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Wave phenomena
analysis and numerics

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CRC Preprint 2016/14, June 2016

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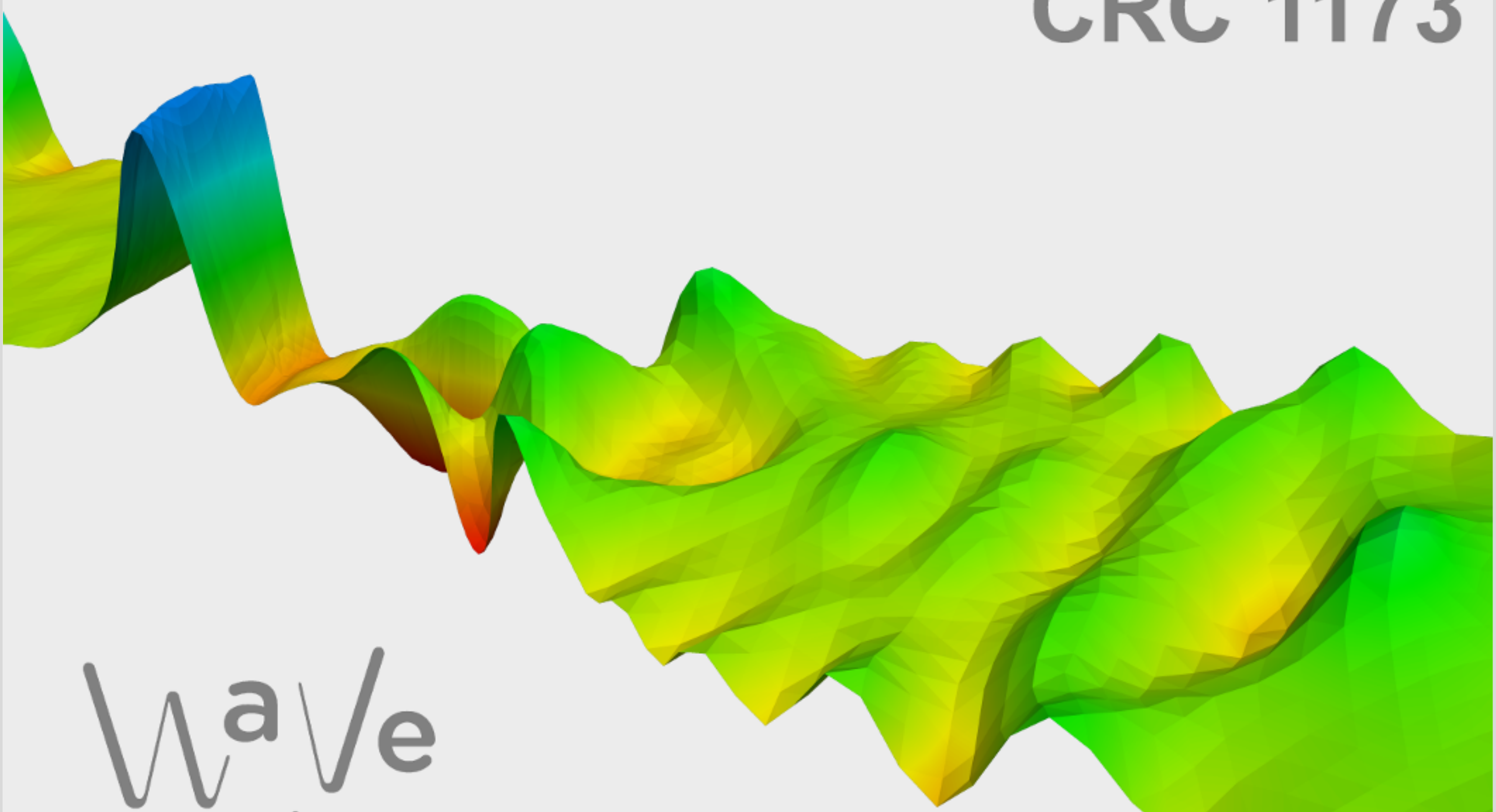
CRC 1173



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Funded by

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ISSN 2365-662X

EXISTENCE OF CYLINDRICALLY SYMMETRIC GROUND STATES TO A NONLINEAR CURL-CURL EQUATION WITH NON-CONSTANT COEFFICIENTS

ANDREAS HIRSCH AND WOLFGANG REICHEL

Abstract. We consider the nonlinear curl-curl problem $\nabla\times\nabla\times U + V(x)U = f(x,|U|^2)U$ in $\mathbb{R}^3$ related to the nonlinear Maxwell equations with Kerr-type nonlinear material laws. We prove the existence of a symmetric ground-state type solution for a bounded, cylindrically symmetric coefficient V and subcritical cylindrically symmetric nonlinearity f. The new existence result extends the class of problems for which ground-state type solutions are known. It is based on compactness properties of symmetric functions [11, 12], new rearrangement type inequalities from [6] and the recent extension of the Nehari-manifold technique from [18].

1. Introduction

We consider the system

$$\nabla\times\nabla\times U + V(x)U = f(x,|U|^2)U \text{ in } \mathbb{R}^3 \tag{1.1}$$

where $V \in L^\infty(\mathbb{R}^3)$ and $f : \mathbb{R}^3\times[0,\infty)\to[0,\infty)$ is a non-negative Carathéodory function growing at infinity with a power at most $\frac{p-1}{2}$ for $p\in(1,5)$. The particular feature of (1.1) is the curl-curl operator. It arises in specific models for standing waves in Maxwell's equations with Kerr-type nonlinear material laws where $f(x,|U|^2)U = \Gamma(x)|U|^2U$. For a detailed physical motivation of (1.1) see [2].

We look for $\mathbb{R}^3$-valued weak solutions U in a cone $K_{4,1}$ of functions with suitable symmetries and $U \in L^2(\mathbb{R}^3)\cap L^{p+1}(\mathbb{R}^3)$, $\nabla\times U \in L^2(\mathbb{R}^3)$. The condition that 0 lies below the spectrum of $\operatorname{curl}\operatorname{curl} + V(x)$ allows us to find ground-state type critical points of a functional $J(u) = \frac{1}{2}\|u\|^2 - I(u)$, cf. (1.4), restricted to the so-called Nehari-manifold. The basic idea of applying symmetrizations to minimizing sequences on the Nehari-manifold goes back to Stuart [17] in the context of the stationary nonlinear Schrödinger equation. Compared to [17] the assumptions on the nonlinearity f can be substantially weakened beyond the classical Ambrosetti-Rabinowitz condition. This is based on three important ingredients:

- the recent extension of the Nehari-manifold method due to Szulkin and Weth [18],
- the weak sequential continuity of functionals $I(u)$ and $I'(u)[u]$ on $K_{4,1}$ due to compactness properties of symmetric functions by Lions [11, 12],
- new rearrangement inequalities for general nonlinearities due to Brock [6].

Using the combination of these ingredients our main result of Theorem 1 substantially extends the know results on the existence of ground-state type solutions for (1.1).

Benci, Fortunato [5] and Azzollini, Benci, D'Aprile, Fortunato in [1] were among the first to consider the constant coefficient case of (1.1) with $V\equiv 0$. Their method was based on cylindrical symmetries of the vector-fields U, cf. [8] for a different class of symmetries. The case where

Date: June 15, 2016.
2000 *Mathematics Subject Classification*. Primary: 35J20, 58E15; Secondary: 47J30, 35Q60.
Key words and phrases. curl-curl problem, nonlinear elliptic equations, cylindrical symmetry, variational methods.

