right\|_{M_{p,q}^s} \lesssim (1 + T)^{\frac{d}{2}} \int_0^t \left\| |u|^2 u(\cdot, \tau) \right\|_{M_{p,q}^s} d\tau \lesssim T (1 + T)^{\frac{d}{2}} \| u \|_X^3,
\]
holding for \( 0 \leq t \leq T \) and \( u \in X \).

Applying the triangle inequality in (10) yields \( \| Tu \|_X \leq C(1 + T)^{\frac{d}{2}} (\| u_0 \|_{M_{p,q}^s} + TR^2) \) for any \( u \in M(R, T) \). Thus, \( T \) maps \( M(R, T) \) onto itself for \( R = 2C \| u_0 \|_{M_{p,q}^s} \) and \( T \) small enough. Furthermore,
\[
|u|^2 - |v|^2 v = (u - v)u + (\overline{u} - \overline{v})v = (u - v)(|u|^2 + |v|^2) + (\overline{u} - \overline{v})v^2
\]

and hence
\[
\| Tu - Tv \|_X \lesssim T (1 + T)^{\frac{d}{2}} R^2 \| u - v \|_X
\]

9
for \( u, v \in M(R, T) \), where we additionally used the algebra property of \( X \) and the homogeneous estimate. Taking \( T \) sufficiently small makes \( T \) a contraction.

Banach’s fixed-point theorem implies the existence and uniqueness of a mild solution up to the minimal time of existence \( T_0 = T \left( \| u_0 \|_{M^{s,q}} \right) \approx \| u_0 \|^2_{M^{s,q}} > 0 \). Uniqueness of the maximal solution and the blow-up alternative now follow easily by the usual contradiction argument.

For the proof of the Lipschitz continuity let us notice that for any \( r > \| u_0 \|_{M^{s,q}}, v_0 \in B_r \) and \( 0 < T \leq T_0(r) \) we have

\[
\| u - v \|_{X(T)} = \| T(u_0)u - T(v_0)v \|_{X(T)} \lesssim (1 + T)^{\frac{d}{2}} \| u_0 - v_0 \|_{M^{s,q}} + T(1 + T)^{\frac{d}{2}} R^2 \| u - v \|_{X(T)},
\]

where \( v \) is the mild solution corresponding to the initial data \( v_0 \) and \( R = 2Cr \) as above. Collecting terms containing \( \| u - v \|_{X(T)} \) shows Lipschitz continuity with constant \( L = L(r) \) for sufficiently small \( T \), say \( T = T(r) \). For arbitrary \( 0 < T' < T^* \) put \( r = 2\| u \|_{X(T')} \) and divide \( [0, T'] \) into \( n \) subintervals of length \( \leq T_i \). The claim follows for \( V = B_r(u_0) \) where \( \delta = \frac{\| u_0 \|_{M^{s,q}}}{T^*_r} \) by iteration. This concludes the proof.

\[ \square \]

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