$f(x,|U|^2)U = \Gamma(x)\,|U|^{p-1}\,U$ with periodic coefficients V and Γ has been treated in [2]. In [14] Mederski considered (1.1) where $f(x,|U|^2)U$ is replaced by, e.g., $\Gamma(x)g(U)$ with $\Gamma > 0$ periodic and bounded, $V \le 0$, $V \in L^{\frac{p+1}{p-1}}(\mathbb{R}^3) \cap L^{\frac{q+1}{q-1}}(\mathbb{R}^3)$ and $g(U) \sim |u|^{p-1}U$ if $|U| \gg 1$ and $g(U) \sim |U|^{q-1}u$ if $|U| \ll 1$ for $1 < p < 5 < q$. A remarkable feature of Mederski's work is that (1.1) can be treated without assuming special symmetries of the field U. The nonlinear curl-curl problem on bounded domains with the boundary condition $\nu \times U = 0$ has been elaborated in [3, 4].

An important feature of [1] is the use of cylindrically symmetric ansatz functions for U. Here we make a slightly different ansatz of the form

$$U(x) = u(r,z)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \text{ where } r = \sqrt{x_1^2 + x_2^2},\ z = x_3. \tag{1.2}$$

Moreover, we assume cylindrically symmetric coefficients $V(x) = V(r,z), f(x,|U|^2) = f(r,z,|U|^2)$. For U of the form (1.2) we see that $\operatorname{div} U = 0$, and hence (1.1) reduces to the scalar equation

$$-\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2} + V(r,z)u = f(r,z,r^2u^2)u \quad \text{for} \quad (r,z) \in \Omega := (0,\infty)\times\mathbb{R}. \tag{1.3}$$

It turns out that a suitable space to consider (1.3) is given by

$$H^1_{\mathrm{cyl}}(r^3drdz) := \left\{v\colon (0,\infty)\times\mathbb{R}\to\mathbb{R} : v, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^2_{\mathrm{cyl}}(r^3drdz)\right\},$$
$$L^2_{\mathrm{cyl}}(r^3drdz) := \left\{v\colon (0,\infty)\times\mathbb{R}\to\mathbb{R} : \int_\Omega v(r,z)^2 r^3 d(r,z) < \infty\right\},$$

cf. Section 2 for more details on these spaces. Weak solutions of (1.3) arise as critical points of the functional

$$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla_{r,z}u|^2 + V(r,z)u^2\right)r^3 d(r,z) - \int_\Omega \frac{1}{2r^2}F(r,z,r^2u^2)r^3 d(r,z),\ \ u \in H^1_{\mathrm{cyl}}(r^3drdz), \tag{1.4}$$

where $F(r,z,t) := \int_0^t f(r,z,s)\,ds$ and $\nabla_{r,z} := \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)$. A ground state u of (1.3) is defined as a weak solution of (1.3) in the Nehari-manifold

$$M := \left\{v \in H^1_{\mathrm{cyl}}(r^3drdz)\setminus\{0\} : \int_\Omega \left(|\nabla_{r,z}v|^2 + V(r,z)v^2\right)r^3 d(r,z) = \int_\Omega f(r,z,r^2v^2)v^2 r^3 d(r,z)\right\}$$

such that

$$J(u) = \inf_{v\in M} J(v),$$

see the classical papers [15], [16]. We find ground states of (1.3) under additional assumptions on V and f. To state these assumptions we need the notion of Steiner-symmetrization, cf. Chapter 3 in [10]. The Steiner-symmetrization (also called symmetric-deacreasing rearrangement) of a cylindrical function $g = g(r,z)$ with respect to z is denoted by $g^\star$. We say that g is *Steiner-symmetric* if g coincides with its Steiner-symmetrization with respect to z, keeping the r-variable fixed. A function $h \in L^\infty(\Omega)$ is *reversed Steiner-symmetric* if $(\operatorname{ess\,sup} h - h)^\star = \operatorname{ess\,sup} h - h$ holds true.

Now we can state our assumptions on f.

(i) $f : \Omega\times[0,\infty)\to\mathbb{R}$ is a Carathéodory function with $0 \le f(r,z,s) \le c(1+s^{\frac{p-1}{2}})$ for some $c > 0$ and $p \in (1,5)$,
(ii) $f(r,z,s) = o(1)$ as $s \to 0$ uniformly in $r,z \in [0,\infty)\times\mathbb{R}$,
(iii) $f(r,z,s)$ strictly increasing in $s \in [0,\infty)$,

(iv) $\frac{F(r,z,s)}{s} \to \infty$ as $s \to \infty$ uniformly in $r, z \in [0, \infty) \times \mathbb{R}$,
(v) for all $r \in [0, \infty)$, $s \geq 0$ and $\sigma > 0$ the function

$$\varphi_\sigma(r, z, s) := f(r, z, (s + \sigma)^2)(s + \sigma)^2 - f(r, z, s^2)s^2$$

is symmetrically nonincreasing in z.

Conditions (ii)–(iv) are inspired by the work of Szulkin and Weth [18]. Namely, if we translate (ii)–(iv) into conditions for $\tilde{f}(r, z, s) := f(r, z, r^2 s^2)s$ then they become identical to (ii)–(iv) of Theorem 20 from [18]. Condition (v) is used to prove the rearrangement inequality of Lemma 9 and it is due to Brock [6].

Next we state our main result.

Theorem 1. *Let $V \in L^\infty(\Omega)$ be reversed Steiner-symmetric such that the map*

$$\|\cdot\| : H^1_{\text{cyl}}(r^3 drdz) \to \mathbb{R}; u \mapsto \left(\int_\Omega \left(|\nabla_{r,z} u|^2 + V(r, z)u^2 \right) r^3 d(r, z) \right)^{\frac{1}{2}} \tag{1.5}$$

is an equivalent norm to $\|\cdot\|_{H^1_{\text{cyl}}(r^3 drdz)}$. Additionally, let f satsify the assumptions (i)-(v). Then (1.3) has a ground state $u \in H^1_{\text{cyl}}(r^3 drdz)$ which is symmetric about $\{z = 0\}$.

Remarks: (1) The assumption of norm-equivalence is for instance satisfied if $V \geq 0$ and $\inf_{B_R^c} V > 0$ for some $R > 0$, where $B_R^c := \{(r, z) \in \Omega : r^2 + z^2 > R^2\}$. For the reader's convenience the proof based on Poincaré's inequality is given in the Appendix. Since Poincaré's inequality is applicable for domains bounded in one direction we can weaken $\inf_{B_R^c} V > 0$ to $\inf_{S^c} V > 0$ for strips $S = [0, \infty) \times [0, \rho]$ with $\rho > 0$ or $S = [r_0, r_1] \times [0, \infty)$ with $0 \leq r_0 < r_1 < \infty$.

(2) The conditions on f are satisfied if for instance $f(r, z, s) = \Gamma(r, z)|s|^{\frac{p-2}{2}} s$ where $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric, $\operatorname{ess\,inf}_\Omega \Gamma > 0$ and $p \in (1, 5)$. This choice of f corresponds to the equation $\nabla \times \nabla \times U + V(r, z)U = \Gamma(r, z) |U|^{p-1} U$ in $\mathbb{R}^3$. Another possible choice is $f(r, z, s) = \Gamma(r, z) \log(1 + s)$ where again $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric and $\operatorname{ess\,inf}_\Omega \Gamma > 0$. This nonlinearity appeared for instance in [13] and it does not satisfy the classical Ambrosetti-Rabinowitz condition.

The paper is structured as follows: In Section 2 we give details on the variational formulation of problem (1.3) and prove pointwise decay estimates of Steiner-symmetric functions in $H^1_{\text{cyl}}(r^3 drdz)$. In Section 3 we give the proof of Theorem 1, and in the Appendix we show an example for the potential V satisfying the equivalent-norm-assumption of Theorem 1.

2. Variational formulation, decay estimates, rearrangements

Let us consider some properties of the space $H^1_{\text{cyl}}(r^3 drdz)$. First, for U of the form (1.2) we have that $U \in H^1(\mathbb{R}^3)$ if and only if $u \in H^1_{\text{cyl}}(r^3 drdz)$. A norm on $H^1_{\text{cyl}}(r^3 drdz)$ is given by

$$\|u\|_{H^1_{\text{cyl}}(r^3 drdz)} := \left(\int_\Omega \left(|\nabla_{r,z} u|^2 + u^2 \right) r^3 d(r, z) \right)^{\frac{1}{2}}.$$

Notice that the space $H^1_{\text{cyl}}(r^3 drdz)$ behaves like a Sobolev-space in dimension 5. Next we show a useful embedding property. For this we need the following Sobolev and Lebesgue spaces in dimension 3 together with their canonical norms:

$$H^1_{\text{cyl}}(rdrdz) := \left\{ v\colon (0, \infty) \times \mathbb{R} \to \mathbb{R} : v, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^2_{\text{cyl}}(rdrdz) \right\},$$

$$L^q_{\mathrm{cyl}}(rdrdz) := \left\{ v\colon (0,\infty)\times\mathbb{R}\to\mathbb{R} : \int_\Omega |v(r,z)|^q r d(r,z) < \infty \right\} \text{ for } q\in[1,\infty).$$

Lemma 2. *For $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ Hardy's inequality holds*

$$\int_\Omega \frac{u^2}{r^2} r^3 d(r,z) \le C_H \int_\Omega \left(\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right) r^3 d(r,z). \tag{2.1}$$

Moreover, if $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ then $ru \in H^1_{\mathrm{cyl}}(rdrdz)$ and there is a constant $C > 0$ such that for $2 \le q \le 6$

$$\|ru\|_{H^1_{\mathrm{cyl}}(rdrdz)}, \|ru\|_{L^q_{\mathrm{cyl}}(rdrdz)} \le C\,\|u\|_{H^1_{\mathrm{cyl}}(r^3drdz)} \tag{2.2}$$

Proof. Hardy's inequality (2.1) is given in Lemma 9 (i) in [2]. For $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ we have $ru, \frac{\partial}{\partial z}(ru)$, $r\frac{\partial u}{\partial r} \in L^2_{\mathrm{cyl}}(rdrdz)$ and by (2.1) also $u \in L^2_{\mathrm{cyl}}(rdrdz)$. Since $\frac{\partial}{\partial r}(ru) = r\frac{\partial u}{\partial r} + u$ we conclude altogether $ru \in H^1_{\mathrm{cyl}}(rdrdz)$. By the Sobolev embedding in three dimensions this implies $ru \in L^q(rdrdz)$ for $q \in [2,6]$ and (2.1) yields

$$\begin{aligned} \|ru\|^2_{H^1_{\mathrm{cyl}}(rdrdz)} &= \int_\Omega \left(|\nabla_{r,z}(ru)|^2 + r^2u^2 \right) rd(r,z) \\ &\le 2\int_\Omega \left(\left(r\frac{\partial u}{\partial z}\right)^2 + \left(r\frac{\partial u}{\partial r}\right)^2 + u^2 + r^2u^2 \right) rd(r,z) \le \tilde{C}\,\|u\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)}. \end{aligned} \tag{2.3}$$

□

Next we show that the functional J from the introduction as well as the functional in the defintion of the Nehari-manifold are well-defined.

Lemma 3. *There is a constant $C > 0$ such that*

$$\int_\Omega f(r,z,r^2u^2)u^2r^3\,drdz, \int_\Omega \frac{1}{2r^2}F(r,z,r^2u^2)r^3 d(r,z) \le C\left(\|u\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)} + \|u\|^{p+1}_{H^1_{\mathrm{cyl}}(r^3drdz)} \right) \tag{2.4}$$

for all $u \in H^1_{\mathrm{cyl}}(r^3drdz)$.

Proof. Clearly assumption (i) and (ii) show that for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$0 \le f(r,z,s) \le \epsilon + C_\epsilon s^{\frac{p-1}{2}}.$$

Hence

$$0 \le f(r,z,r^2u^2)u^2r^3 \le \left(\epsilon r^2u^2 + C_\epsilon |ru|^{p+1}\right) r, \tag{2.5}$$

$$0 \le \frac{1}{2r^2}F(r,z,r^2u^2)r^3 \le \left(\epsilon r^2u^2 + \tilde{C}_\epsilon |ru|^{p+1}\right) r. \tag{2.6}$$

Due to (2.2) this implies the claim. □

In order to find critical points of J we need uniform decay estimates of Steiner-symmetric functions in $H^1_{\mathrm{cyl}}(r^3drdz)$. These estimates are given in [12] in much more generality but for the sake of completeness we give them here together with the simple proof. We start with a well-known fact concerning radially symmetric functions and afterwards extend the result to cylindrically symmetric functions. Let

$$H^1_{\mathrm{rad}}(\mathbb{R}^n) := \left\{ u \in H^1(\mathbb{R}^n) : u \text{ is radially symmetric} \right\}.$$

Lemma 4. *(see* [12]*) Let $n \geq 2$. Then there is a constant $C > 0$ such that*

$$|u(x)| \leq C \, \|\nabla u\|^{1/2}_{L^2(\mathbb{R}^n)} \, \|u\|^{1/2}_{L^2(\mathbb{R}^n)} \, |x|^{-(n-1)/2} \text{ for almost all } x \in \mathbb{R}^n \text{ and all } u \in H^1_{\mathrm{rad}}(\mathbb{R}^n).$$

Proof. By density it is sufficient to prove the estimate for $u \in H^1_{\mathrm{rad}}(\mathbb{R}^n) \cap C^\infty_c(\mathbb{R}^n)$. Let $r := |x|$. Then

$$\frac{d}{dr}\left(r^{n-1}\,|u|^2\right) = (n-1)r^{n-2}\,|u|^2 + r^{n-1} 2u\frac{\partial u}{\partial r} \geq -2\,|u|\left|\frac{\partial u}{\partial r}\right| r^{n-1}.$$

Integrating from r to ∞ and expanding the domain of integration to all of $\mathbb{R}^n$ yields

$$r^{n-1}\,|u(x)|^2 \leq C \int_{\mathbb{R}^n} |u|\,|\nabla u|\,dy \leq C\,\|\nabla u\|_{L^2(\mathbb{R}^n)}\,\|u\|_{L^2(\mathbb{R}^n)}\,.$$

□

Now we give an extension of Lemma 4 to cylindrically symmetric functions which are Steiner-symmetric in the non-radial component. We make use of the following notation: Let $t \in \mathbb{N}_{\geq 2}$ and $s \in \mathbb{N}$ such that $n = t + s$. We write points in $\mathbb{R}^n$ as (x, y) with $x \in \mathbb{R}^t$ and $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$. Furthermore, let

$$K_{t,s} := \left\{ u \in H^1(\mathbb{R}^n) \text{ s.t. } \begin{cases} u(\cdot, y) & \text{is a radially symmetric function for every } y \in \mathbb{R}^s \text{ and} \\ u(x, \cdot) & \text{is Steiner-symmetric w.r.t. } y_i, i = 1, \ldots, s, \text{ for every } x \in \mathbb{R}^t \end{cases} \right\}.$$

In particular, if $u \in K_{t,s}$ then necessarily $u \geq 0$. In this setting we have the following extension of Lemma 4.

Lemma 5. *(see* [12]*) There is a constant $C > 0$ such that*

$$0 \leq u(x, y) \leq C\,\|\nabla_x u\|^{1/2}_{L^2(\mathbb{R}^n)}\,\|u\|^{1/2}_{L^2(\mathbb{R}^n)}\,|x|^{-(t-1)/2}\,|y_1 \cdots y_s|^{-1/2} \text{ for almost all } (x, y) \in \mathbb{R}^n \text{ and all } u \in K_{t,s}.$$

Proof. Let $u \in K_{t,s}$ and fix $y \in \mathbb{R}^s$. W.l.o.g. let $y_i > 0$ for all $i = 1, \ldots, s$. We define

$$v(x) := \int_0^{y_1} \cdots \int_0^{y_s} u(x, z)dz \text{ for } x \in \mathbb{R}^t.$$

By Hölder's inequality we obtain $v^2(x) \leq y_1 \cdots y_s \int_0^{y_1} \cdots \int_0^{y_s} u^2(x, z)dz$, i.e.,

$$\|v\|_{L^2(\mathbb{R}^t)} \leq (y_1 \cdots y_s)^{1/2}\,\|u\|_{L^2(\mathbb{R}^n)}\,. \tag{2.7}$$

In the same manner we receive

$$\|\nabla v\|_{L^2(\mathbb{R}^t)} \leq (y_1 \cdots y_s)^{1/2}\,\|\nabla_x u\|_{L^2(\mathbb{R}^n)}\,. \tag{2.8}$$

Since $v \colon \mathbb{R}^t \to \mathbb{R}$ is radially symmetric we can apply Lemma 4 and get from (2.7) and (2.8)

$$0 \leq v(x) \leq C\,\|\nabla v\|^{1/2}_{L^2(\mathbb{R}^t)}\,\|v\|^{1/2}_{L^2(\mathbb{R}^t)}\,|x|^{-(t-1)/2} \leq C(y_1 \cdots y_s)^{1/2}\,\|\nabla_x u\|^{1/2}_{L^2(\mathbb{R}^n)}\,\|u\|^{1/2}_{L^2(\mathbb{R}^n)}\,|x|^{-(t-1)/2}\,. \tag{2.9}$$

Due to the monotonicity-property in y-direction we also have $v(x) \geq y_1 \cdots y_s u(x, y)$ and thus (2.9) gives the desired inequality. □

We prove three additional lemmas which are used in the next section.

Lemma 6. *The set $K_{t,s}$ is a weakly closed cone in $H^1(\mathbb{R}^n)$.*

Proof. Take a sequence $(u_k)_{k\in\mathbb{N}} \subset K_{t,s}$ such that $u_k \rightharpoonup u \in H^1(\mathbb{R}^n)$ as $k \to \infty$. By the Sobolev embedding on bounded domains we deduce that a subsequence of u_k converges pointwise almost everywhere on $\mathbb{R}^n$ to u. Since every u_k enjoys the radial symmetry in the first component and the non-increasing property in the second variable, the pointwise convergence implies that also u enjoys these properties, i.e., $u \in K_{t,s}$. □

Lemma 7. *The functionals*

$$I(v) = \int_\Omega \frac{1}{2r^2} F(r,z,r^2v^2) r^3\, d(r,z), \qquad I'(v)[v] = \int_\Omega f(r,z,r^2v^2)v^2 r^3\, d(r,z)$$

are weakly sequentially continuous on the set $K_{4,1} \subset H^1_{\mathrm{cyl}}(r^3 drdz)$.

Remark: In the proof we use twice the following principle: if $S \subset \mathbb{R}^m$ is a set of finite measure and $w_k : S \to \mathbb{R}$ a sequence of measurable functions such that $\|w_k\|_{L^r(S)} \le C$ and $w_k \to w$ pointwise a.e. as $k \to \infty$ then $\|w_k - w\|_{L^q(S)} \to 0$ as $k \to \infty$ for $1 \le q < r$. The proof is as follows: Egorov's theorem allows to choose $\Sigma \subset S$ such that $w_k \to w$ uniformly on Σ and $|S \setminus \Sigma| \le \epsilon$ arbitrary small. By Hölder's inequality the remaining integral is estimated by $\int_{S\setminus\Sigma} |w_k - w|^q\, dx \le \epsilon^{1-\frac{q}{r}} \|w_k - w\|^q_{L^r(S)}$.

Proof. Let us take a weakly convergent sequence $(v_k)_{k\in\mathbb{N}}$ in $K_{4,1}$ such that $v_k \rightharpoonup v$ in $H^1_{\mathrm{cyl}}(r^3 drdz)$ and $v_k \to v$ pointwise a.e. in Ω. By Lemma 6 one gets $v \in K_{4,1}$ and using Lemma 5 there exists a constant $C > 0$ such that

$$0 \le v_k(r,z), v(r,z) \le C r^{-\frac{3}{2}} |z|^{-\frac{1}{2}} \text{ for all } k \in \mathbb{N} \text{ and almost all } (r,z) \in \Omega. \tag{2.10}$$

Our goal is now to show at least for a subsequence

$$\int_\Omega \frac{1}{r^2} F(r,z,r^2v_k^2) r^3 d(r,z) \to \int_\Omega \frac{1}{r^2} F(r,z,r^2v^2) r^3 d(r,z) \text{ as } k \to \infty \tag{2.11}$$

and

$$\int_\Omega f(r,z,r^2v_k^2) v_k^2 r^3 d(r,z) \to \int_\Omega f(r,z,r^2v^2) v^2 r^3 d(r,z) \text{ as } k \to \infty. \tag{2.12}$$

By (2.6) we find

$$\frac{1}{r^2} \left| F(r,z,r^2v_k^2) - F(r,z,r^2v^2) \right| r^3 \le \epsilon r^2 (v_k^2 + v^2) r + C_\epsilon \left(|rv_k|^{p+1} + |rv|^{p+1} \right) r$$

and hence

$$\left(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2 + v^2) \right)^+ r \le C_\epsilon \left(|rv_k|^{p+1} + |rv|^{p+1} \right) r. \tag{2.13}$$

Inspired by [11] and [12] the idea is to show

$$rv_k \to rv \text{ in } L^{p+1}(rdrdz) \text{ as } k \to \infty. \tag{2.14}$$

Once (2.14) is established we obtain a majorant $|rv_k|, |rv| \le w \in L^{p+1}(r\, drdz)$ (cf. Lemma A.1 in [19]). Together with (2.13) this majorant allows to apply Lebesgue's dominated convergence theorem and yields

$$\lim_{k\to\infty} \int_\Omega \left(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2 + v^2) \right)^+ r\, drdz = 2\epsilon \|v\|^2_{L^2(r^3drdz)}. \tag{2.15}$$

If we set

$$a_k := \int_\Omega |F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| r\, drdz$$

and

$$b_k := \epsilon \|r^2(v_k^2 + v^2)\|_{L^1(rdrdz)} = \epsilon (\|v_k\|^2_{L^2(r^3drdz)} + \|v\|^2_{L^2(r^3drdz)}) \le C\epsilon$$

then

$$\begin{aligned} \limsup_{k\in\mathbb{N}} a_k &\le \limsup_{k\in\mathbb{N}} b_k + \limsup_{k\in\mathbb{N}} (a_k - b_k)^+ \\ &\le C\epsilon + \limsup_{k\in\mathbb{N}} \left(\int_\Omega \left(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2 + v^2) \right) rdrdz \right)^+ \end{aligned}$$

$$\leq C\epsilon + \limsup_{k\in\mathbb{N}} \int_\Omega \Big(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2+v^2)\Big)^+ \, rdrdz$$
$$\leq \epsilon(C + 2\|v\|^2_{L^2(r^3drdz)}) \text{ by (2.15).}$$

Since $\epsilon > 0$ was arbitrary this shows that $\lim_{k\to\infty} a_k = 0$ and therefore (2.11) holds. The proof of (2.12) is similar since $\Big(f(r,z,r^2v_k^2)r^2v_k^2 - f(r,z,r^2v^2)r^2v^2 - \epsilon r^2(v_k^2+v^2)\Big)^+ r$ satisfies an estimate just like (2.13) if we use (2.5) instead of (2.6).

It remains to prove (2.14). For this, we split our domain Ω into four parts $\Omega_1, \dots, \Omega_4$ and show (2.14) on each of these parts separately. The definitions of $\Omega_1, \dots, \Omega_4$ are as follows: For $R > 0$ let

$$\Omega_1 := \{(r,z)\in\Omega : r < R, |z| < R\}, \quad \Omega_2 := \{(r,z)\in\Omega : r \geq R, |z| \geq R\},$$
$$\Omega_3 := \{(r,z)\in\Omega : r < R, |z| \geq R\}, \quad \Omega_4 := \{(r,z)\in\Omega : r \geq R, |z| < R\}.$$

Convergence on Ω_1: Follows from $rv_k \to rv$ in $L^q(K; r\,drdz)$ for every compact subset $K \subset [0,\infty)\times\mathbb{R}$ and every $q \in [1,6)$. This step works independently of the choice of $R > 0$.

Convergence on Ω_2: Let $\varepsilon > 0$. With the help of (2.10) we calculate

$$\begin{aligned}\int_{\Omega_2} |rv_k - rv|^{p+1}\, rd(r,z) &\leq 2^{p+1}\int_{\Omega_2} r^{p+1}\Big(|v_k|^{p+1} + |v|^{p+1}\Big) rd(r,z)\\ &\leq 2^{p+1}C^{p-1}\int_{\Omega_2} r^{-\frac{p-1}{2}}\,|z|^{-\frac{p-1}{2}}\Big(|v_k(r,z)|^2 + |v(r,z)|^2\Big) r^3 d(r,z)\\ &\leq C_1\Big(\|v_k\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)} + \|v\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)}\Big)R^{-(p-1)} \leq C_2 R^{-(p-1)}\end{aligned}$$

which is less or equal ε if we choose $R > 0$ large enough.

Convergence on Ω_3: Due to symmetry in z-direction it is enough to focus on $\tilde\Omega_3 := \{(r,z)\in\Omega : r < R, z \geq R\}$. Let $\alpha > 0$ be arbitrary. Again by (2.10) we obtain

$$\{(r,z)\in\tilde\Omega_3 : v_k(r,z) > \alpha\} \subset \{(r,z)\in\tilde\Omega_3 : r\, z^{\frac{1}{3}} \leq C_\alpha\} =: S_\alpha,$$

where $C_\alpha = (C/\alpha)^{2/3}$ and C is the constant from (2.10). The set S_α has finite measure since

$$|S_\alpha| \leq \int_R^\infty \int_0^{C_\alpha z^{-1/3}} r^3 dr\, dz = \frac{C_\alpha^4}{4}\int_R^\infty z^{-\frac{4}{3}}dz = \frac{3}{4}C_\alpha^4 R^{-\frac{1}{3}} < \infty.$$

By the convergence principle from the remark above and since by (2.3) $\|rv_k\|_{L^6(rdrdz)} \leq \|v_k\|_{H^1_{\mathrm{cyl}}(r^3drdz)}$ is bounded we obtain $\int_{S_\alpha} r^{p-1}|v_k - v|^{p+1}r^3 d(r,z) \to 0$ as $k\to\infty$ for $1 \leq p < 5$. It remains to prove the convergence on $\tilde\Omega_3 \setminus S_\alpha$. For allmost all $(r,z)\in\tilde\Omega_3\setminus S_\alpha$ we have that $v(r,z) = \lim_{k\to\infty} v_k(r,z) \leq \alpha$. Hence,

$$\int_{\tilde\Omega_3\setminus S_\alpha} r^{p-1}|v_k - v|^{p+1}r^3 d(r,z) \leq R^{p-1}(2\alpha)^{p-1}\int_\Omega |v_k - v|^2 r^3 d(r,z) \leq C\alpha^{p-1}.$$

In summary, since $\alpha > 0$ is arbitrary this shows (2.14) on Ω_3.

Convergence on Ω_4: Again it is enough to focus on $\tilde\Omega_4 := \{(r,z)\in\Omega : r \geq R, 0 \leq z < R\}$. Fix $z \in (0,R)$. Let us first show that

$$\int_{\{r\geq R\}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1}r^3 dr \to 0 \text{ as } k\to\infty. \tag{2.16}$$

Since $v_k(r,\cdot)$ is nonincreasing in its last component we deduce

$$\int_0^\infty r^q v_k^q(r,z) r\,dr \le \frac{1}{z}\int_0^z\int_0^\infty r^q v_k^q(r,\zeta) r\,dr d\zeta \le \frac{1}{z}\int_\Omega r^q v_k^q(r,\zeta) r d(r,\zeta) \le \frac{C}{z} \tag{2.17}$$

for all $q \in [2,6]$ by (2.3). Thus for $q \in [2,6]$ the sequence $\|\cdot v_k(\cdot,z)\|_{L^q((0,\infty),rdr)}$ is uniformly bounded in $k \in \mathbb{N}$. Moreover, (2.10) implies $v_k(r,z) \le C(z) r^{-\frac{3}{2}}$ uniformly in $k \in \mathbb{N}$. Hence for $\tilde{R} > R$

$$\begin{aligned}\int_{\tilde{R}}^\infty r^{p-1}|v_k(r,z)-v(r,z)|^{p+1} r^3 dr &\le (2C(z))^{p-1}\int_{\tilde{R}}^\infty r^{-\frac{p-1}{2}}|v_k(r,z)-v(r,z)|^2 r^3 dr\\ &\le (2C(z))^{p-1}\tilde{R}^{\frac{1-p}{2}}\frac{C}{z} \text{ by (2.17).}\end{aligned}$$

The last term can be made arbitrarily small provided $\tilde{R}$ is chosen big enough. To finish the proof of (2.16) it remains to prove $\int_R^{\tilde{R}} r^{p-1}|v_k(r,z)-v(r,z)|^{p+1} r^3 dr \to 0$ as $k \to \infty$. Since for almost all $z \in (0,R)$ we have $v_k(\cdot,z) \to v(\cdot,z)$ pointwise almost everywhere on $(R,\tilde{R})$ as well as the boundedness of $\|\cdot v_k(\cdot,z)\|_{L^6((0,\infty),rdr)}$ by (2.17) we can apply the convergence principle from the remark above and deduce

$$\int_R^{\tilde{R}} r^{p-1}|v_k(r,z)-v(r,z)|^{p+1} r^3 dr \to 0 \text{ as } k \to \infty.$$

Hence (2.16) is accomplished for almost all $z \in (0,R)$.

Defining $\varphi_k(z) := \int_{\{r\ge R\}} r^{p-1}|v_k(r,z)-v(r,z)|^{p+1} r^3 dr$ we have $\varphi_k \to 0$ as $k \to \infty$ pointwise almost everywhere in $[0,R)$. The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $L^1([0,R),dz)$ since by (2.2)

$$\int_0^R\int_{\{r\ge R\}} r^{p-1}|v_k(r,z)-v(r,z)|^{p+1} r^3 drdz \le C\int_\Omega r^{p-1}\left(|v_k|^{p+1}+|v|^{p+1}\right) r^3 d(r,z) \le \tilde{C}.$$

Moreover, for $p \in (1,3]$, the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}([0,R),dz)$ since

$$\begin{aligned}\left\|\frac{\partial\varphi_k}{\partial z}\right\|^2_{L^1([0,R],dz)} &\le \left(\int_0^R\int_R^\infty (p+1)r^{p-1}|v_k-v|^p\left|\frac{\partial v_k}{\partial z}-\frac{\partial v}{\partial z}\right| r^3 drdz\right)^2\\ &\le \left(\int_\Omega (p+1)r^{p-1}|v_k-v|^p\left|\frac{\partial v_k}{\partial z}-\frac{\partial v}{\partial z}\right| r^3 d(r,z)\right)^2\\ &\le C\int_\Omega r^{2p-2}|v_k-v|^{2p} r^3 d(r,z)\int_\Omega\left|\frac{\partial v_k}{\partial z}-\frac{\partial v}{\partial z}\right|^2 r^3 d(r,z)\\ &= C\|r(v_k-v)\|^{2p}_{L^{2p}(rdrdz)}\int_\Omega\left|\frac{\partial v_k}{\partial z}-\frac{\partial v}{\partial z}\right|^2 r^3 d(r,z) \le C.\end{aligned}$$

Hence, by the compact embedding $W^{1,1}([0,R),dz) \hookrightarrow L^1([0,R),dz)$ we conclude that at least a subsequence of $(\varphi_k)_{k\in\mathbb{N}}$ is converging in $L^1([0,R),dz)$ to a limit function, which must be 0 since we have already asserted the pointwise a.e. convergence to 0 on $[0,R)$. This shows (2.14) on Ω_4 for $p \in (1,3]$. For $p \in (3,5)$ we make use of Hölder's interpolation, namely,

$$\|rv_k - rv\|^{p+1}_{L^{p+1}_{\mathrm{cyl}}(\Omega_4,rdrdz)} \le \|rv_k - rv\|^{4\theta}_{L^4_{\mathrm{cyl}}(\Omega_4,rdrdz)}\|rv_k - rv\|^{6(1-\theta)}_{L^6_{\mathrm{cyl}}(\Omega_4,rdrdz)} \le \tilde{C}\,\|rv_k - rv\|^{4\theta}_{L^4_{\mathrm{cyl}}(\Omega_4,rdrdz)} \to 0$$

as $k \to \infty$, where $\theta \in (0,1)$ is chosen such that $p+1 = 4\theta + 6(1-\theta)$, i.e., $\theta = \frac{5-p}{2}$.

The combination of convergences on $\Omega_1,\dots,\Omega_4$ finally proves (2.14). □

For our last lemma we need the notion of cylindrical C_c^∞-functions which we introduce now.

Definition 8. *A function $u = u(r, z)$ belongs to $C_c^\infty([0,\infty)\times\mathbb{R})$ if and only if $u \in C^\infty([0,\infty)\times\mathbb{R})$, $\operatorname{supp} u$ is compact in $[0,\infty)\times\mathbb{R}$ and $\frac{\partial^j u}{\partial r^j}(0,z) = 0$ for all odd integers $j \in 2\mathbb{N}-1$.*

Remark: Since $u \in C_c^\infty([0,\infty)\times\mathbb{R})$ is equivalent to $\tilde{u} \in C_c^\infty(\mathbb{R}^5)$ with $\tilde{u}(x) := u(|(x_1,\ldots,x_4)|, x_5)$ we see that $C_c^\infty([0,\infty)\times\mathbb{R})$ is dense in $H^1_{\mathrm{cyl}}(r^3drdz)$.

Lemma 9. *For $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ we have $\|u^\star\| \le \|u\|$ where $\star$ denotes Steiner-symmetrization with respect to z and $\|\cdot\|$ is the equivalent norm from Theorem 1. Moreover*

$$I(u) \le I(u^\star) \quad \text{and} \quad I'(u)[u] \le I'(u^\star)[u^\star].$$

Proof. We begin by recalling several classical rearrangement inequalities from [9], [10]. Recall first the Pólya-Szegö inequality

$$\int_{\mathbb{R}^n} |\nabla f^\circledast|^2 dx \le \int_{\mathbb{R}^n} |\nabla f|^2 dx \tag{2.18}$$

for $f \in H^1(\mathbb{R}^n)$ and $^\circledast$ denoting Schwarz-symmetrization (also called symmetrically decreasing rearrangement). Furthermore we have for $0 \le f, g \in L^2(\mathbb{R}^n)$ the classical rearrangement inequality

$$\int_{\mathbb{R}} fg\,dx \le \int_{\mathbb{R}} f^\circledast g^\circledast dx \tag{2.19}$$

and the nonexpansivity of rearrangement

$$\int_{\mathbb{R}^n} |f^\circledast - g^\circledast|^2 dx \le \int_{\mathbb{R}^n} |f-g|^2\,dx. \tag{2.20}$$

From (2.18) we immediately receive for $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ that

$$\int_{\mathbb{R}} |\nabla_z u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_z u|^2 dz. \tag{2.21}$$

Next we want to establish a similar inequality for $\nabla_r u$. We do this first for $u \in C_c^\infty([0,\infty)\times\mathbb{R})$. With the help of (2.20) we find that

$$\int_{\mathbb{R}} \left|\frac{u^\star(r+t,z) - u^\star(r,z)}{t}\right|^2 dz \le \int_{\mathbb{R}} \left|\frac{u(r+t,z)-u(r,z)}{t}\right|^2 dz$$

for almost all $r, t \in [0,\infty)$. Sending $t \to 0$ and using Fatou's lemma on the left side of the inequality yields

$$\int_{\mathbb{R}} |\nabla_r u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_r u|^2 dz \tag{2.22}$$

for $u \in C_c^\infty([0,\infty)\times\mathbb{R})$ and almost all $r \in [0,\infty)$. Since Steiner Symmetrization is continuous in H^1 (see Theorem 1 in [7]) we obtain by approximation that (2.22) is indeed valid for all $u \in H^1_{cyl}(r^3drdz)$. Together with (2.21) we obtain $\int_{\mathbb{R}} |\nabla_{r,z} u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_{r,z} u|^2 dz$ for almost all $r \ge 0$ and integration leads to

$$\int_{\mathbb{R}}\int_0^\infty |\nabla_{r,z} u^\star|^2 r^3 drdz \le \int_{\mathbb{R}}\int_0^\infty |\nabla_{r,z} u|^2 r^3 drdz. \tag{2.23}$$

Fixing $r \in [0,\infty)$ and applying (2.19) to $f(\cdot) = \operatorname{ess\,sup} V - V(r,\cdot)$ and $g(\cdot) = u^2(r,\cdot)$ gives

$$\begin{aligned}\int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))\, u^2(r,\cdot) dz &\le \int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))^\star\, (u^2)^\star(r,\cdot) dz \\ &= \int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))\, (u^\star)^2\,(r,\cdot) dz.\end{aligned}$$

Using $\|u(r,\cdot)\|_{L^2(\mathbb{R})} = \|u^\star(r,\cdot)\|_{L^2(\mathbb{R})}$ this results in

$$\int_{\mathbb{R}}\int_0^\infty V(r,z)\,(u^\star)^2\, r^3 drdz \le \int_{\mathbb{R}}\int_0^\infty V(r,z)u^2r^3drdz. \tag{2.24}$$

The combination of (2.23) and (2.24) yields the claimed inequality $\|u^\star\|^2 \le \|u\|^2$.

Assumption (v) on f allows to apply Theorem 5.1 in [6] and to deduce

$$I'(u)[u] = \int_\Omega f(r,z,r^2u^2)u^2r^3 d(r,z) \le \int_\Omega f(r,z,r^2u^{\star 2})u^{\star 2}r^3 d(r,z) = I'(u^\star)[u^\star]. \tag{2.25}$$

Moroever, using (v) with $s=0$ shows that for all $r\in[0,\infty)$, $\sigma\ge 0$ the function $z\mapsto f(r,z,\sigma^2)$ is symmetrically nonincreasing in z and hence

$$\Phi_\sigma(r,z,s) := F(r,z,r^2(s+\sigma)^2) - F(r,z,r^2s^2) = \int_{r^2s^2}^{r^2(s+\sigma)^2} f(r,z,t)\,dt$$

is symmetrically nonincreasing in z. Applying once more Theorem 5.1 in [6] yields

$$I(u) = \int_\Omega \frac{1}{2r^2}F(r,z,r^2u^2)r^3 d(r,z) \le \int_\Omega \frac{1}{2r^2}F(r,z,r^2u^{\star 2})r^3 d(r,z) = I(u^\star).$$

This finishes the proof of the lemma. □

3. Proof of Theorem 1

Proof. Recall from Lemma 7 the definition $I(u) := \int_\Omega \frac{1}{2r^2}F(r,z,r^2u^2)r^3 d(r,z)$ for $u\in H^1_{\text{cyl}}(r^3drdz)$. We show that the assumptions (i)-(iii) of Theorem 12 in [18] are satisfied. Let $\varepsilon>0$. The growth assumptions (i) and (ii) on f imply that for every $\epsilon>0$ there exists $C_\epsilon>0$ such that the global estimate $0\le f(r,z,s)\le \epsilon + C_\epsilon|s|^{\frac{p-1}{2}}$ holds. Together with (2.2) we obtain

$$\begin{aligned}
|I'(u)[v]| &= \left|\int_\Omega f(r,z,r^2u^2)uvr^3 d(r,z)\right| \\
&\le \varepsilon\int_\Omega |ru||rv|r d(r,z) + C_\epsilon \int_\Omega |ru|^p|rv|r d(r,z) \\
&\le \varepsilon C\,\|u\|_{H^1_{cyl}(r^3drdz)}\|v\|_{H^1_{cyl}(r^3drdz)} + \tilde{C}_\epsilon\,\|u\|^p_{H^1_{cyl}(r^3drdz)}\|v\|_{H^1_{cyl}(r^3drdz)}
\end{aligned}$$

Taking the supremum over all $v\in H^1_{cyl}(r^3drdz)$ with $\|v\|_{H^1_{cyl}(r^3drdz)}=1$ we see that

$$I'(u) = o(\|u\|) \text{ as } u\to 0. \tag{3.1}$$

Moreover, due to assumption (iii) on f the map

$$s\mapsto \frac{I'(su)[u]}{s} = \int_\Omega f(r,z,s^2r^2u^2)u^2r^3 d(r,z) \text{ is strictly increasing for all } u\neq 0 \text{ and } s>0. \tag{3.2}$$

Next we claim that

$$\frac{I(su)}{s^2}\to\infty \text{ as } s\to\infty \text{ uniformly for } u \text{ on weakly compact subsets } W \text{ of } H^1_{cyl}(r^3drdz)\setminus\{0\}. \tag{3.3}$$

Suppose not. Then there are $(u_k)_{k\in\mathbb{N}}\subset W$ and $s_k\to\infty$ as $k\to\infty$ such that $\frac{I(s_ku_k)}{s_k^2}$ is bounded as $k\to\infty$. But along a subsequence we have $u_k\rightharpoonup u\neq 0$ and $u_k(x)\to u(x)$ pointwise almost everywhere. Let

$\Omega^\sharp := \{(r,z) \in \Omega : u(r,z) \neq 0\}$. Then $|\Omega^\sharp| > 0$ and on $\Omega^\sharp$ we have $|s_k u_k(r,z)| \to \infty$ as $k \to \infty$. Fatou's lemma and assumption (iv) on F imply

$$\frac{I(s_k u_k)}{s_k^2} = \int_\Omega \frac{F(r,z,s_k^2 r^2 u_k^2)}{2s_k^2 r^2} r^3 d(r,z) \geq \int_{\Omega^\sharp} \frac{F(r,z,s_k^2 r^2 u_k^2)}{2s_k^2 r^2 u_k^2} u_k^2 r^3 d(r,z) \to \infty \text{ as } k \to \infty,$$

a contradiction. In summary, (3.1), (3.2), (3.3) imply that (i)-(iii) of Theorem 12 in [18] are satisfied.

Now we take a sequence $(u_k)_{k\in\mathbb{N}} \subset M$ such that $J(u_k) \to \inf_M J$ as $k \to \infty$. Since $\|\nabla_{r,z}|u_k|\|_{L^2} = \|\nabla_{r,z} u_k\|_{L^2}$ we can assume that $u_k \geq 0$ for all $k \in \mathbb{N}$. Then Theorem 12 in [18] guarantees that for every k there is a unique $t_k > 0$ such that $v_k := t_k u_k^\star \in M$. We show next that $t_k \leq 1$ for all $k \in \mathbb{N}$. Assume $t_k > 1$. Then

$$\begin{aligned}
\int_\Omega f(r,z,r^2 u_k^{\star 2}) u_k^{\star 2} r^3 d(r,z) &< \int_\Omega f(r,z,t_k^2 r^2 u_k^{\star 2}) u_k^{\star 2} r^3 d(r,z) \quad \text{by assumption (iii)}\\
&= \|u_k^\star\|^2 \quad \text{since } t_k u_k^\star \in M\\
&\leq \|u_k\|^2 \quad \text{by Lemma 9}\\
&= \int_\Omega f(r,z,r^2 u_k^2) u_k^2 r^3 d(r,z) \quad \text{since } u_k \in M.
\end{aligned}$$

This contradicts the inequality $I'(u_k)[u_k] \leq I'(u_k^\star)[u_k^\star]$ from Lemma 9 and thus $t_k \leq 1$ for all $k \in \mathbb{N}$.

Next notice that for fixed $(r,z,s) \in [0,\infty) \times \mathbb{R} \times [0,\infty)$ and $t \in (0,1]$ one has

$$\frac{d}{dt}\Big(t^2 f(r,z,s^2)s^2 - F(r,z,t^2 s^2)\Big) = 2ts^2\Big(f(r,z,s^2) - f(r,z,t^2 s^2)\Big) > 0$$

since f is strictly increasing in its last variable by assumption (iii). This shows that the map $t \mapsto t^2 f(r,z,s^2)s^2 - F(r,z,t^2 s^2)$ is strictly increasing for $t \in [0,1]$. From this monotonicity and the inequality $I(t_k u_k) \leq I(t_k u_k^\star)$ from Lemma 9 we conclude

$$\begin{aligned}
2J(v_k) &= \int_\Omega \left(t_k^2 |\nabla_{r,z} u_k^\star|^2 + V(r,z) t_k^2 u_k^{\star 2} - \frac{1}{r^2} F(r,z,r^2 t_k^2 u_k^{\star 2})\right) r^3 d(r,z)\\
&\leq \int_\Omega \left(t_k^2 |\nabla_{r,z} u_k|^2 + V(r,z) t_k^2 u_k^2 - \frac{1}{r^2} F(r,z,r^2 t_k^2 u_k^2)\right) r^3 d(r,z)\\
&= \int_\Omega \frac{1}{r^2}\Big(f(r,z,r^2 u_k^2) t_k^2 r^2 u_k^2 - F(r,z,r^2 t_k^2 u_k^2)\Big) r^3 d(r,z)\\
&\leq \int_\Omega \frac{1}{r^2}\Big(f(r,z,r^2 u_k^2) r^2 u_k^2 - F(r,z,r^2 u_k^2)\Big) r^3 d(r,z)\\
&= 2J(u_k).
\end{aligned} \tag{3.4}$$

So $(v_k)_{k\in\mathbb{N}} \subset M$ is also a minimizing sequence for J which belongs to $K_{4,1}$. The boundedness of $(v_k)_{k\in\mathbb{N}}$ is established in Proposition 14 in [18]. Hence, we find $v_\infty \in H^1_{\mathrm{cyl}}(r^3 drdz)$ such that $v_k \rightharpoonup v_\infty$ in $H^1_{\mathrm{cyl}}(r^3 drdz)$ along a subsequence as $k \to \infty$. In addition, $v_\infty \in K_{4,1}$ due to Lemma 6 and $v_\infty \neq 0$ by Proposition 14 in [18] where instead of the weak sequential continuity of I on all of $H^1_{\mathrm{cyl}}(r^3 drdz)$ we use it only on $K_{4,1}$ as stated in Lemma 7.

Let us show that $v_\infty \in M$. Since $v_\infty \neq 0$ we can choose $t_\infty > 0$ such that $t_\infty v_\infty \in M$. In the same manner as before for the sequence t_k we can show that $t_\infty \leq 1$. Assume $t_\infty < 1$. Then as in (3.4) and using the weak sequential continuity on $K_{4,1}$ as shown in Lemma 7 we find

$$2J(t_\infty v_\infty) < \int_\Omega \frac{1}{r^2}\Big(f(r,z,r^2 v_\infty^2) r^2 v_\infty^2 - F(r,z,r^2 v_\infty^2)\Big) r^3 d(r,z)$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2v_k^2)r^2v_k^2 - F(r,z,r^2v_k^2)\right) r^3 d(r,z)\\
&= 2\inf_M J \le 2J(t_\infty v_\infty)
\end{aligned}
$$

which is a contradiction. So $t_\infty = 1$ and thus $v_\infty \in M$. Then by the weak lower semi-continuity of $\|\cdot\|$ and once again the weak sequential continuity of I we conclude

$$J(v_\infty) \le \liminf_{k\to\infty} J(v_k) = \inf_M J \le J(v_\infty).$$

Hence, $v_\infty \in K_{4,1}$ is a minimizer of J on M, i.e., a ground state of (1.3) which is Steiner symmetric in z with respect to $\{z=0\}$. □

Appendix

Here we prove that the condition $V \ge 0$ and $\inf_{B_R^c} V > 0$ for some $R > 0$ implies that on $H^1_{\mathrm{cyl}}(r^3 drdz)$ the expression $\left(\int_\Omega \left(|\nabla_{r,z}u|^2 + V(r,z)u^2\right) r^3 d(r,z)\right)^{\frac{1}{2}}$ is an equivalent norm. Suppose not. Then there is a sequence $(u_k)_{k\in\mathbb{N}}$ such that $\|u_k\|_{L^2(r^3drdz)} = 1$ and $\int_\Omega \left(|\nabla_{r,z}u_k|^2 + V(r,z)u_k^2\right) r^3 d(r,z) \to 0$ as $k\to\infty$. In particular,

$$\int_\Omega |\nabla_{r,z}u_k|^2 r^3 d(r,z) \to 0 \text{ and } \int_{B_R^c} u_k^2 r^3 d(r,z) \to 0 \text{ as } k\to\infty. \tag{3.5}$$

Let χ denote a smooth cut-off function such that $\chi(r,z) = 1$ for $0 \le \sqrt{r^2+z^2} < R$ and $\chi(r,z) = 0$ for $\sqrt{r^2+z^2} \ge R+1$. Then $v_k := \chi u_k \in H^1_{0,\mathrm{cyl}}(B_{R+1}, r^3 drdz)$ and

$$|\nabla_{r,z}v_k|^2 = \chi^2|\nabla_{r,z}u_k|^2 + |\nabla_{r,z}\chi|^2 u_k^2 + 2u_k\chi\nabla_{r,z}u_k\cdot\nabla_{r,z}\chi.$$

Hence, by (3.5)

$$
\begin{aligned}
\int_\Omega |\nabla_{r,z}v_k|^2 r^3 d(r,z) &\le 2\int_\Omega \chi^2|\nabla_{r,z}u_k|^2 r^3 d(r,z) + 2\int_\Omega u_k^2|\nabla_{r,z}\chi|^2 r^3 d(r,z)\\
&\le 2\int_\Omega |\nabla_{r,z}u_k|^2 r^3 d(r,z) + 2\|\nabla_{r,z}\chi\|_\infty^2 \int_{B_{R+1}\setminus B_R} u_k^2 r^3 d(r,z) \to 0 \text{ as } k\to\infty.
\end{aligned}
\tag{3.6}
$$

In particular, $\int_{B_{R+1}} |\nabla_{r,z}v_k|^2 r^3 d(r,z) \to 0$ as $k\to\infty$. By Poincaré's inequality, $\|u_k\|_{L^2(r^3drdz)} = 1$ and (3.5) we see

$$C_P \int_{B_{R+1}} |\nabla_{r,z}v_k|^2 r^3 d(r,z) \ge \int_{B_{R+1}} v_k^2 r^3 d(r,z) \ge \int_{B_R} u_k^2 r^3 d(r,z) = 1 - o(1),$$

contradicting (3.6). □

Acknowledgement

We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

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